O teorema de Mader

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Este documento reproduz, em notação compatível com a de Diestel[1], uma nova demonstração do teorema de Mader (Teorema 3.4.1 em Diestel), descoberta por Alex Schrijver (*A short proof of Mader's $\mathcal{S}$-paths theorem*)[2]. O teorema de Mader é uma generalização perfeita do teorema de Menger (Teorema 3.3.1), embora Diestel não deixe isso muito claro.

3.4 Mader's theorem

Let G be a graph and $\mathcal{A}$ a disjoint collection of subsets of $V(G)$. We may sometimes refer to the members of $\mathcal{A}$ as *terminal sets*. An *$\mathcal{A}$-path* is a path whose endvertices belong to two distinct members of $\mathcal{A}$ and whose internal vertices belong to[3] $\overline{\bigcup \mathcal{A}}$.

terminal sets
$\mathcal{A}$-path

We wish to characterize maximum disjoint collections of $\mathcal{A}$-paths. This will be done in terms of splits[4], where a *split* is just a disjoint collection of subsets of $V(G)$.[5] For a set U of vertices, the *U-capacity* of a split $\mathcal{Y}$ is the number

split
capacity
cap_U

$$\mathrm{cap}_U(\mathcal{Y}) := |X| + \sum_{Y \in \mathcal{Y}} \left\lfloor \frac{|\partial_U Y|}{2} \right\rfloor ,$$

where $X := \overline{\bigcup \mathcal{Y}}$ and $\partial_U Y := (Y \cap U) \cup (N(\overline{Y \cup X}) \setminus X)$, i.e., $\partial_U Y$ is the set of vertices in Y that either belong to U or have a neighbour in $\overline{Y \cup X}$.

For a disjoint collection $\mathcal{A}$ of subsets of $V(G)$, a split $\mathcal{Y}$ is *$\mathcal{A}$-separating* if every $\mathcal{A}$-path disjoint from $\overline{\bigcup \mathcal{Y}}$ has at least one edge[6] in some member of $\mathcal{Y}$. A split $\mathcal{Y}$ is *strong* if no edge of G links two different members of $\mathcal{Y}$; every strong split is, of course, $\mathcal{A}$-separating.

$\mathcal{A}$-separating split
strong split

[1] *Graph Theory*, 2nd. edition, Springer, 1999

[2] tomei conhecimento por intermédio de José Coelho de Pina Jr., que também me ajudou a compreender a prova

[3] for any set U of vertices, $\overline{U}$ denotes $V(G) \setminus U$

[4] this term is not standard

[5] you may identify each member of the split with a different color; some vertices may, of course, remain colorless

[6] it is important to require an edge (rather than just two vertices) so that we may test, in polynomial time, whether a given split is $\mathcal{A}$-separating

Lemma 3.4.1 *For any disjoint collection $\mathcal{P}$ of $\mathcal{A}$-paths, and any $\mathcal{A}$-separating split $\mathcal{Y}$,*

$$|\mathcal{P}| \leq \text{cap}_U(\mathcal{Y}) \ , \ \textit{where } U := \bigcup \mathcal{A}.$$

Proof. Let $X := \overline{\bigcup \mathcal{Y}}$. Each $\mathcal{A}$-path has a vertex in X or at least two consecutive vertices in some member Y of $\mathcal{Y}$; in the latter case, it has at least two vertices in $\partial_U Y$. With each P in $\mathcal{P}$, we may associate *(1)* a vertex in $V(P) \cap X$ if this set is not empty or *(2)* a pair of vertices in $\partial_U Y$, where Y is an appropriate member of $\mathcal{Y}$. Each vertex in X will have been associated with at most one member of $\mathcal{P}$, since $\mathcal{P}$ is disjoint. On the other hand, in each set of the form $\partial_U Y$ a disjoint collection of pairs of vertices will have been defined; each of these pairs will have been associated with a different member of $\mathcal{P}$, since $\mathcal{P}$ is disjoint. The inequality $|\mathcal{P}| \leq \text{cap}_U(\mathcal{Y})$ follows. □

Theorem 3.4.2 (Mader 1978) [7] *There exists a disjoint collection $\mathcal{P}$ of $\mathcal{A}$-paths and an $\mathcal{A}$-separating split $\mathcal{Y}$ such that*

$$|\mathcal{P}| = \text{cap}_U(\mathcal{Y}) \ , \ \textit{where } U := \bigcup \mathcal{A}.$$

In other words, the capacity of a minimum-capacity $\mathcal{A}$-separating split is equal to the maximum number of disjoint $\mathcal{A}$-paths.

Let's examine some special cases of Theorem 3.4.2. Two of these are particularly important because they are the basis of induction in Schrijver's proof of the theorem.

- Suppose G has no $\mathcal{A}$-paths (this is the case, for example, if $|\mathcal{A}| \leq 1$ or $\| G \| = 0$). Let $\mathcal{Y}$ be the collection of all singleton subsets of $V(G)$ and observe that $\mathcal{Y}$, is an $\mathcal{A}$-separating split. The capacity of this split is 0. Hence, the assertion of Theorem 3.4.2 is true in this case.

- Suppose $|\mathcal{A}| = 2$. Let A and B be the two members of $\mathcal{A}$ and let X be a minimum set of vertices separating A from B. By Theorem 3.3.1 (Menger's (3.3.1) theorem), there are $|X|$ disjoint A—B paths. Now let $\mathcal{Y}$ be the collection of all singleton subsets of $\overline{X}$. It is easy to see that the split $\mathcal{Y}$ is $\mathcal{A}$-separating and $\text{cap}_{A \cup B}(\mathcal{Y}) = |X|$. Hence, the assertion of Theorem 3.4.2 is true in this case.

- Suppose every member of $\mathcal{A}$ has at most one vertex. In this case, Theorem 2.4.2 (2.4.2) (Gallai's theorem) asserts the existence of a disjoint collection $\mathcal{P}$ of U-paths and a strong split $\mathcal{Y}$ such that

$$|\mathcal{P}| = |\overline{\bigcup \mathcal{Y}}| + \textstyle\sum_{Y \in \mathcal{Y}} \lfloor \frac{1}{2} |Y \cap U| \rfloor \ . \tag{3.1}$$

  Since the split is strong, $Y \cap U = \partial_U Y$ for each Y in $\mathcal{Y}$. Hence, the right side of (3.1) can be written as $\text{cap}_U(\mathcal{Y})$. So, the assertion of Theorem 3.4.2 is true in this case.

[7] This statement of the theorem is somewhat different from that of Theorem 3.4.1 in Diestel (page 56). To begin with, Diestel considers *independent* rather than *disjoint* paths. But the two forms of the theorem are equivalent.

- Suppose $U = V(G)$, i.e., every vertex belongs to some member of $\mathcal{A}$. Let E be the set of edges of G that have both ends in the same member of $\mathcal{A}$. By the Theorem 2.2.3 (maximum matching theorem) there exists a matching M and a strong split $\mathcal{Y}$ in $G - E$ such that (2.2.3)

$$|M| = |\overline{\bigcup \mathcal{Y}}| + \textstyle\sum_{Y \in \mathcal{Y}} \lfloor \frac{1}{2}|Y| \rfloor \,. \tag{3.2}$$

The matching M is, essentially, a disjoint collection of $\mathcal{A}$-path, since each edge in M has ends in two different members of $\mathcal{A}$. On the other hand, every edge of G joining two different members of $\mathcal{Y}$ is in E, whence $\mathcal{Y}$ is $\mathcal{A}$-separating as a split in G. Moreover, $\partial_U Y = Y$ for each Y in $\mathcal{Y}$, whence the right side of (3.2) is equal to $\mathrm{cap}_U(\mathcal{Y})$. So, the assertion of Theorem 3.4.2 is true in this case too.

Proof of Theorem 3.4.2. Let k be the minimum of $\mathrm{cap}_U(\mathcal{Y})$ over all $\mathcal{A}$-separating splits $\mathcal{Y}$. We must prove the existence of k disjoint $\mathcal{A}$-paths in G. k

The proof is by induction. In order to describe the parameters of the induction, we need some new terminology. A *vertex-pair* is an element vw (not necessarily an edge) of $[V(G)]^2$. A vertex-pair vw is *internal* if v and w belong to the same member of $\mathcal{A}$ and *external* if at least one of v and w is in $\overline{U}$. The number of internal vertex-pairs will be denoted by $\alpha(\mathcal{A})$, while the number of external vertex-pairs will be denoted by $\beta(\mathcal{A})$. *internal vertex-pair* *external vertex-pair*

The induction is on $\alpha(\mathcal{A}) + \beta(\mathcal{A}) + \parallel G \parallel$. If $\alpha(\mathcal{A}) = 0$ then every member of $\mathcal{A}$ has at most one vertex and the theorem is true for the reasons discussed above. If (2.4.2) $\beta(\mathcal{A}) = 0$ then $U = V(G)$ or $|G| \leq 1$ and the theorem is true again for the reasons (2.2.3) discussed above. If there are no $\mathcal{A}$-paths, in particular if $\parallel G \parallel = 0$, the assertion of the theorem is true once more for the reasons discussed above. This takes care of the basis of the induction.

If some edge e has both ends in some member of $\mathcal{A}$, we may assume, as induction hypothesis, the existence in $G - e$ of a disjoint collection $\mathcal{P}$ of $\mathcal{A}$-paths and an $\mathcal{A}$-separating split $\mathcal{Y}$ such that $|\mathcal{P}| = \mathrm{cap}_U(\mathcal{Y})$. Every member of $\mathcal{P}$ is, of course, an $\mathcal{A}$-path in G. Moreover, the split $\mathcal{Y}$ is $\mathcal{A}$-separating in G, whence $\mathrm{cap}_U(\mathcal{Y}) \geq k$. So, G has k disjoint $\mathcal{A}$-paths in this case.

We assume in what follows that no edge has both ends in the same member of $\mathcal{A}$, i.e., every member of $\mathcal{A}$ is a stable set. We also assume that $\alpha(\mathcal{A})$ and $\beta(\mathcal{A})$ are not null and that there exists at least one $\mathcal{A}$-path.

1. Since $\alpha(\mathcal{A}) > 1$, some member A of $\mathcal{A}$ has at least two vertices. Let a be one of these vertices and let A, a

$$\mathcal{A}' \;:=\; (\mathcal{A} \setminus \{A\}) \;\cup\; \{A \setminus \{a\}, \{a\}\} \,.$$

Since $\alpha(\mathcal{A}') < \alpha(\mathcal{A})$ while $\beta(\mathcal{A}') = \beta(\mathcal{A})$, we may assume, as induction hypothesis, the existence of a disjoint collection $\mathcal{P}$ of $\mathcal{A}'$-paths and an $\mathcal{A}'$-separating split $\mathcal{Y}$ such that $|\mathcal{P}| = \mathrm{cap}_U(\mathcal{Y})$ (note that $\bigcup \mathcal{A}' = U$). Since every $\mathcal{A}$-path is also an $\mathcal{A}'$-path, the split $\mathcal{Y}$ is $\mathcal{A}$-separating and therefore $\mathrm{cap}_U(\mathcal{Y}) \geq k$. If every path in $\mathcal{P}$ is an $\mathcal{A}$-path, we reached our goal and the proof is finished. $\mathcal{P}$

2. Now suppose some member P_a of $\mathcal{P}$ is not an $\mathcal{A}$-path. Then a is an end of P_a, P_a the other end being in $A \setminus \{a\}$. Since A is a stable set, P_a has an internal vertex, say v; of course v is not in U. Let

$$\mathcal{B} \ := \ (\mathcal{A} \setminus \{A\}) \ \cup \ \{A \cup \{v\}\} \, .$$

Since $\mathcal{A}$ has at least two members (because there exists at least one $\mathcal{A}$-path), $\alpha(\mathcal{B}) + \beta(\mathcal{B}) < \alpha(\mathcal{A}) + \beta(\mathcal{A})$. We may assume therefore, as induction hypothesis, the existence of a disjoint collection $\mathcal{Q}$ of $\mathcal{B}$-paths and a $\mathcal{B}$-separating split $\mathcal{Z}$ such that $|\mathcal{Q}| =$ $\mathcal{Q}$ $\mathrm{cap}_W(\mathcal{Z})$, where $W = \bigcup \mathcal{B} = U \cup \{v\}$. We may, of course, choose $\mathcal{Q}$ so that W

$$\text{the set of edges used by } \mathcal{Q} \text{ but not by } \mathcal{P} \text{ is minimal.} \tag{3.3}$$

Since $\partial_W Z \supseteq \partial_U Z$ for each Z in $\mathcal{Z}$, we must have $\mathrm{cap}_W(\mathcal{Z}) \geq \mathrm{cap}_U(\mathcal{Z})$. On the other hand, the split $\mathcal{Z}$ is $\mathcal{A}$-separating, since every $\mathcal{A}$-path includes an $\mathcal{B}$-path; therefore $\mathrm{cap}_W(\mathcal{Z}) \geq k$. The proof is finished if every element of $\mathcal{Q}$ is an $\mathcal{A}$-path. The proof is also finished if $|\mathcal{Q}| > k$, since at most one of the elements of $\mathcal{Q}$ is not an $\mathcal{A}$-path.

3. Suppose now that $|\mathcal{Q}| = k$ and some member of $\mathcal{Q}$ is not an $\mathcal{A}$-path. Of course v is one of the endvertices of this path, which we call Q_v. Since v is not an endvertex Q_v of an element of $\mathcal{P}$ and $|\mathcal{P}| \geq k = |\mathcal{Q}|$, there exists a path P in $\mathcal{P}$ one of whose endvertices, say b, is not on any path in $\mathcal{Q}$. Of course P is an $\mathcal{A}$-path unless $P = P_a$. b If P intersects no path in $\mathcal{Q}$ (whence, in particular, $P \neq P_a$) then $(\mathcal{Q} \setminus \{Q_v\}) \cup \{P\}$ is a collection of the desired kind and the proof terminates.

4. Now suppose some vertex x of P belongs to some path in $\mathcal{Q}$. Choose x as x close as possible to b and let Q be the path in $\mathcal{Q}$ that uses x. Let q_1 and q_2 be the q_1, q_2 endvertices of Q. Let B be the member of $\mathcal{B}$ that contains b. We prove first that B

$$P = P_a \, . \tag{3.4}$$

The proof is by contradiction. Suppose, for a while, that $P \neq P_a$, whence P has no internal vertices in W. Let p be the endvertex of P distinct from b. Adjust notation so that $q_1 \notin B$. Then the endvertices of the path $Q' := q_1 Q x P b$ are in two different members of $\mathcal{B}$. Moreover, Q' is disjoint from every member of $\mathcal{Q} \setminus \{Q\}$, due to our choice of x. Since the collection $(\mathcal{Q} \setminus \{Q\}) \cup \{Q'\}$ must be consistent with (3.3), it must be the case that *(i)* Q' has an internal vertex in W or *(ii)* every edge of xQq_2 is used by $\mathcal{P}$. The first alternative is only possible if $x = q_2$ and therefore $x = p$. The second is only possible if every edge of xQq_2 is in xPp, whence q_2 is in xPp, and therefore $q_2 = p$. So, in any case,

$$q_2 = p \, .$$

But then $q_2 \notin B$ and so the same argument we have just used to prove that $q_2 = p$, with the rôles of q_1 and q_2 interchanged, will show that $q_1 = p$. So, our assumption that $P \neq P_a$ implies $q_1 = q_2$, which is impossible. This contradiction proves our claim (3.4). We prove next that

$$Q = Q_v \, . \tag{3.5}$$

Since $P = P_a$, we have $B = A \cup \{v\}$ and x is a vertex of vP_ab. Adjust notation so that $q_1 \notin B$. Then the endvertices of $Q' := q_1QxP_ab$ are in two different members of $\mathcal{B}$. Moreover, Q' is disjoint from every member of $\mathcal{Q} \setminus \{Q\}$, due to our choice of x. Since the collection $(\mathcal{Q} \setminus \{Q\}) \cup \{Q'\}$ must be consistent with (3.3), it must be the case that *(i)* Q' has an internal vertex in W or *(ii)* every edge of xQq_2 is used by $\mathcal{P}$. The first alternative is only possible if $x = q_2$ and therefore $x = v$, since vP_ab has no internal vertices in W. The second is only possible if every edge of xQq_2 is in xP_av, whence q_2 is in xP_av, and therefore $q_2 = v$. So, in any case,

$$q_2 = v\,.$$

Hence, $Q = Q_v$, proving the claim (3.5). Now consider the $\mathcal{A}$-path

$$P' := bP_axQ_vq_1\,.$$

It is disjoint from every member of $\mathcal{Q} \setminus \{Q\}$. Since every member of $\mathcal{Q} \setminus \{Q\}$ is also an $\mathcal{A}$-path, the collection $(\mathcal{Q} \setminus \{Q\}) \cup \{P'\}$ has k members. □

Exercises

1. Let G be a K^6 with vertices $v_1, \ldots, v_6$. Let A, A', A'' be the pairs $\{v_1, v_2\}$, $\{v_3, v_4\}$, and $\{v_5, v_6\}$ respectively. Let M be the matching $\{v_1v_2, v_3v_4, v_5v_6\}$. Find the terms of Theorem 3.4.2 for the graph $G - M$ and the terminal sets A, A', A''.

2. Show that Theorem 3.4.1 in Diestel (page 56) and Theorem 3.4.2 are equivalent.

3. Describe an algorithm for deciding whether a split $\mathcal{Y}$ is $\mathcal{A}$-separating.

4. Replace "$\mathcal{A}$-separating" by "strong" in the statement of Lemma 3.4.1. The claim remains valid?

5. Start the proof of Theorem 3.4.2 by defining k as the width of a minimum-width strong split (the width of a split $\mathcal{Y}$ is the number $|X| + \sum_{Y \in \mathcal{Y}} \lfloor \frac{1}{2} |Y \cap U| \rfloor$, where $X = \overline{\bigcup \mathcal{Y}}$ and $U = \bigcup A$). Where and why does the proof break down?

6. Suppose every member of $\mathcal{A}$ is a singleton and $\mathcal{Y}$ is a minimum-capacity $\mathcal{A}$-separating split. Can you extract from $\mathcal{Y}$ a minimum-width strong split (as in Theorem 2.4.2)?

7. Suppose $\mathcal{A} = \{A, B\}$ and $\mathcal{Y}$ is an $\mathcal{A}$-separating split that minimizes $\text{cap}_{A \cup B}(\mathcal{Y})$. Can you extract from $\mathcal{Y}$ a set of minimum cardinality that separates A from B in the manner of Menger's theorem?

8. Let's say that an $\mathcal{A}$-separator is a pair X, F where X is a set of vertices, F is a set of edges, and every $\mathcal{A}$-path has a vertex in X or an edge in F. Show

that Theorem 3.4.2 could have been stated as follows: There exists a disjoint collection $\mathcal{P}$ of $\mathcal{A}$-path and an $\mathcal{A}$-separator X, F such that

$$|\mathcal{P}| = |X| + \sum_{C} \left\lfloor \frac{|\partial C \cap A|}{2} \right\rfloor,$$

where the sum is taken over all components C of $(G - X)[F]$ and ∂C is the set of vertices in C that belong to $\bigcup \mathcal{A}$ or have a neighbour in $(G - X) - C$.

9. Generalize Theorem 3.4.2 to remove the requirement that $\mathcal{A}$ be disjoint.

10. [Disconnecting sets of vertices] A *disconnecting set* is a set of vertices that meets every $\mathcal{A}$-path. Show that there exists a disconnecting set with no more than $2k$ vertices, where k is the maximum number of disjoint $\mathcal{A}$-paths. Give an example in which no set with fewer than $2k$ vertices is disconnecting.

11. [Edge-disjoint version of Mader's theorem] For a subset A of $V(G)$, an *A-path* is any path whose ends are two distinct vertices in A. We wish to find a maximum collection of edge-disjoint A-paths.

    An *edge-split* is an assignment of colors to the edges of the graph in which each edge receives at most one color. An edge-split is *separating* if every A-path has a colorless edge or two consecutive edges of the same color. The *capacity* of an edge-split is the number $|X| + \sum_i \lfloor |\partial Z_i|/2 \rfloor$, where X is the set of colorless edges and ∂Z_i is the set of edges of color i that are incident to a vertex in A or have a vertex in common with an edge of a different color. Show that the cardinality of a maximum collection of edge-disjoint A-paths is equal to the capacity of a minimum-capacity separating split.

    (Challenge: Find a simpler definition of separating split for which the minimax holds.)

    A *disconnecting set* is a set of edges that meets every A-path. Show that there exists a disconnecting set with at most $2k$ edges, where k is the size of a maximum collection of edge-disjoint A-paths.