Graph Theory Exercises

https://www.ime.usp.br/~pf/graph-exercises/

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Contents

Preface

Graph theory studies combinatorial objects called *graphs*. These objects are a good model for many problems in mathematics, computer science, and engineering. Graph theory is not really a *theory*, but a *collection of problems*. Many of those problems have important practical applications and present intriguing intellectual challenges.

The present text is a collection of exercises in graph theory. Most exercises have been extracted from the books by Bondy and Murty [BM08, BM76], Wilson [Wil79], Diestel [Die00, Die05], Bollobás [Bol98], Lovász [Lov93], Melnikov *et alii* [MST+98], Lucchesi [Luc79] and Lovász and Plummer [LP86]. Some are byproducts of research projects. Others were born from conversations with teachers (especially my supervisor Dan Younger), colleagues and students.

The text has many hyperlinks, which take you from one part of the text to another and point to supplementary material. To take advantage of those links, you should read the text on your computer screen (rather than printed on paper).

The website *www.ime.usp.br/~pf/graph-exercises/* has an up-to-date version of the text.

Organization. Chapter 1 deals with basic concepts. Each of the other chapters deals with some classical problem. Many of the problems have a computational character: they ask for an efficient algorithm that will extract some information from a given graph. Some problems are easy, others are hard; some have already been solved, while others are still open.[1]

In what order should the chapters be read? After studying the first section of chapter 1, I guess the reader should move immediately to chapters 2 and following, returning to chapter 1 only when that becomes necessary. There is a good index to help find the definitions of technical terms and concepts.

[1] For many of these problems, a fast algorithm is not (yet?) known; the only known algorithms are not much better than patiently checking a very long list of would-be solutions. In technical terms, such a problem is NP-*complete* or NP-*hard*. See the books by Garey–Johnson [GJ79], Harel [Har92], and Sipser [Sip97].

Classification of the exercises. Most exercises have a generic prefix **E**. Some have more specific prefixes:

$\circ$ **E**	...	particularly easy drill
! E	...	hard
!! E	...	very hard
$\star$ **E**	...	important
$\star$**! E**	...	important and hard
$\star\circ$ **E**	...	important but easy
$\triangleright$ **E**	...	useful as a technical tool
E $\heartsuit$	...	particularly good
D	...	challenge, open problem

But this classification is not very reliable and was not made in a very systematic way.

Acknowledgments. I am thankful to Rogério Brito for solving various typographical issues.

IME–USP, São Paulo, January 2013
P. F.

I am thankful to Murilo Santos de Lima for the excellent translation of the text into English. (All the errors in English usage are due to my meddling with the text after Murilo finished his work.)

IME–USP, São Paulo, March 2019
P. F.

Chapter 1

Basic concepts

This chapter formalizes the notion of a graph and introduces some basic concepts, such as vertex degree, cut, subgraph, connection, component, bridge, articulation, union, intersection, complement, minor, etc. The chapter also introduces some important types of graphs, such as

paths,
circuits,
trees,
bipartite graphs,
edge-biconnected graphs,
biconnected graphs,
planar graphs,
interval graphs,
random graphs, etc.

Various examples of graphs that are found in nature are also presented. That is the case of

grids,
cubes,
Petersen's graph,
graphs for various chess pieces,
graphs of chemical compounds, etc.

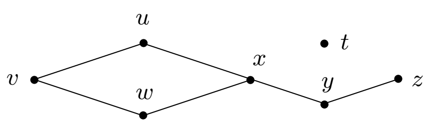
These examples will be useful in the other chapters of the text.

After studying the first section of this chapter the reader may move to the following chapters (beginning with chapter 2, which deals with isomorphism). She/he can return to chapter 1 and study the other sections when this becomes necessary. The index may be useful in this process.

1.1 Graphs

A **graph**[1] is a structure made up of two types of objects: **vertices** and **edges**. Each edge is an unordered pair of vertices, i.e., a set with exactly two vertices.[2] vw An edge like $\{v, w\}$ will simply be denoted by vw or wv; we will say that the edge is **incident** to v and to w; we will also say that v and w are the **endpoints** of the edge; we will say, moreover, that vertices v and w are **neighbors**, or **adjacent**.

> EXAMPLE: the vertices of the graph are t, u, v, w, x, y, z and the edges are vw, uv, xw, xu, xy and yz. The picture below is a graphical representation of this graph.



According to our definition, a graph cannot have two different edges with the same pair of endpoints (i.e., cannot have "parallel" edges). Moreover, the two endpoints of any edge are distinct (i.e., there are no "loops"). Some books like to emphasize this aspect of the definition by saying that the simple graph is "simple"; we will not use this adjective.

(V, E) A graph with vertex set V and edge set E is denoted by (V, E). It is often convenient to name the graph as a whole. If the name of the graph is G, its V_G, E_G vertex set is denoted by V_G and its edge set by E_G. The number of vertices $n(G), m(G)$ of G is denoted by $n(G)$ and the number of edges by $m(G)$; hence,

$$n(G) := |V_G| \quad \text{and} \quad m(G) := |E_G| \,.$$

$V^{(2)}$ The **complement** of a graph (V, E) is the graph $(V, V^{(2)} \setminus E)$, where $V^{(2)}$ is the set of all unordered pairs[3] of elements of V. The complement of G is $\overline{G}$ usually denoted by $\overline{G}$.

K_n A graph G is **complete** if $E_G = V_G^{(2)}$. The expression "G is a K_n" is an abbreviation for "G is a complete graph on n vertices." A graph G is **empty** $\overline{K_n}$ if $E_G = \emptyset$. The expression "G is a $\overline{K_n}$" is an abbreviation for "G is an empty graph on n vertices."

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[1] The term was used for the first time (in the sense that concerns us here) by James Joseph Sylvester (1814 – 1897). (See Wikipedia article.)

[2] We will assume that the vertex and edge sets of a graph are finite and mutually disjoint. We will also assume that the vertex set is not empty.

[3] Diestel [Die05] writes "$[V]^2$." Other authors write "$\binom{V}{2}$."

Exercises

E 1.1 List all graphs that have $\{a, b, c\}$ as their vertex set.[4] Organize the list so that each graph appears besides its complement.

E 1.2 Draw a figure of a K_5 and another of a $\overline{K_5}$. How many edges does a K_n have? What about a $\overline{K_n}$?

E 1.3 The **adjacency matrix** of a graph G is the matrix A defined as follows: for any two vertices u and v,

adjacency matrix

$$A[u,v] \;=\; \begin{array}{ll} 1 & \text{if } uv \in E_G\,, \\ 0 & \text{otherwise.} \end{array}$$

Clearly, the matrix is indexed by $V_G \times V_G$. (The adjacency matrix is a kind of "figure" of the graph. It has some advantages over the points-and-lines figure we used above.)

Write the adjacency matrix of the graph defined in the example that appears on page 8. Write the adjacency matrix of a K_4. What is the relationship between the adjacency matrix of a graph and the adjacency matrix of its complement?

E 1.4 The **incidence matrix** of a graph G is the matrix M defined as follows: for every vertex u and every edge e,

incidence matrix

$$M[u,e] \;=\; \begin{array}{ll} 1 & \text{if } u \text{ is one of the endpoints of } e\,, \\ 0 & \text{otherwise.} \end{array}$$

Clearly, the matrix is indexed by $V_G \times E_G$. (The incidence matrix is a kind of "figure" of the graph. It has some advantages over the points-and-lines figure we used above.)

Write the incidence matrix of the graph defined in the example that appears on page 8. Write the incidence matrix of a K_4. What is the value of the sum of all elements of the incidence matrix of a graph? What is the relationship between the incidence matrix of a graph and the incidence matrix of its complement?

E 1.5 The hydrocarbons known as alkanes have chemical formula C_pH_{2p+2}, where C and H represent atoms of carbon and hydrogen, respectively. Alkane molecules can be represented by graphs such as those in figure 1.1.

alkanes

Draw a figure of a methane C_1H_4 molecule. How many "different" C_3H_8 molecules are there?

[4] In a set, the order in which the elements are presented is irrelevant. Thus, $\{a, b, c\} = \{b, c, a\} = \{c, b, a\}$.

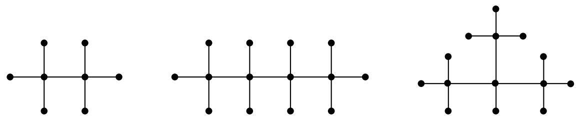

Figure 1.1: Ethane (C_2H_6), butane (C_4H_{10}) and isobutane (C_4H_{10}). The vertices with only one incident edge represent hydrogen (H) atoms; the others represent carbon (C) atoms. (See exercise 1.5.)

E 1.6 Let V be the Cartesian product $\{1, 2, \ldots, p\} \times \{1, 2, \ldots, q\}$, i.e., the set of all ordered pairs[5] (i, j) with i in $\{1, \ldots, p\}$ and j in $\{1, \ldots, q\}$. Let us say that two elements (i, j) and (i', j') of V are adjacent if

$$i = i' \text{ and } |j - j'| = 1 \qquad \text{or} \qquad j = j' \text{ and } |i - i'| = 1\,.$$

grid This adjacency relation defines a graph over the set V of vertices. The graph is known as the p**-by-**q **grid**.

How many edges does the p-by-q grid have? Write the adjacency and incidence matrices of a 4-by-5 grid.

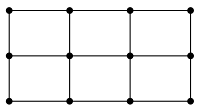

Figure 1.2: A 3-by-4 grid (see exercise 1.6).

E 1.7 Given integer numbers p and q, let V be the set $\{1, 2, 3, \ldots, pq-2, pq-1, pq\}$. Let us say that two elements k and k' of V, with $k < k'$, are adjacent if $k' = k + q$ or[6]

$$k \bmod q \neq 0 \ \text{ and } \ k' = k + 1.$$

This adjacency relation defines a graph with vertex set V.

Draw a figure of the graph with parameters $p = 3$ and $q = 4$. Draw a figure of the graph with parameters $p = 4$ and $q = 3$. What is the relationship between these graphs and the grid defined in exercise 1.6?

queen **E 1.8** The graph **of the queen's moves**, or simply the **queen** graph, is defined like this: the vertices of the graph are the squares of a chess board with t rows and t columns (the usual board has $t = 8$) and two vertices are adjacent if a queen in the game of chess can jump from one of them to the other in a single

[5] An ordered pair is a sequence of length 2. In a sequence, the *order* of the elements is essential. Thus, $(1, 2) \neq (2, 1)$ and $(1, 2, 1) \neq (1, 1, 2)$.

[6] The expression "$k \bmod q$" denotes the remainder of the division of k by q, i.e., $k/q - \lfloor k/q \rfloor$.

move. To make explicit the number of rows and columns, we may say that this is t**-by-**t queen graph. (See figure 1.3.)

Draw a figure of the 4-by-4 queen graph. Write the adjacency and incidence matrices of the 4-by-4 queen graph. How many edges does the 8-by-8 queen graph have? How many edges does the t-by-t queen graph have?

E 1.9 The t-by-t **knight** graph is defined like this: the vertices of the graph are the squares of a chess board with t rows and t columns; two vertices are adjacent if a knight in the game of chess can jump from one of them to the other in a single move. (See figure 1.3.) knight

Draw a figure of the 3-by-3 knight. Write the adjacency and incidence matrices of the 3-by-3 knight graph. How many edges does the 8-by-8 knight graph have? How many edges does the t-by-t knight graph have?

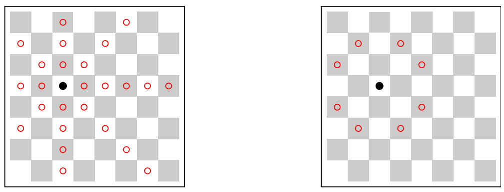

Figure 1.3: 8-by-8 chess boards. The figure on the left indicates all the neighbors of vertex • in the queen graph (see exercise 1.8). The one on the right indicates all the neighbors of vertex • in the knight graph (see exercise 1.9).

E 1.10 The t-by-t **bishop** graph is defined as follows: the vertices of the graph are the squares of a chess board with t rows and t columns; two vertices are adjacent if a bishop in the game of chess can jump from one of them to the other in a single move. bishop

Draw a figure of the 4-by-4 bishop graph. Write the adjacency and incidence matrices of the 4-by-4 bishop graph. How many edges does the 8-by-8 bishop graph have? How many edges does the t-by-t bishop graph have?

E 1.11 The t-by-t **rook** graph is defined as follows: the vertices of the graph are the squares of a chess board with t rows and t columns; two vertices are adjacent if a rook in the game of chess can jump from one of them to the other in a single move. rook

Draw a figure of the 4-by-4 rook graph. Write the adjacency and incidence matrices of the 4-by-4 rook graph. How many edges does the 8-by-8 rook graph have? How many edges does the t-by-t rook graph have?

E 1.12 The t-by-t **king** graph is defined like this: the vertices of the graph are the squares of a chess board with t rows and t columns; two vertices are adjacent if a king in the game of chess can jump from one of them to the other in a single move. king

Draw a figure of the 4-by-4 king graph. Write the adjacency and incidence matrices of the 4-by-4 king graph. How many edges does the 8-by-8 king graph have? How many edges does the t-by-t king graph have?

words **E 1.13** The **words** graph is defined like this: each vertex is a word in the English language and two words are adjacent if they differ in exactly one position. (This graph is adapted from `ladders` in *Stanford GraphBase* [Knu93].) For example, `cords` and `corps` are adjacent, while `corps` and `crops` are not. Draw a figure of the part of the graph defined by the following words:

```
words cords corps coops crops drops drips grips gripe
grape graph
```

Write the adjacency and incidence matrices of this graph.

cube **E 1.14** For any positive integer k, a k-dimensional **cube** (or k-**cube**) is the graph defined as follows: the vertices of the graph are all the sequences[7] $b_1 b_2 \cdots b_k$ of bits[8]; two vertices are adjacent if and only if they differ in exactly one position. For example, the vertices of the 3-dimensional cube are 000, 001, 010, 011, 100, 101, 110, 111; vertex 000 is adjacent to vertices 001, 010, 100 and to no other; and so on. The k-dimensional cube will be denoted by Q_k. Q_k

Draw figures of the cubes Q_1, Q_2 and Q_3. Write the adjacency and incidence matrices of Q_3. How many vertices does Q_k have? How many edges?

E 1.15 Let X be the set $\{1, 2, 3, 4, 5\}$ and V the set $X^{(2)}$ (therefore, V is the set of all subsets of X that have exactly 2 elements). Let us say that two elements v and w of V are adjacent if $v \cap w = \emptyset$. This adjacency relation on V defines the **Petersen** graph.[9] Draw a figure of the graph. Write the adjacency and incidence matrices of the graph. How many vertices and how many edges does the graph have? Petersen

E 1.16 Let V be the set of all subsets of $\{1, 2, \ldots, n\}$ that have exactly k elements, with $k \leq n/2$. Let us say that two elements v and w of V are adjacent if $v \cap w = \emptyset$. This adjacency relation on V defines the **Kneser** graph $K(n, k)$.[10] Kneser

[7] The expression "$b_1 b_2 \cdots b_k$" is an abbreviation for "$(b_1, b_2, \ldots, b_k)$."
[8] Therefore, each b_i belongs to the set $\{0, 1\}$.
[9] Reference to Julius Petersen (Denmark, 1839 – 1910). (See article in Wikipedia.)
[10] Lásló Lovász used this graph in 1978 to prove a conjecture proposed by M. Kneser in 1955.

In particular, $K(5,2)$ is the Petersen graph. Draw figures for $K(n,1)$, $K(n,n)$, $K(n,n{-}1)$, $K(4,2)$, $K(5,3)$, $K(6,2)$ and $K(6,3)$.

E 1.17 The graph of Europe is defined like this: each vertex is one of the countries of Europe; two countries are adjacent if they have a common border. Draw a figure of the graph. How many vertices does the graph have? How many edges? countries

E 1.18 Consider the large cities and the major roads in a country. Let us say that a city is *large* if it has at least 300 thousand inhabitants. Let us say the a road is a *major road* if it is a divided highway. Let us say that two large cities are adjacent if a major road or a concatenation of major roads connects those two cities directly (i.e., without passing through a third large city). Draw a figure of the graph of the large cities defined by the adjacency relation we have just described. cities

E 1.19 Let V be a set of points in the plane. Let us say that two of these points are adjacent if the distance between them is smaller than 2. This adjacency relation defines the graph of **poins in the plane** (on the set V). Draw a figure of the graph defined by the following points. points in the plane

$$\begin{array}{ccc} (0,2) & (1,2) & (2,2) \\ (0,1) & (1,1) & (2,1) \\ (0,0) & (1,0) & (2,0) \end{array}$$

Write the adjacency and incidence matrices of the graph.

E 1.20 Given a set V, let E be the set defined in the following manner: for each unordered pair of elements of V, flip a coin; if the result is heads, add the pair to E. The graph (V,E) so defined is **random**. random

Take your favorite coin and draw a figure of a random graph with vertices $1,\ldots,6$. Now repeat the exercise with a biased coin which produces heads with probability $1/3$ and tails with probability $2/3$.

E 1.21 Let S be a square matrix of integer numbers. Suppose that the rows of S are indexed by a set V and that the columns are indexed by the same set V. The graph of matrix S is defined as follows: the set of vertices of the graph is V and two vertices i and j are adjacent if $S[i,j] \neq 0$.

Is the graph of S well-defined? Which conditions must be imposed on the matrix to make the graph well-defined?

E 1.22 Consider k intervals of finite length, I_1, I_2, …, I_k, on the real line. Let us say that two intervals I_i and I_j are adjacent if $I_i \cap I_j \neq \emptyset$. This adjacency relation defines a graph with vertex set $\{I_1, I_2, \ldots, I_k\}$. This is an **interval** interval graph.

Draw a figure of the graph defined by the intervals $[0,2]$, $[1,4]$, $[3,6]$, $[5,6]$ and $[1,6]$. Write the adjacency and incidence matrices of the graph.

E 1.23 Let $\preceq$ be a partial order relation on a finite set V. Therefore, the relation is transitive (if $x \preceq y$ and $y \preceq z$, then $x \preceq z$), antisymmetric (if $x \preceq y$ and $y \preceq x$, then $x = y$) and reflexive ($x \preceq x$ for every x). Let us say that two distinct elements x and y of V are adjacent if they are comparable, i.e., if
compa-rability $x \preceq y$ or $y \preceq x$. This adjacency relation defines the **comparability** graph of the relation $\preceq$.

Draw a figure of the comparability graph of the usual inclusion relation $\subseteq$ over the subsets of $\{1,2,3\}$.

E 1.24 Two edges of a graph G are **adjacent** if they have a common endpoint.
line This adjacency relation defines the line graph of G. More formally, the **line graph** of a graph G is the graph (E_G, A) in which A is the set of all pairs
$L(G)$ of adjacent edges of G. The line graph of G will be denoted by $L(G)$. (See figure 1.4.)

Draw a figure of $L(K_3)$. Draw a figure of $L(K_4)$. Write the adjacency and incidence matrices of $L(K_4)$. How many vertices and edges does $L(K_n)$ have? Draw a figure of the graph $L(P)$, where P is the Petersen graph (see exercise 1.15).

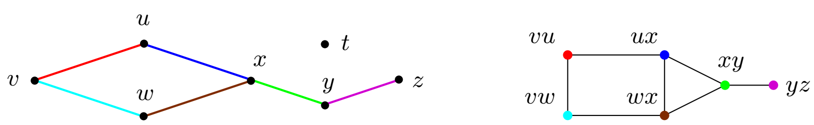

Figure 1.4: A graph (left) and its line graph (right).

1.2 Bipartite graphs

A graph G is **bipartite** if there is a bipartition[11] $\{U, W\}$ of V_G such that every edge of G has one endpoint in U and the other in W. To make the partition explicit, we can say that the graph is $\{U, W\}$-bipartite.

If G is a $\{U, W\}$-bipartite graph, we can say, informally, that the elements of U are the **white vertices** and that those of W are the **black vertices** of the graph.

A $\{U, W\}$-bipartite graph is **complete** if every white vertex is adjacent to every black vertex. A $K_{p,q}$ is a complete bipartite graph with p white and q black vertices. $K_{p,q}$

A **star** is a $K_{1,q}$. If $q \geq 2$, the **center** of the star is the only vertex which is incident to two or more edges. (If $q < 2$, the star has no center.) star

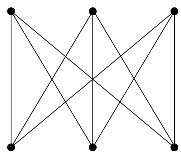

Figure 1.5: A complete bipartite graph.

Exercises

○ **E 1.25** A small factory has five machines — 1, 2, 3, 4 and 5 — and six workers — A, B, C, D, E and F. The table specifies which machines each worker is allowed to operate:

A	$2, 3$	B	$1, 2, 3, 4, 5$
C	3	D	
E	$2, 4, 5$	F	$2, 5$

Draw a figure of the bipartite graph which represents the relationship between workers and machines.

○ **E 1.26** How many edges can a $\{U, W\}$-bipartite graph have?

[11] A **bipartition** of a set V if a pair $\{U, W\}$ of non-empty sets such that $U \cup W = V$ and $U \cap W = \emptyset$. More generally, a **partition** of a set V is a collection $\{X_1, X_2, \ldots, X_k\}$ of pairwise disjoint (i.e., $X_i \cap X_j = \emptyset$ for every $i \neq j$) non-empty sets whose union is V (i.e., $X_1 \cup X_2 \cup \cdots \cup X_k = V$). It makes no sense to say "X_1 is *one of the partitions* of V"; this is equivalent to mistake *herd* for *cow* or *choir* for *singer*. You should say "X_1 is one of the elements of the partition" or "X_1 is one of the parts of the partition."

∘ **E 1.27** How many edges does a $K_{p,q}$ have? How many edges does a $\overline{K_{p,q}}$ have?

E 1.28 Draw a figure of a $K_{3,4}$. Write the adjacency and incidence matrices of a $K_{3,4}$. Draw a figure of a star with 6 vertices.

E 1.29 Is it true that the t-by-t knight graph is bipartite?

E 1.30 What is the shape of the adjacency matrix of a bipartite graph?

E 1.31 The **bipartition matrix** of a $\{U, W\}$-bipartite graph is defined like this: each row of the matrix is an element of U, each column of the matrix is an element of W, and at the intersection of line u and column w we have a 1 if uw is an edge, and a 0 otherwise.

Write the bipartition matrix of the graph from exercise 1.25. Adopt the obvious bipartition: $U = \{A, \ldots, F\}$ and $W = \{1, \ldots, 5\}$.

1.3 Neighborhoods and vertex degrees

The **neighborhood of a vertex** v in a graph G is the set of all neighbors of v. This set will be denoted by

$$\mathrm{N}_G(v)$$

or simply by $\mathrm{N}(v)$.[12] The **degree** of a vertex v in a graph G is the number of edges that are incident to v. The degree of v will be denoted by $\mathrm{N}(v)$

$$\mathrm{d}_G(v)$$

or simply $\mathrm{d}(v)$. Clearly, $\mathrm{d}(v) = |\mathrm{N}(v)|$ for every vertex v. A vertex v is **isolated** if $\mathrm{d}(v) = 0$. $\mathrm{d}(v)$

The **minimum degree** and the **maximum degree** of the vertices of a graph[13] G are the numbers $\delta(G)$ $\Delta(G)$

$$\delta(G) := \min_{v \in V_G} \mathrm{d}_G(v) \quad \text{and} \quad \Delta(G) := \max_{v \in V_G} \mathrm{d}_G(v)$$

respectively. The average of the degrees of G, i.e., $\frac{1}{|V|}\sum_{v\in V} \mathrm{d}(v)$, will be denoted by $\mu(G)$.[14] As we will see in exercise 1.43, $\mu(G)$

$$\mu(G) = 2m(G)/n(G)\ .$$

A graph is **regular** if all its vertices have the same degree, i.e., if $\delta = \Delta$. A graph is r-**regular** if $\mathrm{d}(v) = r$ for every vertex v. A **cubic** graph is the same as a 3-regular graph.

Exercises

∘ **E 1.32** What are the degrees of the vertices of a star (see section 1.2)?

∘ **E 1.33** If G is a K_n, what are the values of $\delta(G)$ and $\Delta(G)$? What are the values of the parameters δ and Δ for a $K_{p,q}$ (see section 1.2)?

∘ **E 1.34** For $r = 1, 2, 3$, draw a figure of an r-regular graph with 12 vertices.

E 1.35 What are the degrees of the vertices of an alcane molecule (see exercise 1.5)?

E 1.36 Calculate the values of the parameters δ, Δ and μ for the k-cube (see exercise 1.14) and for the Petersen graph (see exercise 1.15 or figure 1.6).

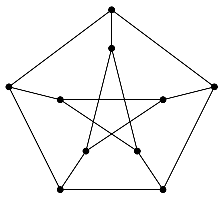

Figure 1.6: Petersen graph. See exercises 1.15 and 1.36.

E 1.37 Calculate the values of the parameters δ and Δ for the graph of European countries (see exercise 1.17).

E 1.38 Calculate the values of the parameters δ, Δ and μ for the queen graph (see exercise 1.8) and for the knight graph (see exercise 1.9).

E 1.39 Let A be the adjacency matrix (see exercise 1.3) and M be the incidence matrix (see exercise 1.4) of a graph G. What is the sum of the elements in row v of A? What is the sum of the elements in row v of M?

▷ **E 1.40** Let G be a $\{U, W\}$-bipartite graph. Suppose G is r-regular, with $r > 0$. Show that $|U| = |W|$.

E 1.41 Is it true that every graph with at least two vertices has two vertices with the same number of neighbors? In other words, if a graph has more than one vertex, is it true that it has two distinct vertices v and w such that $|\mathrm{N}(v)| = |\mathrm{N}(w)|$? (An informal way to say this: is it true that every town with at least two inhabitants has two people with exactly the same number of friends in the town?)

⋆ **E 1.42** Show[15] that, in every graph, the sum of the degrees of the vertices is equal to twice the number of edges. I.e., every graph (V, E) satisfies the identity

$$\textstyle\sum_{v \in V} \mathrm{d}(v) \;=\; 2|E|\,. \tag{1.1}$$

∘ **E 1.43** Show that $\mu(G) = 2m(G)/n(G)$ for every graph G.

[12] Some authors write "$Adj(v)$" instead of "$\mathrm{N}(v)$." Others write "$\Gamma(v)$."

[13] The expression "minimum degree of a graph" is not very grammatical, since "degree of a graph" makes no sense.

[14] Unlike δ and Δ, the notation μ is not universal. Diestel [Die05] and Bondy and Murty [BM08], for example, write d instead of my μ.

[15] Show = prove.

∘ **E 1.44** Show that every graph G has a vertex v such that $\mathrm{d}(v) \le 2m(G)/n(G)$, and a vertex w such that $\mathrm{d}(w) \ge 2m(G)/n(G)$. Is it true that every graph G has a vertex x such that $\mathrm{d}(x) < 2m(G)/n(G)$?

E 1.45 Show that every graph has $\delta \le 2m/n \le \Delta$.

E 1.46 Show that every graph with n vertices has at most $n(n-1)/2$ edges.

▷ **E 1.47** Show that, in any graph, the number of vertices with odd degree is necessarily even.

E 1.48 How many edges does the 8-by-8 queen graph have (see exercise 1.8)? How many edges does the t-by-t queen graph have?

E 1.49 How many edges does the 4-by-4 knight graph have (see exercise 1.9)? How many edges does the t-by-t knight graph have?

E 1.50 How many edges does a r-regular graph with n vertices have?

E 1.51 How many edges does the k-dimensional cube have?

E 1.52 How many edges does the line graph (see exercise 1.24) of a graph G have?

E 1.53 Let $\overline{G}$ be the complement of graph G. Calculate $\delta(\overline{G})$ and $\Delta(\overline{G})$ as a function of $\delta(G)$ and $\Delta(G)$.

E 1.54 Let G be a graph such that $m(G) > n(G)$. Show that $\Delta(G) \ge 3$.

E 1.55 Suppose that a graph G has less edges than vertices, i.e., that $m(G) < n(G)$. Show that G has (at least) one vertex of degree 0 or (at least) two vertices of degree 1.

! E 1.56 Pick two natural numbers n and k and consider the following game for two players, A and B. Each iteration of the game begins with a graph G on n vertices. At the beginning of the first iteration, we have that E_G is empty. In every odd iteration (first, third, etc.), player A picks two non-adjacent vertices u and v and adds uv to the edge set of the graph. In every even iteration (second, fourth, etc.), player B does an analogous move: picks two non-adjacent vertices u and v and adds uv to the edge set of the graph. The first player to yield a graph G such that $\delta(G) \ge k$ loses the game. Problem: determine a winning strategy for A and a winning strategy for B.

1.4 Paths and circuits

This section introduces two simple but very important types of graphs: paths and circuits.[16]

A graph G is a **path** if V_G admits a permutation[17] $(v_1, v_2, \ldots, v_n)$ such that

$$E_G = \{v_i v_{i+1} : 1 \leq i < n\} .$$

The vertices v_1 and v_n are the **endpoints** of the path; the other vertices are **internal**.[18] We will say that this path **connects** v_1 **to** v_n.

$v_1 v_2 \cdots v_n$ The path we have just described can be denoted simply by $v_1 v_2 \cdots v_n$. For example, if we say "the path $xywz$", we are referring to the graph whose vertices are x, y, w, z and whose edges are xy, yw and wz.

A graph G is a **circuit** (= polygon)[19] if V_G has 3 or more elements and admits a permutation $(v_1, v_2, \ldots, v_n)$ such that

$$E_G = \{v_i v_{i+1} : 1 \leq i < n\} \cup \{v_n v_1\} .$$

$v_1 v_2 \cdots v_n v_1$ This circuit can be denoted simply by $v_1 v_2 \cdots v_n v_1$. Thus, if we say "the circuit $xywzx$", we are referring to the graph whose vertices are x, y, w, z and whose edges are xy, yw, wz and zx.

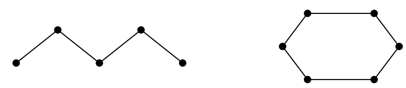

Figure 1.7: A path and a circuit.

The **length** of a path[20] or circuit G is the number $m(G)$. Of course, a path of length m has $m + 1$ vertices and a circuit of length m has m vertices.

A **triangle**, **square**, **pentagon** and **hexagon** is the same as a circuit of length 3, 4, 5 and 6 respectively.

Exercises

○ **E 1.57** Draw a figure of a path of length 0, of a path of length 1 and of a path of length 2. Draw a figure of a circuit of length 3 and of a circuit of length 4. Why does the definition of a circuit require "$n \geq 3$"?

[16] It should be stressed that, for us, paths and circuits are graphs. In some books, paths and circuits are treated as sequences of vertices and edges rather than as graphs.

[17] A *permutation* of a set X is a sequence in which each element of X appears once and only once.

[18] Some authors [Per09] say that a path is only a path if it has 2 or more vertices. For us, however, the graph $(\{v\}, \emptyset)$ is a path. This detail is not as irrelevant as it may seem.

[19] Some authors say "cycle" instead of "circuit."

[20] The expression "*size* of a path" is ambiguous: one cannot tell if we are talking about the number of vertices or the number of edges in the path.

∘ **E 1.58** Let V be the set $\{a, b, c, d, e\}$ and E be the set $\{de, bc, ca, be\}$. Verify that the graph (V, E) is a path. Now suppose that F is the set $\{bc, bd, ea, ed, ac\}$ and verify that the graph (V, F) is a circuit.

∘ **E 1.59** Draw a figure of the path $1\,2\,4\,3\,5$. Draw a figure of the path $1\,3\,2\,4\,3\,5$. Draw a figure of the circuit $1\,2\,4\,3\,5\,1$.

∘ **E 1.60** Verify that the path $u\,v\,w\,x\,y\,z$ can also be denoted by $z\,y\,x\,w\,v\,u$. Verify that those two expressions represent *the same* path.

∘ **E 1.61** Consider the circuit $u\,v\,w\,x\,y\,z\,u$. Show that $z\,y\,x\,w\,v\,u\,z$ is also a circuit. Show that any cyclic permutation — such as $w\,x\,y\,z\,u\,v\,w$, for example — is also a circuit. Show that all those expressions represent *the same* circuit.

∘ **E 1.62** Give the adjacency and incidence matrices of a path of length 4. Give the adjacency and incidence matrices of a circuit of length 5.

∘ **E 1.63** Is it true that the 3-by-3 knight graph is a circuit?

∘ **E 1.64** Verify that the 1-by-n grid is a path of length $n - 1$. Which grids are circuits?

∘ **E 1.65** Suppose that P is a path of length $n-1$ and O is a circuit of length n. What is the value of $\delta(P)$, $\Delta(P)$, $\delta(O)$ and $\Delta(O)$?

∘ **E 1.66** Draw a figure of the complement of a path of length 3. Draw a figure of the complement of a path of length 4. Draw a figure of the complement of a circuit of length 5. Draw a figure of the complement of a circuit of length 6.

E 1.67 How many different paths with vertex set $\{1, 2, 3\}$ are there? How many different circuits with vertex set $\{1, 2, 3\}$ are there? How many different circuits with vertex set $\{1, 2, 3, 4\}$ are there?

E 1.68 Is it true that every 2-regular graph is a circuit?

E 1.69 Let G be a graph with $n(G) \geq 3$, $\Delta(G) = 2$ and $\delta(G) = 1$. If G has exactly two vertices of degree 1, is it true that G is a path?

1.5 Union and intersection of graphs

The **union** of two graphs G and H is the graph $(V_G \cup V_H, E_G \cup E_H)$. It is natural to denote this graph by $G \cup H$.

$G \cup H$

The **intersection** of two graphs G and H is the graph $(V_G \cap V_H, E_G \cap E_H)$. It is natural to denote this graph by $G \cap H$. We will only consider the intersection $G \cap H$ if $V_G \cap V_H$ is not empty.

$G \cap H$

Two graphs G and H are **disjoint** if the sets V_G and V_H are disjoint, i.e., if $V_G \cap V_H = \emptyset$. It is evident that E_G and E_H are also disjoint in that case.

Exercises

◦ **E 1.70** Let G be a complete graph with vertex set $\{1, 2, 3, 4, 5\}$ and H be a complete graph with vertex set $\{4, 5, 6, 7, 8\}$. Draw figures of the graphs $G \cup H$ and $G \cap H$.

E 1.71 Let G be the bishop graph and H be the rook graph (see exercises 1.10 and 1.11). Show that $G \cup H$ is the queen graph.

◦ **E 1.72** Let G be the circuit $1\,2\,3\,4\,5\,6\,1$ and H be the path $4\,7\,8\,5$. Draw figures of the graphs $G \cup H$ and $G \cap H$.

E 1.73 Let P be a set with endpoints u and v and Q be a path with endpoints v and w. Show that if $V_P \cap V_Q = \{v\}$ then the graph $P \cup Q$ is a path.

E 1.74 Suppose that paths P and Q have the same endpoints, say u and v. Suppose also that $V_P \cap V_Q = \{u, v\}$. Under which conditions is the graph $P \cup Q$ a circuit?

E 1.75 Let A, B and C be the sets $\{1, 2, 3, 4\}$, $\{5, 6, 7\}$ and $\{9, 10, 11\}$. Let G be the complete $\{A, B\}$-bipartite graph. Let H be the complete $\{B, C\}$-bipartite graph. Draw figures of the graphs $G \cup H$ and $G \cap H$.

E 1.76 A **wheel** is any graph in the form $G \cup H$, where G is a circuit and H is a star (see section 1.2) with center v such that $V_H \setminus \{v\} = V_G$. Draw figures of the wheels with 4, 5 and 6 vertices. What is the value of the parameters m, δ and Δ for a wheel with n vertices?

1.6 Planar graphs

A graph is **planar** if it can be drawn in the plane in such a way that the lines representing edges do not cross. This definition is not precise, but it is sufficient for now. We will give a better definition in section 1.17.

Exercises

∘ **E 1.77** Check that every path is planar. Check that every circuit is planar.

∘ **E 1.78** Show that every grid (see exercise 1.6) is planar.

E 1.79 Show that the graph of European countries (see exercise 1.17) is planar.

E 1.80 Is the graph of points in the plane described in exercise 1.19 planar?

E 1.81 Show that every K_4 is planar. Is it true that every K_5 is planar?

E 1.82 Show that every $K_{2,3}$ is planar. Is it true that every $K_{3,3}$ is planar?

E 1.83 Show that the graph Q_3 (see exercise 1.14) is planar. Is the graph Q_4 planar as well? Is the graph Q_5 planar?

E 1.84 Is the t-by-t bishop graph (see exercise 1.10) planar?

E 1.85 Is the t-by-t queen graph (see exercise 1.8) planar? Is the t-by-t knight graph (see exercise 1.9) planar?

E 1.86 Show that the complement of a circuit of length 6 is planar.

1.7 Subgraphs

A **subgraph** of a graph G is any graph H such that $V_H \subseteq V_G$ and $E_H \subseteq E_G$. It
$H \subseteq G$ is natural to write "$H \subseteq G$" to say that H is a subgraph of G.

A subgraph H of G is **spanning** if $V_H = V_G$. A subgraph H of G is **proper**
$H \subset G$ if $V_H \neq V_G$ or $E_H \neq E_G$. Sometimes it is convenient to write "$H \subset G$" to say that H is a proper subgraph of G.[21]

The subgraph of G **induced** by a subset X of V_G is the graph (X, F),
$G[X]$ where F is the set $E_G \cap X^{(2)}$. This subgraph is denoted by

$$G[X]\,.$$

$G - X$ For any subset X of V_G, we will denote by $G - X$ the subgraph $G[V_G \smallsetminus X]$. In particular, for any vertex v,

$$G - v$$

is an abbreviation for $G - \{v\}$. For any edge e of G,

$$G - e$$

$G - A$ is the graph $(V_G, E_G \smallsetminus \{e\})$. More generally, if A is a subset of E_G, then $G - A$ is the graph $(V_G, E_G \smallsetminus A)$. Of course $G - A$ is a spanning subgraph of G.

Exercises

◦ **E 1.87** Let H be a subgraph of G. If $V_H = V_G$, is it true that $H = G$? If $E_H = E_G$, is it true that $H = G$?

◦ **E 1.88** Let G be a graph, V' be a subset of V_G and E' be a subset of E_G. Is it true that (V', E') is a subgraph of G?

E 1.89 Repeat exercise 1.42: *Use induction*[22] *on the number of edges of the graph to prove that every graph* (V, E) *satisfies the identity*

$$\textstyle\sum_{v \in V} \mathrm{d}(v) \;=\; 2|E|\,.$$

◦ **E 1.90** Let v be a vertex and e be an edge of a circuit O. Show that the graph $O - v$ is a path. Show that the graph $O - e$ is a path.

E 1.91 Show that every subgraph of a planar graph is planar. In other words, show that if G has a non-planar subgraph then G is not planar.

---

[21] We write "$X \subset Y$" or "$Y \supset X$" to say that the set X is a proper subset of Y, i.e., that $X \subseteq Y$ but $X \neq Y$.

[22] Induction is the art of reducing a problem to a smaller version of itself.

E 1.92 Let v and w be two vertices of a graph G. Suppose that $\mathrm{d}(v) = \delta(G)$ and $\mathrm{d}(w) = \Delta(G)$. Is it true that $\delta(G-v) = \delta(G)-1$? Is it true that $\Delta(G-w) = \Delta(G) - 1$?

∘ **E 1.93** Verify that the t-by-t bishop graph is a subgraph of the t-by-t queen graph. Verify that the t-by-t rook graph is a subgraph of the t-by-t queen graph.

E 1.94 Is the graph Q_3 a subgraph of Q_4?

∘ **E 1.95** Let G be a $\{U, W\}$-bipartite graph. Show that the induced subgraphs $G[U]$ and $G[W]$ are empty.

∘ **E 1.96** Show that every induced subgraph of a complete graph is complete. Is it true that every induced subgraph of a path is a path? Is it true that every induced subgraph of a circuit is a circuit?

∘ **E 1.97** Let G be the graph represented in figure 1.8 and X be the set $\{a, b, f, e, g, l\}$. Draw a figure of $G[X]$.

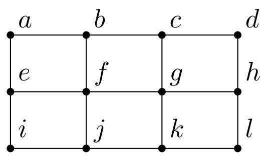

Figure 1.8: See exercises 1.97, 1.116 and 1.117.

E 1.98 ♡ Let H be the line graph (see exercise 1.24) of a graph G (whence $H = L(G)$). Show that H does not contain $K_{1,3}$ as an induced subgraph, i.e., show that there is no subset X of V_H such that $H[X]$ is a $K_{1,3}$. Show that the converse is not true.

E 1.99 Let H be the line graph (see exercise 1.24) of a graph G (whence $H = L(G)$). Let H' be an induced subgraph of H. Show that H' is the line graph of some graph G'.

E 1.100 Given a graph G and an integer k, find a maximum subset X of V_G such that $\delta(G[X]) \geq k$. (I.e., among all subsets X of V_G that have the property $\delta(G[X]) \geq k$, find one with maximum cardinality.)

E 1.101 Let G be a graph such that $n(G) > 1$ and $\delta(G) \leq \frac{1}{2}\mu(G)$. Show that G has a vertex x such that

$$\mu(G - x) \geq \mu(G)\ .$$

In other words, show that it is possible to remove a vertex without reducing the average of the degrees in the graph.

E 1.102 Show that every graph G with at least one edge has a subgraph H such that

$$\delta(H) > \mu(H)/2 \quad \text{but} \quad \mu(H) \geq \mu(G)\ .$$

1.8 Cuts

Let X be a set of vertices of a graph G. The **cut** induced by X (or the **fringe** of X) is the set of all the edges with an endpoint in X and another in $V_G \setminus X$. The cut induced by X will be denoted by $\partial(X)$

$$\partial_G(X)$$

or simply by $\partial(X)$.[23] (Some authors prefer to write, $\delta(X)$ or even $\nabla(X)$.)

We say that the cuts $\partial(\emptyset)$ and $\partial(V_G)$ are **trivial**. Trivial cuts are, of course, empty.

Clearly, $|\partial(\{v\})| = \mathrm{d}(v)$ for every vertex v. For every set X of vertices, we will say that $|\partial(X)|$ is the **degree** of X, and we will denote this number by $\mathrm{d}(X)$: $\mathrm{d}(X)$

$$\mathrm{d}(X) := |\partial(X)| \,.$$

A **cut** or **coboundary** in a graph G is any set in the form $\partial(X)$, where X is a subset of V_G. (A cut is, therefore, a set of edges rather than a set of vertices.)

Exercises

◦ **E 1.103** Let X be a set of vertices of a graph G. Show that $(V_G, \partial(X))$ is a bipartite spanning subgraph of G.

E 1.104 Let G be the graph represented in figure 1.8. Is it true that the set $\{ae, ef, fj, jk, cd, dh\}$ is a cut?

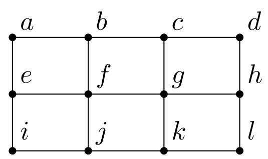

Figure 1.9: See exercise 1.104.

E 1.105 Find the smallest non-trivial cut you can in the 8-by-8 queen graph. Find the largest non-trivial cut you can in the queen graph.

E 1.106 Find the smallest non-trivial cut you can in the t-by-t bishop graph.

[23] Do not confuse ∂ with the Greek letter δ.

E 1.107 Find the smallest cut you can in the Petersen graph. Find the largest cut you can in the Petersen graph.

$\mathrm{N}(X)$ ∘ **E 1.108** For any set X of vertices, we denote by $\mathrm{N}(X)$, the set of vertices in $V_G \setminus X$ that have one or more neighbors in X. Is it true that $\mathrm{d}(X) = |\mathrm{N}(X)|$ for every X?

It is clear that $\mathrm{N}(X) \subseteq \bigcup_{x \in X} \mathrm{N}(x)$.[24] Is it true that both sets are equal?

⋆ **E 1.109** *Show that, for any graph G and any subset X of V_G, we have that*

$$\textstyle\sum_{x \in X} \mathrm{d}_G(x) = 2m(G[X]) + \mathrm{d}_G(X)\,. \tag{1.2}$$

(This is a generalization of exercise 1.42.)

E 1.110 Suppose that all the vertices of a graph G have even degree. Is it true that $\mathrm{d}(X)$ is even for every subset X of V_G?

Suppose that all the vertices of a graph G have odd degree. Is it true that $\mathrm{d}(X)$ is odd for every proper non-empty subset X of V_G?

E 1.111 (LARGE CUT) Show that every graph has a cut containing at least half the edges of the graph. In other words, show that every graph G has a set X of vertices such that $\mathrm{d}(X) \geq \frac{1}{2}\,m(G)$.

E 1.112 Show that every graph G has a bipartite spanning subgraph H that satisfies the condition $\mathrm{d}_H(v) \geq \mathrm{d}_G(v)/2$ for every vertex v.

Operations on cuts

E 1.113 (SYMMETRIC DIFFERENCE) Show that $\partial(X \oplus Y) = \partial(X) \oplus \partial(Y)$ for
$A \oplus B$ any sets X and Y of vertices of a graph. Here, $A \oplus B$ denotes the symmetric difference[25] of sets A and B.

E 1.114 (SUBMODULARITY) *Show that, in any graph G, for any subsets X and Y of V_G,*

$$\mathrm{d}(X \cup Y) + \mathrm{d}(X \cap Y) \leq \mathrm{d}(X) + \mathrm{d}(Y)\,.$$

---

[24] If $X = \{x_1, x_2, \ldots, x_k\}$, then $\bigcup_{x \in X} \mathrm{N}(x)$ is the set $\mathrm{N}(x_1) \cup \mathrm{N}(x_2) \cup \cdots \cup \mathrm{N}(x_k)$, where $\mathrm{N}(x_i)$ is the set of neighbors of x_i, as defined in section 1.3.

[25] The *symmetric difference* of two sets A and B is the set $(A \setminus B) \cup (B \setminus A)$. It is easy to check that $A \oplus B = (A \cup B) \setminus (A \cap B)$.

E 1.115 (Consequence of 1.114) Let v and w be two vertices of a graph G. An *isolator*[26] is any subset of V_G that contains v but does not contain w. An isolator X is *minimum* if $\mathrm{d}(X) \leq \mathrm{d}(X')$ for every isolator X'. *Show that, if X and Y are minimum isolators, then $X \cup Y$ and $X \cap Y$ are also minimum isolators.*

[26] The term *isolator* is not standard. We use it here (and in chapter 15) for lack of a better word.

1.9 Paths and circuits in graphs

If a path $v_1 \cdots v_p$ is a subgraph of G, then we may simply say that $v_1 \cdots v_p$ is a path **in** G, or that G **contains** the path $v_1 \cdots v_p$. For example, if we say that $u\,v\,w\,z$ is a path in G, we must understand that $(\{u,v,w,z\},\{uv,vw,wz\})$ is a subgraph of G. An analogous convention holds for circuits which are subgraphs of G.[27]

If v and w are the two endpoints of a path in G, then it is convenient to say that the path **goes from** v **to** w, or that it **begins at** v and **ends in** w. But those expressions must be used with caution, because paths are static and have no orientation.

maximum maximal

A path P in a graph G is **maximum** if G does not contain a path of length greater than that of P. A path P in G is **maximal** if there is no path P' in G such that $P \subset P'$.

Exercises

∘ **E 1.116** Let G be the graph represented in figure 1.8. Is it true that $e\,a\,b\,f\,g\,k$ is a path in G? Is it true that $e\,a\,b\,f\,c\,d$ is a path in G? Is it true that $e\,a\,b\,f\,g\,k\,j\,i\,e$ is a circuit in G?

E 1.117 Let G be the graph in figure 1.8. Is it true that G contains a circuit of length 6? Is it true that G contains an induced circuit of length 6? (I.e., is it true that there exists a subset X of V_G such that $G[X]$ is a circuit of length 6?) Show an induced path of length 3 in G. (I.e., show a set X of vertices such that $G[X]$ is a path of length 3.) Show a path of length 3 in G which is not induced.

▷ **E 1.118** Let P be a path with endpoints x and x', and let Q be a path with endpoints y and y'. Suppose that $V_P \cap V_Q \neq \emptyset$. Show that there exists a path with endpoints x and y in the graph $P \cup Q$ (see section 1.5).

Additional question: If z is a vertex in $V_P \cap V_Q$, is it true that there exists, in the graph $P \cup Q$, a path from x to y that passes through z?

E 1.119 Find a circuit of minimum length in the Petersen graph (see exercise 1.15 or figure 1.6). Find a circuit of maximum length in the Petersen graph. Find a path of maximum length in the Petersen graph.

∘ **E 1.120** Verify that the 3-by-3 knight graph contains a circuit. Find the longest circuit you can in the 4-by-4 knight graph.

[27] I would like to say "subpath of G" and "subcircuit of G", but these expressions are not used in the literature.

E 1.121 Find the longest path you can in the queen graph. Find the longest circuit you can in the queen graph.

E 1.122 The **Heawood**[28] graph has vertex set $\{0, 1, 2, \ldots, 13\}$. Each vertex i is a neighbor of $(i + 1) \bmod 14$ and of $(i + 13) \bmod 14$.[29] Furthermore, each i is a neighbor of a third vertex, which depends on the parity of i: if i is even, then it is neighbor of $(i + 5) \bmod 14$, and if i is odd, then it is neighbor of $(i + 9) \bmod 14$. Draw a figure of the graph. Find a circuit of minimum length in the Headwood graph.

E 1.123 Suppose that a graph G has an odd circuit. Show that G also has an induced odd circuit, i.e., show that there exists a set X of vertices such that $G[X]$ is an odd circuit. Does something analogous hold for even circuits?

E 1.124 Give an example of a graph G and a path in G that is maximal but not maximum.

▷ **E 1.125** ♡ Suppose that $\mathrm{d}(v) \geq k$ for every vertex v of a graph. Show that the graph has a path of length at least k. (Hint: take a maximal path.)[30]

The problem could be formulated like this: show that every graph G contains a circuit with at least $\delta(G) + 1$ vertices.

▷ **E 1.126** Let G be a graph such that $\delta(G) \geq 2$. Prove that G has a circuit.

E 1.127 Let G be a graph such that $\delta(G) \geq 3$. Prove that G has a circuit of even length.

E 1.128 Let k be a natural number greater than 1. Suppose that $\mathrm{d}(v) \geq k$ for every vertex v of a graph G. Show that G has a circuit of length at least $k + 1$. In other words, show that G has a circuit with at least $\delta(G) + 1$ vertices, as long as $\delta(G) > 1$. (See exercise 1.125.)

▷ **E 1.129** Let P be a maximal path in a graph G. Let u and w be the endpoints of P and suppose that $\mathrm{d}(u)+\mathrm{d}(w) \geq |V_P| \geq 3$. Show that G has a circuit whose vertex set is V_P.

E 1.130 Let G be a graph with $n > 1$ vertices and at least $2n$ edges. Show that G has a circuit of length $\leq 2 \log_2 n$.

[28] Percy John Heawood (1861 – 1955). (See article in Wikipedia.)
[29] The expression "$i \bmod j$" denotes the remainder of the division of i by j.
[30] Chapter 17 discusses the important but difficult problem of finding a path of *maximum* length in a graph.

E 1.131 Let G be a graph without circuits of length smaller than 5. Show that $n(G) \geq \delta(G)^2 + 1$.

E 1.132 Show that every graph G with at least $k\, n(G)$ edges contains a path of length k. (Combine exercises 1.102 and 1.125.)

Paths and circuits versus cuts

We say that a cut $\partial(X)$ **separates** a vertex x from a vertex y if X contains x but not y. If $\partial(X)$ separates x from y then, of course, $\overline{X}$ separates y from x.

E 1.133 Let P be a path in a graph G. Let X be a set of vertices that contains one and only one of the endpoints of P. Show that $E_P \cap \partial(X) \neq \emptyset$.

$\star$ **E 1.134** Prove that, for any pair (x, y) of vertices of a graph, one and only one of the following statements is true: (1) a path connects x to y or (2) an empty cut separates x from y. (Another formulation of the same exercise: prove there exists a path from x to y if and only if no empty cut separates x from y.)

E 1.135 (ALGORITHM) Write an efficient algorithm that will receive vertices v and w of a graph G and find a path from v to w or show that no such path exists.

Walks, trails and cycles

A **walk** in a graph is any finite sequence $(v_0, v_1, v_2, \ldots, v_{k-1}, v_k)$ of vertices such that v_i is adjacent to v_{i-1} for every i between 1 and k. (The vertices in the walk may not be pairwise distinct.) We say that vertex v_0 is the **origin** of the walk and that v_k is the **terminus** of the walk. We also say that the walk **goes from** v_0 **to** v_k and that the walk **connects** v_0 to v_k.

The **edges** of the walk are v_0v_1, v_1v_2, …, $v_{k-1}v_k$. (These edges may not be pairwise distinct.) The **length** of the walk is the number k.

A **trail** is a walk without repeated edges, i.e., a walk whose edges are pairwise distinct. It is clear that the length of a trail is equal to the cardinality of its edge set.

A walk is **simple** if its vertices are pairwise distinct, i.e., if it has no repeated vertices. Every simple walk is, of course, a trail.

A walk $(v_0, \ldots, v_k)$ is **closed** if its origin coincides with its terminus, i.e., if $v_0 = v_k$. A **cycle** is a closed trail, i.e., a closed walk without repeated edges.[31]

▷ **E 1.136** Let $(v_0, v_1, v_2, \ldots, v_k)$ be a walk with origin r and terminus s in a graph G. Show that G has a path with endpoints r and s. More specifically, show that there is a path with endpoints r and s in the subgraph $(\{v_0, v_1, v_2, \ldots, v_k\}\,,\, \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\})$ of G.

E 1.137 Suppose that $(v_0, \ldots, v_k)$ is a closed walk in a graph G. Is it true that G has a circuit?

▷ **E 1.138** Let $(v_0, v_1, v_2, \ldots, v_k)$ be a cycle in a graph G. Show that there exists a circuit in the subgraph $(\{v_1, v_2, \ldots, v_k\}, \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\})$ of G.

∘ **E 1.139** Let v_0, …, v_5 be some vertices (not necessarily pairwise distinct) of a graph G. Which of the following statements are true: (1) if $v_0v_1v_2v_3v_4v_5$ is a path in G, then $(v_0, v_1, v_2, v_3, v_4, v_5)$ is a simple walk; (2) if $v_0v_1v_2v_3v_4v_5v_0$ is a circuit in G, then $(v_0, v_1, v_2, v_3, v_4, v_5, v_0)$ is a cycle; (3) if $(v_0, v_1, v_2, v_3, v_4, v_5)$ is a trail, then $v_0v_1v_2v_3v_4v_5$ is a path; (4) if $(v_0, v_1, v_2, v_3, v_4, v_5, v_0)$ is a cycle, then $v_0v_1v_2v_3v_4v_5v_0$ is a circuit.

[31] According to this definition, a cycle may have length 0. On the other hand, every circuit has length at least 3.

1.10 Connected graphs

In any graph G, we say that a vertex v is **connected** to a vertex w if G contains a path with endpoints v and w. The relation is, of course, symmetric: if v is connected to w, then w is connected to v.

A graph is **connected** if its vertices are pairwise connected. In other words, a graph is connected if v is connected to w for each pair (v, w) of its vertices.

A graph G is connected if and only if $\partial(X) \neq \emptyset$ for every proper non-empty subset X of V_G. (See exercise 1.148.)

Exercises

E 1.140 Is the 3-by-3 knight graph connected? Is the t-by-t bishop graph connected?

E 1.141 Show that the graph Q_k is connected (whatever the value of k).

E 1.142 Show that every path is a connected graph. Show that every circuit is a connected graph.

▷ **E 1.143** Let P and Q be two paths such that $V_P \cap V_Q \neq \emptyset$. Show that the graph $P \cup Q$ (see section 1.5) is connected.

▷ **E 1.144** *Let G and H be two connected graphs such that $V_G \cap V_H \neq \emptyset$. Show that the graph $G \cup H$ (see section 1.5) is connected.*

◦ **E 1.145** Let G and H be two graphs. Which of the following statements are true? 1. If $V_G \cap V_H = \emptyset$ then $G \cup H$ is not connected. 2. If $G \cup H$ is connected then $V_G \cap V_H \neq \emptyset$. 3. If $G \cup H$ is not connected then $V_G \cap V_H = \emptyset$.

▷ **E 1.146** ♡ Suppose that some vertex x of a graph G is connected to each of the other vertices. Show that G is connected.

◦ **E 1.147** Suppose that a spanning subgraph H of a graph G is connected. Show that G is connected.

⋆ **E 1.148** *Show that a graph G is connected if and only if $\partial(X) \neq \emptyset$ for every X such that $\emptyset \subset X \subset V_G$.*

◦ **E 1.149** Let G be a graph and X be a proper non-empty subset of V_G (i.e., $\emptyset \subset X \subset V_G$). Show that the graph $G - \partial(X)$ is not connected.

∘ **E 1.150** Which of the following statements are true for any graph G? 1. If G is connected, then $\partial(X) \neq \emptyset$ for every X such that $\emptyset \subset X \subset V_G$. 2. If G is connected, then $\partial(X) \neq \emptyset$ for some X such that $\emptyset \subset X \subset V_G$. 3. If $\partial(X) \neq \emptyset$ for every X such that $\emptyset \subset X \subset V_G$, then G is connected. 4. If $\partial(X) \neq \emptyset$ for some X such that $\emptyset \subset X \subset V_G$, then G is connected.

E 1.151 Prove that if a graph G is not connected then its complement $\overline{G}$ is connected.

E 1.152 (ALGORITHM) Devise an efficient algorithm to decide whether a graph is connected. What does your algorithm return (i.e., what is the "output" of the algorithm)?

E 1.153 Let x, y and z be three vertices of a connected graph G. Is it true that G has a path that contains x, y and z?

∘ **E 1.154** Let X be a set of vertices from a connected graph G. Is it true that $G[X]$ is connected?

⋆∘ **E 1.155** Let e be an edge and v be a vertex in a circuit O. Show that the graph $O - e$ is connected. Show that $O - v$ is connected.

∘ **E 1.156** Let e be an edge and v be a vertex in a path P. Under what conditions is $P - e$ connected? Under what conditions is $P - v$ connected?

▷ **E 1.157** Let O be a circuit in a connected graph G. Show that $G - e$ is connected for every edge e of O.

E 1.158 Let v be a vertex of degree 1 in a connected graph G. Show that the graph $G - v$ is connected.

E 1.159 Suppose that G is a connected graph with at least one edge. Is it true that there exists an edge a such that $G - a$ is connected?

∘ **E 1.160** Let G be a connected graph and let v be one of the endpoints of a maximal path (see page 30) in G. Is it true that $G[\mathrm{N}(v)]$ is connected?

E 1.161 ♡ Show that every connected graph G with two or more vertices has a vertex v such that $G - v$ is connected.

⋆ **E 1.162** *Prove that every connected graph on n vertices has at least $n - 1$ edges. In other words, show that in every connected graph G we have that*

$$m(G) \geq n(G) - 1\,.$$

E 1.163 Let k be a natural non-null number and G a $\{U, W\}$-bipartite graph. Suppose that $|U| \leq k$ and $|W| \leq k$. Show that if $\delta(G) > k/2$ then G is connected.

E 1.164 Let G be a graph such that $\delta(G) \geq n(G)/2$. Show that G is connected.

E 1.165 Let G be a graph such that $\delta(G) \geq \lfloor n(G)/2 \rfloor$.[32] Show that G is connected. (Show that the result is the best possible in the following sense: there are disconnected graphs with $\mathrm{d}(v) \geq \lfloor n/2 \rfloor - 1$ for every vertex v.)

◦ **E 1.166** Suppose that $\mathrm{d}(v) + \mathrm{d}(w) \geq n - 1$ for every pair (v, w) of non-adjacent vertices of a graph G. Show that G is connected.

E 1.167 Show that every graph with n vertices and more than $\frac{1}{2}(n-1)(n-2)$ edges is connected.

◦ **E 1.168** Let G be a graph and k be a natural number. Show that $\mathrm{d}(X) \geq k$ for every X such that $\emptyset \subset X \subset V_G$ if and only if $G - F$ is connected for every subset F of E_G such that $|F| < k$.

E 1.169 Let G be a connected graph. Prove that the line graph $L(G)$ is also connected.

E 1.170 (MAXIMUM PATHS ARE NOT DISJOINT) Let P^* and Q^* be two paths of maximum length in a connected graph G. Show that P^* and Q^* have a common vertex.

[32] By definition, $\lfloor x \rfloor$ is the only integer i such that $i \leq x < i + 1$.

1.11 Components

A connected subgraph H of a graph G is **maximal** (with regard to the property of being connected) if it is not part of a greater connected subgraph, i.e., if there is no connected graph H' such that $H \subset H' \subseteq G$.

A **component** of a graph G is any maximal connected subgraph of G. The number of components in a graph G will be denoted by $c(G)$

$$c(G) \, .$$

Clearly a graph G is connected if and only if $c(G) = 1$.

The number of components of any graph G is at least $n(G) - m(G)$. (See exercise 1.192.)

Exercises

E 1.171 How many components does the 3-by-3 knight graph have? How many components does the t-by-t bishop graph have?

E 1.172 Let a be an edge and v be a vertex of a path P. Show that $P - a$ has exactly two components. Show that $P - v$ has one or two components.

E 1.173 Let a be an edge and v be a vertex of a circuit O. Show that $O - a$ has only one component. Show that $O - v$ has only one component.

E 1.174 Let P be a path and S be a proper subset of V_P. Show that $c(P - S) \leq |S| + 1$.

E 1.175 Let O be a circuit and S be a subset of V_O such that $0 < |S| < n(O)$. Prove that $c(O - S) \leq |S|$.

E 1.176 Suppose that exactly two vertices, say u and v, of a graph G have odd degree. Show that there exists a path in G whose endpoints are u and v.

▷ **E 1.177** Let G be a graph such that $\Delta(G) \leq 2$. Describe the components of G.

E 1.178 Let G be a 2-regular graph. Show that each component of G is a circuit.

∘ **E 1.179** Show that, in any graph, every vertex belongs to one and only one component. In other words, show that the sets of vertices of all components form a partition of V_G.

∘ **E 1.180** Let H be a component of a graph G. Show that $\partial_G(V_H) = \emptyset$.

∘ **E 1.181** Let X be a set of vertices of a graph G. Prove or disprove the following claim: If $\emptyset \subset X \subset V_G$ and $\partial_G(X) = \emptyset$, then $G[X]$ is a component of G.

E 1.182 Let X be a non-empty set of vertices of a graph G. Show that $G[X]$ is a component of G if and only if $G[X]$ is connected and $\partial_G(X) = \emptyset$.

⋆ **E 1.183** Let x be a vertex of a graph G. Let X be the set of all vertices that are connected to x. Show that $G[X]$ is a component of G.

E 1.184 (ALGORITHM) Devise an efficient algorithm that receives a vertex x of a graph G and calculates the set of vertices of the component of G that contains x.

E 1.185 (ALGORITHM) Devise an efficient algorithm to calculate the number of components in any given graph.

E 1.186 Let H be a spanning subgraph of a graph G. Show that $c(H) \geq c(G)$.

E 1.187 Let e be an edge of a graph G. Show that $c(G) \leq c(G - e) \leq c(G) + 1$ for every edge e of G.

∘ **E 1.188** Let X be a set of vertices of a graph G. Suppose that $c(G - X) > |X| + 1$. Is it true that G is not connected?

E 1.189 Let v be a vertex of a connected graph G. Show that the number of components of $G - v$ is not greater than $\mathrm{d}(v)$.

E 1.190 Let G be a connected graph and suppose that $\mathrm{d}(v)$ is even for every vertex v of G. Show that, for every vertex v, the number of components of $G - v$ is not greater than $\frac{1}{2}\mathrm{d}(v)$.

E 1.191 (ALGORITHM) Devise an efficient algorithm for the following problem: Given a graph G and a natural number k, find a set X with no more than k vertices which maximizes the number of components of $G - X$.

⋆ **E 1.192** *Show that for every graph G we have that*

$$m(G) \geq n(G) - c(G)\ .$$

E 1.193 Let n, m and c be the numbers of vertices, edges and components, respectively, of a graph G. Show that

$$m \leq \tfrac{1}{2}(n-c)(n-c+1) .$$

1.12 Bridges

A **bridge** (or **isthmus**, or **cut edge**) in a graph G is any edge e such that $c(G - e) > c(G)$, i.e., such that $G - e$ has more components than G.

An edge e is a bridge if and only if the set $\{e\}$ is a cut in the graph. (See exercise 1.197.)

There is an interesting dichotomy between bridges and circuits: in any graph, every edge is either a bridge or belongs to a circuit. (See exercise 1.199.)

Exercises

∘ **E 1.194** Does the t-by-t bishop graph have bridges?

E 1.195 Suppose that a graph G has a bridge uv. What does the adjacency matrix of G look like? What does the incidence matrix of G look like?

∘ **E 1.196** Let uv be an edge of a graph G. Show that uv is a bridge if and only if $\langle u\, v \rangle$ is the only path in G with endpoints u and v.

∘ **E 1.197** Let e be an edge of a graph G. Show that e is a bridge if and only if $\{e\}$ is a cut, i.e., $\{e\} = \partial(X)$ for some set X of vertices. (See also exercise 1.187.)

∘ **E 1.198** Let G be the graph with vertices $a, b, \ldots, g$ and edges ab, bc, cd, de, ec, bf, gb, ag. Which edges belong to circuits? Which edges are bridges?

⋆ **E 1.199** (BRIDGES/CIRCUITS DICHOTOMY) Prove that, in any graph, every edge is of one and only one of two types: either it belongs to a circuit or it is a bridge.

E 1.200 What is the shape of a graph all of whose edges are bridges? What is the shape of a graph all of whose edges belong to circuits?

E 1.201 Suppose that all the vertices of a graph G have even degree. Show that G has no bridges.

E 1.202 Let r be a natural number greater than 1 and let G be a r-regular bipartite graph. Show that G has no bridges.

E 1.203 Let G be a connected graph and X a subset of V_G such that $\mathrm{d}(X) = 1$. Show that the induced subgraphs $G[X]$ and $G[\overline{X}]$ are both connected.

E 1.204 (ALGORITHM) Devise an efficient algorithm to find all bridges of a given graph.

1.13 Edge-biconnected graphs

A graph is **edge-biconnected** if it is connected, has three or more vertices, and has no bridges.[33]

A graph with three or more vertices is edge-biconnected if and only if each pair of its vertices is connected by two paths that have no common edges. (See exercise 1.208.) This property explains the name "edge-biconnected."

Exercises

◦ **E 1.205** Show that every circuit is edge-biconnected. Show that paths are not edge-biconnected.

E 1.206 Show that one of the two components of the 3-by-3 bishop graph is edge-biconnected.

◦ **E 1.207** Let G be an edge-biconnected graph. Show that $\mathrm{d}(X) \geq 2$ for every non-empty proper subset X of V_G.

⋆ **E 1.208** (Two paths without common edges) Let G be a graph with at least three vertices that has the following property: every pair of vertices of G is connected by two paths that have no common edges. In other words, suppose that, for every pair (r, s) of vertices of G, there are paths P and Q, both with endpoints r and s, such that $E_P \cap E_Q = \emptyset$. Show that G is edge-biconnected.

Let G be an edge-biconnected graph. Show that every pair of vertices of G is connected by two paths that have no common edges.[34] (Compare to exercise 1.134.)

E 1.209 Show that $m(G) \geq n(G)$ for every edge-biconnected graph G.

[33] In some branches of Computer Science, we say that such a graph is "fault tolerant."
[34] See a generalization in chapter 15, exercise 15.7.

1.14 Articulations and biconnected graphs

An **articulation** or **cut vertex** in a graph G is a vertex v such that $c(G - v) > c(G)$, i.e., such that $G - v$ has more components than G.

A graph is **biconnected** if it is connected, has no articulations, and has three or more vertices.[35]

A graph with three or more vertices is biconnected if and only if each pair of its vertices is connected by two internally disjoint paths (i.e., two paths without common internal vertices). (See exercise 1.218.) This property explains the name "biconnected."

From this, it follows that a graph is biconnected if and only if each pair of its vertices belongs to a circuit. (See exercise 1.219.)

Exercises

E 1.210 Let v be a vertex in a graph G. Show that v is an articulation if and only if there are two vertices x and y in $V_G \setminus \{v\}$ such that (1) some path in G connects x to y and (2) every path between x and y contains v.

E 1.211 Let v be an articulation of a graph G. What is the shape of the adjacency matrix of G? What is the shape of the incidence matrix of G?

∘ **E 1.212** Is it true that every graph without articulations has no bridges? Is it true that every graph without bridges has no articulations?

∘ **E 1.213** Let T be a tree and v be a vertex of T such that $\mathrm{d}(v) \geq 2$. Is it true that v is an articulation?

E 1.214 (ALGORITHM) Write an algorithm that finds all the articulations of a graph.

E 1.215 Show that every circuit is biconnected.

E 1.216 The 3-by-3 bishop graph has two components. Show that only one of them is biconnected.

∘ **E 1.217** Show that not all edge-biconnected graphs are biconnected. Show that every biconnected graph is edge-biconnected.

[35] In some branches of Computer Science, we may say that such a graph is "fault tolerant."

$\star$ **E 1.218** (TWO INTERNALLY DISJOINT PATHS) Let G be a graph with at least three vertices that has the following property: every pair of vertices of G is connected by two internally disjoint paths. In other words, suppose that, for each pair (r, s) of vertices of G, there are paths P and Q, both with endpoints r and s, such that $V_P \cap V_Q = \{r, s\}$. Show that G is biconnected.

Let G be a biconnected graph. Show that every pair of vertices of G is connected by two internally disjoint paths.[36]

E 1.219 (ARTICULATIONS VERSUS CIRCUITS) Suppose that every pair of vertices of a graph G belongs to some circuit. Show that G has no articulations.

Let G be a biconnected graph. Show that every pair of vertices of G belongs to some circuit. (See exercise 1.218.)

E 1.220 Show a graph that satisfies the following property: every two vertices of the graph belong to a common circuit, but there are three vertices that do not belong to a common circuit.

E 1.221 Let G be a connected graph with no articulations. Assuming that $\delta(G) \geq 3$, show that G has a vertex v such that $G - v$ is connected and has no articulations. (Compare to exercise 1.161 in section 1.10.)

[36] See a generalization in chapter 16, exercise 16.8.

1.15 Trees and forests

This section deals with two important kinds of graphs: the forests and the trees. Trees are a generalization of paths (see exercises 1.224 and 1.225) and forests generalize trees.

A **forest**, or **acyclic graph**, is a graph with no circuits. This definition can be restated as follows: a graph is a *forest* if each of its edges is a bridge (see exercise 1.223). A **tree** is a connected forest. Clearly every component of a forest is a tree.[37] A **leaf** of a forest is any vertex of degree 1.

A graph G is a forest if and only if $m(G) = n(G) - c(G)$. (See exercise 1.231.)

Any two of the following properties imply the third: "G is a forest", "G is connected" and "$m(G) = n(G) - 1$." (See exercises 1.228, 1.229 and 1.230.)

Exercises

∘ **E 1.222** Show that every path is a tree. Show that every star (see section 1.2) is a tree.

⋆ **E 1.223** Show that a graph is a forest if and only if each of its edges is a bridge. (See exercise 1.199.)

E 1.224 Let $(v_1, v_2, v_3, \ldots, v_n)$ be a sequence of pairwise distinct objects. For each j, let $i(j)$ be an element of $\{1, \ldots, j-1\}$. Show that the graph $\big(\{v_1, v_2, v_3, \ldots, v_n\}, \{v_2 v_{i(2)}, v_3 v_{i(3)}, \ldots, v_n v_{i(n)}\}\big)$ is a tree. (Compare this construction to the definition of a path in section 1.4.) (Compare to exercise 1.225.)

E 1.225 Let T be a tree. Show that there exists a permutation $(v_1, v_2, \ldots, v_n)$ of V_T with the following property: for $j = 2, \ldots, n$, vertex v_j is adjacent to exactly one of the vertices of the set $\{v_1, \ldots, v_{j-1}\}$. (Compare to exercise 1.224.)

⋆ **E 1.226** *Show that a graph is a forest if and only if it has the following property: for every pair (x, y) of its vertices, there exists at most one path with endpoints x and y in the graph.*

[37] In Computer Science, the word "tree" brings to mind the ideas of *parent* and *child*. In the present context, however, the expressions "parent of a vertex" and "child of a vertex" do not make sense. (They only acquire meaning if one of the vertices in the tree is chosen to play the role of a "root." If r is the root of the tree, then the **parent** of any other vertex v is the vertex adjacent to v in the only path (see exercise 1.226) that connects v to r. Every neighbor of v which is not its parent is a **child** of v.)

E 1.227 (ALGORITHM) Devise an efficient algorithm to decide whether a given graph is a tree.

$\star$ **E 1.228** *Prove that* $m(T) = n(T) - 1$ *for every tree* T. (Compare to exercise 1.162.)

$\star$ **E 1.229** *Let* G *be a connected graph such that* $m(G) = n(G) - 1$. *Prove that* G *is a tree.*

E 1.230 *Let* F *be a forest such that* $m(F) = n(F) - 1$. *Prove that* F *is a tree.*

$\star$ **E 1.231** Show that a graph G is a forest if and only if $m(G) = n(G) - c(G)$. (Compare to exercise 1.192.)

$\triangleright$ **E 1.232** Show that every tree with at least one edge has at least two leaves.

E 1.233 Show that every forest F has at least $\Delta(F)$ leaves.

E 1.234 Let T be a tree with two or more vertices. Let X be the set of vertices whose degree is greater than 2. Show that T has $2 + \sum_{x \in X} (\mathrm{d}(x) - 2)$ leaves.

E 1.235 Let T be a tree with vertices $1, \ldots, n$. Suppose that the degrees of the vertices $1, 2, 3, 4, 5, 6$ are $7, 6, 5, 4, 3, 2$ respectively, and that vertices $7, \ldots, n$ are leaves. Determine n (and therefore the number of leaves of the tree).

E 1.236 Let T be a tree with $p + q$ vertices. Suppose that p of the vertices have degree 4 and q are leaves. Show that $q = 2p + 2$.[38]

E 1.237 Let T be a tree with at least three vertices. Is it true that the complement $\overline{T}$ of T is connected unless T is a star?

E 1.238 Let T be a tree and U be a subset of V_T. Assuming that $|U|$ is even, show that there exists a subset X of E_T such that $\mathrm{d}_{T-X}(u)$ is odd for every u in U, and $\mathrm{d}_{T-X}(v)$ is even for every v in $V_T \setminus U$.

E 1.239 (HELLY PROPERTY[39]) Let P, Q, R be three paths in a tree T. Suppose that $V_P \cap V_Q \neq \emptyset$, $V_Q \cap V_R \neq \emptyset$ and $V_P \cap V_R \neq \emptyset$. Prove that $V_P \cap V_Q \cap V_R \neq \emptyset$.

E 1.240 Prove that every forest is planar.

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[38] Imagine that the vertices of degree 4 are carbon atoms and those of degree 1 are hydrogen atoms. The graph represents then a molecule of the hydrocarbon C_pH_q. See exercise 1.5.

[39] Reference to mathematician Eduard Helly (1884 – 1943).

1.16 Graph minors

This section introduces the concept of *minor*, which can be seen as a generalization of the idea of subgraph. We can say that a minor describes the "gross" structure of the graph, while a subgraph describes the "fine" structure. Minors have an important role in the study of planarity (chapter 19), vertex coloring (chapter 8) and many other problems.

A graph H is a **minor**, or **subcontraction**, of a graph G if V_H is a subpartition[40] $\{V_1, \ldots, V_p\}$ of V_G such that

- every subgraph $G[V_i]$ is connected and
- if V_i is adjacent to V_j in H then there exists an edge from V_i to V_j in G

(but there may exist an edge from V_i to V_j in G even if V_i is not adjacent to V_j in H). The expression "H is a minor of G" is also used, in a broader sense, to say that H is *isomorphic to* a minor of G.

In a very informal way, we can say that H is a minor of G if H can be obtained from a subgraph of G by successive "edge contraction" operations. (The "contraction" of an edge uv makes vertices u and v coincide.)

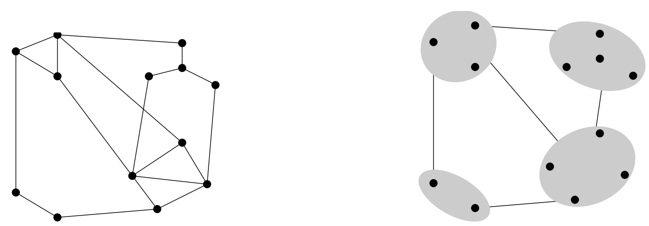

Figure 1.10: The graph on the right, H, is a minor of the graph on the left, G. (This is a very special minor, since $V_1 \cup \cdots \cup V_p = V_G$.)

A graph H is a **topological minor** of a graph G if $V_H \subseteq V_G$ and there exists a function P that maps each edge in H to a path in G in such a way that

- for every edge xy of H, the path $P(xy)$ has endpoints x and y and has no internal vertices in V_H, and
- if xy and uv are two distinct edges in H, then $P(xy)$ and $P(uv)$ have no common internal vertices.

The expression "H is a topological minor of G" is also used, in a broader sense, to say that H is *isomorphic to* some topological minor of G.

[40] A **subpartition** of a set V is a collection $\{V_1, \ldots, V_p\}$ of non-empty subsets of V such that $V_i \cap V_j = \emptyset$ whenever $i \neq j$.

A topological minor is a special type of minor, although this is not immediately apparent. (See exercise 1.250.)

If H is a topological minor of G, we also say that G **contains a subdivision** of H, because it is possible to obtain a subgraph of G from H by successive "edge subdivision" operations. (Each "subdivision" of an edge creates a new vertex of degree two in the "interior" of the edge.)

Exercises

∘ **E 1.241** Let H be a subgraph of a graph G. Show that H is a topological minor of G.

∘ **E 1.242** Let H be a subgraph of a graph G. Show that H is isomorphic to a minor of G.

E 1.243 Show that a graph G has a topological minor which is isomorphic to K_3 if and only if G contains a circuit.

E 1.244 Show that a graph G has a minor isomorphic to K_3 if and only if G contains a circuit.

E 1.245 Let G be the 4-by-4 king graph. Let u be the vertex with coordinates $(2, 2)$, and v be the vertex with coordinates $(3, 3)$. Show that $G + uv$ does not have a subgraph isomorphic to K_4 but has a topological minor isomorphic to K_4.

E 1.246 Let G be the 5-by-5 king graph. Let u be the vertex with coordinates $(2, 2)$ and v be the vertex with coordinates $(4, 4)$. Show that $G + uv$ has no subgraph isomorphic to K_4 but has a minor isomorphic to K_4.

Let x be the vertex with coordinates $(2, 4)$ and y be the vertex with coordinates $(4, 2)$. Show that $G + uv + xy$ has a minor isomorphic to K_4.

E 1.247 Show that the Petersen graph has a minor isomorphic to K_5 (but has neither a subgraph isomorphic to K_5 nor a topological minor isomorphic to K_5).

E 1.248 Show that the Petersen graph has a topological minor isomorphic to $K_{3,3}$. Show that the Petersen graph has a minor isomorphic to $K_{3,3}$.

E 1.249 Show that $K_{3,3}$ is isomorphic to a topological minor of the cube Q_4. Show that K_5 is a topological minor of Q_4.

⋆ **E 1.250** *Show that if H is a topological minor of G then H is isomorphic to a minor of G.* Show that the converse is not true.

⋆ **E 1.251** Let H be a minor of a graph G. Suppose that $\Delta(H) \leq 3$. Prove that H is isomorphic to a topological minor of G. Give a good example to show that the condition "$\Delta(H) \leq 3$" is essential.

E 1.252 If H is (isomorphic to) a minor of G, we write $H \preceq G$. Show that $\preceq$ is an order relation. More precisely, show that $H \preceq G$

1. $G \preceq G$,
2. if $H \preceq G$ and $G \preceq H$, then $H \cong G$,
3. if $H \preceq G$ and $G \preceq F$, then $H \preceq F$.

Show also that the is-a-topological-minor-of relation is an order relation.

1.17 Plane maps and their faces

We have said in section 1.6 that, roughly speaking, a graph is **planar** if it can be drawn in the plane[41] in such a way that edges do not cross. The exact definition involves the concepts of polygonal arc and plane map, which we discuss next.

A **polygonal arc** is any finite union of line segments in the plane $\mathbb{R}^2$ that is topologically homeomorphic to the closed interval $[0,1]$ of the line $\mathbb{R}$. In other words, a finite union c of line segments is a **polygonal arc** if there is a topologically continuous bijection of the interval $[0,1]$ into c. The images of 0 and 1 under this continuous bijection are the **endpoints** of the polygonal arc.[42]

$\mathbb{V}$ $\mathbb{E}$ A **plane map**[43] is a pair $(\mathbb{V},\mathbb{E})$ of finite sets, where $\mathbb{V}$ is a set of points in the plane $\mathbb{R}^2$, and $\mathbb{E}$ is a set of polygonal arcs such that

- the endpoints of each polygonal arc are elements of $\mathbb{V}$,
- the interior of each polygonal arc is disjoint from $\mathbb{V}$,
- the interior of each polygonal arc is disjoint from every other polygonal arc,
- two different polygonal arcs have at most one endpoint in common.

points lines The elements of $\mathbb{V}$ are the **points**[44] of the map, and the elements of $\mathbb{E}$ are the **lines**[45] **of the map**.

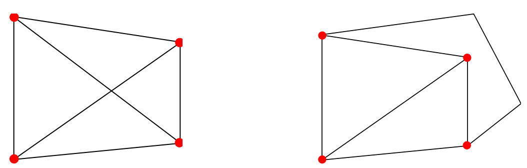

Figure 1.11: The map on the left is not plane. The plane map on the right represents a K_4.

graph of a map The **graph of** a plane map $(\mathbb{V},\mathbb{E})$ is defined in the obvious way: the vertex set of the graph is $\mathbb{V}$, and two vertices v and w are adjacent in the graph if a line on the map has endpoints v and w. (We must be careful with the notation, since the letter ”V” is being used to name both the set of points of a plane map and the set of vertices of the corresponding graph. Similarly, the letter “E” is

[41] From a technical point of view, it would be more convenient to use the surface of the sphere instead of the plane. But the results are equivalent.
[42] By definition, the two endpoints are distinct.
[43] Some authors say “plane graph.” Do not mistake this expression with “planar graph.”
[44] I prefer not to say “vertices” to avoid confusion with the vertices of a graph.
[45] I prefer not to say “edges” to avoid confusion with the edges of a graph.

being use to name both the set of lines of a plane map and the set of edges of the corresponding graph.)

We say that a plane map $\mathbb{M}$ **represents** a graph G if the graph of $\mathbb{M}$ is isomorphic (see chapter 2) to G. Generally, a graph can be represented by many different plane maps.

map represents graph

A graph G is **planar** if it is representable by a plane map, i.e., if there is a plane map whose graph is isomorphic to G. This is the precise version of the vague definition we gave in section 1.6.

Exercises

E 1.253 See the "planarity game" in planarity.net.

E 1.254 Is the Petersen graph (see figure 1.6) planar?

E 1.255 Let G be a K_5 (i.e., a complete graph with 5 vertices). Show that $G-e$ is planar for every edge e of G. Repeat the exercise for $K_{3,3}$ (see figure 19.1) in place of K_5.

∘ **E 1.256** Show that a graph is planar if and only if each of its components is planar.

E 1.257 Let e be a bridge of a graph G. Show that G is planar if and only if $G-e$ is planar.

Let v be an articulation of G. Show that G is planar if and only if $G-v$ is planar.

Faces and planar duality

The **support** of plane map $(\mathbb{V}, \mathbb{E})$ is the set $\mathbb{V} \cup \bigcup \mathbb{E}$ (this is, obviously, a subset of $\mathbb{R}^2$).[46] In other words, the support of the map is the set of all the points of $\mathbb{R}^2$ that are points of the map or belong to lines of the map.

A **face** of a plane map $(\mathbb{V}, \mathbb{E})$ is any region of the complement of the support of a map, i.e., any connected component — in the topological sense[47] — of the set $\mathbb{R}^2 \setminus (\mathbb{V} \cup \bigcup \mathbb{E})$. The **boundary** of each face is formed by lines of the map; the number of lines in the boundary of a face F is the **degree** of F.

[46] If $X = \{X_1, X_2, \ldots, X_k\}$, then $\bigcup X$ denotes the set $X_1 \cup X_2 \cup \cdots \cup X_k$.

[47] The topological concept of connection is formally analogous to the concept of connection in graph theory: a subset X of the plane is **connected** if, for any points x and x' in X, there is a polygonal arc with endpoints x and x' whose points are all in X.

Let G be the graph of a plane map $\mathbb{M}$ with 3 or more points. If G is edge-biconnected, then the faces of $\mathbb{M}$ are "well-behaved": each face is topologically homeomorphic to a disk, and each line belongs to the boundary of two different faces. If G is biconnected, then the faces of $\mathbb{M}$ are even more "well-behaved": the boundary of each face corresponds to a circuit of G.

The **graph of the faces**, or **dual graph**, of a plane map $\mathbb{M}$ is defined as follows: the vertices of the graph are the faces of the map, and two faces are adjacent if their boundaries share at least one line.[48] (Note that the dual graph is defined for a map rather than for the graph of the map. A planar graph can be represented by many different plane maps,[49] and each of these maps has its dual graph.)

An example: the graph of countries of Europe we examined in exercise 1.17 is the dual graph of the map (in the ordinary, geographic, sense) of Europe.

Exercises

E 1.258 Let $\mathbb{M}$ be a plane map, and suppose that the graph of the map is a path. Show that $\mathbb{M}$ has only one face.

Let $\mathbb{M}$ be a plane map, and suppose that the graph of the map is a circuit. Show that $\mathbb{M}$ has exactly two faces (and the two faces have the same boundary).

E 1.259 Show that a plane map has exactly one face if and only if the graph of the map is a forest.

E 1.260 Consider a plane map that represents the p-by-q grid. How many faces does the map have?

E 1.261 Let $\mathbb{M}$ be a plane map that represents the 3-by-4 grid. Draw a figure of the graph of the faces (i.e., of the dual graph) of $\mathbb{M}$.

E 1.262 How many faces does a plane map representing the cube Q_3 have?

E 1.263 Draw a figure of the graphs of the faces of each of the plane maps in figures 1.12 and 1.13.

---

[48] The study of the planar duality is "cleaner" if the definition of a graph is relaxed to allow "parallel edges" (i.e., two or more different edges with the same pair of endpoints) and "loops" (i.e., an edge with two equal endpoints). Of course the definition of plane map should be relaxed accordingly. I prefer not to adopt such a relaxation in the present text. To make up for that, it will be often necessary to avoid graphs with articulations or vertices of degree 2.

[49] But see exercise 1.282.

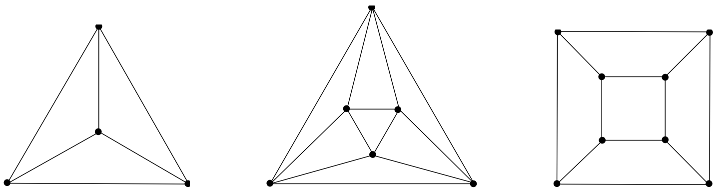

Figure 1.12: Draw a figure of the dual graph of each of the plane maps in the figure.

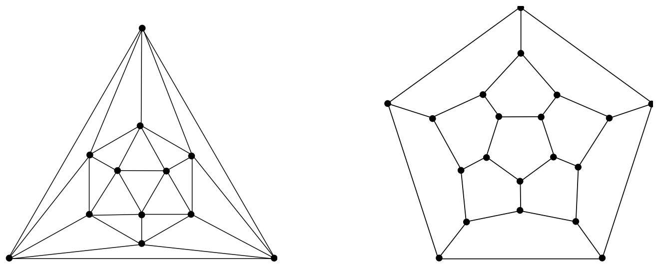

Figure 1.13: Draw a figure of the dual graph of each of the plane maps in the figure.

E 1.264 Let G be a K_5, and e be an edge of G. Let $\mathbb{M}$ be a plane map that represents $G - e$. Let G^* be the graph of the faces (i.e., the dual graph) of $\mathbb{M}$. Draw a figure of G^*. Check that G^* is planar. Show a plane representation, say $\mathbb{M}^*$, of G^*. Draw a figure of the graph of the faces of $\mathbb{M}^*$.

E 1.265 Give an example of a connected planar graph that can be represented by two plane maps with different numbers of faces.

$\star$ **E 1.266** (EULER'S FORMULA[50]) *Let* $(\mathbb{V}, \mathbb{E})$ *be a plane map whose graph is connected. Show that*

$$|\mathbb{V}| - |\mathbb{E}| + |\mathbb{F}| = 2\,, \tag{1.3}$$

where $\mathbb{F}$ *is the set of faces of the map.* (Check that the formula is false for maps whose graphs are not connected.) (Compare to exercise 1.228.)

E 1.267 Let G be an edge-biconnected planar graph. Let $(\mathbb{V}, \mathbb{E})$ be a plane map that represents G, and let $\mathbb{F}$ be the set of faces of the map. Show that $\sum_{F \in \mathbb{F}} \mathrm{d}(F) = 2|\mathbb{E}|$, where $\mathrm{d}(F)$ is the degree of face F. (Compare to exercise 1.42.)

[50] Leonhard Euler (1707 – 1783). See article in Wikipedia.

⋆ **E 1.268** *Let G be a connected planar graph with three or more vertices. Show that*

$$m(G) \leq 3n(G) - 6\,. \tag{1.4}$$

(Hint: Make an induction on the number of bridges.) Deduce from this that the inequality holds for any planar graph with three or more vertices. What do the faces of a plane map with n points and exactly $3n-6$ lines look like?

∘ **E 1.269** Is it true that every graph G with $m(G) \leq 3n(G) - 6$ is planar?

E 1.270 Deduce from inequality (1.4) that K_5 is not planar.

E 1.271 Let G be an edge-biconnected planar graph. Suppose that the girth (see chapter 14) of G is at least 4. Show that $m(G) \leq 2n(G) - 4$. (Compare to exercise 1.268.) Deduce from this inequality that $K_{3,3}$ is not planar. Deduce from this inequality that Q_4 is not planar.

E 1.272 Let G be a bipartite graph with three or more vertices. Assuming G is planar, show that $m(G) \leq 2n(G) - 4$. (See exercise 1.271.)

E 1.273 Let G be a $\{U, W\}$-bipartite graph. Assuming G is planar, show that $m(G) \leq 2|U| + 2|W| - 4$.

E 1.274 Let G be a graph and k be a natural number not less than 3. Suppose that G has at least $\frac{1}{2}(k+2)$ vertices and girth at least k. Assuming G is planar, show that $m(G) \leq (n(G) - 2)k/(k-2)$. (Compare to exercise 1.271.)

⋆ **E 1.275** Show that every planar graph has at least one vertex with degree at most 5. In other words, show that $\delta(G) \leq 5$ for every planar graph G.

Give an example of a planar graph that has no vertices of degree smaller than 5.

E 1.276 Let G be a bipartite planar graph. Show that $\delta(G) \leq 3$.

∘ **E 1.277** A plane map "of type (n, m, d, g)" is a plane map with n points and m lines whose points have degree d and whose faces have degree g. Show a plane map of type $(4, 6, 3, 3)$. Show a plane map of type $(6, 12, 4, 3)$. Show a plane map of type $(8, 12, 3, 4)$.

∘ **E 1.278** Let G be a biconnected cubic graph with 10 vertices. Show that G cannot be represented by a plane map all of whose faces have the same degree.

E 1.279 Let G be a graph with 11 or more vertices. Show that G and its complement $\overline{G}$ cannot be both planar.

E 1.280 A plane map is **self-dual** if its graph is isomorphic to its dual graph. Show that $2|\mathbb{V}| = |\mathbb{E}| + 2$ if $(\mathbb{V}, \mathbb{E})$ is self-dual. Show that not every plane map $(\mathbb{V}, \mathbb{E})$ such that $2|\mathbb{V}| = |\mathbb{E}| + 2$ is self-dual.

$\star$ **E 1.281** Let G be the graph of a plane map $\mathbb{M}$. Suppose that G is biconnected and has no vertices of degree 2 (i.e., $\delta(G) \geq 3$). Let G^* be the graph of the faces (i.e., the dual graph) of the map $\mathbb{M}$. Show that G^* is planar.

Let $\mathbb{M}^*$ be a plane map that represents G^*. Let G^{**} be the graph of the faces of $\mathbb{M}^*$. Show that $G^{**} \cong G$, i.e., that G^{**} is isomorphic to G.

! E 1.282 (WHITNEY'S THEOREM) Every 3-connected planar graph (see page 129) has essentially a unique plane map. Here "essentially" means that all the plane maps are equivalent. Two maps of a same graph are **equivalent** if the set of edges of the corresponding face boundaries are equal.

E 1.283 (OUTERPLANAR) A plane map $\mathbb{M}$ is **outerplane** if all its points are on the boundary of the same face. A graph G is **outerplanar** if it is representable by an outerplane map.

Show that K_4 is not outerplanar. Show that $K_{2,3}$ is not outerplanar.

E 1.284 Show that the dual graph of an outerplane map (see exercise 1.283) may not be planar.

E 1.285 Let $\mathbb{M}$ be an outerplane map (see exercise 1.283). Let F be the face whose boundary contains all the points of $\mathbb{M}$. Let G^* be the graph of the faces (i.e., the dual graph) of $\mathbb{M}$. Show that $G^* - F$ is a tree.

E 1.286 Let e be an edge of an outerplanar graph G (see exercise 1.283). Is it true that there is an outerplane map of G in which the representation of e belongs to the boundary of the face that contains all the vertices?

1.18 Random graphs

V $\mathcal{G}(n)$ Let V be the set $\{1, \ldots, n\}$ and let $\mathcal{G}(n)$ be the collection[51] of all the graphs with vertex set V. It is obvious that

$$|\mathcal{G}(n)| = 2^N \,, \quad \text{com } N := \tbinom{n}{2} \,.$$

N Any graph property (like, for example, the property of being connected)[52] defines a subcollection of $\mathcal{G}(n)$. Thus, we do not distinguish the concepts of "property" and "subcollection" of $\mathcal{G}(n)$. We will say that **almost every graph** has a given property $\mathcal{P}(n)$ if

$$\lim_{n\to\infty} \frac{|\mathcal{P}(n)|}{|\mathcal{G}(n)|} \;=\; 1 \,.$$

One way to study the set $\mathcal{G}(n)$ is based on the introduction of a probability measure in this set. Let p be a number in the interval $(0, 1)$, and choose each element of $V^{(2)}$, independently, with probability p. (See exercise 1.20.) If A is the set of chosen pairs, then (V, A) is a **random graph** in $\mathcal{G}(n)$. The probability that a graph (V, A) built in this way is identical to a given element of $\mathcal{G}(n)$ with m edges is

$$p^m \, (1-p)^{N-m} \,.$$

If $p = \frac{1}{2}$, then all the 2^N graphs in $\mathcal{G}(n)$ are equiprobable: the probability of obtaining any one of them is $1/2^N$.

Exercises

∘ **E 1.287** Show that almost every graph in $\mathcal{G}(n)$ has more than 10000 edges.

E 1.288 Show that almost every graph G in $\mathcal{G}(n)$ is connected. (See section 1.18.)

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[51] "Collection" is the same as "set."

[52] Naturally, we are only interested in properties that are invariant under isomorphism (see chapter 2).

Chapter 2

Isomorphism

Two graphs are isomorphic if they have the same "structure." The exact definition of the concept is a bit laborious, as we will see next.

An **isomorphism** between two graphs G and H is a bijection[1] f from V_G to V_H such that, for every pair (v, w) of elements of V_G, v and w are adjacent in G if and only if $f(v)$ and $f(w)$ are adjacent in H.

Two graphs G and H are **isomorphic** if there exists an isomorphism between them. In other words, two graphs are isomorphic if it is possible to change the names of the vertices in one of them in such a way that the two graphs become equal. The expression "$G \cong H$" is an abbreviation for "G is isomorphic to H." $\cong$

> ISOMORPHISM PROBLEM: Decide whether two given graphs are isomorphic.

Exercises

E 2.1 A graph G has vertex set $\{a, b, c, d\}$ and edge set $\{ab, bc, cd, da\}$. A graph H has vertex set $\{a, b, c, d\}$ and edge set $\{ab, bd, dc, ca\}$. Are graphs G and H equal?

E 2.2 Are the graphs G and H described below isomorphic?

$$\begin{array}{ll} V_G = \{a, b, c, d, e, f, g\} & E_G = \{ab, bc, cd, cf, fe, gf, ga, gb\} \\ V_H = \{h, i, j, k, l, m, n\} & E_H = \{hk, nj, jk, lk, lm, li, ij, in\} \end{array}$$

What if we replace hk with hn in E_H?

[1] A *bijection* is a function f from a set A to a set B such that (1) $f(a) \neq f(a')$ whenever $a \neq a'$, and (2) for every b in B there exists a in A such that $b = f(a)$.

E 2.3 Are the graphs in figure 2.1 isomorphic?

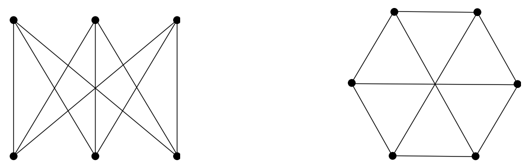

Figure 2.1: Are these graphs isomorphic?

E 2.4 Show that two paths are isomorphic if and only if they have the same length. Show that two circuits are isomorphic if and only if they have the same length.

E 2.5 List all the graphs with vertex set $\{1, 2, 3, 4\}$ that are pairwise non-isomorphic. (In other words, make the list in such a way that every graph with 4 vertices is isomorphic to one and only one of the graphs in the list.)

E 2.6 For $n = 1$, 2, 3, $\ldots$, make a list of all the trees with vertex set $\{1, 2,$ $3, \ldots, n\}$ that are pairwise non-isomorphic.

E 2.7 Are the graphs in figure 2.2 pairwise isomorphic?

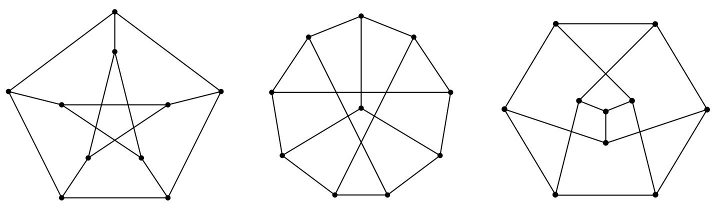

Figure 2.2: Are these graphs pairwise isomorphic?

E 2.8 Are the graphs in figure 2.3 isomorphic? Justify your answer.

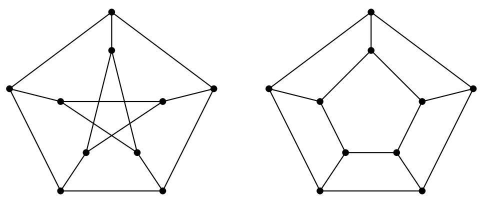

Figure 2.3: Are these graphs isomorphic?

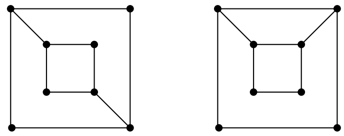

Figure 2.4: Are these graphs isomorphic?

E 2.9 Are the graphs in figure 2.4 isomorphic? Justify your answer.

E 2.10 Let G be the 3-by-4 grid and H be the 4-by-3 grid (see exercise 1.6). Are graphs G and H equal? are they isomorphic?

E 2.11 Show that the p-by-q grid and the graph defined in exercise 1.7 are isomorphic.

E 2.12 Show that the cube Q_3 is isomorphic to some subgraph of Q_4.

E 2.13 Show that the cube Q_4 has a subgraph isomorphic to the 4-by-4 grid.

E 2.14 For which values of t are the two components of the t-by-t bishop graph isomorphic?

D 2.15 Characterize the graphs that are isomorphic to a subgraph of Q_k.

E 2.16 Suppose that graphs G and H are isomorphic. Show that $n(G) = n(H)$ and $m(G) = m(H)$. Show that, for any isomorphism f from G to H, we have $\mathrm{d}_G(v) = \mathrm{d}_H(f(v))$ for every v in V_G. Assuming G has a circuit of length k, show that H also has a circuit of length k.

E 2.17 (ALGORITHM) The following algorithm is supposed to decide whether graphs G and H are isomorphic:

> examine all bijections from V_G to V_H;
> if one of them is an isomorphism, then G is isomorphic to H;
> otherwise, G and H are not isomorphic.

Discuss the algorithm.

E 2.18 (ALGORITHM) The following algorithm is supposed to decide whether graphs G and H are isomorphic:

> if $n(G) \neq n(H)$ then G and H are not isomorphic;
> if $m(G) \neq m(H)$ then G and H are not isomorphic;
> if $|\{ v \in V_G : \mathrm{d}_G(v) = i \}| \neq |\{ v \in V_H : \mathrm{d}_H(v) = i \}|$ for some i

then G and H are not isomorphic;
otherwise G and H are isomorphic.

Discuss the algorithm.

E 2.19 Let G and H be two graphs and f be a bijection from V_G to V_H such that $\mathrm{d}_G(v) = \mathrm{d}_H(f(v))$ for every v in V_G. Is it true that $G \cong H$?

∘ **E 2.20** True or false? To show that two graphs G and H with the same number of vertices are *not* isomorphic, it suffices to exhibit a bijection f from V_G to V_H and a pair of vertices u and v in V_G such that (1) $uv \in E_G$ but $f(u)f(v) \notin E_H$ or (2) $uv \notin E_G$ but $f(u)f(v) \in E_H$.

E 2.21 Let $\mathcal{A}$ be the set of all graphs that represent the alkanes that have formula C_4H_{10}. (See exercise 1.5.) Each of these alkanes has 4 vertices of degree 4 and 10 vertices of degree 1. How many pairwise non-isomorphic graphs are there in $\mathcal{A}$?

E 2.22 Show that the line graph (see exercise 1.24) of a K_5 is isomorphic to the complement of the Petersen graph.

E 2.23 Is it true that every graph is isomorphic to the line graph (see exercise 1.24) of some other graph?

E 2.24 Let X be a set of vertices in a graph G. Suppose that the induced subgraph $G[X]$ is a star (see section 1.2) with 4 vertices. Show that G is not isomorphic to the line graph (see exercise 1.24) of another graph, i.e., that there is no graph H such that $G \cong L(H)$.

E 2.25 Let G be the Petersen graph. Given any two vertices u and v in G, show that there exists an isomorphism from G to G (isomorphisms of this kind are known as *automorphisms*) which maps u to v. Given any two edges uv and xy of G, show that there exists an isomorphism from G to G that maps uv to xy (i.e., maps u to x and v to y, or maps u to y and v to x).

E 2.26 A graph is **self-complementary** if it is isomorphic to its complement. Assuming G is a self-complementary graph, show that $n(G) = 0 \pmod 4$ or $n(G) = 1 \pmod 4$.[2]

[2] The expression "$x = i \pmod 4$" means that the remainder of the division of x by 4 and the remainder of the division of i by 4 are equal, i.e., that $x \bmod 4 = i \bmod 4$. In other words, $x - i$ is divisible by 4.

D 2.27 Find an efficient characterization of non-isomorphic graphs. In other words, find a property $\mathcal{X}$ which is easy to verify and makes the following statement true: "Two graphs G and H are not isomorphic if and only if $\mathcal{X}$."

D 2.28 (ALGORITHM) Sketch a fast algorithm to solve the isomorphism problem (i.e., to decide whether two given graphs are isomorphic).

Chapter 3

Graph design with given degrees

A graph G **realizes** a sequence $(g_1, g_2, \ldots, g_n)$ of natural numbers[1] if the vertices of the graph are $1, 2, \ldots, n$ and $\mathrm{d}(i) = g_i$ for every i.

A sequence $(g_1, g_2, \ldots, g_n)$ of natural numbers is **graphic** if some graph realizes it. graphic sequence

> GRAPH DESIGN PROBLEM: Given a sequence of natural numbers, decide if it is a graphic or not.

Exercises

E 3.1 Consider the sequences $(2,2,0)$, $(1,1,2,2)$, $(0,3,1,0)$, $(0,1,2,2,3)$, $(3,3,2,2,1)$, $(6,6,5,4,3,3,1)$ and $(7,6,5,4,3,3,2)$. Which of these sequences are graphic?

E 3.2 Suppose that $(g_1, g_2, \ldots, g_n)$ is a graphic sequence. Show that $g_i \leq n-1$ for every i and that $\sum g_i$ is even. Formulate the converse; is it true?

E 3.3 Show that $(g_1, g_2, \ldots, g_n)$ is a graphic sequence if and only if $(n-g_1-1, n-g_2-1, \ldots, n-g_n-1)$ is a graphic sequence. (This fact can be used as a basis for an algorithm to recognize graphic sequences.)

E 3.4 Prove that, for each $n \geq 5$, there is a 4-regular graph with n vertices.

E 3.5 Is it true that, for every number r, there is a r-regular graph? Is it true that, for every pair (r, n) of numbers such that $r < n$, there is a r-regular graph with n vertices?

[1] The set of the natural numbers is $\{0, 1, 2, \ldots\}$.

E 3.6 Suppose that $(g_1, g_2, \ldots, g_n)$ is a graphic sequence and that k is a natural number $\leq n$. Show that

$$(k, g_1{+}1, g_2{+}1, \ldots, g_k{+}1, g_{k+1}, \ldots, g_n)$$

is also a graphic sequence. Formulate the converse statement; is it true?

⋆ **E 3.7** (THEOREM OF HAVEL AND HAKIMI[2]) *Let $(k, g_1, g_2, \ldots, g_n)$ be a sequence of natural numbers such that $k \geq g_1 \geq g_2 \geq \ldots \geq g_n$. Suppose that the sequence*

$$(g_1 - 1,\ g_2 - 1,\ \ldots,\ g_k - 1,\ g_{k+1},\ \ldots,\ g_n)$$

is graphic. Show that the first sequence is also graphic.

⋆ **E 3.8** (THEOREM OF ERDŐS[3] AND GALLAI[4]) *Show that a sequence $(g_1, g_2, \ldots, g_n)$ of natural numbers is graphic if and only if $\sum_i g_i$ is even and*

$$\sum_{i=1}^{k} g_i \;\leq\; k(k-1) \;+ \sum_{i=k+1}^{n} \min\,(k, g_i)$$

for each k such that $1 \leq k \leq n$.

E 3.9 (ALGORITHM) Sketch an efficient algorithm to solve the graph design problem stated above, i.e., to decide whether a given sequence of natural numbers is graphic. Seek for inspiration in exercises 3.6 and 3.7, or in exercise 3.8.

E 3.10 Let $g_1, \ldots, g_n$ be positive integer numbers. Suppose that $\sum_{i=1}^{n} g_i = 2(n-1)$. Show that there is a tree (see section 1.15) T with vertices $1, \ldots, n$ such that $\mathrm{d}(i) = g_i$ for each i.

2 Published in 1955 by Václav Havel, and in 1962 by S. Louis Hakimi.
3 Paul Erdős (1913 – 1996). (See article in Wikipedia.)
4 Tibor Gallai (1912 – 1992).

Chapter 4

Bicolorable graphs

A **bicoloring** of a graph G is a bipartition[1] $\{U, W\}$ of V_G such that every edge of G has an endpoint in U and another in W. (You can imagine that all the vertices in U are red and all the vertices in W are blue.) For example, every bipartite graph (see section 1.2) has an obvious bicoloring.

BICOLORING PROBLEM: Find a bicoloring of a given graph.

Deciding whether the problem has a solution at all — i.e., whether the graph admits[2] a bicoloring — is part of the problem.[3] We will say that a graph is **bicolorable** if it admits a bicoloring. Therefore, a bicolorable graph is the same as a bipartite graph.[4]

As we will see later (exercise 4.15), a graph is bicolorable if and only if it does not contain an odd circuit. We say that a circuit is **odd** if its length is an odd number.

Exercises

$\star$ **E 4.1** Show that a graph G is bicolorable if and only if E_G is a cut.

E 4.2 Show that the t-by-t knight graph is bicolorable.

---

[1] A **bipartition** of a set V is a pair $\{U, W\}$ of non-empty subsets of V such that $U \cup W = V$ and $U \cap W = \emptyset$. The bipartition is the pair $\{U, W\}$; it makes no sense to say "U is one of the bipartitions of V." Instead, say "U is one of the elements of the bipartition."

[2] The expression "admits a bicoloring" means the same as "has a bicoloring."

[3] Many problems in graph theory are of the kind "show that this graph does *not* have property X." In the present case, the question is "show that this graph does not admit a bicoloring." The answer "try all possible bipartitions of V_G" is not satisfactory, because the number of bipartitions is huge: a set of size n has 2^{n-1} different bipartitions.

[4] Actually, there is a subtle distinction between the two concepts: a bicolorable graph only becomes bipartite after one of its bicolorings is explicitly given.

E 4.3 Is it true that the t-by-t bishop graph is bicolorable?

E 4.4 Show that every cube Q_k is bicolorable.

E 4.5 Show that every grid is bicolorable.

◦ **E 4.6** Show that every path is bicolorable. Show that every even-length circuit is bicolorable.

E 4.7 Show that a graph may have two or more different bicolorings. Show that connected graphs have at most one bicoloring.

▷ **E 4.8** Show that every forest is bicolorable.

E 4.9 Let $\{U, W\}$ be a bicoloring of a forest such that $|U| = |W|$. Show that the forest has at least one leaf in U and one in W.

◦ **E 4.10** Suppose that a graph G is bicolorable. Is it true that every subgraph of G is bicolorable? Is it true that every induced subgraph of G is bicolorable?

E 4.11 Are the graphs in figure 4.1 bicolorable?

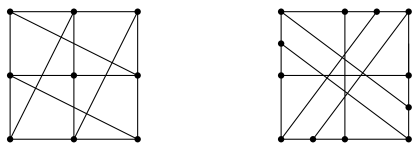

Figure 4.1: Exercise 4.11. Are these graphs bicolorable?

E 4.12 ♡ How many edges can a bicolorable graph with n vertices have?

◦ **E 4.13** Suppose that a graph G has an odd circuit. Show that G is not bicolorable.

⋆ **E 4.14** Show that every graph without odd circuits is bicolorable.[5]

⋆ **E 4.15** (SOLVING THE BICOLORING PROBLEM) *Deduce from 4.13 and 4.14 that a graph is bicolorable if and only if it has no odd circuits.*

[5] Therefore, an odd circuit is a *certificate of inexistence* of a bicoloring of the graph. Conversely, a bicoloring of the graph is a *certificate of absence* of odd circuits.

E 4.16 We say that a graph G has an *induced* circuit if there exists $X \subseteq V_G$ such that $G[X]$ is a circuit. Show that a graph is bicolorable if and only if it has no induced odd circuits.

E 4.17 (ALGORITHM) Devise an efficient algorithm to decide whether a given graph is bicolorable. The algorithm must return a bicoloring of the graph or an odd circuit.

E 4.18 (EVEN/ODD PATH) Let G be a biconnected graph. Let r and s be two vertices of G. There is an even-length path from r to s and an odd-length path from r to s (the two paths are not necessarily disjoint) if and only if G is not bicolorable.

E 4.19 (ODD CIRCUIT) Let G be a biconnected graph and suppose that G has an odd-length circuit. Show that every vertex of G belongs to an odd-length circuit.

Characterization of cuts

E 4.20 Characterize the sets of edges of a graph that are cuts, i.e., give conditions under which a set of edges of a graph is a cut.

E 4.21 (ALGORITHM) Sketch an efficient algorithm for the following task: Given a graph G and a subset C of E_G, the algorithm must decide whether C is a cut. In the "yes" case, the algorithm should return a set X of vertices such that $\partial(X) = C$. What should your algorithm return in the "no" case?

E 4.22 Let D be a cut and O a circuit in a graph. Show that $|D \cap E_O|$ is even.

E 4.23 (Converse of 4.22.) Let D be a set of edges of a graph G. Suppose $|D \cap E_O|$ is even for every circuit O in G. Show that D is a cut.

E 4.24 Let D be a cut and P a path in a graph G. What can you say about the parity of $|D \cap E_P|$?

E 4.25 A *signed graph* is a triple (G, P, N) in which G is a graph and (P, N) is a partition of the set E_G. The edges in P are *positive* and the others are *negative*. A signed graph (G, P, N) is *balanced* if every circuit in G has an even number of negative edges. Prove that a signed graph (G, P, N) is balanced if and only if there is a bipartition $\{U, W\}$ of V_G such that $\partial(U) = N$.

Chapter 5

Stable sets

A set X of vertices of a graph is **stable**, or **independent**, if its elements are pairwise non-adjacent. In other words, X is stable if no edge of the graph has both endpoints in X, i.e., if the induced graph $G[X]$ is empty. For example, if $\{U, W\}$ is a bicoloring of the graph, then sets U and W are stable.

A stable set X^* is **maximum** if there is not stable set X such that $|X| > |X^*|$. In other words, X^* is maximum if $|X^*| \geq |X|$ for every stable set X. maximum

> MAXIMUM STABLE SET PROBLEM: Find a maximum stable set in a given graph.

It is important not to confuse *maximum* and *maximal*. A stable set X' is **maximal** if it is not part of a bigger stable set, i.e., if there is no stable set X such that $X \supset X'$.[1] maximal

A variant of the problem is the following: Given a graph and a natural number k, find a stable set with k or more vertices. (Clearly this variant does not always have a solution.)

The size of a maximum stable set in a graph G is known as **stability number**, or **independence number**, of G and is denoted by

$$\alpha(G) .$$

Exercises

E 5.1 Show that the stability number is invariant under isomorphism. In other words, if G and H are isomorphic graphs, then $\alpha(G) = \alpha(H)$.

◦ **E 5.2** Find a maximum stable set in a K_n. Find a maximum stable set in

[1] The expression "$A \supset B$" means "B is a proper subset of A", i.e., $B \subseteq A$ but $B \neq A$.

a $\overline{K_n}$.

E 5.3 In the graph of figure 5.1, show a maximal stable set which is not maximum.

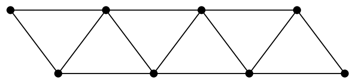

Figure 5.1: See exercise 5.3.

∘ **E 5.4** Suppose that X and Y are maximal stable sets of a graph. Is it true that X and Y are disjoint (i.e., that $X \cap Y = \emptyset$)?

∘ **E 5.5** Calculate a maximum stable set in a path. Calculate a maximum stable set in a circuit.

E 5.6 Find a maximum stable set in the p-by-q grid.

E 5.7 Show a maximum stable set in the cube Q_k.

E 5.8 Find a maximum stable set in the knight graph.

E 5.9 Find a maximum stable set in the bishop graph.

E 5.10 Find a maximum stable set in the queen graph. (In other words, arrange the greatest possible number of queens on the board so that they do not attack each other.)

E 5.11 Find a maximum stable set in the Petersen graph.

E 5.12 Find a maximum stable set in the Kneser graph $K(n, k)$ (see exercise 1.16).

E 5.13 Find a maximum stable set in the graph of Europe (see exercise 1.17).

⋆ **E 5.14** Let G be a bicolorable graph with a bicoloring $\{U, W\}$, and suppose that $|U| \geq |W|$. Is it true that U is a maximum stable set?

E 5.15 Suppose that a graph G admits a bicoloring. Is it true that every maximal stable set of G is maximum? What if G is a tree?

∘ **E 5.16** Let H be a subgraph of a graph G. What is the relationship between $\alpha(H)$ and $\alpha(G)$?

∘ **E 5.17** Let G and H be two graphs such that $V_G \cap V_H = \emptyset$. Show that $\alpha(G \cup H) = \alpha(G) + \alpha(H)$.

∘ **E 5.18** Let A be the adjacency matrix of a graph G (see exercise 1.3). Let X be a stable set of G. What does the restriction of A to $X \times X$ look like?

E 5.19 (ALGORITHM) Discuss the following algorithm for the maximum stable set problem:

> given a graph G, examine each of the subsets of V_G;
> discard the non-stable subsets;
> find a largest among the remaining.

D 5.20 (ALGORITHM) *Devise a fast algorithm to solve the maximum stable set probem. Devise, at least, an algorithm that yields a large stable set.*[2]

∘ **E 5.21** (ALGORITHM) Devise an algorithm that finds a maximal stable set in any given graph. (Hint: use a "greedy" strategy: in each iteration, pick any reasonable vertex.[3])

E 5.22 (ALGORITHM) The following greedy algorithm receives a graph G and returns a stable set X:

```
X ← ∅
H ← G
while V_H ≠ ∅ do
    pick v in V_H such that |N_H(v)| is minimum
    X ← X ∪ {v}
    Z ← {v} ∪ N_H(v)
    H ← H − Z
return X
```

Is it true that this algorithm returns a maximum stable set for any given graph G? What if G is bipartite? What if G is a tree?

E 5.23 ♡ Prove that every maxi*mal* stable set of any graph G has at least

$$\left\lceil \frac{n(G)}{\Delta(G)+1} \right\rceil$$

---

[2] A fast algorithm to find a maximum stable set in any graph is not (yet?) known. In technical terms, this problem is NP-hard. See observation in page 5.

[3] Greedy algorithms grab the object that seems best in the current iteration, without regard for the consequences of this choice in the long term.

vertices.[4] From this, deduce that $\alpha(G) \geq \frac{n(G)}{\Delta(G)+1}$ for every graph G.

E 5.24 $\heartsuit$ Prove that every graph G satisfies the inequality

$$\alpha(G) \geq \sum_{v \in V_G} \frac{1}{\mathrm{d}(v)+1}\,.$$

I.e., prove that G has a stable set with $\left\lceil \sum \frac{1}{\mathrm{d}(v)+1} \right\rceil$ vertices.

E 5.25 Let G_t be the t-by-t queen graph. Use exercise 5.24 to estimate $\alpha(G_t)$.

E 5.26 Let X be the stable set produced by the algorithm in exercise 5.22. Show that $|X| \geq \sum_{v \in V_G} 1/(\mathrm{d}(v)+1)$.

E 5.27 Prove that every graph G satisfies the inequality

$$\alpha(G) \geq \frac{n}{\mu+1}\,,$$

where n is the number of vertices, m is the number of edges, and μ is the average degree of the vertices of G.

E 5.28 Let us say that a *path cover* of a graph G is a collection $\{P_1, \ldots, P_k\}$ of paths in G such that $V_{P_1} \cup \cdots \cup V_{P_k} = V_G$. Suppose that every path cover of a graph G has at least k paths. Show that $\alpha(G) \geq k$. In other words, show that G has a stable set with at least k vertices.

E 5.29 (ALGORITHM) Sketch an efficient algorithm that receives a bipartite graph and returns a maximum stable set.[5]

$\alpha \leq$ **E 5.30** Let G be a graph without isolated vertices. Show that

$$\alpha(G) \leq m(G)/\delta(G)\,.$$

In other words, show that G has no stable set with more than $\lfloor m(G)/\delta(G) \rfloor$ vertices.[6]

E 5.31 Let n and a be two positive integers. Let $k = \lfloor n/a \rfloor$ and $r = n - ka$. Let H be the graph resulting from the union of r copies of K_{k+1} and $a - r$ copies of K_k (the vertex sets of the copies are pairwise disjoint). Observe that

$$n(H) = n, \quad m(H) = r\binom{k+1}{2} + (a-r)\binom{k}{2} \quad \text{and} \quad \alpha(H) = a\,.$$

[4] By definition, $\lceil x \rceil$ is the only integer number j such that $j - 1 < x \leq j$.
[5] We will discuss this algorithm in detail in chapter 10.
[6] By definition, $\lfloor x \rfloor$ is the only integer i such that $i \leq x < i + 1$.

Show that $\alpha(G) > \alpha(H)$ for any graph G such that $n(G) = n$ and $m(G) < m(H)$.[7]

Random graphs

E 5.32 *Show that the stability number of most graphs is not much more than* $2\log_2 n(G)$*. More precisely, show that, for any positive real number* ε*, one has*

$$\alpha(G) < (2+\varepsilon)\,\log_2 n$$

for almost every graph G *in* $\mathcal{G}(n)$*.* (See section 1.18.)

[7] It can be shown that H is the only graph (up to isomorphism) with n vertices, $m(H)$ edges and stability number a. This fact is a known theorem by Paul Turán (1910 – 1976). (See article in Wikipedia.) The complement of H is known as the **Turán graph**.

Chapter 6

Cliques

A **clique** in a graph is any set of pairwise adjacent vertices. In other words, X is a clique if the induced graph $G[X]$ is complete.

> MAXIMUM CLIQUE PROBLEM: Find a maximum clique in a given graph.

Here is a variant of the problem: given a graph and a natural number k, find a clique with k or more vertices.

The size of a maximum clique of a graph G is known as the **clique number** of G and denoted by

$$\omega(G) .$$

Exercises

◦ **E 6.1** Find a maximum clique in a K_n. Find a maximum clique in a $\overline{K_n}$.

◦ **E 6.2** Find a maximum clique in a path. Find a maximum clique in a circuit.

E 6.3 Let G be a circuit of length 6. Find a maximum clique in $\overline{G}$.

E 6.4 Show a graph and a clique that is maximal but not maximum.

E 6.5 Find a maximum clique in the knight graph.

E 6.6 Show a maximum clique in the cube Q_k.

◦ **E 6.7** Suppose that G is a bipartite graph. How many vertices does a maximum clique in G have?

E 6.8 Find a maximum clique in the bishop graph.

E 6.9 Find a maximum clique in the queen graph.

E 6.10 Show that every maximal clique of the Kneser graph $K(n, k)$ (see exercise 1.16) has $\lfloor n/k \rfloor$ vertices. Show a maximum clique in $K(n, k)$.

E 6.11 What is the relationship between the maximum clique problem and the maximum stable set problem (see chapter 5)? How is it possible to use an algorithm that solves one of these problems to solve the other?

∘ **E 6.12** Prove that $\omega(G) \leq \Delta(G) + 1$ for every graph G.

E 6.13 Show that the following relation is valid for any set X of vertices of a graph G: X is a clique in G if and only if X is a stable set in $\overline{G}$. Deduce that $\omega(G) = \alpha(\overline{G})$.

E 6.14 Suppose that a graph G has the following property: for each vertex v, the set $\mathrm{N}(v)$ is a clique. Show that each component of G is a clique.

E 6.15 Show that $\omega(G) \geq 3$ for every graph G with more than $n(G)^2/4$ edges. (See exercise 4.12.)

E 6.16 Suppose that $\omega(G) \leq k$. What is the maximum number of edges (in relation to the number of vertices) that G can have?

E 6.17 Is it true that every maximal clique in a tree is maximum?

∘ **E 6.18** (ALGORITHM) Devise an algorithm that finds a maximal clique in any given graph.

D 6.19 (ALGORITHM) *Devise a fast algorithm to solve the maximum clique problem. Devise, at least, an algorithm to find a* large *clique in any given graph.*[1]

E 6.20 Let $L(G)$ be the line graph (see exercise 1.24) of a graph G. Show that, for each vertex v of G, the set $\partial_G(v)$ is a clique in $L(G)$. Show that the edge set of any triangle in G is a clique in $L(G)$. Show that any clique in $L(G)$ whose size is not 3 is a subset of some set of the form $\partial_G(v)$.

[1] A fast algorithm for the maximum clique problem is not (yet?) known. In technical terms, this problem is NP-hard. See observation in page 5.

E 6.21 Given a graph G, compute a maximum clique in the line graph $L(G)$. (See exercise 6.20.)

E 6.22 Show that a graph H is isomorphic to the line graph (see exercise 1.24) of some other graph G if and only if there is a collection of cliques of H such that each edge of H has both endpoints in one and only one of the cliques, and every vertex of H belongs to at most two of the cliques.

E 6.23 Let G be a planar graph (see section 1.6). Show that $\omega(G) \leq 4$.

Chapter 7

Vertex covers

A **vertex cover**[1], or simply **cover**, of a graph is any set of vertices that contains at least one of the endpoints of each edge. In other words, a set X of vertices is a cover if every edge has at least one of its endpoints in X.

MINIMUM COVER PROBLEM: Find a minimum cover in a given graph.

The cardinality of a minimum cover of a graph G is denoted by $\beta(G)$.

There a simple and intimate relation between vertex covers and stable sets (see exercise 7.6). This relation makes the minimum cover problem equivalent to the maximum stable set problem.

Exercises

E 7.1 An art gallery consists of a large number of straight corridors that interconnect small halls. A guard standing in a hall can watch over all the corridors that radiate from the hall. What is the smallest number of guards needed to watch over the whole gallery?

E 7.2 Find a minimum cover in the knight graph. Find a minimum cover in the bishop graph.

E 7.3 Find a minimum cover in the cube Q_k.

∘ **E 7.4** Show that a graph G is bicolorable if and only if V_G has a subset U which is, at the same time, an independent set and a cover.

∘ **E 7.5** What is a minimal cover? Use the graph in figure 5.1 to give an

[1] Maybe it would be better to say "cover of edges by vertices."

example of a minimal cover. Is it true that every minimum cover is minimal? Is it true that every minimal cover is minimum?

$\star$ **E 7.6** Prove the following relationship between covers and stable sets: in any graph G, a set X of vertices is a cover if and only if $V_G \setminus X$ is stable. Deduce that $\beta(G) = n(G) - \alpha(G)$.

E 7.7 Show that the minimum cover problem is equivalent to the maximum stable set problem. (In other words, show that any algorithm for one of the problems can be converted into an algorithm for the other.)

E 7.8 Suppose that T is a tree. Is it true that every minimal cover of T is minimum?

$\circ$ **E 7.9** Let $\{U, W\}$ be a bicoloring of a graph G. Let $\{X, X'\}$ be a partition of U and $\{Y, Y'\}$ be a partition of W. Suppose that $\mathrm{N}(X) \subseteq Y$. (See the definition of $\mathrm{N}(X)$ in exercise 1.108.) Show that $Y \cup X'$ is a cover.

$\star\circ$ **E 7.10** (APPROXIMATED SOLUTION) Specify an algorithm that will give an approximate solution to the minimum cover problem: given a graph G, the algorithm must return a cover X such that $|X| \leq 2\beta(G)$.

E 7.11 Consider the equivalence discussed in exercise 7.7, and the approximate solution discussed in exercise 7.10. Is it possible to obtain a similar approximation for the maximum stable set problem (see chapter 5)?

Chapter 8

Vertex coloring

A **vertex coloring**, or a **covering by stable sets**, of a graph is a collection of stable sets that covers the set of vertices of the graph. More precisely, a vertex coloring of a graph G is a collection $\{X_1, X_2, \ldots, X_k\}$ of stable sets such that $X_1 \cup X_2 \cup \cdots \cup X_k = V_G$.

Although not essential, it is convenient to assume that sets $X_1, \ldots, X_k$ are pairwise disjoint. We can thus say that a vertex coloring of G is a *partition* of V_G into stable sets. Each vertex of the graph will be in one and only one of those sets. (Clearly the concept of vertex coloring is a generalization of the concept of bicoloring, discussed in chapter 4.)

If we imagine that each stable set X_i corresponds to a color, we can say that a vertex coloring is an assignment of colors to the vertices such that adjacent vertices get different colors.

We say that a coloring $\{X_1, X_2, \ldots, X_k\}$ **uses** k **colors**.[1] We also say that this is a k-**coloring**. If a graph has a k-coloring, then it also has a k'-coloring for all $k' > k$.

A vertex coloring is **minimum** if it uses the smallest possible number of colors, i.e., if no other coloring uses fewer colors.

> VERTEX COLORING PROBLEM: Find a minimum vertex coloring of a given graph.

The **chromatic number** of a graph G is the number of colors in a minimum vertex coloring of G. This number is denoted by

$$\chi(G) .$$

A graph G is k-**colorable** if $\chi(G) \leq k$. In particular, "2-colorable" is the same as "bicolorable", according to chapter 4.

[1] Even if some X_i is empty.

See the WWW site *Graph Coloring Page* by Joseph Culberson at the University of Alberta, Canada.

A **clique cover** of G is any partition $\{X_1, \ldots, X_k\}$ of V_G such that each X_i is a clique. Clearly a clique cover of G is the same as a vertex cover of $\overline{G}$. The corresponding CLIQUE COVER PROBLEM consists of finding the smallest clique cover of V_G.

Exercises

E 8.1 A factory must store a set of chemicals. For safety reasons, certain pairs of chemicals should be kept in separate compartments of the warehouse. How many compartments, at least, should the warehouse have?

∘ **E 8.2** Show that the chromatic number is invariant under isomorphism. In other words, if G and H are isomorphic graphs, then $\chi(G) = \chi(H)$.

∘ **E 8.3** Let $\{X_1, \ldots, X_k\}$ be a vertex coloring of a graph G. Show that there is a coloring $\{Y_1, \ldots, Y_k\}$ such that sets $Y_1, \ldots, Y_k$ are pairwise disjoint.

∘ **E 8.4** Find a minimum vertex coloring of a path. Repeat with a circuit in place of the path. Repeat with a grid in place of the path.

E 8.5 Let R_t be the t-by-t rook graph. Find a minimum vertex coloring of R_t.

E 8.6 Let B_t be the t-by-t bishop graph. Find a minimum vertex coloring of B_t.

E 8.7 Let B_t be the t-by-t bishop graph. Find a minimum clique cover of the graph B_t.

E 8.8 Let K_t be the t-by-t knight graph. Find a minimum vertex coloring of K_t. Find a minimum clique cover of K_t. (Start with the cases $t = 2, \ldots, 6$.)

E 8.9 Let Q_t be the t-by-t queen graph. Find a minimum vertex coloring of Q_t. (Consider first the cases $t = 2, \ldots, 6$.)

E 8.10 Let K_t be the t-by-t king graph. Find a minimum vertex coloring of K_t.

E 8.11 Find a minimum vertex coloring of the graph of Europe (see exercise 1.17).

E 8.12 Find a minimum vertex coloring of the cube Q_k. Find a minimum clique cover of the cube Q_k.

E 8.13 Find a minimum vertex coloring of the Petersen graph.

E 8.14 Find a minimum vertex coloring of the graph G defined in the following manner: begin with five mutually disjoint copies, say $B_1, \ldots, B_5$, of a complete graph on 3 vertices; for each i, add edges connecting all vertices of B_i to all vertices of B_{i+1}; finally, add edges connecting all vertices of B_5 to all vertices of B_1. (This graph was used by Catlin[2] as a counterexample for Hajós' conjecture. See exercise 8.71.)

E 8.15 We are given machines $1, \ldots, n$ and time intervals $I_1, \ldots, I_n$. For each i, a worker must operate machine i during the interval I_i. If $I_i \cap I_j \neq \emptyset$, the same worker cannot operate both i and j. How many workers are necessary and suficient to operate all the machines? (See exercise 1.22.)

∘ **E 8.16** Show a graph with two different minimum vertex colorings.

∘ **E 8.17** Suppose that a graph G has a vertex coloring with k colors. Is it true that G has a coloring $\{X_1, \ldots, X_k\}$ such that X_1 is a maximum independent set?

∘ **E 8.18** Let G be a graph with at least one edge. Prove that there is a partition $\{A, B\}$ of V_G such that $\chi(G[A]) + \chi(G[B]) = \chi(G)$.

∘ **E 8.19** Let G and H be two graphs such that $V_G \cap V_H = \emptyset$. Show that $\chi(G \cup H) = \max\{\chi(G), \chi(H)\}$.

∘ **E 8.20** Let H be a subgraph of a graph G. What is the relationship between $\chi(H)$ and $\chi(G)$?

E 8.21 Let e be a bridge of a graph G with two or more edges. Show that $\chi(G - e) = \chi(G)$.

E 8.22 Let v be an articulation of a graph G. Is it true that $\chi(G) = \chi(G - v)$?

E 8.23 Let v be a vertex of a graph G. Suppose that $\mathrm{d}(v) < \chi(G) - 1$. Show that $\chi(G) = \chi(G - v)$.

E 8.24 Show that, for every graph G, every vertex coloring of G uses at least $\lceil n(G)/\alpha(G) \rceil$ colors. In other words, show that $\chi(G) \geq n(G)/\alpha(G)$. $\chi \geq$

E 8.25 (GENERALIZATION OF 8.24) For each vertex v of a graph G, let α_v be the cardinality of a maximum stable set among those that contain v. Show
$\chi \geq$ that $\chi(G) \geq \sum_v 1/\alpha_v$.

E 8.26 Show that every graph with n vertices and chromatic number k has at
$\chi \geq$ most $\frac{1}{2}(n^2 - n^2/k)$ edges. Deduce that $\chi(G) \geq n^2/(n^2 - 2m)$ for every graph G with n vertices and m edges.[3] Deduce that $\chi(G) \geq n/(n-r)$ if G is r-regular.[4]

E 8.27 Show that $\chi(G) \leq \frac{1}{2} + \sqrt{2m(G) + \frac{1}{4}}$ for every graph G.

E 8.28 (ALGORITHM) Does the following algorithm solve the vertex coloring problem? For an input graph G, the algorithm does the following:

> Picks a maximum stable set, say X_1, in G. Then, picks a maximum stable set X_2 in $G - X_1$. Then, picks a maximum stable set X_3 in $(G - X_1) - X_2$. And so on, until there are no more vertices to pick.

E 8.29 (ALGORITHM) The following greedy algorithm receives a graph G and returns a vertex coloring $X_1, \ldots, X_k$. Each iteration begins with a collection $X_1, \ldots, X_k$ of stable sets; the first iteration may begin with the empty collection, i.e., with $k = 0$. Each iteration consists of the following:

> CASE 1: $X_1 \cup \ldots \cup X_k = V_G$.
> Return $X_1, \ldots, X_k$ and stop.
>
> CASE 2: $X_1 \cup \ldots \cup X_k \neq V_G$.
> Pick a vertex v in $V_G \setminus (X_1 \cup \ldots \cup X_k)$.
> If $X_i \cup \{v\}$ is stable for some i between 1 and k,
> then start a new iteration with $X_i \cup \{v\}$ replacing X_i.
> Otherwise, make $X_{k+1} = \{v\}$ and
> start a new iteration with $k+1$ replacing k.

Does this algorithm solve the vertex coloring problem?

E 8.30 ♡ Prove that every graph G has a vertex coloring with only $\Delta(G) + 1$ colors. In other words, prove that $\chi(G) \leq \Delta(G) + 1$ for every graph G.

∘ **E 8.31** Is it true that $\chi(G) \geq \Delta(G)$ for every graph G? In other words, is it true that every vertex coloring of G uses at least $\Delta(G)$ colors?

---

[2] P. A. Catlin, 1979.

[3] Some examples: if $m > 0$ then $\chi > 1$; if $m > n^2/4$ then $\chi > 2$; if $m > 4n^2/9$ then $\chi > 9$.

[4] Examples: if $r > 0$ then $\chi > 0$; if $r > n/2$ then $\chi > 2$; if $r > 2n/3$ then $\chi > 3$.

E 8.32 Prove that every graph G admits a clique cover of size at most $n(G) - \delta(G)$. In other words, show that $\chi(\overline{G}) \leq n(G) - \delta(G)$.

E 8.33 ♡ Let G be a non-regular connected graph. Show that $\chi(G) \leq \Delta(G)$. (Compare to exercise 8.30.)

E 8.34 (BROOKS' THEOREM[5]) Let G be a connected graph which is not complete and different from an odd circuit. Show that $\chi(G) \leq \Delta(G)$. (Compare to exercise 8.33.)

E 8.35 Show that the difference $\Delta(G) - \chi(G)$ may be arbitrarily large. Show that the quotient $\Delta(G)/\chi(G)$ may be arbitrarily large. (Compare to exercise 8.30.)

E 8.36 (GENERALIZATION OF 8.30) Let v_1, v_2, …, v_n be the vertices of a graph G and suppose that $\mathrm{d}(v_1) \geq \mathrm{d}(v_2) \geq \cdots \geq \mathrm{d}(v_n)$. Show that $\chi(G) \leq \max_{i=1}^{n} \min\{i, \mathrm{d}(v_i) + 1\}$. (Hint: the right side of the inequality is equal to $\max\{\, i : \mathrm{d}(v_i) + 1 \geq i \,\}$.)

E 8.37 Let G be a graph with the following property: every pair of odd circuits has (at least) one vertex in common. Prove that G has a 5-coloring.

⋆ **E 8.38** Suppose a graph G has a clique with k vertices. Show that every vertex coloring of G uses at least k colors. In other words, show that $\chi \geq$

$$\chi(G) \geq \omega(G)$$

for every graph G. (See exercises 6.8 and 6.9.)

E 8.39 Suppose a graph G has a clique with k vertices and a vertex coloring that uses k colors. Prove that the clique is maximum and that the coloring is minimum.[6]

E 8.40 Prove that $\chi(G) = \omega(G)$ for every bipartite graph G.

⋆ **E 8.41** (ALGORITHM[7]) Let $v_1, v_2, \ldots, v_n$ be the vertices of a graph G. Suppose that, for $i = 2, \ldots, n$, the set

$$\{v_1, \ldots, v_{i-1}\} \cap \mathrm{N}(v_i)$$

[5] Published by R. L. Brooks in 1941.

[6] Such a clique can be used as a *minimality certificate* for the coloring. Conversely, such coloring can be used as a *maximality certificate* for the clique.

[7] *Perfect Elimination Scheme*.

is a clique. Show that $\chi(G) = \omega(G)$. (Hint: Write an algorithm to calculate a minimum vertex coloring and a maximum clique.)

E 8.42 Show that, for every k, there is a graph without cliques of size k that does not admit a coloring with k (or less) colors. In other words, show that there are graphs G such that $\chi(G) > \omega(G)$. Show that for every k there is a graph G such that $\chi(G) = k$ and $\omega(G) = 2$.

E 8.43 Suppose a graph G has a stable set with k vertices. Show that every clique cover of V_G uses at least k cliques. Deduce that $\chi(\overline{G}) \geq \alpha(G)$.

E 8.44 Suppose a graph G has no induced subgraph isomorphic to a path on 4 vertices. Show that $\chi(G) = \omega(G)$.

E 8.45 Suppose a graph G has no induced subgraph isomorphic to either $K_{1,3}$ or $K_4 - e$. Show that $\chi(G) \leq \omega(G) + 1$.

E 8.46 Let G be a graph (not necessarily bicolorable), and let $\{R, S\}$ be a partition of V_G. Suppose that $\mathrm{d}(R) < k$ (i.e., there are less than k edges with an endpoint in R and another in S). Suppose also that graphs $G[R]$ and $G[S]$ admit vertex colorings with only k colors. Show that G admits a vertex coloring with k colors.

E 8.47 Let P be a maximum-length path in a graph G. Show that $\chi(G) \leq n(P)$. (In other words, if a graph has no path with more than k vertices, then it can be colored with only k colors.)

E 8.48 (GALLAI–ROY THEOREM[8]) For any acyclic orientation[9] D of a graph G, let $l(D)$ be the length of a maximum directed path[10] in D. Then

$$\chi(G) = 1 + \min_{D} l(D) \, .$$

D 8.49 (ALGORITHM) *Devise a fast algorithm to solve the vertex coloring problem.*[11]

[8] Published in 1966 by Tibor Gallai and in 1967, independently, by Bernard Roy.

[9] An **orientation** of a graph consists of the substitution of each edge vw by the ordered pair (v, w) or by the ordered pair (w, v). Such ordered pair is called an **arc**. The result is a **directed graph**. A directed graph D is **acyclic** if it has no directed circuits. A circuit is **directed** if all its arcs are oriented in the same direction.

[10] A path is **directed** if all its arcs are oriented in the same direction.

[11] A fast algorithm for the problem is not (yet?) known. In technical terms, this problem is NP-hard. See observation in page 5.

Coloring with a given number of colors

If the number of available colors is fixed, we have the following variant of the coloring problem:

> COLORING PROBLEM WITH GIVEN NUMBER OF COLORS: Given a natural number k and a graph G, find a k-coloring of G.

(Clearly, the problem does not always have a solution.) The 2-coloring problem, for example, is equivalent to the problem of deciding whether a given graph is bicolorable (see exercise 4.15).

E 8.50 The following algorithm receives a graph G and promises to return a bicoloring of G. Each iteration begins with a pair (X_1, X_2) of stable sets; the first iteration may begin with $X_1 = X_2 = \emptyset$. Each iteration consists of the following:

CASE 1: $X_1 \cup X_2 = V_G$.
  Return X_1, X_2 and stop.

CASE 2: $X_1 \cup X_2 \neq V_G$.
  Pick a vertex v in $V_G \setminus (X_1 \cup X_2)$.
  Pick i in $\{1, 2\}$ such that $X_i \cup \{v\}$ is stable.
  Start a new iteration with $X_i \cup \{v\}$ in the role of X_i.

Does this algorithm deliver on its promise (i.e., does it produce a 2-coloring of the graph)?

E 8.51 The following greedy algorithm receives a graph G and promises to return a 3-coloring of G. Each iteration begins with stable sets X_1, X_2 and X_3; the first iteration may begin with $X_1 = X_2 = X_3 = \emptyset$. Each iteration consists of the following:

CASE 1: $X_1 \cup X_2 \cup X_3 = V_G$.
  Return X_1, X_2, X_3 and stop.

CASE 2: $X_1 \cup X_2 \cup X_3 \neq V_G$.
  Pick a vertex v in $V_G \setminus (X_1 \cup X_2 \cup X_3)$.
  Pick i in $\{1, 2, 3\}$ such that $X_i \cup \{v\}$ is stable.
  Start a new iteration with $X_i \cup \{v\}$ in place of X_i.

Does the algorithm deliver on its promise?

E 8.52 The following greedy algorithm receives a graph G and promises to return a 3-coloring of G:

```
W ← V_G
while W ≠ ∅ do
    pick w in W
    i ← 1
```

while $\mathrm{N}(w) \cap X_i \neq \emptyset$ do $i \leftarrow i+1$
$X_i \leftarrow X_i \cup \{w\}$
$W \leftarrow W \setminus X_i$
return X_1, X_2, X_3

Does the algorithm keep its promise?

E 8.53 What is the maximum number of edges in a graph with n vertices that admits a 3-coloring?

E 8.54 Imagine a grid in which all the vertices except for one are colored. Each colored vertex has one of 3 possible colors. Devise a "color swapping heuristic in bicolored components" (compare to exercise 12.22) to obtain, from the given partial coloring, a coloring of all the vertices with only 3 colors.

E 8.55 Let I and J be stable sets in a graph G and suppose $I \cap J = \emptyset$. Let X be the set of vertices in one component of the bipartite graph $G[I \cup J]$. Show that the set $I \oplus X$ is stable in graph G.

E 8.56 (ALGORITHM) Describe an heuristic[12] for vertex coloring based on exercise 8.55. (At the beginning of each iteration, we have a partial vertex coloring; each iteration picks a uncolored vertex and tries to assign to it one of the previously used colors.)

D 8.57 As we know, 2-colorable graphs are characterized by the absence of odd circuits. Devise a good characterization of 3-colorable graphs. Devise a good characterization of k-colorable graphs.

D 8.58 (ALGORITHM) Devise a fast algorithm to solve the 3-coloring problem.[13]

D 8.59 (ALGORITHM) Let k be a natural number greater than 3. Devise a fast algorithm to solve the k-coloring problem.

Coloring planar graphs

Planar graphs can be colored with few colors.

[12] According to Wilf (in *Algorithms and Complexity*, Prentice-Hall, 1986), heuristics are "methods that seem to work well in practice, for reasons nobody understands."

[13] A fast algorithm to decide whether a given graph is 3-colorable is not (yet?) known. In technical terms, this decision problem is NP-complete. See observation in page 5.

E 8.60 Show that $\chi(G) \leq 6$ for every planar graph G. (See exercises 1.275 and 8.30.)

E 8.61 (HEAWOOD'S THEOREM[14]) Show that $\chi(G) \leq 5$ for every planar graph G. (See exercises 1.275, 8.54 and 8.56.)

E 8.62 (ALGORITHM) Devise an algorithm that produces a 5-coloring of any given planar graph.

!! E 8.63 (FOUR COLOR THEOREM[15]) *Show that every planar graph admits a vertex coloring with* 4 *or less colors. In other words, show that*

$$\chi(G) \leq 4$$

for every planar graph G.

◦ **E 8.64** Show that there are planar graphs that do not admit a vertex coloring with only 3 colors.

E 8.65 Is it true that $\chi(G) = \omega(G)$ for every planar graph G? (See exercises 8.38 and 8.39.)

E 8.66 Let G be the graph of Europe (see exercise 1.17). Show that $\chi(G) = 4$.

!! E 8.67 (ALGORITHM) Devise an algorithm to produce a 4-coloring of any given planar graph.

E 8.68 Show that $\alpha(G) \geq \frac{1}{4}n(G)$ for every planar graph G. (It would be very interesting to have a proof of this fact that did not rely on the four color theorem, exercise 8.63.)

E 8.69 A **face coloring** of a plane map is an assignment of colors to the faces of the map (see subsection 1.17) such that adjacent faces receive different colors.

Deduce from the Four Color Theorem (exercise 8.63) that the faces of any planar map whose graph is edge-biconnected can be colored with 4 or less colors.

E 8.70 Show that the set of faces of every outerplanar graph (see exercise 1.283) is 3-colorable.

[14] Percy John Heawood (1861 – 1955).

[15] Proved in 1976 by Kenneth Appel and Wolfgang Haken. Proof simplified in 1997 by Neil Robertson, Daniel P. Sanders, Paul D. Seymour and Robin Thomas. See the pages "The four color theorem", "Four-Color Theorem" and "Four-Color Theorem."

Coloring versus minors

Before going into this section, study chapter 19 (Planarity).

! E 8.71 Prove that the following conjecture by Hajós[16] is false: For every graph G, if $\chi(G) \geq k$, then G has a topological minor (see section 1.16) isomorphic to K_k.

E 8.72 Let G be a graph such that $\chi(G) \geq 3$. Show that G has a minor (see section 1.16) isomorphic to K_3. (Compare to exercise 8.38.)

E 8.73 Let G be a graph such that $\chi(G) \geq 4$. Show that G has a minor isomorphic to K_4.

!! E 8.74 Let G be a graph such that $\chi(G) \geq 5$. Show that G has a minor isomorphic to K_5. (This is equivalent to the Four Color theorem, exercise 8.63.)

D 8.75 (HADWIGER CONJECTURE[17]) *For every natural number $k \geq 2$ and every graph G, if $\chi(G) \geq k$ then G has a minor isomorphic to K_k.* (This is a profound generalization of the Four Color theorem, exercise 8.63.)

Coloring random graphs

E 8.76 Let ε be a positive real number. Show that

$$\chi(G) > \frac{1}{2+\varepsilon}\,\frac{n}{\log_2 n}$$

for almost every graph G in $\mathcal{G}(n)$. (See section 1.18.)

! E 8.77 *Let ε be a positive real number smaller than 2. Show that*

$$\chi(G) < \frac{1}{2-\varepsilon}\,\frac{n}{\log_2 n}$$

for almost every graph G in $\mathcal{G}(n)$. (Compare to exercise 8.76.)

16 The conjecture was proposed by G. Hajós, in 1961.
17 The conjecture was proposed by H. Hadwiger, in 1943.

Perfect graphs

A graph is **perfect** if $\chi(G[X]) = \omega(G[X])$ for every subset X of V_G. (See exercise 8.38.)

E 8.78 Show a non-perfect graph G such that $\chi(G) = \omega(G)$.

E 8.79 Show that every bicolorable graph is perfect.

! E 8.80 (LOVÁSZ'S THEOREM[18]) Show that the complement of every perfect graph is perfect.

!! E 8.81 (STRONG PERFECT GRAPH THEOREM) An **odd hole** is an induced circuit with an odd ≥ 5 number of vertices.

Prove that a graph G is perfect if and only if neither G nor $\overline{G}$ contains an odd hole.[19] (This characterization of perfect graphs had been conjectured by Claude Berge in 1960.)

[18] Published by Lásló Lovász in 1972.
[19] This theorem was proved in 2002 by Maria Chudnovsky and Paul D. Seymour, based on previous work by Neil Robertson and Robin Thomas.

Chapter 9

Matchings

Two edges of a graph are **adjacent** if they have a common endpoint. A **matching** is a set of pairwise non-adjacent edges. In other words, a matching in a graph is a set M of edges such that $|M \cap \partial(v)| \leq 1$ for each vertex v.

> MAXIMUM MATCHING PROBLEM: Find a maximum matching in a given graph.

A matching M^* is **maximum** if there is no matching M such that $|M| > |M^*|$. The cardinality of a maximum matching in a graph G is denoted by

$$\alpha'(G) .$$

By the way, a matching M' is **maximal** if it is not part of a larger matching, i.e., if there is no matching M such that $M \supset M'$.

The matching problem is a particular case of the stable set problem (see exercise 9.15). Although we do not know how to solve the latter in an efficient way, we know how to solve the former.

A matching M is **perfect** if each vertex of the graph is the endpoint of some element of M. The following is an interesting specialization of the problem above: Find a perfect matching in a given graph. It is clear that not every graph has a perfect matching; the difficulty of the problem is to decide whether the given graph has such a matching.

The following concepts are important when studying matchings:

1. A matching M **saturates** a vertex v if $\partial(v) \cap M \neq \emptyset$, i.e., if some edge of M is incident to v.
2. A matching M **saturates** a set X of vertices if M saturates each vertex in X.
3. A path is **alternating** with regard to a matching M if its edges are alternately in M and not in M. Sometimes it is more convenient to say "M-alternating" than "alternating with regard to M".

4. An **augmenting path** for a matching M is an M-alternating path of non-null length whose endpoints are not saturated by M.

Exercises

∘ **E 9.1** Let M be a set of edges of a graph G. Let H be the graph (V_G, M). Show that M is a matching in G if and only if $\mathrm{d}_H(v) \leq 1$ for every vertex v of H.

∘ **E 9.2** How many edges does a maximum matching in a complete graph with n vertices have?

∘ **E 9.3** How many edges does a maximum matching in a complete bipartite graph have?

∘ **E 9.4** Calculate a maximum matching in a path. Calculate a maximum matching in a circuit.

∘ **E 9.5** Suppose that a graph G has a perfect matching. Show that $n(G)$ is even.

E 9.6 Let G be a K_6 and M be a perfect matching in G. Show that $G - M$ is planar. Show that $G - M$ has a perfect matching, say M'. Show that the complement of $(G - M) - M'$ is a circuit of length 6.

E 9.7 Calculate a maximum matching in a cubic graph which has a Hamiltonian circuit (see chapter 17).

∘ **E 9.8** Is it true that every regular graph has a perfect matching?

∘ **E 9.9** Find a maximum matching in the t-by-t queen graph.

E 9.10 Find a maximum matching in the t-by-t bishop graph.

E 9.11 Find a maximum matching in the t-by-t knight graph.

E 9.12 How many edges does a maximum matching in the cube Q_k have?

E 9.13 Show a graph G and a matching in G which is maximal but not maximum.

E 9.14 Is it true that, in any tree, every maximal matching is maximum?

E 9.15 Show that the maximum matching problem is a special case of the maximum stable set problem.

E 9.16 Is it true that, in any graph, every non-isolated vertex is saturated by some maximum matching? Is it true that every edge belongs to some perfect matching?

▷ **E 9.17** ♡ Let M and M' be two matchings in a graph G. Describe the graph $(V_G, M \cup M')$. Describe the graph $(V_G, M \oplus M')$.[1] What happens if matchings M and M' are perfect? $M \oplus M'$

E 9.18 Suppose that a graph has a bridge a. Show that either every perfect matching contains a, or no perfect matching contains a.

E 9.19 Prove that every forest has at most one perfect matching.

∘ **E 9.20** Let M be a matching in a graph, and let P be an M-alternating path. Show that every path in P is also M-alternating.

E 9.21 Let M be a matching in a graph, and let P be a maximal M-alternating path. Suppose that the two extreme edges of P are not in M. Is it true that P is an augmenting path?

⋆ **E 9.22** (AUGMENTING PATH) *Suppose that P is an augmenting path for a matching M. Prove that*

$$M \oplus E_P$$

is a matching. Prove that $|M \oplus E_P| > |M|$.

⋆ **E 9.23** *Let M be a matching in a graph G. Suppose that M is not maximum. Prove that there is an augmenting path for M.*

⋆ **E 9.24** (BERGE'S THEOREM[2]) Prove that a matching M is maximum if and only if there is no augmenting path for M. (Follows from exercises 9.22 and 9.23.)

[1] For any sets A and B, we denote by $A \oplus B$ the set $(A \setminus B) \cup (B \setminus A)$. It is easy to check that $A \oplus B = (A \cup B) \setminus (A \cap B)$.

[2] Published in 1957 by Claude Berge (1926 – 2002).

! E 9.25 (ALGORITHM) Let M be a matching in a graph G. Let a and b be two vertices not saturated by M. Write an algorithm to find an alternating path with origin a and terminus b (or to decide that no such path exists).

E 9.26 ♡ Let M be a matching. Let $(v_0, v_1, \ldots, v_k)$ be a walk (see the end of section 1.9) whose edges are alternately in M and not in M, and suppose that v_0 and v_k are not saturated by M. Let A be the edge set of that walk. Show that the set $M \oplus A$ may not be a matching. (Compare to exercise 9.22.)

E 9.27 (SLITHER) Two players, say A and B, alternately pick vertices of a graph G. First, A picks a vertex v_0. Then, B picks a vertex v_1 which is adjacent to v_0. Then, A picks a vertex which is adjacent to v_1, but different from v_0 and from v_1. And so on.

Here is a clearer description of the game: The chosen vertices form a path $v_0v_1v_2 \cdots v_k$. If k is odd, A picks a vertex v_{k+1} different from the others but adjacent to v_k. If k is even, B does an analogous move. The last player able to move wins the game.

Prove that B has a winning strategy if G has a perfect matching. Prove that A has a winning strategy otherwise.

E 9.28 Let M be a matching, and X a set of vertices of a graph. Assuming X is saturated by M, show that X is also saturated by some maximum matching. (Compare with exercise 9.16.)

E 9.29 Let X and Y be two sets of vertices of a graph G. Suppose that X is saturated by some matching, and that Y is saturated by some matching (not necessarily the same).

If y is an element of $Y \smallsetminus X$, is it true that $X \cup \{y\}$ is saturated by some matching?

If $|Y| > |X|$, is it true that there is some y in $Y \smallsetminus X$ such that $X \cup \{y\}$ is saturated by some matching?

⋆ E 9.30 (MATCHINGS VERSUS COVERS) *Show that, in any graph, for any matching M and any cover K* (see chapter 7),

$$|M| \leq |K| .$$

Deduce that, if $|M| = |K|$, then M is a maximum matching, and K is a minimum cover.[3] Give an example of a graph that *does not* have a pair (M, K) such that $|M| = |K|$. (See also exercise 9.33.)

$\alpha' \leq$ **⋆ E 9.31** Show that $\alpha'(G) \leq \beta(G)$ for every graph G.

[3] Thus a cover of the same size of a matching is a *maximality certificate* of the matching.

∘ **E 9.32** Let K be a minimal cover of a graph (see chapter 7). Is it true that every edge of any matching has only one of its endpoints in K?

E 9.33 Let M be a matching and K be a cover (see chapter 7) such that $|M| = |K|$. Show that M saturates K, and that every element of M has only one endpoint in K. (See exercise 9.30.)

E 9.34 Suppose that M is a maximal matching in a graph. Let $V(M)$ be the set of vertices saturated by M. Show that $V(M)$ is a cover (see chapter 7).

Pick one of the endpoints of each edge in M. Let W be the resulting set. Is it true that W is a cover?

E 9.35 (ALMOST MAXIMUM MATCHING) Let M be a maximal matching and M^* a maximum matching in a graph. Of course $|M| \leq |M^*|$. Show that $|M| \geq \frac{1}{2}|M^*|$. Is it true that $|M| > \frac{1}{2}|M^*|$ for every graph?

E 9.36 Suppose that a graph G has a perfect matching. Show that, for every vertex v, the graph $G - v$ has exactly one component with an odd number of vertices.

E 9.37 Let G be a graph with $n(G) \geq 2k$ and $\delta(G) \geq k$. Show that G has a matching with a least k edges.

Chapter 10

Matchings in bipartite graphs

When restricted to bipartite graphs, the maximum matching problem (see chapter 9) has a very elegant and efficient solution. Two theorems (see exercises 10.7 and 10.23) sum up the solution:

1. In a bipartite graph, a maximum matching has the same size as a minimum cover.
2. A graph with a bipartition $\{U, W\}$ has a matching that saturates U if and only if $|\mathrm{N}(Z)| \geq |Z|$ for every subset Z of U.

The expression $\mathrm{N}(Z)$ denotes the set of all vertices that are not in Z but have some neighbor in Z. (See exercise 1.108.) $\mathrm{N}(X)$

Exercises

E 10.1 Show a maximum matching in the graph shown on figure 10.1.

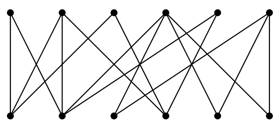

Figure 10.1: Find a maximum matching.

E 10.2 Find a maximum matching in the cube Q_k.

E 10.3 Find a maximum matching in the graph shown on figure 10.2.

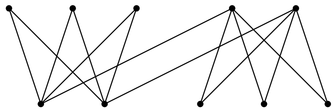

Figure 10.2: Find a maximum matching.

▷ **E 10.4** (DE CAEN'S LEMMA[1]) Let G be a bipartite graph with at least one edge. Show that some vertex of G is saturated by *all* maximum matchings. (Used in the inductive solution of exercise 10.6.)

∘ **E 10.5** Is it true that every bipartite graph has an edge that belongs to all maximum matchings? (Compare to exercise 10.4.) Is it true that every bipartite graph has an edge that does not belong to any maximum matching?

⋆ **E 10.6** *Show that every bipartite graph has a matching M and a cover K such that*

$$|M| = |K| .$$

(Compare to exercise 9.30. See exercise 10.4.)

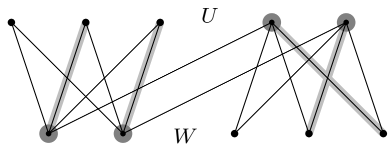

Figure 10.3: The gray circles indicate a cover. The gray lines indicate a matching. (Exercise 10.6.)

⋆ **E 10.7** (KŐNIG–EGERVÁRY'S THEOREM[2]) *Let M^* be a maximum matching and K_* a minimum cover in a bicolorable graph. Show that* $|M^*| = |K_*|$. (Follows from 9.30 and 10.6.)

E 10.8 Show that $\alpha'(G) = \beta(G)$ in every bicolorable graph G.

E 10.9 Find a maximum matching and a minimum cover in the graph shown on figure 10.1.

E 10.10 Let G be a bicolorable graph. Prove that $\chi(\overline{G}) = \omega(\overline{G})$.

[1] Published in 1988.

[2] Published in 1931 by Dénes Kőnig (1884 – 1944). The theorem is also attributed to Eugene Egerváry (1891 – 1958).

E 10.11 Give a necessary and sufficient condition for a bipartite graph to have a matching with k edges.

E 10.12 Let G be a $\{U, W\}$-bipartite graph. Suppose that $|U| = |W|$ and $m(G) > (k-1)\,|U|$ for some positive integer k. Prove that G has a matching of cardinality k.

Algorithms

∘ **E 10.13** Let G be a $\{U, W\}$-bipartite graph. Let $\{U', U''\}$ be a partition of U, and $\{W', W''\}$ be a partition of W. Show that $\mathrm{N}(U') \subseteq W'$ if and only if $\mathrm{N}(W'') \subseteq U''$. Show that if $\mathrm{N}(U') \subseteq W'$ then $W' \cup U''$ is a cover.

For any cover K of G, show that $\mathrm{N}(U \setminus K) \subseteq W \cap K$ and $\mathrm{N}(W \setminus K) \subseteq U \cap K$.

▷ **E 10.14** Let M be a matching in a $\{U, W\}$-bipartite graph G. Let $V(M)$ be the set of vertices that M saturates. Let X be the vertex set of all M-alternating paths that have one endpoint in $U \setminus V(M)$. Prove that

$$(W \cap X) \cup (U \setminus X)$$

is a cover of G. (Used to solve exercise 10.15.)

▷ **E 10.15** (ALGORITHM) Devise an efficient algorithm that will receive a $\{U, W\}$-bipartite graph and a matching M and return (1) a matching M' such that $|M'| > |M|$ or (2) a cover K such that $|K| = |M|$. (See exercise 10.14.)

⋆ **E 10.16** (HUNGARIAN ALGORITHM[3]) *Devise an efficient algorithm to receive a bipartite graph G and return a matching M and a cover K such that $|M| = |K|$.* (See exercise 10.15.) (This is the algorithmic version of exercise 10.6.)

E 10.17 (ALGORITHM) Let M be a matching in a $\{U, W\}$-bipartite graph G. Let $V(M)$ be the set of vertices saturated by M. Let a be a vertex in $U \setminus V(M)$, and b be a vertex in $W \setminus V(M)$. Write an algorithm to find an alternating path from a to b (or decide that no such path exists).

E 10.18 Use the Hungarian algorithm (exercise 10.16) to prove the Kőnig–Egerváry theorem (exercise 10.7).

E 10.19 (ALGORITHM) Show how to find a maximum stable set in a bipartite graph.

[3] Reference to the Hungarian mathematicians Kőnig, Egerváry, etc.

Semi-perfect matchings

A matching in a bipartite graph is *semi-perfect* if it saturates one of the "sides" of the graph. Clearly, every semi-perfect matching is maximum, but not every maximum matching is semi-perfect.

> PROBLEM: Given a $\{U, W\}$-bipartite graph, find a matching that saturates U.

◦ **E 10.20** Let G be a $\{U, W\}$-bipartite graph. Let M be a matching that saturates U. Prove that M is a maximum matching. Prove that U is a minimum cover.

◦ **E 10.21** Let G be a $\{U, W\}$-bipartite graph. Suppose that $|\mathrm{N}(Z)| < |Z|$ for some subset Z of U. Show that G has no matching that saturates U.

⋆ **E 10.22** *Let G be a $\{U, W\}$-bipartite graph. Suppose that*

$$|\mathrm{N}(Z)| \geq |Z|$$

for every subset Z of U. Prove that G has a matching that saturates U.

⋆ **E 10.23** (HALL'S THEOREM[4]) *Show that a $\{U, W\}$-bipartite graph has a matching that saturates U if and only if $|\mathrm{N}(Z)| \geq |Z|$ for every subset Z of U.* (Follows from 10.21 and 10.22.)

◦ **E 10.24** Which of the following statements hold for every $\{U, W\}$-bipartite graph G? (1) If G has a matching that saturates U, then $|\mathrm{N}(Z)| \geq |Z|$ for every subset Z of U. (2) If G has a matching that saturates U, then $|\mathrm{N}(Z)| \geq |Z|$ for some subset Z of U. (3) If $|\mathrm{N}(Z)| < |Z|$ for some subset Z of U, then G does not have a matching that saturates U. (4) If $|\mathrm{N}(Z)| \geq |Z|$ for every subset Z of U, then G has a matching that saturates U. (5) If $|\mathrm{N}(Z)| < |Z|$ for every subset Z of U, then G does not have a matching that saturates U. (6) If $|\mathrm{N}(Z)| \geq |Z|$ for some subset Z of U, then G has a matching that saturates U.

E 10.25 (ALGORITHM) Devise an efficient algorithm that will receive a bipartite graph and its bicoloring $\{U, W\}$ and return (1) a matching that saturates U or (2) a subset Z of U such that $|\mathrm{N}(Z)| > |Z|$. (This is the algorithmic version of Hall's theorem, exercise 10.23.)

[4] Published in 1935 by Philip Hall (1904 – 1982). (See article in Wikipedia.)

E 10.26 Deduce Kőnig–Egerváry's theorem (exercise 10.7) from Hall's theorem (exercise 10.23).

E 10.27 Let M be a set of men, W be a set of women, and k be a positive integer. Suppose that each man knows at most k women, and each women knows at least k men. Prove that it is possible for each women to marry a man she knows.

E 10.28 Let G be a $\{U, W\}$-bipartite graph with at least one edge. Suppose that $\mathrm{d}(u) \geq \mathrm{d}(w)$ for every u in U and w in W. Prove that there is a matching that covers U.

E 10.29 Let G be an r-regular bipartite graph with $r > 0$. Show that G has a perfect matching.

E 10.30 Prove that a bicolorable graph G has a perfect matching if and only if $|\mathrm{N}^*(Z)| \geq |Z|$ for every subset Z of V_G, where $\mathrm{N}^*(Z)$ is the set $\bigcup_{z \in Z} \mathrm{N}(z)$. (Clearly $\mathrm{N}^*(Z)$ contains $\mathrm{N}(Z)$.)

Give an example of a (non-bicolorable) graph that does not have a perfect matching, but satisfies inequality $|\mathrm{N}^*(Z)| \geq |Z|$ for every set Z of vertices.

E 10.31 Let G be a $\{U, W\}$-bipartite graph, and X be a subset of U. Give a necessary and sufficient condition for G to have a matching that saturates X.

▷ **E 10.32** Let G be a $\{U, W\}$-bipartite graph, X be a subset of U, and Y be a subset of W. Let M be a matching that saturates X, and N be a matching that saturates Y. Show that there is a matching that saturates $X \cup Y$.

E 10.33 Let G be a $\{U, W\}$-bipartite graph with at least one edge. Let X be the set of vertices in U that have degree $\Delta(G)$. Show that G has a matching that saturates X.

▷ **E 10.34** ♡ Let G be a bicolorable graph with at least one edge. Show that there is a matching that saturates all vertices of degree $\Delta(G)$. (See exercises 10.32 and 10.33.)

E 10.35 Let K be a complete $\{U, W\}$-bipartite graph with $|U| = |W|$. Let G be a subgraph of K and set $r = \Delta(G)$. Show that there is an r-regular graph H such that $G \subseteq H \subseteq K$.

E 10.36 Let G be a bicolorable graph and set $r = \Delta(G)$. Show that G is a subgraph of some r-regular bicolorable graph.

E 10.37 (GENERALIZATION OF HALL'S THEOREM) Let G be a $\{U, W\}$-bipartite graph. Prove that every maximum matching in G has cardinality

$$\min_{Z \subseteq U} |U| - |Z| + |\mathrm{N}(Z)| \, .$$

E 10.38 Let G be a graph (not necessarily bicolorable), and let $\{A, B\}$ be a partition of V_G. Suppose that there are less than k edges with one endpoint in A and another in B (i.e., $\mathrm{d}(A) < k$). Suppose also that graphs $G[A]$ and $G[B]$ admit vertex colorings (see chapter 8) with k colors. Show that G admits a vertex coloring with k colors.

Chapter 11

Matchings in arbitrary graphs

A component of a graph is **odd** if it has an odd number of vertices. The number of odd components of a graph G will be denoted in this section by $o(G)$. $o(G)$

Exercises

E 11.1 Let T be a tree and suppose that $o(T - v) = 1$ for each vertex v of T. Show that T has a perfect matching. (See exercise 9.36.)

E 11.2 Suppose that a graph G has a perfect matching. Show that $o(G - S) \leq |S|$ for every set S of vertices.

$\circ$ **E 11.3** Suppose that a graph G satisfies the condition $o(G - S) \leq |S|$ for every set S of vertices. Prove that $n(G)$ is even.

$\star$**! E 11.4** *Suppose that a graph G has the following property: for every set S of vertices,*

$$o(G - S) \; \leq \; |S| \, .$$

Show that G has a perfect matching.

$\star$ **E 11.5** (TUTTE'S THEOREM[1]) *Show that a graph G has a perfect matching if and only if $o(G - S) \leq |S|$ for every set S of vertices.* (Follows from exercises 11.2 and 11.4.)

$\star$**! E 11.6** (ALGORITHM) Sketch an efficient algorithm to decide whether a graph has a perfect matching.

[1] Published in 1947 by William T. Tutte (1917 – 2002). (see article in Wikipedia.)

E 11.7 Deduce Hall's theorem (exercise 10.23) from Tutte's theorem (exercise 11.4).

$\star$ **E 11.8** Let M be a matching and S be a set of vertices of a graph G. Prove that the number of vertices not saturated by M is at least $o(G-S)-|S|$. Deduce that

$$|M| \leq \gamma(G,S)\,,$$

$\gamma()$ where $\gamma(G,S)$ is the number $\frac{1}{2}n(G)-\frac{1}{2}\big(o(G-S)-|S|\big)$.

E 11.9 Let M be a matching and S be a set of vertices of a graph G. Suppose that $|M| = \gamma(G,S)$, where $\gamma(G,S)$ is the number defined in exercise 11.8. Show that matching M is maximum.[2]

$\star$! **E 11.10** Show that, in any graph G, there is a matching M and a subset S of V_G such that M saturates all but $o(G-S)-|S|$ vertices, i.e., such that

$$|M| \geq \gamma(G,S)\,,$$

where $\gamma(G,S)$ is the number defined in exercise 11.8.

$\star$! **E 11.11** (TUTTE–BERGE'S THEOREM[3]) Prove that, in any graph G,

$$\alpha'(G) = \gamma(G)\,,$$

where $\gamma(G)$ is the minimum value of $\gamma(G,S)$ for all subsets S of V_G, and $\gamma(G,S)$ is the number defined in exercise 11.8. (See exercise 11.10.)

E 11.12 Deduce exercise 9.30 from exercise 11.8. Deduce Kőnig–Egerváry's theorem (exercise 10.7) from Tutte–Berge's theorem (exercise 11.11).

E 11.13 Let G be a cubic bridgeless graph. Show that G has a perfect matching. Show that not every cubic graph has a perfect matching.

$\star$! **E 11.14** (GALLAI–EDMONDS DECOMPOSITION[4]) Let G be a graph and D the set of vertices that are not saturated by every maximum matching. (See exercise 10.4.) Let A be the set $\mathrm{N}(D)$. (See the definition of N in exercise 1.108.) Let C be the set $V_G \smallsetminus (D \cup A)$. Show that, for every maximum matching M^* in G, we have that

$$2|M^*| = n(G) - c(G[D]) + |A|\,,$$

where $c(H)$ denotes the number of components of graph H.

[2] The set S serves as a *maximality certificate* of the matching.

[3] This is a combination of Tutte's theorem (exercise 11.4) with a theorem by Claude Berge (1926 – 2002) published in 1958.

[4] Published in 1963 and 1964 by Tibor Gallai (1912 – 1992), and in 1965 by Jack Edmonds.

! E 11.15 (EDMONDS' ALGORITHM[5]) Sketch an efficient algorithm to find a maximum matching in any given graph.

! E 11.16 An **edge cover**[6] of a graph is a set F of edges such that every vertex of the graph is an endpoint of some element of F. (Do not confuse this with the concept of vertex cover.) Devise an efficient algorithm to produce a minimum edge cover.

! E 11.17 (MAXIMUM-WEIGHT MATCHING ALGORITHM) Let K be a complete graph and π be a function from E_K to $\{0, 1, 2, 3, \ldots\}$. For each edge e of the graph, we say that $\pi(e)$ is the *weight* of e. The *weight* of a subset F of E_K is $\sum_{e \in F} \pi(e)$. Sketch an algorithm to find a maximum-weight matching in K.

[5] Jack Edmonds.

[6] Maybe I should say "cover of vertices by edges".

Chapter 12

Edge coloring

An **edge coloring**, or **covering by matchings**, of a graph is a collection of matchings that covers the edge set. More precisely, an edge coloring of a graph G is a collection M_1, M_2, …, M_k of matchings such that $M_1 \cup M_2 \cup \cdots \cup M_k = E_G$. (We could require the matchings to be pairwise disjoint; this is convenient but not essential.)

If we imagine that each matching M_i corresponds to a color, we can say that an edge coloring is an assignment of colors to the edges of the graph such that adjacent edges get different colors.

If $M_1, \ldots, M_k$ is an edge coloring, we say that k is the **number of colors** of the coloring (even if some M_i is empty). We also say that this is a k-**coloring**. An edge coloring is **minimum** if its number of colors is the smallest possible, i.e., if no other coloring uses fewer colors.

> EDGE COLORING PROBLEM: Find a minimum edge coloring of a given graph.

The **chromatic index** of a graph G is the minimum number of colors necessary to color the edges of G. This number is denoted by

$$\chi'(G) .$$

(Do not confuse this with the chromatic number $\chi(G)$, defined in chapter 8.)

Exercises

∘ **E 12.1** Let $M_1, \ldots, M_k$ be an edge coloring of a graph. Show that there is a coloring $M'_1, \ldots, M'_k$ such that matchings $M'_1, \ldots, M'_k$ are pairwise disjoint.

E 12.2 An industrial process consists of a certain set of tasks. Each task is executed by a worker on a machine and takes 1 day of work. Each worker is qualified to operate only some of the machines. How many days are needed to complete the process?

E 12.3 A school can be represented by a $\{U, W\}$-bipartite graph: each vertex in U is a teacher, each vertex in W is a class of student, and each teacher is adjacent to the classes she must teach. A school week is divided into periods (Monday from 8 to 10, Monday from 10 to 12, etc.), and each period is represented by a color. An edge coloring of the graph is a scheduling of the classes for a week. How many periods are necessary and sufficient to fulfill the week's program?[1]

E 12.4 Show that the edge coloring problem is a special case of the vertex coloring problem (see chapter 8).

E 12.5 Show a graph with two different minimum edge colorings.

∘ **E 12.6** Find a minimum edge coloring of a complete graph. Find a minimum edge coloring of a complete bipartite graph.

∘ **E 12.7** Compute the chromatic index of a path and of a circuit. Compute the chromatic index of graphs with $\Delta = 0$, with $\Delta = 1$, and with $\Delta = 2$.

E 12.8 Calculate the chromatic index of the Petersen graph.

E 12.9 Let G be a cubic graph that has a Hamiltonian circuit. (A circuit C in G is Hamiltonian if $V_C = V_G$. See chapter 17.) Prove that $\chi'(G) = 3$.

E 12.10 Show that $\chi'(G) \geq m(G)/\alpha'(G)$ for every graph G.

⋆ **E 12.11** Show that any edge coloring of a graph G uses at least $\Delta(G)$ colors.

$\chi' \geq$ In other words, show that

$$\chi'(G) \geq \Delta(G)$$

for every graph G. Show that this is a special case of the inequality $\chi \geq \omega$ in exercise 8.38.

∘ **E 12.12** Let G be an r-regular graph with an odd number of vertices. Show that $\chi'(G) > r$.

E 12.13 Let G be an r-regular graph with $r \geq 2$. Suppose that G has a bridge and show that $\chi'(G) > r$. (See exercise 9.18.)

[1] This is the "timetabling problem."

E 12.14 Let G be an r-regular graph with $r \geq 1$. Suppose that G has an articulation and show that $\chi'(G) > r$.

E 12.15 Suppose that $n(G)$ is odd and $m(G) > \Delta(G)\,(n(G) - 1)/2$. Show that $\chi'(G) > \Delta(G)$.

E 12.16 Let G be an r-regular graph, $r \geq 1$, with an odd number of vertices. Let H be the subgraph obtained by removing at most $(r - 1)/2$ edges from G. Show that $\chi'(H) > \Delta(H)$.

E 12.17 Show that the edge set of every tree T can be colored with (only) $\Delta(T)$ colors. (Compare to exercise 12.11.)

E 12.18 Show that $\chi'(G) \leq 2\Delta(G) - 1$ for every graph G. (Hint: induction on the number of edges of G.)

E 12.19 (MINIMAL COLORING ALGORITHM) Consider the following greedy algorithm for coloring the edges of a graph G. Each iteration of the algorithm starts with a collection $M_1, \ldots, M_j$ of matchings. In each iteration,

> pick an edge e which is not in $M_1 \cup \cdots \cup M_j$; if there is an i such that $M_i \cup \{e\}$ is a matching, then start a new iteration with $M_1, \ldots, M_{i-1}$, $M_i \cup \{e\}$, $M_{i+1}, \ldots, M_j$; else, start a new iteration with $M_1, \ldots, M_j$, $\{e\}$.

Show that the algorithm uses at most $2\Delta(G) - 1$ matchings. Show that the algorithm uses at most $2\chi'(G) - 1$ matchings. Does the algorithm produce a minimum coloring?

E 12.20 (MINIMAL COLORING ALGORITHM) Consider the following greedy algorithm for coloring the edges of a graph G:

> the j-th iteration starts with a collection $M_1, M_2, \ldots, M_{j-1}$ of matchings and calculates a maximal matching M_j in the graph $G - (M_1 \cup M_2 \cup \cdots \cup M_{j-1})$.

Show that the algorithm uses at most $2\Delta(G) - 1$ colors. Show that the algorithm uses at most $2\chi'(G) - 1$ colors. Does the algorithm produce a minimum coloring?

E 12.21 (ALGORITHM) Consider the following algorithm for coloring the edges of a graph G:

> the j-th iteration starts with a collection $M_1, M_2, \ldots, M_{j-1}$ of matchings and calculates a *maximum* matching M_j in the graph $G - (M_1 \cup M_2 \cup \cdots \cup M_{j-1})$.

Does this algorithm produce a minimum coloring?

E 12.22 (COLOR EXCHANGE HEURISTIC) Consider the following "alternating path color exchange" heuristic[2] that tries to solve the edge coloring problem:

> At the beginning of each iteration we have a partial coloring, i.e., a collection M_1, M_2, ..., M_k of pairwise disjoint matchings. Let vw be an uncolored edge, i.e., an edge which is not in $M_1 \cup \cdots \cup M_k$. Let M_i be a "missing" color at v, and let M_j be a "missing" color at w. Let P be the component of graph $(V_G, M_i \cup M_j)$ that contains v. Substitute $M_i \oplus E_P$ for M_i. Then, substitute $M_j \cup \{vw\}$ for M_j and start a new iteration.

Complete the details and discuss the heuristic. Does it solve the edge coloring problem?

E 12.23 Show that every bipartite r-regular graph admits an edge coloring with only r colors. (See exercise 10.29.)

E 12.24 Pick 16 squares in an 8-by-8 chess board so that each row and each column of the board contains exactly two of the chosen squares. Prove that it is possible to place 8 white and 8 black pawns in the 16 chosen squares so that each row and each column contains exactly one white and one black pawn.

⋆ **E 12.25** *Let G be a bicolorable graph. Show that the edge set of G can be colored with no more than $\Delta(G)$ colors.* (See heuristic 12.22 or exercise 10.34.)

⋆ **E 12.26** (KŐNIG'S THEOREM[3]) *Show that $\chi'(G) = \Delta(G)$ for every bicolorable graph G.* (Follows from exercises 12.11 and 12.25.)

E 12.27 Show minimum edge colorings of the graphs in figures 10.1 and 10.2.

E 12.28 Let M and N be two matchings of a graph G. Suppose that $\big||M| - |N|\big| \geq 2$. Show that there are matchings M' and N' such that $M \cup N = M' \cup N'$ and $\big||M'| - |N'|\big| < \big||M| - |N|\big|$.

E 12.29 Let G be a graph and set $k = \chi'(G)$. Show that there is a k-coloring M_1, M_2, ..., M_k of the edges such that $\big||M_i| - |M_j|\big| \leq 1$ for every pair i,j of colors. Write a "formula" for $|M_i|$ in terms of $m(G)$. (See exercise 12.28.)

[2] Wilf says that heuristics are "methods that seem to work well in practice, for reasons nobody understands" (*Algorithms and Complexity*, Prentice-Hall, 1986).

[3] Dénes Kőnig (1884 – 1944). (See article in Wikipedia.)

⋆ **E 12.30** (VIZING'S THEOREM[4]) *Show that*

$$\chi'(G) \leq \Delta(G) + 1$$

for every graph G.[5] (If we combine this with exercise 12.11, we can say that $\Delta \leq \chi' \leq \Delta + 1$ for every graph.)

E 12.31 Show that the color exchange heuristic suggested in exercise 12.22 is not sufficient to prove Vizing's theorem (exercise 12.30).

E 12.32 (ERDŐS AND WILSON THEOREM) Let $\mathcal{G}^1(n)$ be the collection of all graphs in $\mathcal{G}(n)$ (see section 1.18) for which $\chi' = \Delta$. Show that

$$\lim_{n\to\infty} \frac{|\mathcal{G}^1(n)|}{|\mathcal{G}(n)|} = 1.$$

D 12.33 (ALGORITHM) Devise a fast algorithm to calculate $\chi'(G)$ for any given graph G.

D 12.34 (ALGORITHM) Devise a fast algorithm to solve the edge coloring problem.

E 12.35 Let X be the set of vertices of a graph G that have degree $\Delta(G)$. Show the following: if $G[X]$ is a forest, then $\chi'(G) = \Delta(G)$.

Planar graphs

E 12.36 Prove that the following statement is equivalent to the Four Color Theorem (exercise 8.63): $\chi'(G) = 3$ for every cubic edge-biconnected planar graph G. (Compare to exercise 12.13. See exercise 8.69.)

[4] Published in 1964–1965 by Vadim G. Vizing (1937 – 2017). Discovered, independently, by Ram Prakash Gupta in 1966.

[5] It is tempting to compare this inequality with inequality $\chi \leq \Delta + 1$ in exercise 8.30. But the reasons behind the two inequalities are very different.

Chapter 13

Connectors and acyclic sets

A **connector** of a graph G is any subset C of E_G such that graph (V_G, C) is connected.[1] A connector C_* is **minimum** if there is no other connector C such that $|C| < |C_*|$.

> MINIMUM CONNECTOR PROBLEM: Find a minimum connector of a given graph.

A connector $\hat{C}$ is **minimal** if there is no connector C such that $C \subset \hat{C}$. Every minimum connector is, of course, minimal. It is a bit surprising (given the case of vertex covers, for example) that the converse is true (see exercise 13.5). Because of this, the minimum connector problem is computationally very easy.

A set F of edges of a graph G is **acyclic** if the graph (V_G, F) has no circuits, i.e., if (V_G, F) is a forest. An acyclic subset F^* of E_G is **maximum** if there is no acyclic subset F such that $|F| > |F^*|$.

> MAXIMUM ACYCLIC SET PROBLEM: Given a graph G, find a maximum acyclic subset of E_G.

An acyclic subset $\check{F}$ of E_G is **maximal** if there is no acyclic subset F of E_G such that $F \supset \check{F}$. Every maximum acyclic subset is, of course, maximal. It is a bit surprising that the converse is true (see exercise 13.6). Because of this, the maximum acyclic subset problem is computationally very easy.

A **spanning tree** of a graph G is any spanning subgraph of G that happens to be a tree.[2] A spanning tree is essentially the same as a minimal connector and a maximal acyclic set (see exercise 13.4).

[1] See Schrijver [Sch03, p.855], for example.

[2] A spanning tree of a graph could be called *skeleton* of the graph. In German, one says *Gerüst* (= scaffold).

Exercises

∘ **E 13.1** Show that a graph has a connector if and only if it is connected. Show that every graph has an acyclic set.

E 13.2 Let C be a minimal connector of a graph. Show that C is a maximal acyclic set.

E 13.3 Let G be a connected graph, and F be a maximal acyclic subset of E_G. Show that F is a minimal connector. (See exercise 1.199.)

E 13.4 Let C be a minimal connector, and F be a maximal acyclic set of a connected graph G. Show that graphs (V_G, C) and (V_G, F) are spanning trees of G.

Let T be a spanning tree of G. Show that E_T is a minimal connector and also a maximal acyclic set of G.

⋆ **E 13.5** Show that every minimal connector of a graph G has exactly $n(G)-1$ edges. (See exercise 1.228.) Deduce that every minimal connector is minimum.

⋆ **E 13.6** Show that every maximal acyclic set of a graph G has exactly $n(G) - c(G)$ edges, where $c(G)$ is the number of components of G. (See exercise 1.231.) Deduce that every maximal acyclic set is maximum.

E 13.7 (ALGORITHM) Devise an efficient algorithm to receive a graph G and return a minimum connector of G (or a proof that G is not connected).

Devise an efficient algorithm to receive a graph G and return a maximum acyclic subset of E_G.

⋆ **E 13.8** (EDGE EXCHANGE) *Let C be a minimal connector of a graph G and b an element of $E_G \setminus C$. Show that there is an element a of C such that*

$$(C \cup \{b\}) \setminus \{a\}$$

is a minimal connector of G.

E 13.9 Let a be an edge in a minimal connector C of a graph G. Give a necessary and sufficient condition for $E_G \setminus C$ to have an edge b such that $(C \setminus \{a\}) \cup \{b\}$ is a connector. (Compare to exercise 13.8.)

E 13.10 (ALGORITHM) Suppose that each edge e of a graph G has a numeric "weight" $\pi(e) \geq 0$. The weight of any set A of edges is then is the number $\sum_{e \in A} \pi(e)$.

Devise an algorithm to find a minimum-weight connector of G.[3]

[3] The famous algorithms of J. B. Kruskal and R. C. Prim solve this problem. They are among the oldest and best known greedy algorithms in graph theory. The proof of correction of these algorithms relies on exercise 13.8.

Chapter 14

Minimum paths and circuits

A path is **shorter** than another one if the length of the former is smaller than that of the latter. A path P_* is **shortest**, or **minimum**, if no shorter path has the same endpoints as P_*.

> SHORTEST PATH PROBLEM: Given vertices v and w of a graph, find a shortest path with endpoints v and w.

The **distance** between two vertices v and w is the length of a shortest path with endpoints v and w.[1] (If there is no path with these endpoints, the distance is not defined.) The distance between vertices v and w of a graph G will be denoted by

$$\mathrm{dist}_G(v, w) .$$

If G is implicit, we will simply write $\mathrm{dist}(v, w)$.

A circuit is **minimum** if the graph has no shorter circuit. The **girth** of a graph is the length of a minimum circuit. (If the graph has no circuit, its girth is not defined.)

Exercises

E 14.1 Compute the distance between vertex x and each of the other vertices in figure 14.1. Then, show a shortest path between x and y.

∘ **E 14.2** Let k be the distance between two vertices v and w in a graph. Show that (1) there is a path of length k from v to w and (2) there is no path of length smaller than k from v to w. Show the converse: if (1) and (2) hold, then the distance between v and w is k.

[1] The expressions "minimum distance" and "shortest distance" are redundant and should be avoided.

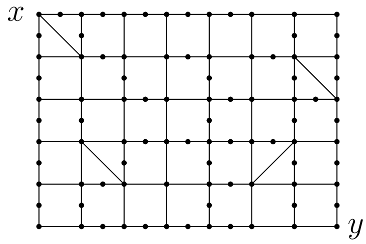

Figure 14.1: Find a shortest path between x and y. See exercise 14.1.

E 14.3 Suppose that $v_0 v_1 \cdots v_k$ is a shortest path (among those with end-points v_0 and v_k). Prove that

$$\operatorname{dist}(v_0, v_j) = j$$

for every index j.

$\star$ **E 14.4** (TRIANGLE INEQUALITY) Let (x, y, z) be a triple of vertices of a connected graph. Prove that

$$\operatorname{dist}(x, y) + \operatorname{dist}(y, z) \;\geq\; \operatorname{dist}(x, z)\,.$$

E 14.5 Let r be a vertex and uv be an edge of a connected graph. Show that

$$|\operatorname{dist}(r, u) - \operatorname{dist}(r, v)| \;\leq\; 1\,.$$

E 14.6 (BREATH-FIRST SEARCH ALGORITHM) Devise an efficient algorithm that will receive two vertices v and w of a graph and compute the distance between v and w. Devise an efficient algorithm to find a shortest path between two given vertices.

E 14.7 (DISTANCE TREE) Let r be a vertex of a connected graph G. Show that G has a spanning tree T such that

$$\operatorname{dist}_G(r, x) \;=\; \operatorname{dist}_T(r, x)$$

for every vertex x (i.e., the distance between r and x in G is equal to the distance between r and x in T).

$\circ$ **E 14.8** Is it true that every connected graph G has a spanning tree T such that $\operatorname{dist}_G(u, v) = \operatorname{dist}_T(u, v)$ for every pair u,v of vertices?

E 14.9 The **eccentricity** of a vertex v in a graph is the number $\mathrm{exc}(v) := \max_{w \in V} \mathrm{dist}(v, w)$. A **center** is a vertex of minimum eccentricity. The **radius** of the graph is the eccentricity of a center.

Show that every tree has at most two centers, and if it has two then they are adjacent.

E 14.10 The **Heawood**[2] graph has vertex set $\{0, 1, 2, \ldots, 13\}$. Each vertex i is adjacent to vertices $(i + 1) \bmod 14$ and $(i + 13) \bmod 14$.[3] In addition, each i is adjacent to a third vertex, which depends on the parity of i: if i is even, then it is adjacent to $(i+5) \bmod 14$, and if i is odd, then it is adjacent to $(i+9) \bmod 14$.

Draw a figure of the Heawood graph. Find a minimum-length circuit in the graph.

E 14.11 Show that every connected graph with $m \geq 3n/2$ has a circuit of length $\leq c \log n$ for some constant c.

E 14.12 (ALGORITHM) Devise an algorithm to calculate the girth of any given graph. Devise an algorithm to find a minimum circuit in any given graph. (See exercise 14.6.)

Parity constraints

We say that a circuit or path **is odd** if its length is an odd number. Similarly, a circuit or path **is even** if its length is an even number.

The **odd girth** of a graph is the length of a minimum odd circuit in the graph. The **even girth** is defined similarly.

E 14.13 Let r be a vertex of a connected graph G. Let u and v be two adjacent vertices such that $\mathrm{dist}(r, u) = \mathrm{dist}(r, v)$. Show that G has an odd circuit of length at most $\mathrm{dist}(r, u) + \mathrm{dist}(r, v) + 1$. (See exercise 14.3.)

E 14.14 Let r be a vertex of a connected graph G. Let x and y be two adjacent vertices such that $\mathrm{dist}(r, x)$ and $\mathrm{dist}(r, y)$ have the same parity (i.e., both are even or both are odd). Show that G has an odd circuit.

E 14.15 (Converse of 14.13) Let r be a vertex of a connected graph G. Let O be an odd circuit in G. Show that O has an edge xy such that $\mathrm{dist}_G(r, x) = \mathrm{dist}_G(r, y)$. (See exercise 14.5.)

[2] Percy John Heawood (1861 – 1955).
[3] The expression "$i \bmod j$" denotes the remainder of the division of i by j.

E 14.16 Let r be a vertex of a connected graph G. Show that G is bicolorable if and only if $\text{dist}(r, u) \neq \text{dist}(r, v)$ for every edge uv. (See exercises 14.13 and 14.15.)

E 14.17 Use the concept of distance to show that a graph is bicolorable if and only if it has no odd circuits. (Compare to exercise 4.15. See exercise 14.14.)

E 14.18 (ALGORITHM) Devise an algorithm to calculate the odd girth of any given graph. Devise an algorithm to find a minimum odd circuit in any given graph. (See exercises 14.13, 14.15 and 14.6. Compare to exercise 1.123.)

! E 14.19 (ALGORITHM) Devise an algorithm to calculate the even girth of any given graph. Devise an algorithm to find a minimum even circuit in any given graph.

! E 14.20 (ALGORITHM) Given two vertices u and v of a graph G, find a shortest path in the collection of all even paths whose endpoints are u and v.

! E 14.21 (ALGORITHM) Give two vertices u and v of a graph G, find a shortest path in the collection of all odd paths whose endpoints are u and v.

Random graphs

E 14.22 The **diameter** of a graph G is the number $\max_{(u,v)} \text{dist}(u, v)$, where the maximum is taken over all the pairs (u, v) of vertices. Prove that the diameter of almost every graph in $\mathcal{G}(n)$ (see section 1.18) is at most 2.

Chapter 15

Flows

A **flow** in a graph is a collection of paths that share no edges.[1] (Perhaps "spaghetti" would be a good alternative for "flow"!) More precisely, a **flow** is a collection $\mathcal{F}$ of paths such that

$$E_P \cap E_Q = \emptyset$$

for every pair (P, Q) of distinct elements of $\mathcal{F}$.

Given vertices a and b of a graph G, we will say that a flow $\mathcal{F}$ **connects** a **to** b if every path in $\mathcal{F}$ has origin a and terminus b. We can also say, under these circumstances, that $\mathcal{F}$ is a flow **between** a **and** b, or **from** a **to** b. (A flow from b to a is essentially the same as a flow from a to b.)

A flow $\mathcal{F}$ from a to b is **maximum** if $|\mathcal{F}| \geq |\mathcal{F}'|$ for every flow $\mathcal{F}'$ from a to b.

> MAXIMUM FLOW PROBLEM: Given two vertices a and b of a graph G, find a maximum flow from a to b.

A set C of edges **separates** a **from** b if every path from a to b has at least one edge in C. According to exercise 15.4, every set that separates a from b is, essentially, a cut $\partial(X)$ such that X contains a but does not contain b.

Exercises

E 15.1 Consider the bishop graph on a 3-by-3 board. Let a be the upper left corner square, and b be the upper right corner square. Find a maximum flow from a to b.

E 15.2 Consider the queen graph on a 3-by-3 board. Let a be the upper

[1] This use of the word "flow" is unorthodox. In most books, the word is used in a somewhat different way.

left corner square, and b be the square in the middle of the board. Find a maximum flow from a to b.

E 15.3 Consider the knight graph on a 3-by-3 board. Let a be the first square of the first row, and b be the last square of the second row. Find a maximum flow from a to b.

E 15.4 (SEPARATORS VERSUS CUTS) Let a and b be two vertices of a graph G. Let X be a subset of V_G that contains a but does not contain b. Show that $\partial(X)$ separates a from b.

Let C be a set of edges that separates a from b. Show that there exists a subset X of V_G such that $a \in X$, $b \notin X$ and $\partial(X) \subseteq C$.

∘ **E 15.5** Let a and b be two vertices of a graph. Suppose that there is a flow of cardinality k between a and b. Show that every set of edges that separates a from b has at least k edges.[2]

⋆ **E 15.6** (FLOW VERSUS CUT) *Let a and b be two vertices of a graph G. Suppose that every cut that separates a from b has at least k edges. Show that some flow that connects a to b in G has cardinality k.* (Compare to exercises 15.5 and 1.208.)

⋆ **E 15.7** (MENGER'S THEOREM[3]) Let a and b be two vertices of a graph. Let $\mathcal{F}^*$ be a maximum flow among those that connect a to b. Let C_* be a minimum set of edges among those that separate a from b. Show that

$$|\mathcal{F}^*| = |C_*| \,.$$

(This is a combination of exercises 15.5 and 15.6.[4])

∘ **E 15.8** Let a and b be two vertices of a graph G. Suppose that every set that separates a from b has at least 2 edges. Let P be a path in G with endpoints a and b. Is it true that $G - E_P$ has a path with endpoints a and b?

E 15.9 (ALGORITHM) Write an algorithm that will receive two vertices a and b of a graph G and return a flow $\mathcal{F}$ from a to b and a set X that contains a but not b such that $|\mathcal{F}| = \mathrm{d}(X)$.

[2] Thus, a cut with k edges *certifies* the inexistence of a flow of size greater than k.

[3] Karl Menger (1902 – 1985). The "g" sounds like "goat" and not like "gentle."

[4] It is also a special case of the Max-Flow Min-Cut Theorem, by Ford, Fulkerson, Elias, Feinstein, and Shannon.

E 15.10 Let G be a graph and let A and B be two non-empty subsets of V_G such that $A \cap B = \emptyset$. A **barrier** is any subset F of E_G such that every path from A to B has one edge in F.

Suppose that every barrier has at least k edges. Show that there is a collection of k paths from A to B such that no edge belongs to two of the paths.

k-edge-connected graphs

The **edge-connectivity** of a graph G is the cardinality of the smallest subset C of E_G such that $G-C$ is disconnected (i.e., has two or more components). The edge-connectivity of G is denoted by

$$\kappa'(G)\,.$$

This definition of connectivity does not apply in case G has only one vertex, because then there is no C such that $G - C$ is disconnected. The convention, in this case, is to say that $\kappa'(G) = 1$. (Maybe ∞ would make more sense.)

We say that a graph G is k**-edge-connected** for every $k \leq \kappa'(G)$. Thus, a 1-edge-connected graph is the same as a connected graph, and a 2-edge-connected graph is the same as an edge-biconnected graph (see section 1.13).

◦ **E 15.11** Calculate the edge-connectivity of a path. Calculate the edge-connectivity of a circuit.

E 15.12 Calculate the edge-connectivity of a complete graph with $n \geq 2$ vertices.

E 15.13 Let G be a k-edge-connected graph, with $k > 0$. Let C be a set of k edges. Show that $G - C$ has at most 2 components.

◦ **E 15.14** Let G be a graph with two or more vertices, and let k be a natural number. Show that G is k-edge-connected if and only if $G - C$ is connected for every subset C of E_G such that $|C| < k$.

E 15.15 Let G be a graph with two or more vertices and k a natural number. Show that G is k-edge-connected if and only if $\mathrm{d}(X) \geq k$ for every set X of vertices such that $\emptyset \subset X \subset V_G$. (See exercise 15.4.)

E 15.16 Let B_t be one of the components of the bishop graph on a t-by-t board. Calculate $\kappa'(B_t)$ for $t = 2, 3, 4$.

E 15.17 Let Q_t be the queen graph on a t-by-t board. Calculate $\kappa'(Q_t)$ for $t = 2, 3, 4$.

E 15.18 Let C_4 be the knight graph on a 4-by-4 board. Calculate $\kappa'(C_4)$.

$\star$ **E 15.19** Let G be a graph with two or more vertices. Show (from Menger's theorem, exercise 15.7) that *G is k-edge-connected if and only if every pair of its vertices is connected by a flow with cardinality k.*

$\star$ **E 15.20** Show that $\kappa'(G) \leq \delta(G)$ for every graph G with two or more vertices. Show that the inequality can be strict.

E 15.21 Let G be a graph with two or more vertices such that $\delta(G) \geq (n(G) - 1)/2$. Show that $\kappa'(G) = \delta(G)$.

$\circ$ **E 15.22** Let G be a k-edge-connected graph. Show that $m(G) \geq k\, n(G)/2$.

E 15.23 Let G be a k-edge-connected graph. Let A and B be two non-empty subsets of V_G such that $A \cap B = \emptyset$. Show that there is a flow $\mathcal{F}$ with cardinality k in G such that every path in $\mathcal{F}$ has an endpoint in A and another in B.

E 15.24 Let G be a $(2k-1)$-edge-connected graph. Show that G has a bipartite spanning subgraph H which is k-edge-connected.

Chapter 16

Internally disjoint flow

A vertex of a path is considered **internal** if it is different from both the original and the terminus of the path. Two paths in a graph are **internally disjoint** if they have no common internal vertices, i.e., if each vertex of the graph is internal to at most one of the paths.

For vertices a and b of a graph, an **internally disjoint flow** is a collection of pairwise internally disjoint paths from a to b. Hence,

$$V_P \cap V_Q = \{a, b\}$$

for every pair (P, Q) of paths in the collection. We will say that such a collection **connects** a **to** b.

> MAXIMUM INTERNALLY DISJOINT FLOW PROBLEM: Given two vertices a and b of a graph, find a maximum internally disjoint flow connecting a to b.

To avoid fruitless discussions, we shall restrict the problem to the case in which a and b are not adjacent.

A **separator** of (a, b) is any set S of vertices such that a and b are in distinct components of $G - S$. In other words, a separator of (a, b) is any subset S of $V_G \setminus \{a, b\}$ such that every path with endpoints a and b has at least one vertex in S. (Of course there is no separator of (a, b) if a and b are adjacent.)

Exercises

E 16.1 Consider the bishop graph on a 4-by-4 board. Let a be the first square of the first row, and b be the last square of the last row. Find a maximum internally disjoint flow connecting a to b. Repeat the exercise with b being the third square of the first row.

E 16.2 Consider the queen graph on a 3-by-3 board. Let a be the top left corner square, and b be the square in the middle of the board. Find a maximum internally disjoint flow connecting a to b.

E 16.3 Consider the knight graph on a 3-by-3 board. Let a be the first square of the first row, and b be the last square of the second row. Find a maximum internally disjoint flow connecting a to b.

◦ **E 16.4** Let v be an articulation in a graph G, and let a and b be vertices in different components of $G - v$. Check that $\{v\}$ separates a from b.

◦ **E 16.5** Criticize the following alternative definition of a separator: "A *separator* of (a, b) is a set S of vertices such that every path with endpoints a and b has at least one vertex in S."

◦ **E 16.6** Let a and b be two non-adjacent vertices of a graph. Suppose that an internally disjoint flow from a to b has k paths. Show that every separator of (a, b) has at least k vertices.[1] (What happens if a and b are adjacent?)

⋆ **E 16.7** (FLOW VERSUS SEPARATOR) *Let a and b be two* non-adjacent *vertices of a graph G. Suppose that every separator of (a, b) has at least k vertices. Show that G has an internally disjoint flow of size k connecting a to b.* (Compare to exercises 16.6 and 1.218.)

⋆ **E 16.8** (MENGER'S THEOREM[2]) Let a and b be two *non-adjacent* vertices of a graph. Let $\mathcal{P}^*$ be a maximum internally disjoint flow among all of those that connect a to b. Let S_* be a minimum separator of (a, b). Show that

$$|\mathcal{P}^*| = |S_*| .$$

(This is a combination of exercises 16.6 and 16.7.)

⋆ **E 16.9** Deduce Kőnig–Egerváry's theorem (exercise 10.7) from Menger's theorem (exercise 16.8).

E 16.10 (FAN LEMMA) Let a be a vertex of a graph G and let B be a non-empty subset of $V_G \setminus \{a\}$. A **fan** is a collection of paths from a to B such that $V_P \cap V_Q = \{a\}$ for every pair (P, Q) of paths in the collection. A **barrier** is any subset S of $V_G \setminus \{a\}$ such that every path from a to B has a vertex in S.

Suppose that every barrier has k or more vertices. Show that there is a fan with k paths.

[1] Thus, a separator with k vertices is a *certificate of inexistence* of an internally disjoint flow of size greater than k.

[2] Karl Menger (1902 – 1985). Pronounce the "g" as in "goat", not as in "gentle."

E 16.11 Let G be a graph, and let A and B be two non-empty subsets of V_G. A collection of paths is **disjoint** if $V_P \cap V_Q = \emptyset$ for each pair (P, Q) of paths of the collection. A **barrier** is any subset S of V_G such that every path from A to B has a vertex in S.

Suppose that every barrier has at least k vertices. Show that there is a disjoint collection of k paths from A to B.

Connectivity

The **connectivity** of a graph G is the cardinality of the smallest subset S of V_G such that $G - S$ is disconnected (i.e., has two or more components). The connectivity of a graph G is denoted by

$$\kappa(G) .$$

This definition does not apply when G is complete, because in that case there is no S such that $G - S$ is disconnected.[3] By convention, we set $\kappa(K_n) = n - 1$ for every $n \geq 2$ and $\kappa(K_1) = 1$.

We say that a graph G is k**-connected** for every $k \leq \kappa(G)$. Thus, a 1-connected graph is the same as a connected graph, and a 2-connected graph is the same as a biconnected graph (see section 1.14).

◦ **E 16.12** Compute the connectivity of a path. Compute the connectivity of a circuit.

◦ **E 16.13** Let G be a complete graph with $n \geq 2$ vertices, and let e be one of its edges. Calculate the connectivity of $G - e$.

E 16.14 Let G be a non-complete graph and k a natural number. Show that G is k-connected if and only if $G - S$ is connected for every subset S of V_G such that $|S| < k$.

E 16.15 Let B_t be one of the components of the bishop graph on a t-by-t board. Calculate $\kappa(B_t)$ for $t = 2, 3, 4$.

E 16.16 Let Q_t be the queen graph on a t-by-t board. Calculate $\kappa(Q_t)$ for $t = 2, 3, 4$.

E 16.17 Let K_4 be the knight graph on a 4-by-4 board. Calculate $\kappa(K_4)$.

[3] We could say, perhaps, that $\kappa(G)$ is ∞ if G is complete.

E 16.18 Let G be a circuit of length 6. Calculate the connectivity of $\overline{G}$.

E 16.19 Calculate the connectivity of the Petersen graph.

$\star$ **E 16.20** Let G be a non-complete graph. Show (from Menger's theorem, exercise 16.8) that G *is k-connected if and only if each pair of non-adjacent vertices of G is connected by an internally disjoint flow of size k.*

$\circ$ **E 16.21** Show that every k-connected graph with two or more vertices is k-edge-connected. Show that the converse is not true, i.e., that not every k-edge-connected graph with two or more vertices is k-connected.

$\star$ **E 16.22** Show that $\kappa(G) \leq \kappa'(G)$ for every graph G with two or more vertices. Show that the inequality may be strict.

E 16.23 Show that $\kappa(G) = \kappa'(G)$ for every cubic graph G.

E 16.24 Let k be a natural number greater than 1, and G be a k-connected graph with $n(G) \geq 2k$. Show that G has a circuit with $2k$ or more vertices.

E 16.25 (DIRAC'S THEOREM[4]) Let k be a natural number greater than 1, and let G be a k-connected graph. Let Z be a set of k vertices of G. Show that some circuit in G contains all the vertices of Z.

[4] Published in 1952 by Gabriel Andrew Dirac (1925 – 1984).

Chapter 17

Hamiltonian circuits and paths

A circuit in a graph is **longest** if the graph does not have a longer circuit. The **circumference** of a graph is the length of a longest circuit in the graph.

> LONGEST CIRCUIT PROBLEM: Find a longest circuit in a given graph.

The problem of finding a longest path is formulated similarly. A path in a graph is **longest** if the graph does not have a longer path.

A circuit is **Hamiltonian**[1] if it contains all the vertices of the graph. Any Hamiltonian circuit is, of course, longest. The longest circuit problem has the following obvious specialization:

> HAMILTONIAN CIRCUIT PROBLEM: Decide whether a given graph has a Hamiltonian circuit.

The concept of **Hamiltonian path** and the Hamiltonian path problem are defined similarly.

Some of the exercises below involve the condition "$\delta(G) \geq k$." Recall that this condition is equivalent to "$|\mathrm{N}(v)| \geq k$ for every vertex v," since $|\mathrm{N}(v)| = \mathrm{d}(v)$ for every vertex v.

Exercises

∘ **E 17.1** Is it true that every complete graph has a Hamiltonian circuit?

∘ **E 17.2** Give necessary and sufficient conditions for a complete bipartite graph to have a Hamiltonian circuit.

[1] Reference to William Rowan Hamilton (1805 – 1865). (See article in Wikipedia.) A reference to Thomas P. Kirkman (1806 – 1895) would be be fairer (see article in Wikipedia).

E 17.3 Find a longest circuit in the graphs shown in figure 17.1.

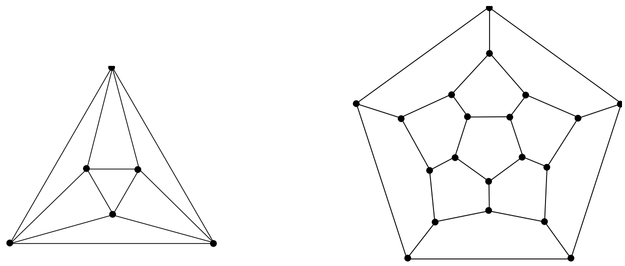

Figure 17.1: Find a longest circuit. See exercise 17.3.

E 17.4 Find a longest circuit in the Petersen graph. Find a longest path in the Petersen graph.

E 17.5 Prove that, for all $k \geq 2$, the graph Q_k has a Hamiltonian circuit. (Hint: Use induction on k.)

E 17.6 Give a necessary and sufficient condition for a grid to have a Hamiltonian circuit.

E 17.7 Find a Hamiltonian circuit in the t-by-t knight graph. (See article in Wikipedia and article in Wolfram MathWorld.)

◦ **E 17.8** Show that $\chi'(G) = 3$ for every cubic graph G that has a Hamiltonian circuit.

E 17.9 (ALGORITHM) Discuss the following algorithm for the Hamiltonian circuit problem: Given a graph G, generate a list of all permutations of V_G; discard the permutations that do not correspond to Hamiltonian circuits; return any of the remaining permutations.

E 17.10 Show that every graph G has a path of length at least $\delta(G)$. (See exercise 1.125.)

E 17.11 Show that every graph G has a circuit with $\delta(G)+1$ or more vertices, provided $\delta(G)$ is greater than 1. (See exercise 1.128 in section 1.9.)

E 17.12 Show that every graph G has a path with at least $\chi(G)$ vertices, where $\chi(G)$ is the chromatic number (see chapter 8) of G. (See exercise 8.47).

E 17.13 Let P^* and Q^* be two longest paths in a connected graph G. Show that P^* and Q^* have a common vertex. (See exercise 1.170.)

$\circ$ **E 17.14** Let G be a graph with a Hamiltonian circuit. Show that G has no bridges. Show that G has no articulations.

E 17.15 Let G be a graph with a Hamiltonian circuit. Show that every edge of G belongs to a circuit.

E 17.16 Let G be a $\{U, W\}$-bipartite graph such that $|U| \neq |W|$. Prove that G has no Hamiltonian circuit. (Another way of posing the question: for every $\{U, W\}$-bipartite graph with a Hamiltonian circuit, one has $|U| = |W|$.)

E 17.17 Suppose that a graph G has a stable set with more than $n(G)/2$ vertices. Show that G has no Hamiltonian circuit.

Is it true that every graph G with $\alpha(G) \leq n(G)/2$ has a Hamiltonian circuit?

E 17.18 (NECESSARY CONDITION) Let S be a set of vertices of a graph G. Suppose that $|S| < n(G)$ and

$$c(G - S) > |S| + 1 \,,$$

where $c(G - S)$ is the number of components of $G - S$. Show that G has no Hamiltonian path. (See exercise 1.174.)

$\star$ **E 17.19** (NECESSARY CONDITION) Let S be a set of vertices of a graph G. Suppose that $0 < |S| < n(G)$ and

$$c(G - S) > |S| \,,$$

where $c(G - S)$ is the number of components of $G - S$. Show that G has no Hamiltonian circuit. (See exercise 1.175.) Another way of posing the question: If G has a Hamiltonian circuit, then $c(G - S) \leq |S|$ for every non-empty proper subset S of V_G.

Show that the condition "$c(G - S) > |S|$" is a generalization of exercises 17.14, 17.16 and 17.17.

E 17.20 Suppose that a graph G satisfies the inequality $c(G - S) \leq |S|$ for every set S of vertices such that $0 < |S| < n(G)$. Is it true that G has a Hamiltonian circuit?

D 17.21 (SUFFICIENT CONDITION: CHVÁTAL'S CONJECTURE[2]) Let G be a graph such that $c(G - S) \leq |S|/2$ for every subset S of V_G such that $2 \leq |S| < n(G)$. Prove that G has a Hamiltonian circuit. (Compare to exercise 17.19.)

E 17.22 Is it true that there is a natural number k such that every k-connected graph (see page 129) has a Hamiltonian circuit? (Compare to exercises 17.19 and 17.35.)

E 17.23 Let G be a connected graph and O be a circuit in G such that, for each edge e of O, the graph $O - e$ is a longest path in G. Prove that G has a Hamiltonian circuit.

E 17.24 Let G be a graph with 4 or more vertices such that $\delta(G) \geq n(G) - 2$. Show that G has a Hamiltonian circuit.

E 17.25 Let G be a graph with n vertices and m edges. Suppose that $m \geq 2 + \frac{1}{2}(n-1)(n-2)$. Prove that G has a Hamiltonian circuit.

⋆ **E 17.26** (SUFFICIENT CONDITION: DIRAC'S THEOREM[3]) Let G be a graph with 3 or more vertices that satisfies the condition

$$\delta(G) \geq n(G)/2\,.$$

Show that G has a Hamiltonian circuit. (Hint: Use exercise 1.129.)

⋆ **E 17.27** (GENERALIZATION OF DIRAC'S THEOREM) Let G be a graph with 3 or more vertices that satisfies the condition

$$\mathrm{d}_G(u) + \mathrm{d}_G(v) \geq n(G)$$

for every pair (u, v) of distinct non-adjacent vertices. Show that G has a Hamiltonian circuit. (Hint: Use exercise 1.129.)

E 17.28 Let G be a graph and $\{V_1, V_2, V_3\}$ be a partition of V_G into non-empty parts. Suppose that (1) each vertex in V_1 is adjacent to all vertices of $V_2 \cup V_3$, and (2) each vertex of V_2 is adjacent to all vertices of V_3.

Prove that, if $|V_2| = 2|V_1|$ and $|V_3| = 3|V_1|$, then G has a Hamiltonian circuit. Prove that, if $|V_2| = 2|V_1|$ and $|V_3| = 3|V_1| + 1$, then G does not have a Hamiltonian circuit.

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[2] Proposed by Vašek Chvátal in 1971.
[3] Published in 1952 by Gabriel Andrew Dirac (1925 – 1984).

E 17.29 Let $\mathcal{A}$ be an algorithm that decides whether a given graph has a Hamiltonian circuit. Use $\mathcal{A}$ to formulate an algorithm to decide whether a given graph has a Hamiltonian path.

Let $\mathcal{B}$ an algorithm that decides whether a given graph has a Hamiltonian path. Use $\mathcal{B}$ to formulate an algorithm to decide whether a given graph has a Hamiltonian circuit.

D 17.30 (NECESSARY AND SUFFICIENT CONDITION?) Discover a necessary and sufficient condition for a graph to have a Hamiltonian circuit. Discover a necessary and sufficient condition for a graph to have a Hamiltonian path.

D 17.31 (ALGORITHM) Devise a fast algorithm to find a Hamiltonian circuit in a graph (or decide that the graph has no such circuit).[4]

D 17.32 (ALGORITHM) Devise a fast algorithm to find a longest circuit in any graph which is not a forest.[5]

D 17.33 (ALGORITHM) Devise a fast algorithm to find a Hamiltonian path in a graph (or to decide that the graph has no such path).[6]

D 17.34 (TRAVELING SALESMAN PROBLEM (TSP)) Let K be a complete graph and φ be a function from E_K into $\{0, 1, 2, 3, \ldots\}$. For each edge e of the graph, we say that $\varphi(e)$ is the *cost* of e. The *cost* of any subgraph H of K is $\sum_{e \in E_H} \varphi(e)$. Devise an algorithm to find a minimum-cost Hamiltonian circuit in K.[7]

Planar Hamiltonian graphs

! E 17.35 (TUTTE'S THEOREM[8]) Show that every 4-connected planar graph (see page 129) has a Hamiltonian circuit. (Compare to exercise 17.22.)

E 17.36 Show that not every 3-connected planar graph (see page 129) has a Hamiltonian circuit.

[4] Such an algorithm has not yet been found. In technical terms, the problem of deciding whether a graph has a Hamiltonian circuit is NP-complete. See the books by Garey–Johnson [GJ79], Harel [Har92], and Sipser [Sip97].

[5] Such an algorithm has not yet been found.

[6] Such an algorithm has not yet been found.

[7] The problem is NP-hard. See the books by Garey–Johnson [GJ79], Harel [Har92] and Sipser [Sip97]. See the website *The Traveling Salesman Problem*, maintained by Bill Cook at Georgia Tech University.

[8] William T. Tutte (1917 – 2002). (See article in Wikipedia.)

D 17.37 (BARNETTE'S CONJECTURE) Prove or disprove the following conjecture: Every 3-connected 3-regular bicolorable planar graph (see page 129) has a Hamiltonian circuit.[9]

[9] D. Barnette proposed the conjecture in 1970.

Chapter 18

Circuit covers

A **circuit cover**, or **covering by circuits**, of a graph G is any collection $\mathcal{O}$ of circuits of G such that $\bigcup_{O \in \mathcal{O}} E_O = E_G$. In other words, a circuit cover is a collection of circuits such that each edge of the graph belongs to a least one of the circuits of the collection.[1]

It would be natural to dedicate this chapter to the minimum circuit cover problem. But this problem is very hard. (See the end of the chapter.) We will deal then with the circuit *decomposition* problem, which is much simpler.

A **circuit decomposition**, or **decomposition into circuits**, or **simple circuit cover**, of a graph is a circuit cover that covers each edge of the graph only once.

> CIRCUIT DECOMPOSITION PROBLEM: Find a circuit decomposition of a given graph.

Due to exercise 18.16, this problem is also known as the "Eulerian cycle problem."

Exercises

E 18.1 Show a circuit decomposition of each graph in figure 18.1.

E 18.2 For which values of p and q does a p-by-q grid have a circuit decomposition?

E 18.3 Find a circuit decomposition of the knight graph.

E 18.4 For which values of k does the cube Q_k have a circuit decomposition?

[1] Coverings by circuits are very different from coverings by matchings (i.e., edge colorings) because a part of a circuit is not a circuit, while every part of a matching is a matching.

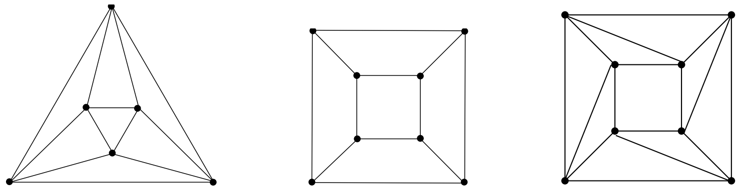

Figure 18.1: Find a circuit decomposition. See exercise 18.1.

E 18.5 Give a necessary and sufficient condition for a complete graph to have a circuit decomposition.

◦ **E 18.6** Suppose that a graph G has a bridge. Show that G has no circuit decomposition. (See exercise 1.199.)

E 18.7 Let F be a set of edges of a graph G with three or more vertices. Suppose that the graph (V_G, F) is connected and has a circuit decomposition. Prove that G is edge-biconnected. Is the converse true?

◦ **E 18.8** Suppose that a graph G has an odd-degree vertex. Show that G has no circuit decomposition.[2] (Another way to say the same thing: if a graph G has a circuit decomposition, then all vertices of G have even degree.)

⋆ **E 18.9** (THEOREM OF VEBLEN[3] AND EULER[4]) Show that a graph has a circuit decomposition if and only if the degree of each of its vertices is even. (Compare to exercise 18.8.) In other words, show that the absence of odd-degree vertices is a necessary and sufficient condition for a graph to have a circuit decomposition.

E 18.10 Show that a graph has a circuit decomposition if and only if all of its cuts are even.

E 18.11 Graphs that have circuit decompositions have no odd vertices. On the other hand, bicolorable graphs have no odd circuits. Is there something behind this parallel?

E 18.12 Let G be the graph of a plane map $\mathbb{M}$. Suppose that G is biconnected and has no vertices of degree 2. Let G^* be the graph of faces (i.e., the dual graph) of the map $\mathbb{M}$. Show that G^* is bicolorable if and only if G has a circuit decomposition.

[2] Therefore, an odd-degree vertex is a *certificate of inexistence* of a circuit decomposition.
[3] Oswald Veblen (1880 – 1960). See article in Wikipedia.
[4] Leonhard Euler (1707 – 1783). See article in Wikipedia.

E 18.13 (ALGORITHM) Devise an algorithm that receives a graph G and returns a circuit decomposition of G, or proves that there is no such decomposition.

Eulerian cycles and trails

As we said in the end of section 1.9, a **walk** in a graph is any sequence $(v_0, v_1, v_2, \ldots, v_{k-1}, v_k)$ of vertices such that v_i is adjacent to v_{i-1} for every i between 1 e k. We say that the walk **goes from** v_0 **to** v_k. The **length** of the walk is the number k.

A **trail** is a walk without repeated edges, i.e., a walk whose edges are pairwise distinct. A trail $(v_0, \ldots, v_k)$ is **closed** if $v_0 = v_k$.

A trail is **Eulerian**[5] if it passes through all the edges of the graph.[6] Thus, a trail $(v_0, v_1, \ldots, v_{k-1}, v_k)$ is Eulerian if and only if $\{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\}$ is the set of (all the) edges of the graph.

A **cycle** is a closed trail.[7] A cycle is Eulerian if and only if it passes through all the edges of the graph.

E 18.14 Find an Eulerian cycle in the graph of figure 18.2.

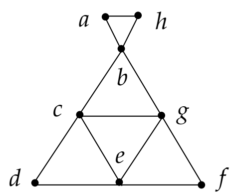

Figure 18.2: Find an Eulerian cycle. See exercise 18.14.

E 18.15 Consider the 21 tiles of the domino game which are not doubles. Each of those tiles corresponds to a subset of cardinality 2 of the set $\{0, 1, 2, \ldots, 6\}$. One is allowed to make a tile $\{i, j\}$ "touch" a tile $\{j, k\}$ so as to produce the sequence (i, j, j, k). Question: Is it possible to form a "wheel" that contains all the 21 tiles? What if we eliminate all tiles that contain a "6"?

[5] Reference to Leonhard Euler (1707 – 1783). See article in Wikipedia.

[6] Some authors also require the trail to pass through all the vertices of the graph. The difference between these two definitions is superficial.

[7] According to this definition, a cycle may have length 0. A circuit, on the other hand, has length at least 3 by definition.

$\star$ **E 18.16** (EULERIAN CYCLES) Show that every graph with an Eulerian cycle has a circuit decomposition.

Show that every connected graph with a circuit decomposition has an Eulerian cycle.

E 18.17 Give a necessary and sufficient condition for a graph to have a non-closed Eulerian trail.

E 18.18 (ALGORITHM) Devise an algorithm tho find an Eulerian trail (closed or not) in any given connected graph.

! E 18.19 (CHINESE POSTMAN PROBLEM[8]) Given a graph, find a shortest walk among those that are closed and pass through all the edges of the graph.

E 18.20 Suppose that a graph G has an Eulerian cycle. Show that the line graph $L(G)$ has a Hamiltonian circuit (see exercise 1.24).

Show that the converse is not true: $L(G)$ may have a Hamiltonian circuit without G having an Eulerian cycle.

E 18.21 Let xy and yz be two edges of a connected graph G that has no odd-degree vertices. Is it true that G has an Eulerian cycle in which xy and yz appear consecutively?

E 18.22 Let G be a connected graph all of whose vertices have even degree. Suppose also that $m(G)$ is even. Prove that E_G admits a partition $\{F_1, F_2\}$ such that $|F_1 \cap \partial\{v\}| = |F_2 \cap \partial\{v\}|$ for each vertex v, i.e., v is incident to the same number of edges of F_1 and F_2.

Minimum circuit covers

As we said in the beginning of the chapter, a **circuit cover** of a graph G is any collection $\mathcal{O}$ of circuits of G such that $\bigcup_{O \in \mathcal{O}} E_O = E_G$.

A circuit cover $\mathcal{O}$ is **minimum** if there is no circuit cover $\mathcal{O}'$ such that $|\mathcal{O}'| < |\mathcal{O}|$.

The **total length** of a circuit cover $\mathcal{O}$ is the sum $\sum_{O \in \mathcal{O}} |E_O|$. It is clear that every circuit decomposition is a cover of minimum total length.

The **thickness** of a circuit cover $\mathcal{O}$ of a graph G is the number $\max_{e \in E_G} |\{ O \in \mathcal{O} : E_O \ni e \}|$. So, if a circuit cover has thickness k, then every edge of the graph belongs to at most k of the circuits. Conversely,

[8] Proposed in 1962 by the Chinese mathematician Mei-Ko Kwan.

if each edge of the graph belongs to $\leq k$ circuits of the cover, then the cover has thickness $\leq k$. It is clear that a circuit decomposition is the same as a cover of thickness 1.

Exercises

∘ **E 18.23** Show that a graph has a circuit cover if and only if it has no bridges. (See exercise 1.199.)

D 18.24 (MINIMUM CIRCUIT COVER) Devise an algorithm to find a minimum circuit cover of any bridgeless graph.

! E 18.25 Show that, for every k even, the cube Q_k can be covered with only $k/2$ circuits.

E 18.26 Find a minimum circuit cover in the Petersen graph.

E 18.27 Find a minimum circuit cover of the first graph in figure 18.1. (This graph can be described as $K_6 - M$, where M is a perfect matching.)

D 18.28 (SHORTEST COVER) Find a circuit cover of minimum total length in any given bridgeless graph.[9]

E 18.29 Find a circuit cover of minimum total length in the Petersen graph.

⋆ **E 18.30** Show that every edge-biconnected planar graph G has a circuit cover of total length $\leq 2m(G)$.

D 18.31 (MINIMUM THICKNESS COVER) Find a minimum-thickness circuit cover of any given bridgeless graph.

(According to the Circuit Double Cover conjecture,[10] every bridgeless graph has a cover of thickness ≤ 2.)

⋆ **E 18.32** Show that every edge-biconnected planar graph has a circuit cover of thickness ≤ 2.

E 18.33 Find a minimum-thickness circuit cover of the Petersen graph.

[9] An efficient algorithm for this problem is not (yet) known. In technical terms, the problem is NP-hard.

[10] The conjecture is attributed to George Szekeres and Paul Seymour.

E 18.34 Find a minimum-thickness circuit cover of K_5.

E 18.35 Find a minimum-thickness circuit cover of $K_{3,3}$.

! E 18.36 (THEOREM OF KILPATRICK AND JAEGER[11]) Show that every 4-edge-connected graph (see page 125) has a circuit cover of thickness ≤ 2.

E 18.37 Show (by means of examples) that the concepts of minimum cover, cover of minimum total length, and minimum-thickness cover are pairwise distinct.

E 18.38 Why doesn't the Chinese postman problem (exercise 18.19) solve the minimum-thickness circuit cover problem (see exercise 18.31)? Why doesn't it solve the problem of the circuit cover of minimum total length (see exercise 18.28)?

[11] Published by Kilpatrick in 1975 and by F. Jaeger in 1976.

Chapter 19

Characterization of planarity

As we said in section 1.17, a graph is **planar** if it is representable by a plane map, i.e., if it is isomorphic to the graph of some plane map.

PLANARITY PROBLEM: Decide whether a given graph is planar or not.

If a graph is not planar, how can we make this evident? A very beautiful answer involves the concept of forbidden minors (see section 1.16): every non-planar graph has a minor which is *obviously* non-planar.

Exercises

$\star$ **E 19.1** Show that $K_{3,3}$ is not planar. (See, for example, exercise 1.271.)

$\star$ **E 19.2** Show that K_5 is not planar. (See, for example, exercise 1.270.)

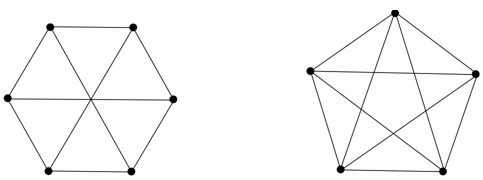

Figure 19.1: $K_{3,3}$ and K_5 are not planar. See exercises 19.1 and 19.2.

$\circ$ **E 19.3** Show that every subgraph of a planar graph is planar. In other words, if a graph G has a non-planar subgraph, then G is not planar.

$\circ$ **E 19.4** Suppose that all the proper subgraphs of a graph G are planar. Is it true that G is planar?

E 19.5 Suppose that a graph G has neither a subgraph isomorphic to K_5 nor a subgraph isomorphic to $K_{3,3}$. Is it true that G is planar?

⋆◦ **E 19.6** Show that every topological minor (see section 1.16) of a planar graph is planar. In other words, if a graph G has a non-planar topological minor, then G is not planar. (In particular, if G contains a subdivision of K_5 or $K_{3,3}$, then G is not planar.)

⋆ **E 19.7** Show that every minor (see section 1.16) of a planar graph is planar. In other words, if a graph G has a non-planar minor, then G is not planar. (In particular, if G has a subcontraction isomorphic to K_5 or to $K_{3,3}$, then G is not planar.)

E 19.8 Show that every proper minor of K_5 is planar. Show that every proper minor of $K_{3,3}$ is planar.

◦ **E 19.9** Show that $K_{3,3}$ is not a minor of K_5. Show that K_5 is not a minor of $K_{3,3}$.

E 19.10 For which values of t is the t-by-t bishop graph planar?

E 19.11 For which values of t is the t-by-t knight graph planar?

E 19.12 Show that the Petersen graph is not planar. (See exercises 1.247 and 1.248.)

E 19.13 Show that the cube Q_4 is not planar. (See exercise 1.249.)

⋆! **E 19.14** (WAGNER'S THEOREM[1]) *Show that a graph is planar if and only if it has neither a minor isomorphic to K_5 nor a minor isomorphic to $K_{3,3}$.* (Compare to exercise 19.7.)

⋆! **E 19.15** (KURATOWSKI'S THEOREM[2]) *Show that a graph is planar if and only if it has neither a topological minor isomorphic to K_5 nor a topological minor isomorphic to $K_{3,3}$.* (Compare to exercise 19.6.)

E 19.16 Discuss the following statement: "Since K_5 is not bicolorable, we can conclude that every non-planar bicolorable graph has $K_{3,3}$ as a topological minor".

[1] Published in 1937 by Klaus W. Wagner (1910 – 2000).
[2] Published in 1930 by Kazimierz Kuratowski (1896 – 1980).

! E 19.17 (ALGORITHM) Write an algorithm to decide whether a given graph is planar.

! E 19.18 Show that every 4-connected non-planar graph has a K_5 minor. Show that every 3-connected non-planar graph with 6 or more vertices has a $K_{3,3}$ minor.

D 19.19 Prove the following conjecture by Dirac:[3] If a graph G does not have a K_5 topological minor, then $m(G) \leq 3n(G) - 6$. (Compare to exercise 1.268.)

E 19.20 Show that a graph is outerplanar (see exercise 1.283) if and only if it has neither a K_4 minor nor a $K_{2,3}$ minor. Show that a graph is outerplanar if and only if it has neither a K_4 topological minor nor a $K_{2,3}$ topological minor.

[3] The conjecture was proposed by G. A. Dirac in 1964.

Appendix A

Hints for some of the exercises

Exercise 1.208. The proof does induction on the distance between r and s. It deals with the case in which r and s are neighbors, then the case in which there is a path of length 2 from r to s, and so on.

Let $v_0\, v_1\, \ldots\, v_k$ be a path from r to s. By induction hypothesis, there are two paths, P and Q, from v_0 to v_{k-1} such that $E_P \cap E_Q = \emptyset$. Let C be a circuit that contains edge $v_{k-1}v_k$. The graph $P \cup Q \cup C$ contains two paths from v_0 to v_k that share no edges.

Exercise 1.226. Suppose that there are two different paths, say P and Q, with endpoints x and y. Find a circuit in the graph $P \cup Q$.

Exercise 1.233. Let v be a vertex such that $\mathrm{d}(v) = \Delta$. For each neighbor w of v, take a maximal path among those that have v as first vertex and w as second vertex.

Exercise 1.228. Write a proof by induction on $m(G)$. Induction step: Let a be an edge of T, and let T_1 and T_2 be the two components of $T-a$. By induction hypothesis, $m(T_1) = n(T_1) - 1$ and $m(T_2) = n(T_2) - 1$.

Exercise 1.229. Induction on $n(G)$. Induction step: Suppose $m(G) = n(G) - 1$. Let v be a vertex such that $\mathrm{d}(v) = 1$. The graph $G - v$ is connected and $m(G - v) = n(G - v) - 1$. By induction hypothesis, $G - v$ has no circuits. Therefore, G has no circuits.

Exercise 1.266. Do an induction on the number of faces. The induction base uses the equality $m = n - 1$ which is valid for trees (see exercise 1.228).

Exercise 1.268. Begin by dealing with the case in which G is edge-biconnected. See exercises 1.266 and 1.267.

Exercise 1.275. See exercise 1.268.

Exercise 4.23. Let us say that the edges in D are red and the others are black. A path is even if it has an even number of red edges, and odd if it has an odd number of red edges.

Fundamental fact: if two paths have the same endpoints, then they have the same parity. Prove this fact by induction on the number of shared vertices.

Exercise 5.27. Show that the algorithm from exercise 5.22 returns a stable set S such that $|S| \geq n/(\mu + 1)$.

Exercise 6.15. Assume $\omega \leq 2$. Build a bicolorable graph H such that $V_H = V_G$ and $\mathrm{d}_G(v) \leq \mathrm{d}_H(v)$ for every vertex v. Use exercise 4.12.

Exercise 8.47. Let $\{X_1, \ldots, X_k\}$ be a minimum coloring which maximizes the number $\sum_{i=1}^{k} i\,|X_i|$. Then there is a path of the form $x_1\, x_2\, \ldots\, x_k$ with $x_i \in X_i$.

Another possibility: see algorithms in exercises 8.28 and 8.29.

Another possibility: remove the last vertex in a maximum-length path and apply induction.

Exercise 8.41. In the beginning of each iteration, vertices $v_1, \ldots, v_j$ have already been colored with k colors, and $G[\{v_1, \ldots, v_j\}]$ has a clique with k vertices.

Exercise 9.23. See exercise 9.17.

Exercise 9.36. Show that $G - v$ has at least one odd component. Then show that $G - v$ cannot have more than one odd component.

Exercise 10.4. Prove that one of the endpoints of each edge is saturated by all maximum matchings. Proof by contradiction: suppose there is an edge uw and maximum matchings M and N such that M does not saturate u and N does not saturate w. Study the component of $(V_G, M \cup N)$ that contains u.

Exercise 10.6. Proof by induction on the number of vertices. Take a vertex u that is saturated by all maximum matchings. Apply the induction hypothesis to $G - u$.

Exercise 10.18. Let M be a maximum matching. Let us say that a path is *good* if it has an endpoint in $U \setminus V(M)$ and is M-alternating. Let X be the set of vertices of all good paths. Let $K := (W \cap X) \cup (U \setminus X)$. Show that K is a cover. Show that $|K| = |M|$.

Exercise 10.12. Let M^* be a maximum matching, and suppose that $|M^*| < k$. According to Kőnig's theorem, G has a cover K_* such that $|K_*| < k$. Thus, $m(G) \leq |U|\,|K_*| \leq |U|\,(k-1)$. Contradiction.

Exercise 10.22. Let M be a matching and K a cover such that $|M| = |K|$ (see exercise 10.6). Show that $|U| \leq |K|$ and see exercise 9.30.

Exercise 10.22. The proof is an induction on the cardinality of U. The induction step has two cases. In the first, $|\mathrm{N}_G(Z)| > |Z|$ for every proper non-empty subset Z of U. In the second, $|\mathrm{N}_G(Y)| = |Y|$ for some proper non-empty subset Y of U. In the first case, take any edge uw with $u \in U$ amd apply the induction hypothesis to $G - \{u, w\}$. In the second, apply the induction hypothesis to $G - (Y \cup \mathrm{N}_G(Y))$ and to $G[(U \smallsetminus Y) \cup (W \smallsetminus \mathrm{N}(Y))]$.

Exercise 10.32. Consider the graph $(V_G, M \cup N)$. See exercise 9.17.

Exercise 12.2. Each worker is a vertex of my graph; each machine is a vertex as well; each edge is a task that assigns a worker to a machine; each color is a work day.

Exercise 12.15. G does not have a perfect matching.

Exercise 12.16. G does not have a perfect matching. Follows from exercise 12.15.

Exercise 12.23. Make an induction on r. See exercise 10.29.

Exercise 12.25. See exercise 10.34.

Exercise 13.4. It suffices to prove that each edge of C is a bridge in the graph (V_G, C). (See also exercises 1.149 and 1.157.)

Exercise 13.8. See exercise 1.226.

Exercise 14.9. Let x and z be two non-adjacent vertices of a tree. Let L be the only path from x to z. Show that if $\mathrm{exc}(x) = \mathrm{exc}(z)$ then $\mathrm{exc}(y) < \mathrm{exc}(x)$ for every internal vertex y of L.

Exercise 14.14. See exercise 14.13.

Exercise 14.18. For each vertex r, analyze the distance tree (exercise 14.7) centered in r. See exercises 14.13, 14.5, 1.123, and 14.3.

Exercise 14.19. For each vertex r, analyze the distance tree 14.7 centered in r.

Exercise 14.20. Find a minimum-weight perfect matching (see exercise 11.17) in an appropriate graph built from (G, u, v).

Exercise 14.21. Find a minimum-weight perfect matching (see exercise 11.17) in an appropriate graph built from (G, u, v).

Exercise 15.6. Do a proof by induction on k. Adopt the following definitions: For every subset B of E_G, let $V(B)$ be the set of vertices that that are incident to elements of B, and let $G(B)$ be the graph $(V(B) \cup \{r\}, B)$. A subset B of E_G is *good* if $G(B)$ is connected and $\mathrm{d}_{G-B}(Y) \geq k-1$ for every Y that contains r but does not contain s.

Start by proving the following lemma: For every good set B, if s is not in $G(B)$, then there is an edge e in $E_G \setminus B$ such that $B \cup \{e\}$ is good. Use exercise 1.115.

Exercise 16.10. Add to G a new vertex y and new edges connecting y to each vertex in S. Show that the new graph is k-connected.

Exercise 16.25. Do an induction on k, starting with $k = 2$. At the induction step, use the fan lemma, exercise 16.10.

Exercise 17.10. Take a maximal path. (See section 1.7.)

Exercise 17.19. See exercises 1.174 and 1.175.

Exercise 17.26. Let u and v be the endpoints of a maximum path P. Show that the graph $(V_P, E_P \cup \partial(u) \cup \partial(v))$ has a Hamiltonian circuit.

Exercise 18.6. See exercises 1.110 and 18.8.

Exercise 18.7. See exercises 1.110, 1.199 and 18.6.

Exercise 18.16. See exercise 1.126.

Exercise 18.19. If there are no odd-degree vertices, it suffices to find an Eulerian cycle. If there are only two odd-degree vertices, it suffices to consider a minimum-length path between those vertices. If there are more than two odd-degree vertices, use the minimum-weight matching algorithm (exercise 11.17) to choose paths connecting odd vertices in pairs.

Exercise 19.6. Follows from 19.7 and 1.250.

Exercise 19.7. This is a generalization of exercise 19.6. See exercises 1.250 and 1.251.

Exercise 19.14. Follows from 19.15 and 1.250.

Exercise 19.15. Follows from 19.14 and 1.251.

Appendix B

The Greek alphabet

Just as other areas of mathematics, graph theory resorts to the Greek alphabet for some of its notation:

α	A	alpha	ν	N	nu
β	B	beta	ξ	Ξ	xi
γ	Γ	gamma	o	O	omicron
δ	Δ	delta	π	Π	pi
ε	E	epsilon	ρ	P	rho
ζ	Z	zeta	σ	Σ	sigma
η	H	eta	τ	T	tau
θ	Θ	theta	υ	Υ	upsilon
ι	I	iota	φ	Φ	phi
κ	K	kappa	χ	X	chi
λ	Λ	lambda	ψ	Ψ	psi
μ	M	mu	ω	Ω	omega

The symbol ∂ does not belong to the Greek alphabet. It is used, in this text, to denote cuts (see section 1.8).

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Index