

Woodall's conjecture on packings of dijoins

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<https://www.ime.usp.br/~pf/dijoins/>

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Abstract

In every digraph, the size of a minimum directed cut is equal to the maximum number of pairwise disjoint dijoins. This is Woodall's conjecture [Sch03]. Discussion of the conjecture stalled for two decades until Cornuéjols and Guenin [CGM00] took up the subject to study an abstract generalization of the conjecture.

This talk presents Woodall's conjecture, its capacitated version, Schrijver's counterexample, Cornuéjols and Guenin's counterexamples, and some results of Williams [Wil04].

1 Introduction

A *digraph* is a pair (V, A) where V is a finite set and A is a set of ordered pairs of elements of V . The elements of V are called *vertices* and those of A are called *arcs*. For each arc vw , the vertex v is the *positive endpoint* and w is the *negative endpoint* of the arc. The sets of vertices and arcs of a digraph D are denoted by $V(D)$ and $A(D)$ respectively. The *transpose*, or *directional dual*, of a digraph D is the digraph obtained by replacing each arc vw by the pair wv .

Cuts. An arc vw *exits* a subset X of $V(D)$ if $v \in X$ and $w \notin X$. An arc vw *enters* X if $v \notin X$ and $w \in X$. A *source* is any subset S of $V(D)$ such that no arc enters S . The sources \emptyset and $V(D)$ are *trivial*. A *sink* is a source in the transpose of D . A *source vertex* is any vertex s such that $\{s\}$ is a source and a *sink vertex* is a source vertex in the transpose of D .

For any set X of vertices, we denote by ∂X the set of arcs that have one endpoint in X and the other outside X . A *directed cut*, or simply *cut*, is any set of the form ∂S where S is either nontrivial source or a nontrivial sink. We say that S is a *positive margin* of the cut and $V(D) \setminus S$ is a *negative margin*. We also say that ∂S is the cut *associated* to S . A cut is *minimal* if none of its proper subsets is a cut.

A digraph is *connected* if \emptyset is not a cut. In a connected digraph, every cut has a unique positive margin and a unique negative margin.

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Joins. A *dijoin*, or simply *join*, is any set of arcs that intersects every cut, i.e., any subset J of $A(D)$ such that $J \cap C \neq \emptyset$ for each cut C . A join is *minimal* if none of its proper subsets is a join.

A digraph has a join if and only if \emptyset is not a cut. On the other hand, \emptyset is a join if and only if the digraph has no cut.

The following characterization is useful: a set J of arcs is a join if and only if for every pair (s, t) of vertices there is a path from s to t whose forward-directed arcs¹ belong to J . This characterization can also be formulated as follows: a set J of arcs is a join if and only if the contraction of all the arcs of J makes the digraph strongly connected.²

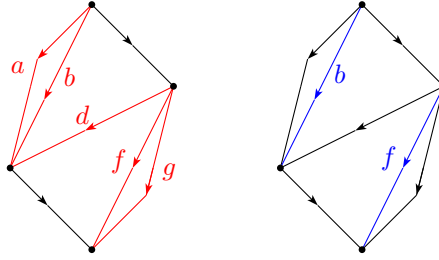


Figure 1: In the digraph on the left, the set of arcs $\{a, b, d, f, g\}$ is a cut. In the digraph on the right, the set of arcs $\{b, f\}$ is a join. (This example has parallel arcs, which is not in accordance with our definition of a digraph. To conform to the definition, we could “subdivide” the parallel arcs; but this would make the example too heavy.)

Cuts versus packings of joins. A set \mathcal{P} of joins is *disjoint* if the elements of \mathcal{P} are pairwise disjoint. In other words, \mathcal{P} is disjoint if each arc of the digraph belongs to at most one element of \mathcal{P} . A *packing of joins* is the same as a disjoint set of joins. There is no harm in assuming that the joins that make up a packing are minimal.

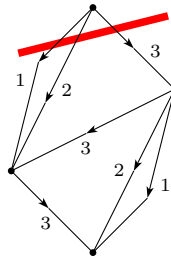


Figure 2: The colored line indicates a cut of size 3. The labels 1, 2 and 3 indicate a packing of three joins.

There is an obvious relationship between the size of a cut and the size of a packing of joins:

Lemma 1.1 For any packing \mathcal{P} of joins and any cut C the inequality $|\mathcal{P}| \leq |C|$ holds. ■

¹ An arc vw of a path is *forward-directed* if the path traverses the arc from v to w and *backward-directed* if the path traverses the arc from w to v .

² A digraph is *strongly connected* if for each ordered pair (s, t) of its vertices there is a path from s to t without backward-directed arcs.

The following conjecture of Woodall [Woo78a, Woo78b, Sch03] remains open:

Conjecture 1 (Woodall) *Every digraph with a cut has a packing \mathcal{P} of joins and a cut C such that $|\mathcal{P}| = |C|$.*

This conjecture is dual to the theorem of Lucchesi–Younger [LY78], according to which every connected digraph has a packing \mathcal{C} of cuts and a join J such that $|\mathcal{C}| = |J|$.

Every arc of a digraph belongs to either a cut or a directed circuit,³ but not both. In particular, $C \cap A(Z) = \emptyset$ for every cut C and every directed circuit Z , where $A(Z)$ is the set of arcs of Z . It follows from this observation that we can restrict the study of Conjecture 1 to DAGs, that is, to digraphs that have no directed circuits.

2 Minimum cut and maximum packing of joins

A cut C is *minimum* if there is no cut C' such that $|C'| < |C|$. A packing \mathcal{P} of joins is *maximum* if there is no packing \mathcal{P}' of joins such that $|\mathcal{P}'| > |\mathcal{P}|$. Woodall's conjecture leads us to consider the following pair of optimization problems:

Problem 1 *Find a minimum cut of a digraph.*

Problem 2 *Find a maximum packing of joins of a digraph.*

There is a polynomial algorithm for Problem 1 (it is a variant of the Max-flow Min-cut algorithm). No polynomial algorithm is known for Problem 2, but there is no evidence that the problem is NP-hard.

It is convenient to adopt a notation for the size of the objects that the two problems deal with. Given a digraph D , we denote by

$$\nu(D)$$

the size of a maximum packing of joins of D and we denote by

$$\tau(D)$$

the size of a minimum cut of D . If D has no cut then $\tau(D) = \infty$ and $\nu(D) = \infty$ (since an unbounded number of copies of \emptyset is a packing of joins). If D has a cut then $\tau(D)$ and $\nu(D)$ are finite. If D is disconnected then $\tau(D) = 0$ (since \emptyset is a cut) and $\nu(D) = 0$ (since there are no joins). If D consists of a path with at least one arc then $\tau(D) = 1$ and $\nu(D) = 1$ (since $A(D)$ is a join).

It follows immediately from Lemma 1.1 that $\nu(D) \leq \tau(D)$ for every digraph D . Conjecture 1 can then be formulated as follows:

Conjecture 2 (Woodall) *Every digraph D satisfies the equality $\nu(D) = \tau(D)$.*

We say that a digraph D *satisfies Woodall's conjecture* if $\nu(D) = \tau(D)$. If $\tau(D) \leq 1$ then it is obvious that D satisfies Woodall's conjecture. It is less obvious that if $\tau(D) = 2$ then D satisfies the conjecture [Sch03, p.968]. It is also known [FY87, Sch82] that every DAG with a single source vertex (or a single sink vertex) satisfies the conjecture.

³ A circuit is *directed* if it has no backward-directed arcs.

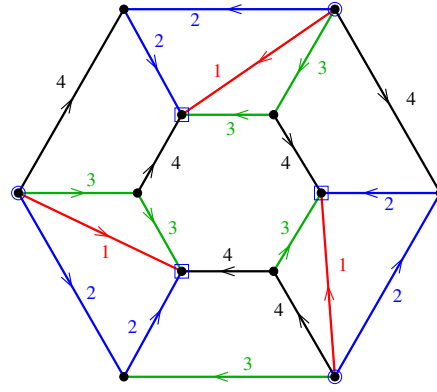


Figure 3: In this digraph, $\nu = 4$ and $\tau = 4$. Therefore, the digraph satisfies Woodall's conjecture. The colors (and the numerical labels) indicate a packing of 4 joins. The digraph is a DAG. The source vertices are marked by circles and the sink vertices by squares.

3 Linear programs

Let \mathcal{J} the set of all the minimal joins of a digraph $D = (V, A)$ and M be the matrix indexed by $\mathcal{J} \times A$ whose rows are the characteristic vectors of the elements of \mathcal{J} . Consider the following dual pair of linear programs:

$$\text{maximize } y1 \quad \text{under the constraints } y \in \mathbb{R}_+^{\mathcal{J}} \text{ and } yM \leq 1, \quad (1)$$

$$\text{minimize } 1x \quad \text{under the constraints } x \in \mathbb{R}_+^A \text{ and } Mx \geq 1. \quad (2)$$

(The “1” represents a vector whose elements are all equal to 1. The vector is indexed by \mathcal{J} or by A , depending on the context.)

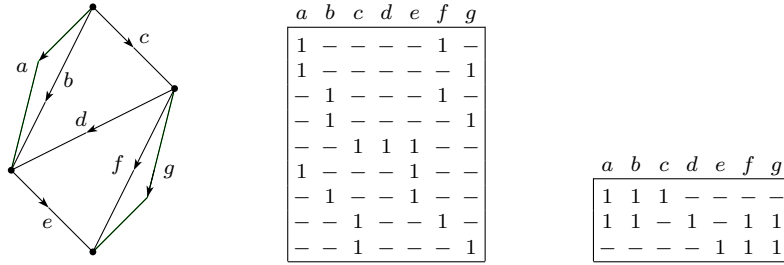


Figure 4: The rows of the first matrix are the characteristic vectors of the minimal joins of the digraph. The digraph is a DAG and has only one source and only one sink. The rows of the second matrix are the characteristic vectors of the minimal cuts.

If we replace “ $y \in \mathbb{R}_+^{\mathcal{J}}$ ” with “ $y \in \{0, 1\}^{\mathcal{J}}$ ” in the linear program (1) we will have an integer program that represents Problem 2. Every vector y in this program will represent a packing of joins and $y1$ will be the size of the packing. The optimum value of the integer program will be $\nu(D)$.

If we replace “ $x \in \mathbb{R}_+^A$ ” with “ $x \in \{0, 1\}^A$ ” in the linear program (2) we will have an integer program that represents Problem 1. Every x in this program will be the characteristic vector of a cut (since a cut is the same as a set of arcs that intersects all the joins) and $1x$ will be the size of the cut. The optimum value of the integer program will be $\tau(D)$.

As already noted, Woodall's conjecture is dual to the Lucchesi–Younger theorem [LY78]. It follows from that theorem (although this is not obvious) that all vertices of the polyhedron $\{x : x \in \mathbb{R}_+^A \text{ and } Mx \geq 1\}$ are integer and therefore every solution of the linear program (2) belongs to $\{0, 1\}^A$. It follows that $\tau(D) = \nu^*(D)$, where $\nu^*(D)$ is the optimum value of the linear program (1).

4 Max-flow analogy

To some extent, Woodall's conjecture is similar to the Max-flow Min-cut theorem [Sch03, chap.10]. This theorem applies to any digraph and any pair (s, t) of its vertices and guarantees that the size of a max-flow from s to t is equal to the size of a minimum half-cut among those separating s from t . Here, a *flow* is a set of directed paths⁴ from s to t with no common arcs; and a *half-cut* is the set of arcs that exit some set X of vertices that contains s but does not contain t .

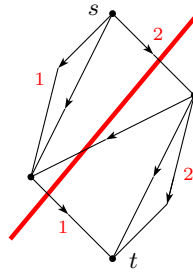


Figure 5: A maximum flow (labels 1 and 2) and a minimum semicut (colored line).

The similarity between Woodall's conjecture and the Max-flow Min-cut theorem is only partial. In the theorem, there are two fixed vertices and the paths are directed. In the conjecture, there are no fixed vertices, the paths (which represent joins) are not necessarily directed, and only the forward-directed arcs of the paths are taken into account.

The Max-flow Min-cut theorem admits a generalization in which each arc a has a capacity (or upper-bound) u_a in the set $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ of natural numbers. An arc a cannot be used more than u_a times by the flow and contributes u_a to the size of each half-cut that contains it. It is difficult to imagine that the Max-flow Min-cut theorem could hold without its capacitated generalization also holding.

The similarity between the Max-flow Min-cut theorem and Woodall's conjecture suggests studying the capacitated generalization of the conjecture.

5 Capacitated generalization of Woodall's conjecture

A *capacitated digraph* is a pair (D, u) where D is a digraph and u is a vector indexed by $A(D)$ with values in $\mathbb{N} \cup \{\infty\}$. This vector assigns a *capacity* u_a to each arc a of D . The arc a is *null* if $u_a = 0$ and *infinite* if $u_a = \infty$. Assigning capacity ∞ to an arc has the same effect as contracting the arc and is a convenient way to avoid contraction itself.

⁴ A path is *directed* if it has no backward-directed arcs.

The presence of infinite arcs calls for a redefinition of the terms “source” and “cut”. Thus, a **source** of a capacitated digraph (D, u) is a source S of D such that ∂S contains no infinite arcs, and a **cut** of (D, u) is a cut of D without infinite arcs. In other words, a cut of (D, u) is a set ∂S such that S is a nontrivial source of (D, u) .

The **capacity** of a cut C of (D, u) is the number $u(C) := \sum_{a \in C} u_a$. A cut C is **minimum** if there exists no cut C' of (D, u) such that $u(C') < u(C)$.

It is also convenient to redefine the term “join”. Thus, a **join** of (D, u) is a set of arcs that intersects all the cuts of (D, u) and does not contain infinite arcs. Of course any join of D without infinite arcs is also a join of (D, u) .

The concepts of directed path and directed circuit also need to be redefined. We will say that a path and a circuit are **directed in** (D, u) if all their backward-directed arcs are infinite. (In other words, infinite arcs can be traversed in either direction.) Under this redefinition, every non-infinite arc of (D, u) belongs to either a cut of (D, u) or a directed circuit in (D, u) , but not both.

In the context of capacitated digraphs, it is natural to use *collections* of joins in place of the sets of joins of Section 1. A collection is a “set” that can have multiple copies of the same element, each copy contributing 1 to the size of the collection. A collection \mathcal{P} of joins of (D, u) is **disjoint in** (D, u) if

$$|\mathcal{P}(a)| \leq u_a$$

for each arc a , where $\mathcal{P}(a) := \{J \in \mathcal{P} : J \ni a\}$ is the collection of the joins that contain a . In other words, \mathcal{P} is disjoint if each arc a belongs to at most u_a elements of \mathcal{P} . If a is null then no element of \mathcal{P} contains a .

A **packing** of joins **in** (D, u) is a disjoint collection of joins of (D, u) . The following relation between packings and cuts generalizes Lemma 1.1:

Lemma 5.1 *In any capacitated digraph (D, u) , for any packing \mathcal{P} of joins and any cut C ,*

$$|\mathcal{P}| \leq u(C).$$

Furthermore, if $|\mathcal{P}| = u(C)$ then $|J \cap C| = 1$ for each J in \mathcal{P} and $|\mathcal{P}(a)| = u_a$ for each a in C .

PROOF: Let \mathcal{P} be a packing of joins and C a cut of (D, u) . For each element J of \mathcal{P} there exists an arc a of C such that $\mathcal{P}(a) \ni J$. Therefore,

$$|\mathcal{P}| \leq \sum_{a \in C} |\mathcal{P}(a)| \leq \sum_{a \in C} u_a = u(C).$$

Suppose now that $|\mathcal{P}| = u(C)$. Then the first “ \leq ” holds as “ $=$ ” and therefore $|J \cap C| = 1$ for each J in \mathcal{P} . The second “ \leq ” also holds as “ $=$ ”, whence $|\mathcal{P}(a)| = u_a$ for each a in C . ■

The definitions of parameters τ and ν needs to be adjusted to take into account the capacities of the arcs. Thus, we denote by

$$\nu(D, u) \text{ and } \tau(D, u)$$

the size of a maximum packing of joins of (D, u) and the capacity of a minimum cut of (D, u) respectively. Lemma 5.1 has the following immediate consequence: every capacitated digraph (D, u) satisfies the inequality

$$\nu(D, u) \leq \tau(D, u). \quad (3)$$

The corresponding generalization of Woodall's conjecture (Conjecture 2) is known as the Edmonds–Giles conjecture [EG77]:

Conjecture 3 (Edmonds–Giles) *Every capacitated digraph (D, u) satisfies the equality $\nu(D, u) = \tau(D, u)$.*

If $\tau(D, u) = 0$ then $\nu(D, u) = 0$ and hence $\nu(D, u) = \tau(D, u)$. If $\tau(D, u) = 1$ then $\nu(D, u) = \tau(D, u)$ since $\{a \in A(D) : 0 < u_a < \infty\}$ is a join. Therefore, the conjecture is correct when restricted to capacitated digraphs in which $\tau(D, u) \leq 1$.

Null arcs. The capacitated generalization of the Max-flow Min-cut theorem (see Section 4) can be reduced to the original, non-capacitated, version. The reduction consists of removing the arcs of capacity 0 and replacing each arc of capacity $k \geq 2$ with k arcs in parallel. At first glance, the same construction could reduce the Edmonds–Giles conjecture to Woodall's conjecture. Indeed, an arc a of capacity $k \geq 2$ can be simulated by k copies of a in parallel, but removing an arc of capacity 0 may create new cuts,⁵ thus changing the instance of the problem. Therefore, the Edmonds–Giles conjecture is not a special case of Woodall's conjecture.

6 Counterexamples

The Edmonds–Giles conjecture is false. The following sections will present several counterexamples. A *counterexample* is any capacitated digraph (D, u) such that $\nu(D, u) < \tau(D, u)$. All known counterexamples have null arcs and therefore do not affect Woodall's conjecture.

We say that a digraph D is *good* if there is no u such that (D, u) is a counterexample. Conjecture 3 could be formulated by saying “every digraph is good”. It is known, for example, that

1. every DAG with a single source vertex is good;
2. every source-sink connected⁶ DAG is good.

The proof of 1 is analogous to that of the Max-flow Min-cut theorem mentioned in Section 4. This proof contains a polynomial algorithm that calculates $\tau(D, u)$. The proof of 2 was obtained by Schrijver [Sch82] and, independently, by F. and Younger [FY87].

7 Schrijver's counterexample

Schrijver [Sch80] found the first counterexample to Conjecture 3. The counterexample is represented in Figure 6 and will be denoted by (D_1, u_1) .

Fact 7.1 $\nu(D_1, u_1) = 1$ and $\tau(D_1, u_1) = 2$.

⁵ The removal of an arc does not create new cuts if and only if the arc is transitive. An arc vw is *transitive* in (D, u) if there is a directed path from v to w in $(D - vw, u')$, where u' is the restriction of u to the set of arcs of $D - vw$.

⁶ A DAG is *source-sink connected* if each source vertex is connected to each sink vertex by a directed path.

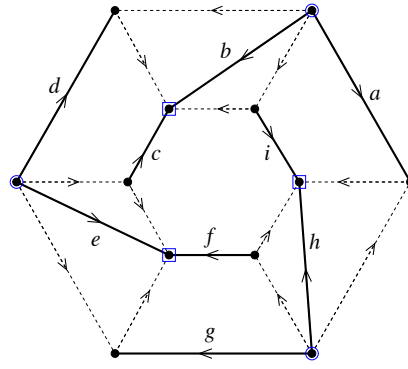


Figure 6: Schrijver's counterexample (D_1, u_1) . The capacity vector u_1 has values in $\{0, 1\}$. The null arcs are indicated by dashed lines; the others are indicated by solid lines. The digraph is a DAG; the source vertices are marked by circles and the sink vertices by squares.

PROOF: The vector u_1 is binary, that is, its components are in $\{0, 1\}$. It is easy to verify that $\tau(D_1, u_1) = 2$ and that one of the two margins of each minimum cut of (D_1, u_1) has a single vertex. Let B_1 be the set of *active* arcs, that is, arcs whose capacity is 1. The subdigraph induced by B_1 consists of three paths, each having length 3. We say that these are the *active paths* of the digraph. We also say that a cut is *critical* if it intersects each active path only once. As seen in Figure 7, there are four critical cuts.

Suppose for a moment that $\nu(D_1, u_1) \geq 2$. Then B_1 includes two mutually disjoint joins, say J and K . The arcs of each active path lie alternately in J and K , since each internal vertex of each active path is a margin of a cut with exactly 2 active arcs. In other words, each active path exhibits either the pattern (J, K, J) or the pattern (K, J, K) . In the set of three active paths, these two patterns can be combined in only 4 different ways, as shown in Figure 7. However, for each of the 4 combinations, either J or K does not intersect one of the critical cuts. Thus, J or K is not a join, contrary to what we had supposed. This contradiction shows that $\nu(D_1, u_1) < 2$. Since B_1 is a join, we have $\nu(D_1, u_1) = 1$. ■

a	b	c	d	e	f	g	h	i
J	K	J	J	K	J	J	K	J
J	K	J	J	K	J	K	J	K
J	K	J	K	J	K	J	K	J
J	K	J	K	J	K	K	J	K

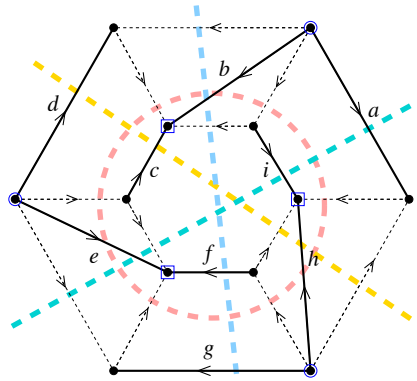


Figure 7: Each row of the table shows a possible arrangement of two potential mutually disjoint joins, J and K , in the capacitated digraph (D_1, u_1) in Figure 6. In each row of the table, one of J and K does not intersect one of the four critical cuts indicated in the drawing. In the first row, for example, J does not intersect the critical cut indicated by the pink circle.

The Schrijver capacitated digraph has the form of a ring of length $2i$, with $i = 3$. The

analogous capacitated digraphs with $i = 5, 7, 9, \dots$ (see Figure 8) are also counterexamples. The analogous capacitated digraphs with $i = 2, 4, 6, 8, \dots$ are not counterexamples.

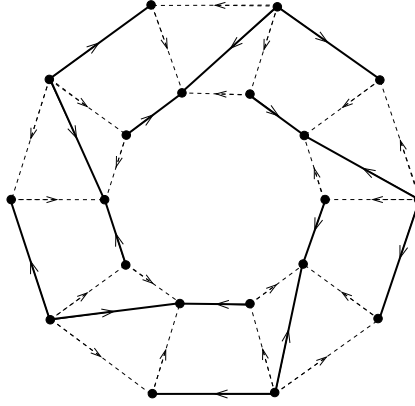


Figure 8: The counterexample (D'_1, u'_1) in the figure is the generalization of (D_1, u_1) in Figure 6 based on a ring of length 2×5 .

7.1 Fractional packing of joins

The following digression is interesting but has no direct bearing on the Edmonds–Giles conjecture. The capacitated digraph (D_1, u_1) in Figure 6 does not have a packing of size 2, but it does have a “fractional packing” of size 2, as we will show.

Let’s say that the joins $\{a, c, d, f, h\}$, $\{d, f, g, i, b\}$, $\{g, i, a, c, e\}$ and $\{b, h, e\}$ are *special*. Assign weight $\frac{1}{2}$ to each special join and weight 0 to all other join of D_1 . Every arc of capacity 1 in (D_1, u_1) belongs to exactly two of the special joins, and every arc of capacity 0 belongs to none of the special joins. Thus, the sum of the weights of all joins that contain a given arc a is no more than the capacity of a . Therefore, we can say that the weighted collection of special joins is “disjoint” in (D_1, u_1) . The size of this weighted collection is the sum of the weights of all joins, i.e., $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$. Thus, (D_1, u_1) has a “fractional packing” of size 2.

This example illustrates a general phenomenon. For any capacitated digraph (D, u) , consider the linear programs

$$\text{maximize } y1 \quad \text{under the constraints } y \in \mathbb{R}_+^{\mathcal{J}} \text{ and } yM \leq u \quad (4)$$

$$\text{minimize } ux \quad \text{under the constraints } x \in \mathbb{R}_+^A \text{ and } Mx \geq 1 \quad (5)$$

which generalize programs (2) and (1) of Section 3. It can be shown that $\nu^*(D, u) = \tau(D, u)$, where $\nu^*(D, u)$ is the optimum value of program (4) and $\tau(D, u)$ is the optimum value of program (5).

8 Cornuéjols and Guenin’s counterexamples

For two decades, (D_1, u_1) was the only known counterexample to Conjecture 3. In 2002, Cornuéjols and his student Guenin [CG02] found two new counterexamples, which we will denote by (D_2, u_2) and (D_3, u_3) . These counterexamples are represented in figures 9 and 10 respectively.

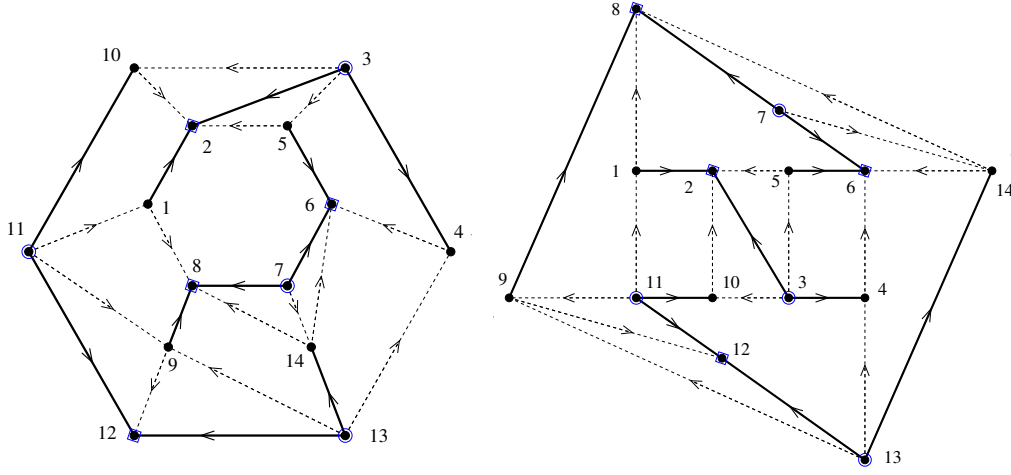


Figure 9: Two drawings of the counterexample (D_2, u_2) of Cornuéjols and Guenin. The capacity vector u_2 has values in $\{0, 1\}$. The null arcs are indicated by dashed lines; the others by solid lines. The digraph is a DAG; the source vertices are marked by circles and the sink vertices by squares.

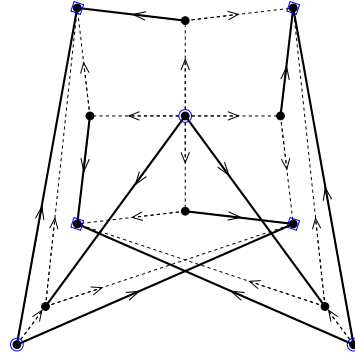


Figure 10: The counterexample (D_3, u_3) of Cornuéjols and Guenin. The capacity vector u_3 has values in $\{0, 1\}$. The null arcs are indicated by dashed lines; the others by solid lines. The digraph is a DAG.

Fact 8.1 $\nu(D_2, u_2) = 1$ and $\tau(D_2, u_2) = 2$. ■

Fact 8.2 $\nu(D_3, u_3) = 1$ and $\tau(D_3, u_3) = 2$. ■

The proofs of Facts 8.1 and 8.2 are similar to the proof of Fact 7.1. Figure 11 shows the critical cuts used in the proofs. (These are the cuts that intersect each active path only once.)

9 Minimal counterexamples

When looking for counterexamples to the Edmonds–Giles conjecture, we can limit ourselves to the counterexamples that, in some sense, do not “include” other counterexamples. We say that such counterexamples are minimal.

We begin by introducing an order relation between capacity vectors. Given two capacity vectors u and u' for a digraph, we say that $u' < u$ if $u'_a \leq u_a$ for every arc a and $u'_a < u_a$

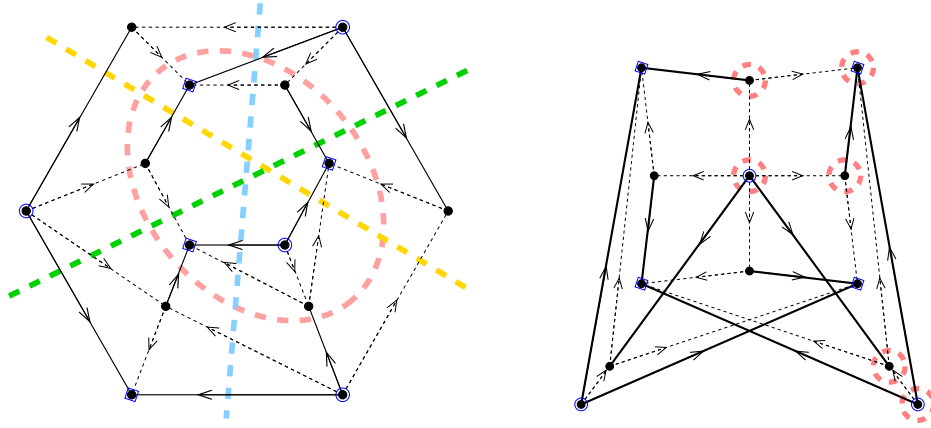


Figure 11: The first drawing represents the four critical cuts of (D_2, u_2) . The second drawing marks the vertices on the positive margin of one of the critical cuts of (D_3, u_3) ; the other three critical cuts are defined by symmetry.

for some arc a . Clearly the relation is transitive (i.e., if $u'' < u'$ and $u' < u$ then $u'' < u$) and antisymmetric (i.e., if $u' < u$ then $u \not< u'$).

We also need some auxiliary notation: for any capacitated digraph (D, u) , we will denote by $I(D, u)$ and $N(D, u)$ the set of infinite arcs and the set of null arcs of the digraph.

We can now define an inclusion relation. We say that a capacitated digraph (D, u) *includes* a capacitated digraph (D', u') if either

- i. $V' \subseteq V$ and $A' \subset A$ or
- ii. $V' = V$ and $A' = A$ and $I' \supset I$ or
- iii. $V' = V$ and $A' = A$ and $I' = I$ and $N' \supset N$ or
- iv. $V' = V$ and $A' = A$ and $I' = I$ and $N' = N$ and $u' < u$,

where V' , A' , I' and N' are abbreviations for $V(D')$, $A(D')$, $I(D', u')$ and $N(D', u')$ respectively and the abbreviations V , A , I and N are defined similarly for (D, u) . This inclusion relation is transitive and antisymmetric. Here are some examples to illustrate the concept:

1. Williams [Wil04] noted that the counterexample (D_2, u_2) of Cornuéjols–Guenin includes another counterexample. Let v and w be the vertices 14 and 8 of Figure 9 and let D'_2 be the digraph $D_2 - vw$. Let u'_2 the restriction of u_2 to the set $A(D'_2)$. Then (D'_2, u'_2) is included in (D_2, u_2) . Moreover, (D'_2, u'_2) is a counterexample, since $\nu(D'_2, u'_2) = \nu(D_2, u_2) < \tau(D_2, u_2) = \tau(D'_2, u'_2)$.
2. The counterexample (D'_1, u'_1) in Figure 8 includes the capacitated digraph (D''_1, u''_1) in Figure 12. The latter is a counterexample because it is “equivalente” to the counterexample (D_1, u_1) in Figure 6.
3. Schrijver’s counterexample (D_1, u_1) (see Figure 6) includes no other counterexample, although this is not obvious.

We say that a counterexample (D, u) is *minimal* if it does not include another counterexample.

If $A(D) = \emptyset$ or $A(D) = I(D, u) \cup N(D, u)$ then (D, u) is not a counterexample. It follows from this observation and from the transitivity and antisymmetry of the inclusion relation that every counterexample includes a minimal counterexample.

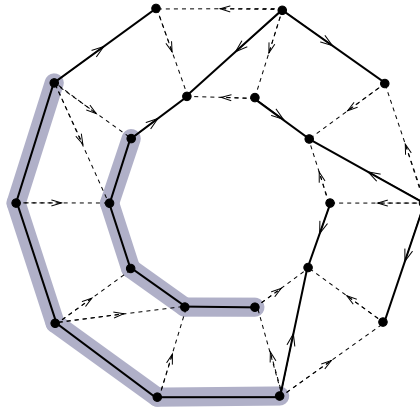


Figure 12: Capacitated digraph (D''_1, u''_1) . (Compare with (D'_1, u'_1) in Figure 8.) The gray bands indicate the infinite arcs. The orientation of these arcs has been omitted because they can be traversed in any direction.

Every minimal counterexample is, of course, connected. We will examine next some other properties of minimal counterexamples. The counterexamples (D_1, u_1) , (D_2, u_2) , and (D_3, u_3) in figures 6, 9, and 10 have these properties, although the latter two are not minimal.

9.1 There are no null transitive arcs

An arc vw is *transitive* in (D, u) if there exists a directed path from v to w in $(D - vw, u')$, where u' is the restriction of u to the set of arcs of $D - vw$. (According to our definitions, the path may have backward-directed infinite arcs.)

Proposition 9.1 *Minimal counterexamples do not have null transitive arcs.*

PROOF: We will show that the removal of a null transitive arc does not create new cuts and does not change the values of the parameters ν and τ .

Let (D, u) be a capacitated digraph and b be a null transitive arc. Let D' the digraph $D - b$ and u' be the restriction of u to the set of arcs of $D - b$. Let v be the positive endpoint and w the negative endpoint of b . Let B the set of arcs of a directed path from v to w in (D', u') .

Note that a cut of D intersects B if and only if it contains b . Furthermore, every cut of D contains at most one arc of B . It follows that D and D' have the same set of sources and therefore also (D, u) and (D', u') have the same set of sources. Furthermore, for each source F , we have $u'(C') = u(C)$, where C' and C are the cuts associated to F in D' and D respectively. Therefore,

$$\tau(D', u') = \tau(D, u). \quad (6)$$

Let \mathcal{P} be a packing of joins of (D, u) and \mathcal{P}' be a packing of joins of (D', u') . Since (D, u) and (D', u') have the same set of sources, every join of (D', u') is also a join of (D, u) . Therefore, \mathcal{P}' is a packing in (D, u) . Conversely, every join of (D, u) that does not contain b is a join of (D', u') . Since $u_b = 0$, none of the joins in \mathcal{P} contains b , and therefore \mathcal{P} is a packing in (D', u') . It follows that

$$\nu(D', u') = \nu(D, u). \quad (7)$$

By virtue of (6) and (7), if (D, u) is a counterexample then (D', u') is also a counterexample. Since $V(D') = V(D)$ and $A(D') \subset A(D)$, the counterexample (D, u) is not minimal. ■

The counterexample (D_1'', u_1'') in Figure 12 has several null transitive arcs and therefore is not minimal.

9.2 The capacity vector is critical

An arc is *active* in a capacitated digraph (D, u) if it is neither null nor infinite. The capacity vector u is *critical* if every active arc belongs to a minimum cut.

Proposition 9.2 *In every minimal counterexample, the capacity vector is critical.*

PROOF: Let (D, u) be a counterexample and suppose that u is not critical. Then some active arc a does not belong to a minimum cut. Let u' be the capacity vector defined by

$$u'_a := u_a - 1 \quad \text{and} \quad u'_e := u_e \quad \text{for each } e \neq a.$$

Clearly $I(D, u') = I(D, u)$ and therefore (D, u') and (D, u) have the same set of cuts. It is also clear that $u'(C) = u(C) - 1$ for every cut C that contains a and $u'(C) = u(C)$ for all the remaining cuts. Since minimum cuts of (D, u) do not contain a , we have

$$\tau(D, u') = \tau(D, u).$$

Now consider the joins. Let \mathcal{P}' be a maximum packing of joins of (D, u') . Since \mathcal{P}' is also a packing in (D, u) , we have

$$\nu(D, u') = |\mathcal{P}'| \leq \nu(D, u).$$

But $\nu(D, u) < \tau(D, u)$, whence $\nu(D, u') < \tau(D, u')$, and therefore (D, u') is a counterexample. As $N(D', u') \supseteq N(D, u)$ and $u' < u$, the counterexample (D, u) is not minimal. ■

9.3 All directed circuits are infinite

Assigning ∞ to the arcs of a directed circuit does not change the set of cuts of the digraph.

Proposition 9.3 *In a minimal counterexample, the arcs of every directed circuit are infinite.*

PROOF: Let (D, u) be a capacitated digraph and O be a directed circuit in (D, u) . (According to our definitions, O may have backward-directed infinite arcs.) Suppose that $u_a < \infty$ for some forward-directed arc a of O . Define a new capacity vector u' as follows:

$$u'_a := \infty \quad \text{and} \quad u'_e := u_e \quad \text{for each } e \neq a.$$

Since O is directed in (D, u) , no cut of (D, u) contains arcs of O . Therefore, the set of cuts of (D, u') is identical to the set of cuts of (D, u) . Thus,

$$\tau(D, u') = \tau(D, u).$$

No minimal join of (D, u) contains a since no cut of (D, u) contains a . Thus, (D, u') and (D, u) have the same minimal joins. Therefore, every packing of minimal joins of (D, u) is also a packing in (D, u') , and vice versa. It follows that

$$\nu(D, u') = \nu(D, u).$$

Now suppose that (D, u) is a counterexample. Then $\nu(D, u) < \tau(D, u)$ and therefore $\nu(D, u') < \tau(D, u')$, that is, (D, u') is a counterexample. Since $I(D, u') \supset I(D, u)$, and therefore the counterexample (D, u) is not minimal. ■

It follows from this proposition that every minimal counterexample is essentially a DAG.

9.4 All minimum cuts are peripheral

A cut C is *peripheral* in (D, u) if, for one of the two margins of C , every arc that has both endpoints on that margin is infinite. For example, C is peripheral if either the positive margin or the negative margin of C has a single vertex.

Proposition 9.4 *In a minimal counterexample, every minimum cut is peripheral.*

PROOF: We will show that any capacitated digraph can be divided, along a minimum non-peripheral cut, into two “independent” capacitated digraphs.

Let C be a minimum cut of a capacitated digraph (D, u) . Let u' be the capacity vector defined as follows:

$$u'_a := \begin{cases} \infty & \text{if } a \text{ has both the endpoints on the negative margin of } C, \\ u_a & \text{otherwise.} \end{cases}$$

(Informally, u' describes the contraction of the negative margin of C to a vertex.) Let u'' be the capacity vector defined as follows:

$$u''_a := \begin{cases} \infty & \text{if } a \text{ has both endpoints on the positive margin of } C, \\ u_a & \text{otherwise.} \end{cases}$$

According to Lemma 9.1 below, if (D, u) is a counterexample then either (D, u') or (D, u'') is a counterexample. Moreover, if C is not peripheral then $I(D, u') \supset I(D, u)$ (since some non-infinite arc has both endpoints on the negative margin of C) and, similarly, $I(D, u'') \supset I(D, u)$. Therefore, if the counterexample (D, u) is minimal, the cut C must be peripheral. ■

To conclude the proof of the proposition, we must establish the following lemma:

Lemma 9.1 *Let C be a minimum cut of a capacitated digraph (D, u) and let u' and u'' be the capacity vectors defined at the beginning of the proof of Proposition 9.4. If (D, u') and (D, u'') are not counterexamples then (D, u) is also not a counterexample.*

PROOF: On the one hand, C is a cut of (D, u') (since C does not contain infinite arcs) and $u'(C) = u(C)$, whence $\tau(D, u') \leq u'(C) = u(C) = \tau(D, u)$. On the other hand, $\tau(D, u') \geq \tau(D, u)$ since the set of cuts of (D, u') is part of the set of cuts of (D, u) and the capacity of a cut of (D, u') is equal to the capacity of that cut in (D, u) . Hence $\tau(D, u') = \tau(D, u)$ and therefore C is a minimum cut of (D, u') . An analogous reasoning shows that $\tau(D, u'') = \tau(D, u)$ and C is a minimum cut of (D, u'') .

1. Suppose that (D, u') is not a counterexample, i.e., that $\nu(D, u') = \tau(D, u')$. Let \mathcal{P}' be a maximum packing of joins of (D, u') . Of course $|\mathcal{P}'| = \nu(D, u') = \tau(D, u')$. Since $\tau(D, u') = u'(C)$, we have $|\mathcal{P}'| = u'(C)$. Lemma 5.1 (see Section 5) then guarantees that

$$|\mathcal{P}'(a)| = u_a \quad \text{for each } a \text{ in } C \text{ and} \quad (8)$$

$$|J' \cap C| = 1 \quad \text{for each } J' \text{ in } \mathcal{P}'. \quad (9)$$

Now suppose that (D, u'') is not a counterexample and let \mathcal{P}'' be a maximum packing of joins of (D, u'') . A reasoning similar to the previous paragraph shows that

$$|\mathcal{P}''(a)| = u_a \quad \text{for each } a \text{ in } C \text{ and} \quad (10)$$

$$|J'' \cap C| = 1 \quad \text{for each } J'' \text{ in } \mathcal{P}''. \quad (11)$$

2. By virtue of (8) and (9), for each nonnull arc a of C , there are elements $J'_{a,1}, \dots, J'_{a,u_a}$ of \mathcal{P}' such that

$$J'_{a,i} \cap C = \{a\} \quad (12)$$

for $i = 1, \dots, u_a$. By virtue of (10) and (11), there are elements $J''_{a,1}, \dots, J''_{a,u_a}$ of \mathcal{P}'' such that $J''_{a,i} \cap C = \{a\}$ for $i = 1, \dots, u_a$. Let

$$J_{a,i} := J'_{a,i} \cup J''_{a,i} \quad (13)$$

for each a in C and each i in $\{1, \dots, u_a\}$. Given any pair (a, i) , let J' , J'' and J be abbreviations for $J'_{a,i}$, $J''_{a,i}$ and $J_{a,i}$ respectively. Our next task is to show that J is a join of (D, u) . Since J' , J'' and J have no infinite arcs, we only need to show that $J \cap B \neq \emptyset$ for every cut B of (D, u) .

3. Let B be a cut of (D, u) and X be the positive margin of B . Let Y be the positive margin of C . If $X \cap Y = \emptyset$ or $X \supseteq Y$ then B is a cut of (D, u'') , whence $J'' \cap B \neq \emptyset$. If $X \cup Y = V$ or $X \subseteq Y$ then B is a cut of (D, u') , whence $J' \cap B \neq \emptyset$. In both cases we have $J \cap B \neq \emptyset$. In the other cases, thanks to (9), (11), (12) and (13), Lemma 9.2 below guarantees that $J \cap B \neq \emptyset$. Hence, J is a join of (D, u) .

4. Let \mathcal{P} be the collection of all joins $J_{a,i}$ such that a is a nonnull arc of C and i belongs to $\{1, \dots, u_a\}$. For every arc e of D , if e has positive endpoint on the positive margin of C then

$$|\mathcal{P}(e)| \leq u_e$$

since \mathcal{P}' is a packing in (D, u') and $u'_e = u_e$. Similarly, if e has negative endpoint on the negative margin of C then $|\mathcal{P}(e)| \leq u_e$. Therefore, \mathcal{P} is a packing in (D, u) .

5. It follows from the previous paragraph that $\nu(D, u) \geq |\mathcal{P}|$. But $|\mathcal{P}| = |\mathcal{P}'| = |\mathcal{P}''| = \tau(D, u)$, and therefore $\nu(D, u) \geq \tau(D, u)$. Hence, (D, u) is not a counterexample. ■

To conclude the proof of the lemma, we must establish the following consequence of the modularity of ∂ :

Lemma 9.2 (modularity) *Let Y be a nontrivial source of a digraph D . Let J be a set of arcs that intersects all cuts ∂X of D for which X is a source such that*

$$X \cup Y = V \quad \text{or} \quad X \cap Y = \emptyset \quad \text{or} \quad X \supseteq Y \quad \text{or} \quad X \subseteq Y.$$

If $|J \cap \partial Y| = 1$ then J is a join of D .

PROOF: Let X be a nontrivial source of D such that $X \cup Y \neq V$ and $X \cap Y \neq \emptyset$. To prove that J is a join of D , it suffices to show that $J \cap \partial X \neq \emptyset$.

It is clear that $X \cup Y$ and $X \cap Y$ are nontrivial sources of D . Therefore, $\partial(X \cup Y)$ and $\partial(X \cap Y)$ are cuts of D . Now observe that the union of $\partial(X \cup Y)$ with $\partial(X \cap Y)$ is equal to the union of ∂X with ∂Y and that the intersection of $\partial(X \cup Y)$ with $\partial(X \cap Y)$ is equal to the intersection of ∂X with ∂Y . Therefore, the sum $|\partial(X \cup Y)| + |\partial(X \cap Y)|$ is equal to the sum $|\partial X| + |\partial Y|$. Similarly,

$$|J \cap \partial(X \cup Y)| + |J \cap \partial(X \cap Y)| = |J \cap \partial X| + |J \cap \partial Y|. \quad (14)$$

Since $X \cup Y \supseteq Y$ and $X \cap Y \subseteq Y$, the assumptions of the lemma guarantee that each term on the left-hand side of (14) is at least 1. Since the second term on the right-hand side of (14) is exactly 1, the first term on the right-hand side must be at least 1. Therefore, $J \cap \partial X \neq \emptyset$, as claimed. ■

9.5 There are no active circuits

An arc is *active* in (D, u) if it is neither null nor infinite. Williams [Wil04] showed that in a minimal counterexample the subdigraph induced by the set of active arcs is a forest:

Proposition 9.5 *No minimal counterexample has a circuit of active arcs.*

PROOF: Let (D, u) be a counterexample and O be a circuit of D whose arcs are active. We will show that the counterexample is not minimal.

Let e be a minimum capacity arc in O and $k := u_e$. Adjust the notation so that e is forward-directed in O . Let u' be the capacity vector defined as follows:

$$u'_a := \begin{cases} u_a - k & \text{if } a \text{ is a forward-directed arc of } O, \\ u_a + k & \text{if } a \text{ is a backward-directed arc of } O, \\ u_a & \text{otherwise.} \end{cases}$$

It is clear that $u'_e = 0$ and therefore $N(D, u') \supset N(D, u)$. Thus, if (D, u') is a counterexample then the counterexample (D, u) is not minimal. We turn now to the case in which (D, u') is not a counterexample.

The set of cuts of (D, u') is identical to the set of cuts of (D, u) . It follows that the sets of joins of (D, u') and (D, u) are identical. Thus, we can say “cut” and “join” without adding “of (D, u) ” or “of (D, u') ”. Note that every cut contains the same number of backward-directed and forward-directed arcs of O . Therefore,

$$u'(C) = u(C) \quad (15)$$

for every cut C , and so

$$\tau(D, u') = \tau(D, u). \quad (16)$$

Let \mathcal{P}' be a maximum packing of joins of (D, u') . Since (D, u') is not a counterexample, $|\mathcal{P}'| = \tau(D, u')$. Let J_0 be an element of \mathcal{P}' . Lemma 9.3 below shows that $u'(C) - |J_0 \cap C| \geq |\mathcal{P}'| - 1$ for every cut C . Therefore,

$$u'(C) - |J_0 \cap C| \geq \tau(D, u') - 1$$

for every cut C . By virtue of (15) and (16), everything holds with u in place of u' , that is,

$$u(C) - |J_0 \cap C| \geq \tau(D, u) - 1 \quad (17)$$

for every cut C . It is clear that J_0 has no null arcs of (D, u') and therefore no null arcs of (D, u) .

Now that we have a join J_0 that satisfies (17), we can discard u' and \mathcal{P}' . Let u'' be the vector defined as follows: for each arc a ,

$$u''_a := \begin{cases} u_a - 1 & \text{if } a \in J_0 \text{ and} \\ u_a & \text{otherwise.} \end{cases} \quad (18)$$

Since J_0 has no null arcs, u'' is a capacity vector. The sets of cuts of (D, u'') and of (D, u) are identical and therefore the sets of joins of (D, u'') and (D, u) are identical. Thus, we can say “cut” and “join” without adding “of (D, u) ” or “of (D, u'') ”. For every cut C , we have $u''(C) = u(C) - |J_0 \cap C|$, whence $u''(C) \geq \tau(D, u) - 1$ by virtue of (17). Therefore,

$$\tau(D, u'') \geq \tau(D, u) - 1.$$

Let \mathcal{P}'' be a maximum packing of joins of (D, u'') . Suppose for a moment that (D, u'') is not a counterexample. Then $|\mathcal{P}''| = \nu(D, u'') = \tau(D, u'')$. Consider now the collection $\mathcal{P} := \mathcal{P}'' \cup \{J_0\}$ and observe that

$$|\mathcal{P}| = |\mathcal{P}''| + 1 = \tau(D, u'') + 1 \geq \tau(D, u) - 1 + 1 = \tau(D, u).$$

Also observe that \mathcal{P} is a packing in (D, u) , since $|\mathcal{P}(a)| = |\mathcal{P}''(a)| + 1 \leq u''_a + 1 = u_a$ for each a in J_0 and $|\mathcal{P}(a)| = |\mathcal{P}''(a)| \leq u''_a = u_a$ for each a outside J_0 . Therefore, $\nu(D, u) \geq |\mathcal{P}| \geq \tau(D, u)$ and so (D, u) is not a counterexample. This contradicts the way (D, u) was chosen at the beginning of the proof. Therefore, contrary to what we had supposed for a moment, (D, u'') is a counterexample. Since $I(D'', u'') = I(D, u)$ and $N(D'', u'') \supseteq N(D, u)$ and $u'' < u$, the counterexample (D, u) is not minimal. ■

To conclude the proof of the proposition, we must establish the following lemma:

Lemma 9.3 *For any packing \mathcal{P} of joins of (D, u) , any element J_0 of \mathcal{P} , and any cut C , the inequality $u(C) - |J_0 \cap C| \geq |\mathcal{P}| - 1$ holds.*

PROOF: Let \mathcal{P} be a packing of joins of (D, u) . Then $|\{J \in \mathcal{P} : J \ni a\}| \leq u_a$ for each a in C and therefore

$$\begin{aligned} u(C) &= \sum_{a \in C} u_a \\ &\geq \sum_{a \in C} |\{J \in \mathcal{P} : J \ni a\}| \\ &= \sum_{J \in \mathcal{P}} |\{a \in C : a \in J\}| \\ &= \sum_{J \in \mathcal{P}} |J \cap C| \\ &= |J_0 \cap C| + \sum_{J \in \mathcal{P} \setminus \{J_0\}} |J \cap C|. \end{aligned}$$

Since $|J \cap C| \geq 1$ for each J , we have $u(C) \geq |J_0 \cap C| + |\mathcal{P} \setminus \{J_0\}|$. Hence, $u(C) - |J_0 \cap C| \geq |\mathcal{P}| - 1$. ■

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