BEAUTIFUL CONJECTURES
IN
GRAPH THEORY

Adrian Bondy

What is a *beautiful* conjecture?

The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics.

G.H. Hardy

Some criteria:

- ▷ *Simplicity*: short, easily understandable statement relating basic concepts.
- ▷ *Element of Surprise*: links together seemingly disparate concepts.
- ▷ *Generality*: valid for a wide variety of objects.
- ▷ *Centrality*: close ties with a number of existing theorems and/or conjectures.
- ▷ *Longevity*: at least twenty years old.
- ▷ *Fecundity*: attempts to prove the conjecture have led to new concepts or new proof techniques.

Reconstruction Conjecture

P.J. Kelly and S.M. Ulam 1942

Every simple graph on at least three vertices is reconstructible from its vertex-deleted subgraphs

STANISLAW ULAM

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
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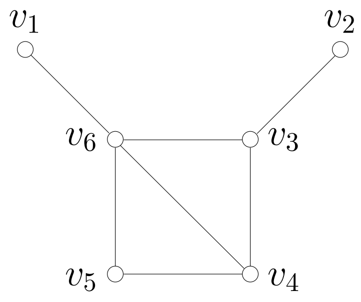

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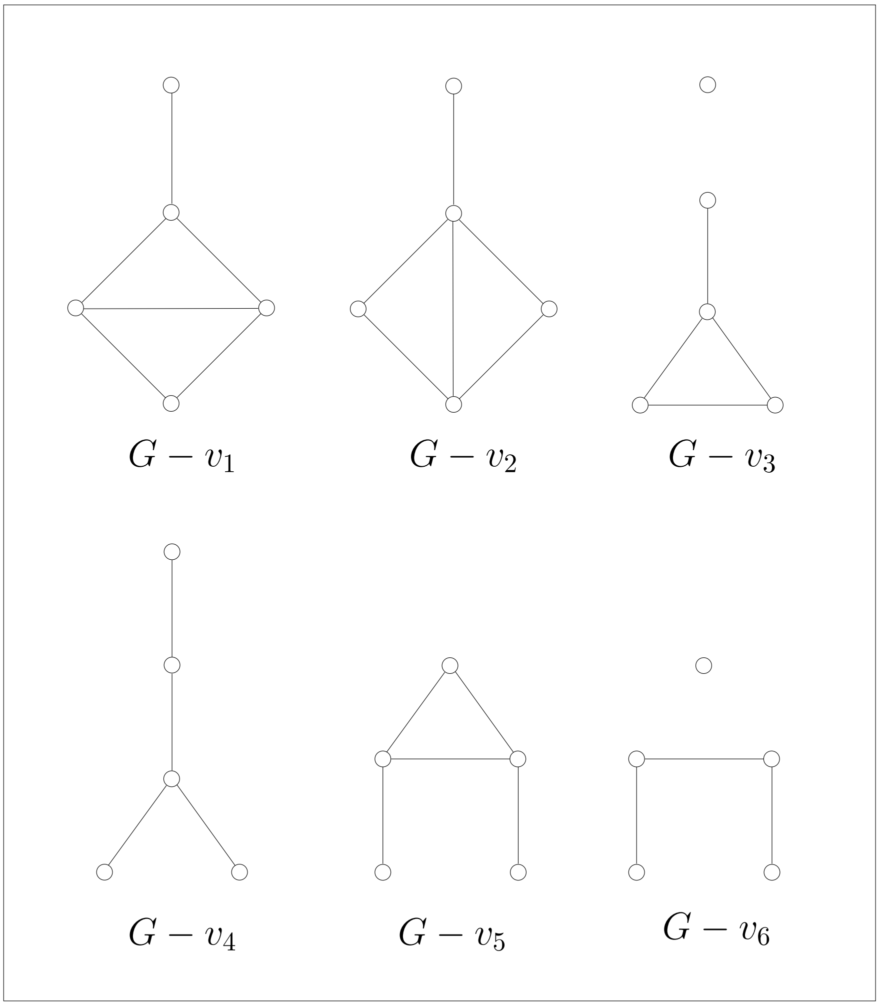

Edge Reconstruction Conjecture

F. Harary 1964

Every simple graph on at least four edges is reconstructible from its edge-deleted subgraphs

FRANK HARARY

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
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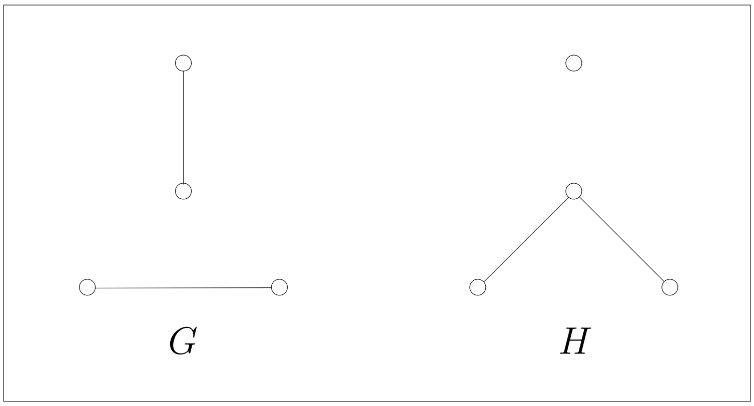

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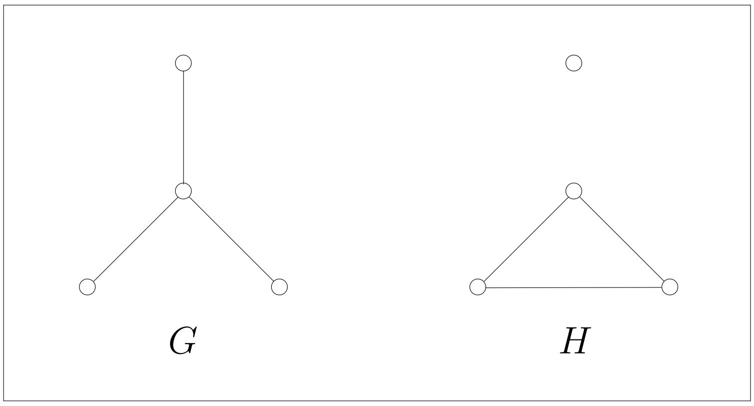

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MAIN FACTS

Reconstruction Conjecture

False for digraphs. There exist infinite families of nonreconstructible tournaments.

P.J. Stockmeyer 1977

Edge Reconstruction Conjecture

True for graphs on n vertices and more than $n \log_2 n$ edges.

L. Lovász 1972, V. Müller 1977

Path Decompositions

T. Gallai 1968

Every connected simple graph on n vertices can be decomposed into at most $\frac{1}{2}(n+1)$ paths

TIBOR GALLAI

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
$***$		$***$	$*$	$**$	$**$

Circuit Decompositions

G. Hajós 1968

Every simple even graph on n vertices can be decomposed into at most $\frac{1}{2}(n-1)$ circuits

György Hajós

Simple	Surprising	General	Central	Old	Fertile
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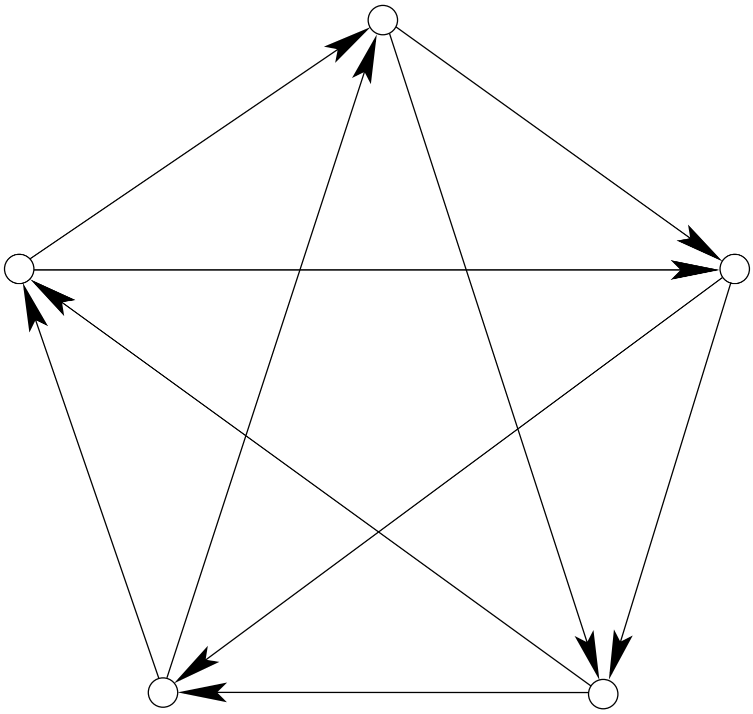
Hamilton Decompositions

P.J. Kelly 1968

Every regular tournament can be decomposed into directed Hamilton circuits.



Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
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MAIN FACTS

Gallai's Conjecture

True for graphs in which all degrees are odd.

L. Lovász 1968

Hajós' Conjecture

True for planar graphs and for graphs with maximum degree four.

J. Tao 1984,

Kelly's Conjecture

Claimed true for very large tournaments.

R. Häggkvist (unpublished)

Circuit Double Cover Conjecture

P.D. Seymour 1979

Every graph without cut edges has a double covering by circuits.

Paul Seymour

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
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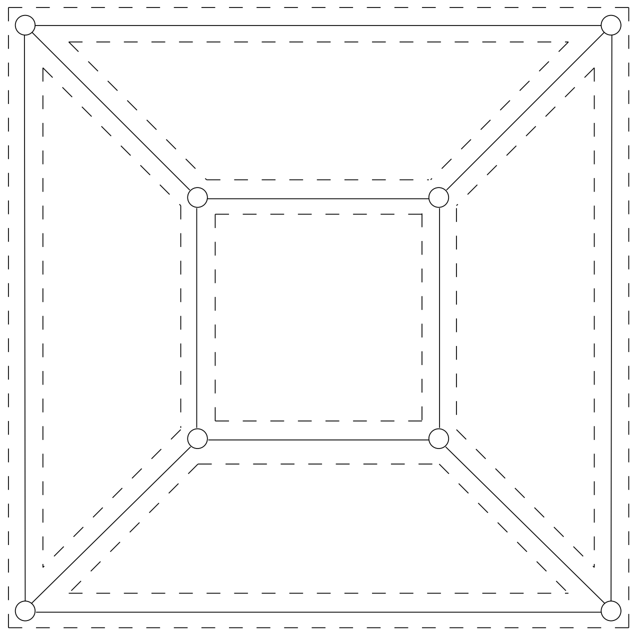
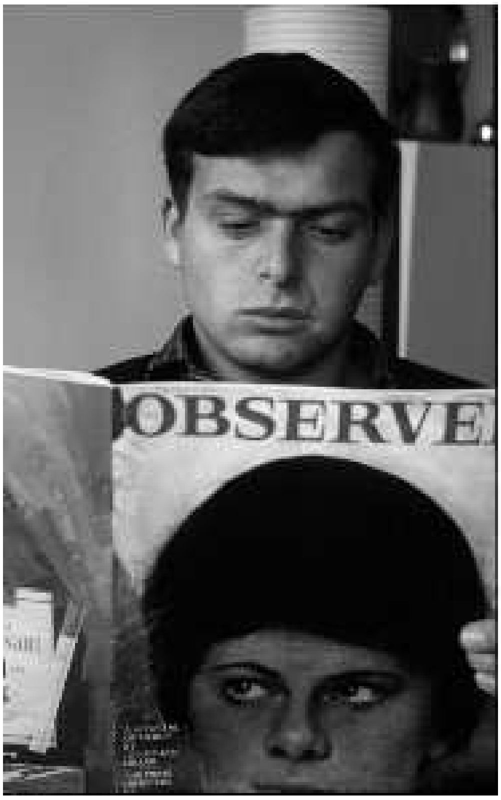


Small Circuit Double Cover Conjecture

JAB 1990

Every simple graph on n vertices without cut edges has a double covering by at most $n-1$ circuits.



JAB

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
$***$	$**$	$***$	$**$	$*$	$*$

Cycle Double Cover Conjecture

M. Preissmann 1981

Every graph without cut edges has a double covering by at most five even subgraphs

Myriam Preissmann

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
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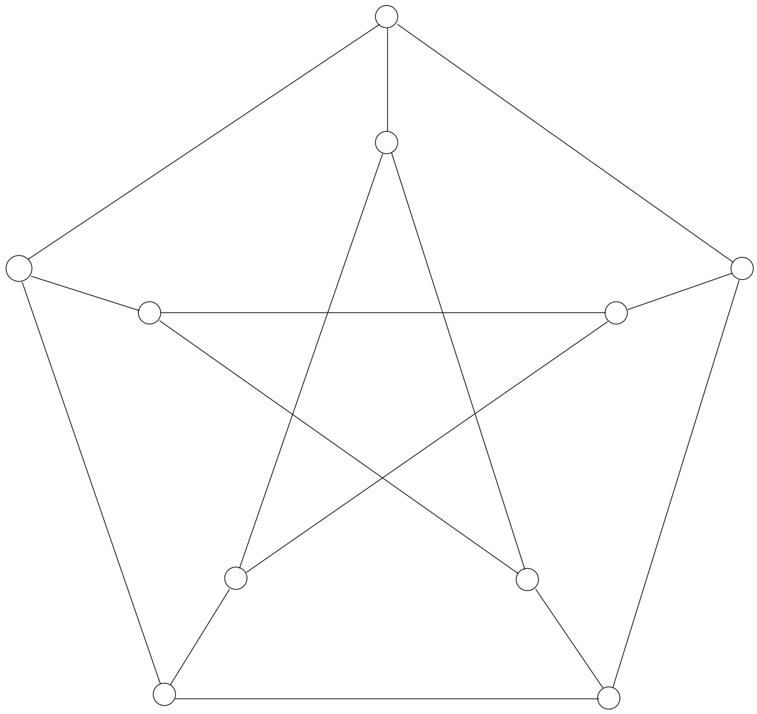


PETERSEN GRAPH

Matching Double Cover Conjecture

R.D. Fulkerson 1971

Every cubic graph without cut edges has a double covering by six perfect matchings

REFORMULATION:

Cycle Quadruple Cover Conjecture

F. Jaeger 1985

Every graph without cut edges has a quadruple covering by six even subgraphs

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
* * *	*	**	*	**	*

MAIN FACTS

Circuit Double Cover Conjecture

If false, a minimal counterexample must have girth at least ten.

L. Goddyn 1988

Small Circuit Double Cover Conjecture

True for graphs in which some vertex is adjacent to every other vertex.

H. Li 1990

Cycle Double Cover Conjecture

True for 4*-edge-connected graphs.*

P.A. Kilpatrick 1975, F. Jaeger 1976

True for various classes of snarks.

U. Celmins 1984

Cycle Quadruple Cover Conjecture

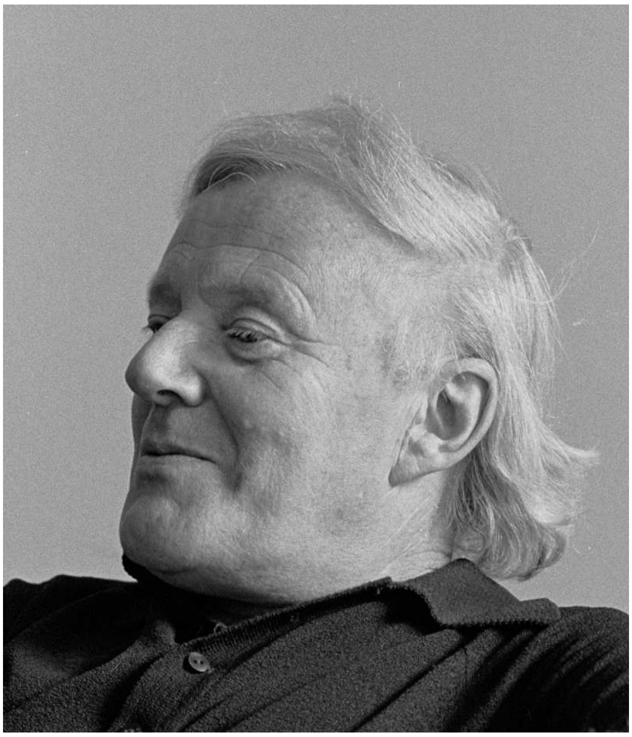
Every graph without cut edges has a quadruple covering by seven *even subgraphs.*

J.C. Bermond, B. Jackson and F. Jaeger 1983

Five-Flow Conjecture

W.T. Tutte 1954

Every graph without cut edges has a 5-flow



Bill Tutte

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
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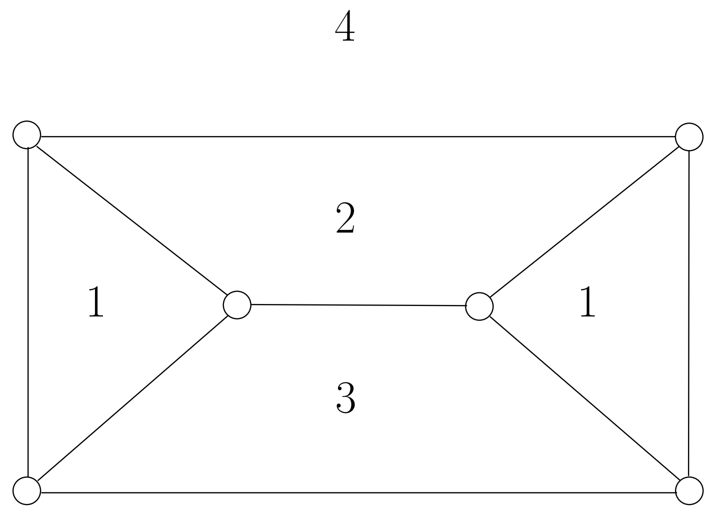
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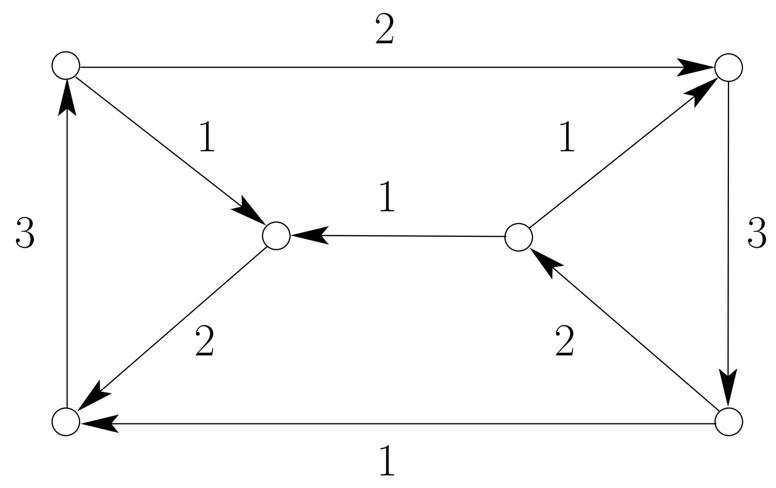
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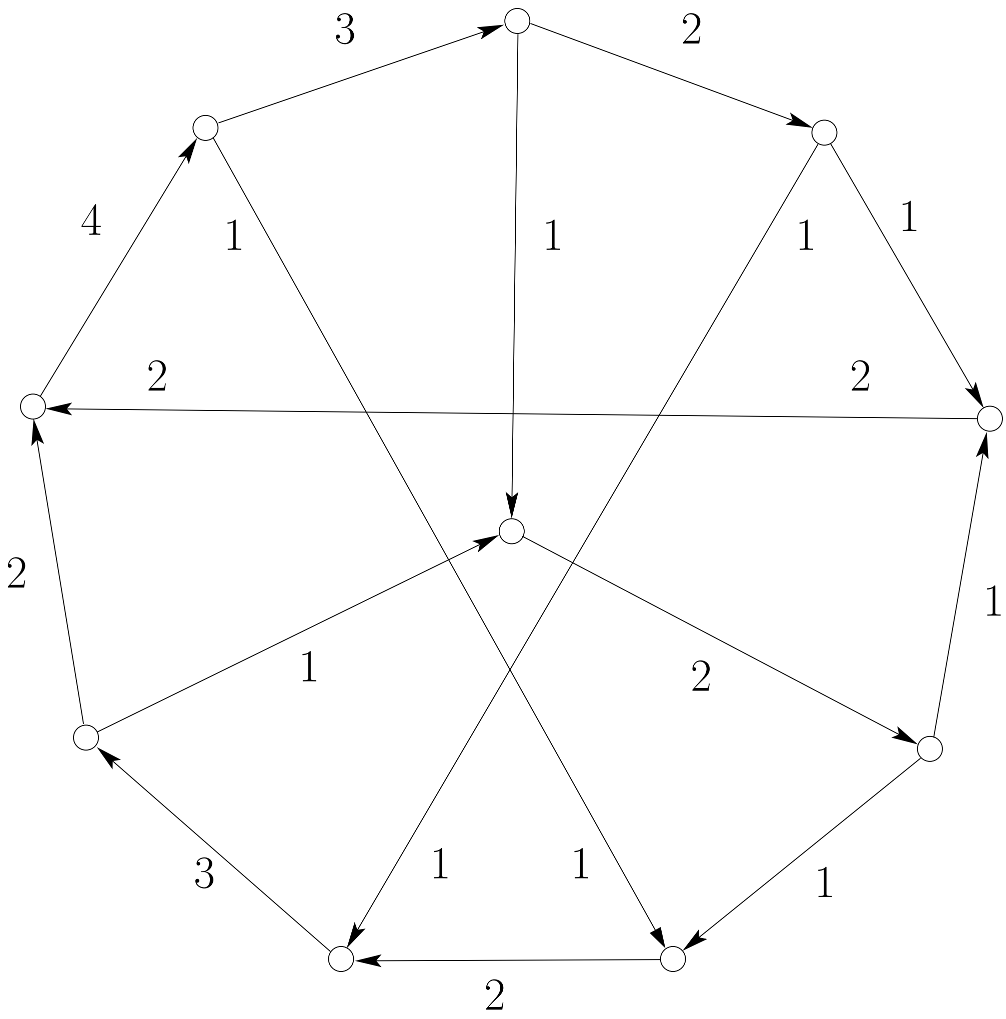
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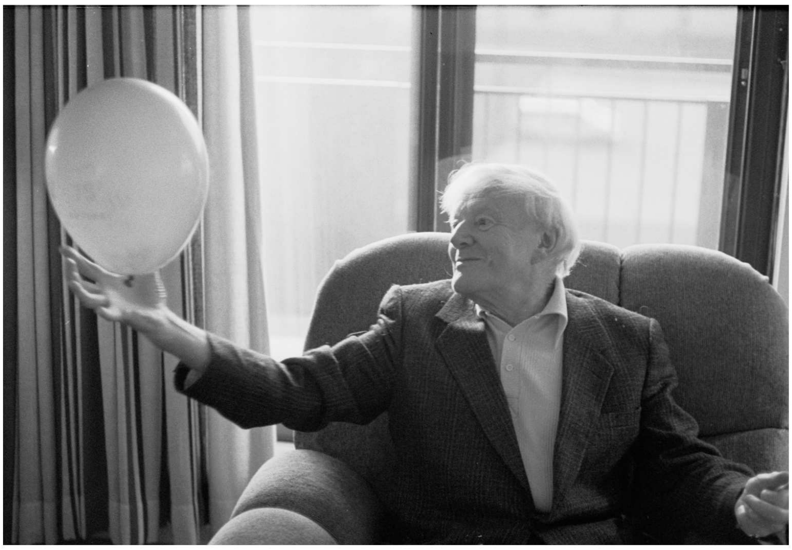
Three-Flow Conjecture

W.T. Tutte 1954

Every 4-edge-connected graph has a 3-flow



Bill Tutte

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
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WEAKER CONJECTURE:

Weak Three-Flow Conjecture

F. Jaeger, 1976

There exists an integer k such that every k-edge-connected graph has a 3-flow

MAIN FACTS

Five-Flow Conjecture

Every graph without cut edges has a 6*-flow.*

P.D. Seymour 1981

Three-Flow Conjecture

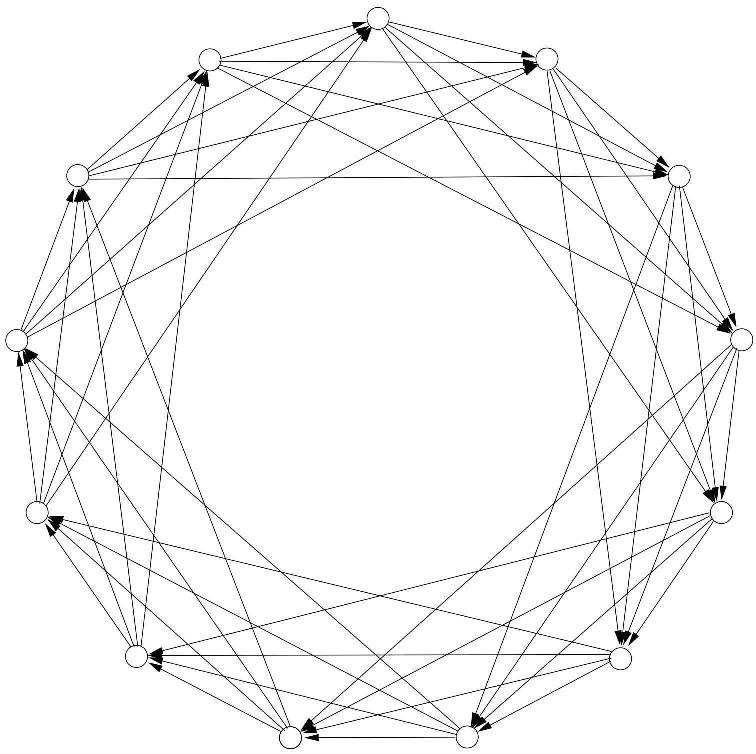
Every 4*-edge-connected graph has a* 4*-flow.*

F. Jaeger 1976

Directed Cages

M. Behzad, G. Chartrand and C.E. Wall 1970

Every d-diregular digraph on n vertices has a directed circuit of length at most $\lceil n/d \rceil$



Extremal graph for $d = \lceil n/3 \rceil$

(directed triangle)

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
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Second Neighbourhoods

P.D. Seymour 1990

Every digraph without 2-circuits has a vertex with at least as many second neighbours as first neighbours

Paul Seymour

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
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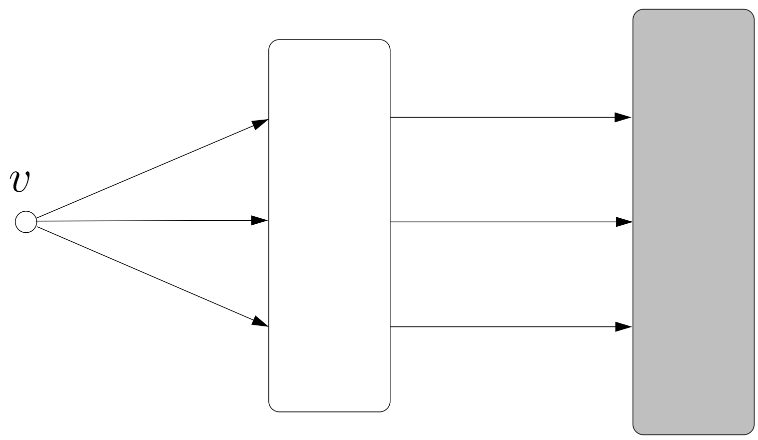
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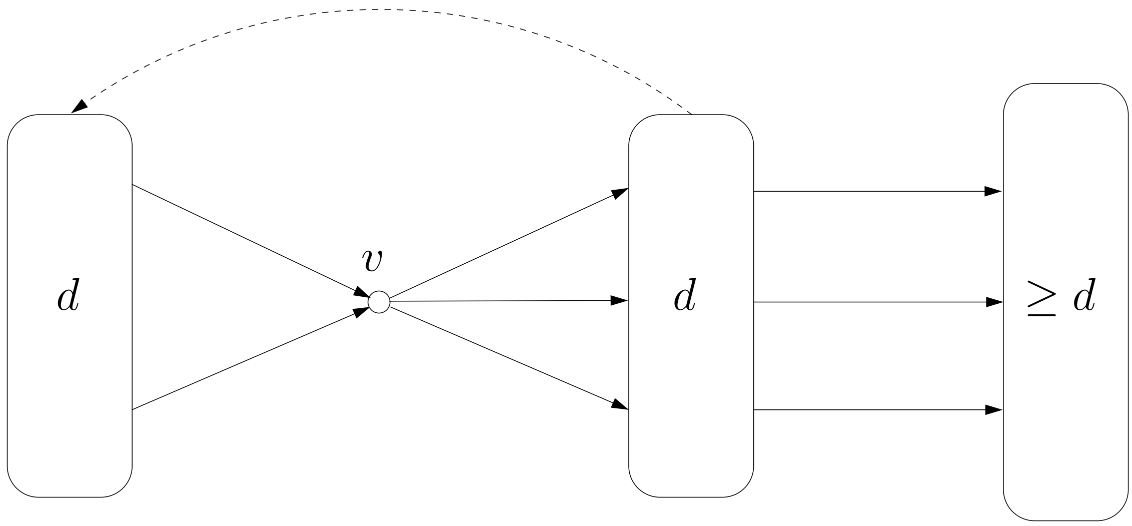
The *Second Neighbourhood Conjecture* implies the case

$$d = \left\lceil \frac{n}{3} \right\rceil$$

of the *Directed Cages Conjecture*:



If no directed triangle

$$n \geq 3d + 1 > n$$

<u>MAIN FACTS</u>

Behzad-Chartrand-Wall Conjecture

Every d-diregular digraph on n vertices has a directed circuit of length at most $n/d + 2500$.

V. Chvátal and E. Szemerédi 1983

True for $d \leq 5$.

C. Hoàng and B.A. Reed 1987

Every cn-diregular digraph on n vertices with $c \geq .34615$ has a directed triangle.

M. de Graaf 2004

Second Neighbourhood Conjecture

True for tournaments.

J. Fisher 1996, F.Havet and S. Thomassé 2000

Chords of Longest Circuits

C. Thomassen 1976

Every longest circuit in a 3-connected graph has a chord

Carsten Thomassen

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
$*\,*\,*$		$**$		$*$	$*$

Smith's Conjecture

S. Smith 1984

In a k-connected graph, where $k \geq 2$, any two longest circuits have at least k vertices in common

Scott Smith

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
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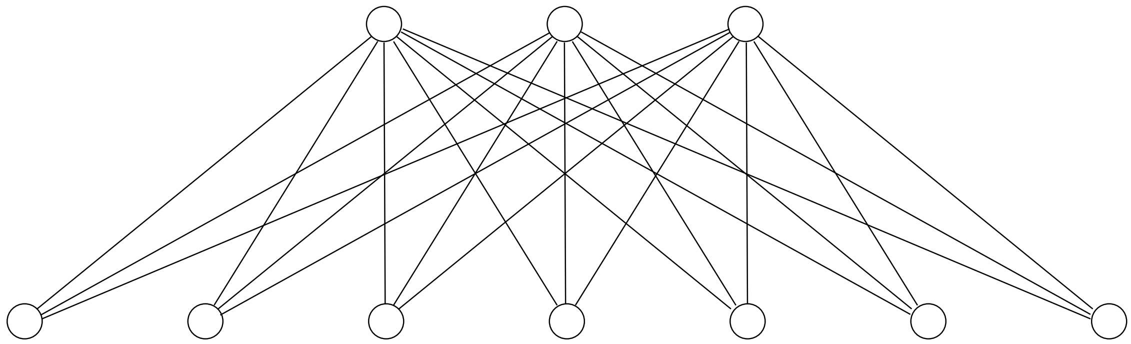

Hamilton Circuits in Line Graphs

C. Thomassen 1986

Every 4-connected line graph is hamiltonian

Carsten Thomassen

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
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Hamilton Circuits in Claw-Free Graphs

M. Matthews and D. Sumner 1984

Every 4-connected claw-free graph is hamiltonian

Simple	*Surprising*	*General*	*Central*	*Old*	*Prolific*
*	*			**	*

MAIN FACTS

Thomassen's Chord Conjecture

True for bipartite graphs.

C. Thomassen 1997

Scott Smith's Conjecture

True for $k \leq 6$.

M. Grötschel 1984

Thomassen's Line Graph Conjecture

Line graphs of 4-edge-connected graphs are hamiltonian.

C. Thomassen 1986

Every 7-connected line graph is hamiltonian.

S.M. Zhan 1991

Hamilton Circuits in Regular Graphs

J. Sheehan 1975

Every simple 4-regular graph with a Hamilton circuit has a second Hamilton circuit

John Sheehan

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
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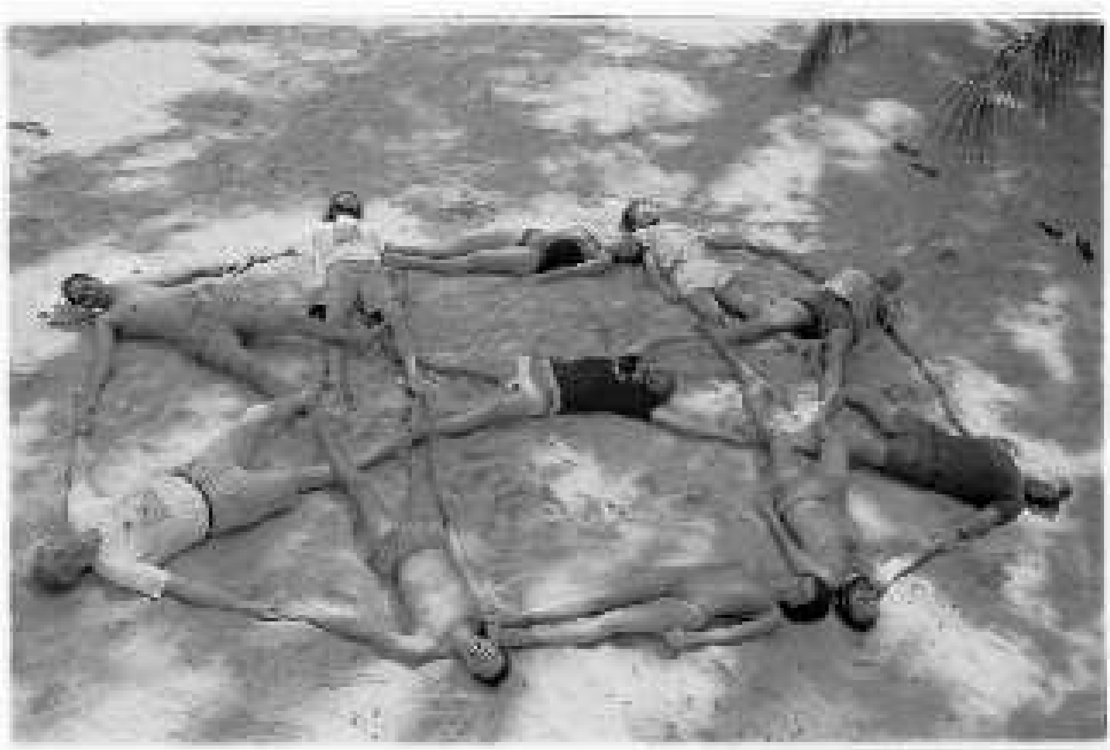


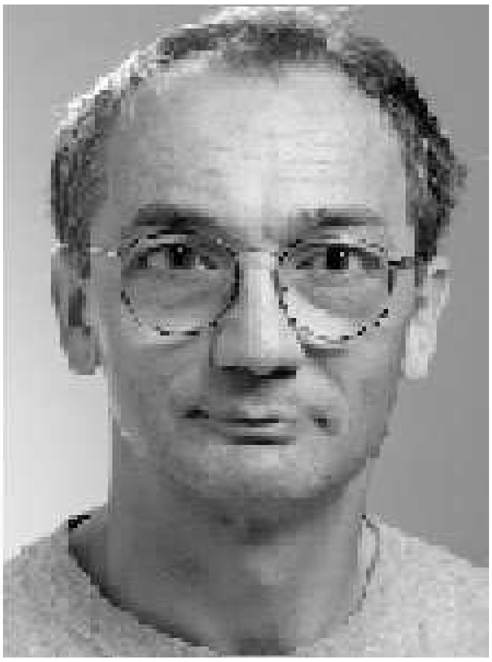
AN INTERESTING GRAPH

Used by Fleischner to construct a 4-regular multigraph with exactly one Hamilton circuit.

Finding a Second Hamilton Circuit

M. Chrobak and S. Poljak 1988

Given a Hamilton circuit in a 3-regular graph, find (in polynomial time) a second Hamilton circuit



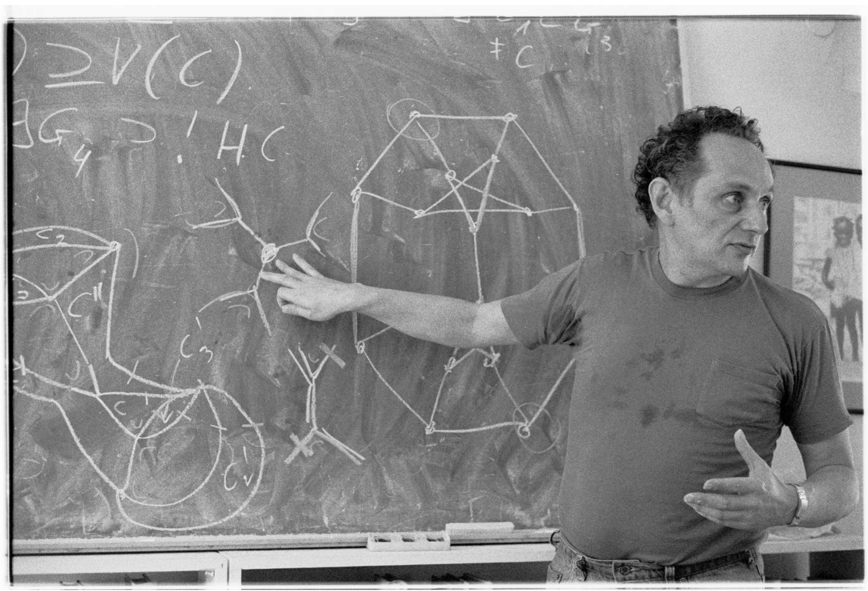
Marek Chrobak and Svatopluk Poljak

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
* * *	*	*	*	*	

Hamilton Circuits in 4-Connected Graphs

H. Fleischner 2004

Every 4-connected graph with a Hamilton circuit has a second Hamilton circuit



Herbert Fleischner

Simple	*Surprising*	*General*	*Central*	*Old*	*Fertile*
* * *	**	*	*		

MAIN FACTS

Sheehan's Conjecture

Every simple 300*-regular graph with a Hamilton circuit has a second Hamilton circuit.*

C. Thomassen 1998

There exist simple uniquely hamiltonian graphs of minimum degree four.

H. Fleischner 2004

Fleischner's Conjecture

True for planar graphs.

W.T. Tutte 1956

What is a *beautiful* theorem?

Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture.

Bertrand Russell

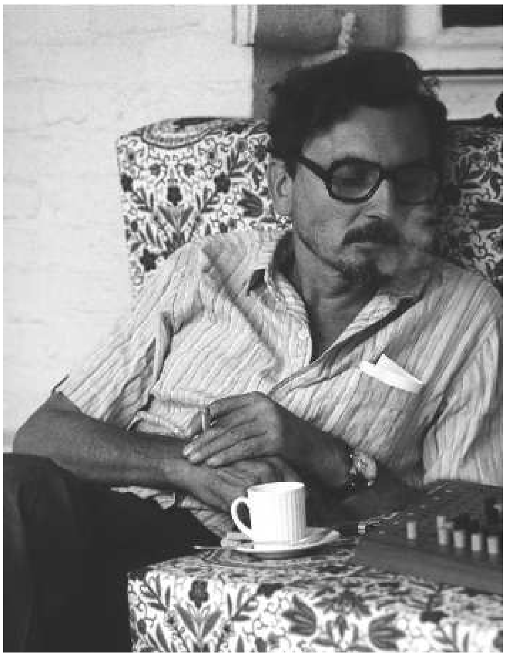
Some criteria:

- ▷ *Simplicity*: short, easily understandable statement relating basic concepts.
- ▷ *Element of Surprise*: links together seemingly disparate concepts.
- ▷ *Generality*: valid for a wide variety of objects.
- ▷ *Centrality*: close ties with a number of existing theorems and/or conjectures.
- ▷ *Fecundity*: has inspired interesting extensions and/or generalizations.
- ▷ *Correctness*: a beautiful theorem should be true!

What is a *beautiful* proof?

. . . an elegant proof is a proof which would not normally come to mind, like an elegant chess problem: the first move should be paradoxical . . .

Claude Berge



Claude Berge

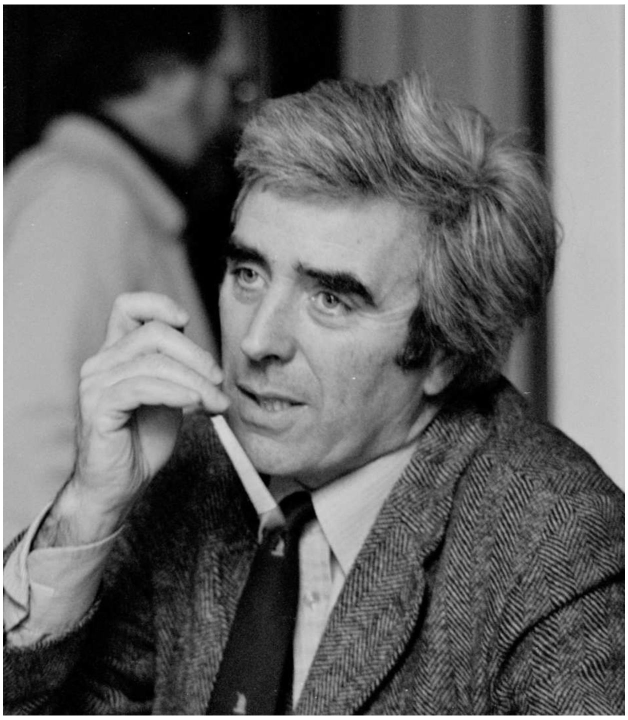
Some criteria:

- ▷ *Elegance*: combination of simplicity and surprise.
- ▷ *Ingenuity*: inspired use of standard techniques.
- ▷ *Originality*: introduction of new proof techniques.
- ▷ *Fecundity*: inspires new proof techniques or new proofs of existing theorems.
- ▷ *Correctness*: a beautiful proof should be correct!

Most Beautiful Conjecture

J.A.B.

Dominic will continue to prove and conjecture for many years to come



HAPPY BIRTHDAY, DOMINIC!

http://www.genealogy.math.ndsu.nodak.edu