# REPRESENTATIONS OF THE SYMMETRIC GROUP AND POLYNOMIAL IDENTITIES

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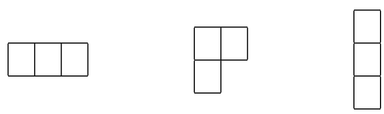
**Abstract.** Let $S_n$ denote the symmetric group on $n$ symbols. When $\mathbb{F}$ has characteristic zero or greater than $n$, the group algebra $\mathbb{F}S_n$ is a direct sum of $p(n)$ matrix algebras over $\mathbb{F}$, where $p(n)$ is the number of partitions of $n$. We present an efficient method due to J. M. Clifton (1981) that calculates the matrix associated to each element of $S_n$, for each partition of $n$. In 1950, A. I. Malcev and W. Specht independently used the representation theory of the symmetric group to classify polynomial identities of algebraic structures. Starting in the 1970's the method was further developed by A. Regev (see the survey paper Regev [50]). The method was implemented as a computer algebra system by I. R. Hentzel in 1977. We present this computational method and give applications to research on identities of Cayley-Dickson algebras, nuclear elements of the free alternative algebra and special identities of Bol algebras. For some applications of the computational method to other algebraic structures see the papers [9]-[14] by M. R. Bremner and the author.

## 1. Representations of the symmetric group

Let $n$ be a positive integer, $S_n$ the symmetric group on $\{1, \dots, n\}$ and $FS_n$ the group algebra over the field $\mathbb{F}$. We remind that if $G$ is a group then the group algebra $\mathbb{F}G$ is the vector space over $\mathbb{F}$ with basis $G$, and the multiplication in $\mathbb{F}G$ is given by extending the multiplication in $G$. The representation theory of $\mathbb{F}S_n$ was determined in 1900 by Alfred Young (1873-1940). It has since been simplified and exposited by many authors. See for example Boerner [4] and James and Kerber [40]. We give an exposition of this theory based on Clifton [15]. In his review (MR0624907 (82j:20024)) of Clifton's paper, James wrote the following: "Most methods for working out the matrices for the natural representation are messy, but this paper gives an approach which is simple both to prove and to apply."

**1.1. Frames and tableaus.** A *partition* of $n$ is an ordered sequence of positive integers $\lambda = \lambda_1 \dots \lambda_k$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \cdots + \lambda_k = n$. The *frame* corresponding to $\lambda$ consists of $n$ boxes in $k$ left-justified rows of lengths $\lambda_1, \dots, \lambda_k$. For example, if $n = 3$ then there are 3 partitions $3,\ 21,\ 111$ with corresponding frames



A *tableau* corresponding to $\lambda$ consists of a bijective assignment of the numbers $1, \ldots, n$ to the $n$ boxes. A *standard tableau* is one in which the assigned numbers increase in each row from left to right and in each column from top to bottom. For example, if $n = 3$ then the standard tableaus are:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

The number of standard tableaus corresponding to partition $\lambda = \lambda_1 \ldots \lambda_k$ of $n$ is given by

$$\frac{n! \prod_{i<j} (l_i - l_j)}{l_1!\, l_2! \ \ldots \ l_k!} \qquad \text{where} \quad l_i = \lambda_i + k - i \quad (i = 1, \ldots, k).$$

If $\mathcal{T}$ is a tableau we denote by $\mathcal{T}(i,j)$ the number in row $i$ and column $j$. The *lexicographical order* on tableaus is defined by $\mathcal{T} < \mathcal{T}'$ if and only if $\mathcal{T}(i,j) < \mathcal{T}'(i,j)$ where $i$ is the least row index for which $\mathcal{T}(i,j) \neq \mathcal{T}'(i,j)$.

For finite groups we have:

**Theorem 1.1. (H. Maschke 1899)** *For any finite group $G$ and any field $\mathbb{F}$, of characteristic 0 or characteristic $p$ not dividing $|G|$, the group algebra $\mathbb{F}G$ is semisimple.*

It follows that $\mathbb{F}G$ is isomorphic to the direct sum of simple ideals, and each simple ideal is isomorphic to the endomorphism algebra of a vector space over a division ring over $\mathbb{F}$. For $G = S_n$, these division rings are all equal to $\mathbb{F}$ and we have:

**Corollary 1.2.** *The* **Wedderburn decomposition** *of $\mathbb{F}S_n$ is given by an isomorphism*

$$\phi \colon \mathbb{F}\, S_n \to \bigoplus_{\lambda} M_{d_\lambda}(\mathbb{F}),$$

*where the summation is over all partitions $\lambda$ of $n$, and $d_\lambda$ is the corresponding number of standard tableaus.*

A permutation $\pi$ of $\{1, 2, \ldots, n\}$ is denoted by the sequence $\pi(1)\pi(2)\ldots\pi(n)$. For $n = 3$ we have the isomorphism

$$\phi \colon \mathbb{F}S_3 \to \mathbb{F} \oplus M_2(\mathbb{F}) \oplus \mathbb{F}.$$

given by

$$\begin{aligned}
\phi(123) &= \left[1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 1\right], & \phi(132) &= \left[1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, -1\right], \\
\phi(213) &= \left[1, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, -1\right], & \phi(231) &= \left[1, \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, 1\right], \\
\phi(312) &= \left[1, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, 1\right], & \phi(321) &= \left[1, \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, -1\right].
\end{aligned}$$

**1.2. Computing the irreducible representations.** Let $\lambda$ be a partition of $n$. Let $\mathcal{T}_1, \dots, \mathcal{T}_{d_\lambda}$ be the standard tableaus for $\lambda$ in lexicographical order. For $\pi \in S_n$ we denote by $R^\lambda_\pi$ the $d_\lambda \times d_\lambda$ matrix representing $\pi$, that is, the projection of $\pi$ onto the summand $M_{d_\lambda}(\mathbb{F})$ in the Wedderburn decomposition given by Corollary 1.2.The map $R^\lambda \colon S_n \to M_{d_\lambda}(\mathbb{F})$ defined by $R^\lambda(\pi) = R^\lambda_\pi$ is the *irreducible representation* of $S_n$ corresponding to the partition $\lambda$. Let $\iota$ denote the identity permutation in $S_n$. The matrix $R^\lambda(\pi)$ is given by

$$R^\lambda_\pi = (A^\lambda_\iota)^{-1} A^\lambda_\pi,$$

where $A^\lambda_\iota$ and $A^\lambda_\pi$ are the Clifton matrices corresponding to $\iota$ and $\pi$.

The *Clifton matrix* $A^\lambda_\pi$ is the $d_\lambda \times d_\lambda$ matrix defined by the following algorithm for computing each entry $(A^\lambda_\pi)_{ij}$:

- Apply $\pi$ to the standard tableau $T_j$ obtaining the (possibly non-standard) tableau $\pi T_j$.
- If there exist two numbers that appear together both in a column of $T_i$ and in a row of $\pi T_j$, then $(A^\lambda_\pi)_{ij} = 0$.
- Otherwise, there exists a *vertical permutation* $\kappa \in S_n$ which
  - leaves the columns of $T_i$ fixed as sets,
  - and takes the numbers of $T_i$ into the rows they occupy in $\pi T_j$.

  In this case, $(A^\lambda_\pi)_{ij} = \epsilon(\kappa)$, the sign of $\kappa$.

For $n \leq 4$ and for all partitions $\lambda$, $A^\lambda_\iota$ is the identity matrix. For $n \geq 5$ and for some partitions $\lambda$, the matrix $A^\lambda_\iota$ is not the identity matrix; however this matrix is always invertible.

**Example 1.3.** Let $n = 3$ and $\lambda = 21$. We have that $d_\lambda = 2$. For $\pi = 213$ we calculate $A_\pi$.

| $(i,j)$ | $T_i$ | $\pi(T_j)$ | | |
|---|---|---|---|---|
| $(1,1)$ | $T_1 = \begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ | $\pi T_1 = \begin{array}{\|c\|c\|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$ | $\kappa = \iota$ | $\epsilon(\kappa) = 1$ |
| $(1,2)$ | $T_1 = \begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ | $\pi T_2 = \begin{array}{\|c\|c\|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$ | $\kappa = 321$ | $\epsilon(\kappa) = -1$ |
| $(2,1)$ | $T_2 = \begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ | $\pi T_1 = \begin{array}{\|c\|c\|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$ | $\kappa$ does not exist | |
| $(2,2)$ | $T_2 = \begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ | $\pi T_2 = \begin{array}{\|c\|c\|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$ | $\kappa = 213$ | $\epsilon(\kappa) = -1$ |

This gives the Clifton matrix $A^\lambda_\pi = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$.

**Example 1.4.** Let $n = 5$ and $\lambda = 32$. In this case $d_\lambda = 5$ and the standard tableaus are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

To calculate the $(1,5)$ entry of the Clifton matrix $A^\lambda_\iota$, we consider $T_1$ and $\iota T_5 = T_5$. In this case the transposition $\kappa = 15342$ interchanging 2 and 5 is the required vertical permutation, so $(A^\lambda_\iota)_{15} = -1$. In fact, we have

$$A^\lambda_\iota = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (A^\lambda_\iota)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is the simplest case in which $A^\lambda_\iota$ is not the identity matrix.

To illustrate the difference that can exist between the Clifton matrix $A^\lambda_\pi$ and the representation matrix $R^\lambda_\pi = (A^\lambda_\iota)^{-1} A^\lambda_\pi$, consider the 5-cycle $\pi = 23451$; in this case we calculate

$$\begin{array}{ccc} R^\lambda_\pi & & (A^\lambda_\iota)^{-1} \qquad\qquad\qquad A^\lambda_\pi \\ \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} & = & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

The Wedderburn decomposition $\phi\colon \mathbb{F}\, S_n \to \bigoplus_\lambda M_{d_\lambda}(\mathbb{F})$ has an inverse

$$\phi^{-1}\colon \bigoplus_\lambda M_{d_\lambda}(\mathbb{F}) \to \mathbb{F}\, S_n$$

given by $\phi^{-1}(E^\lambda_{ij}) = U^\lambda_{ij}$, where $\lambda$ is a partition of $n$ and $E^\lambda_{ij}$ is the $d_\lambda \times d_\lambda$ matrix unit. We describe how the elements $U^\lambda_{ij} \in FS_n$ can be calculate.

Let $\mathcal{T}^\lambda_i$ be the $i$-th standard tableau in lexicographical order. We denote by $\mathcal{R}^\lambda_i$ the subgroup of $S_n$ which permutes the elements of $T^\lambda_i$ within each row but leaves the rows fixed as sets. We denote by $\mathcal{C}^\lambda_i$ the subgroup of $S_n$ which permutes the elements of $T^\lambda_i$ within each column but leaves the columns fixed as sets. We define the elements $D^\lambda_{ii}$ by

$$D^\lambda_{ii} = \frac{d_\lambda}{n!} \sum_{\sigma\, \in\, \mathcal{R}^\lambda_i} \; \sum_{\tau\, \in\, \mathcal{C}^\lambda_i} \epsilon(\tau)\; \sigma\, \tau.$$

If $T^\lambda_j$ is the $j$-th standard tableau we denote by $s^\lambda_{ij} \in S_n$ the permutation such that $s^\lambda_{ij}\; T^\lambda_i = T^\lambda_j$. We define the elements $D^\lambda_{ij}$ by

$$D^\lambda_{ij} = D^\lambda_{ii}\; (s^\lambda_{ij})^{-1}.$$

Let $(A^\lambda_\iota)^{-1} = (a^\lambda_{ij})$. We finally define the elements $U^\lambda_{ij}$ by

$$U^\lambda_{ij} = \sum_{k=1}^{d_\lambda} a^\lambda_{jk}\; D^\lambda_{ik}.$$

**Theorem 1.5.** *The elements $U^\lambda_{ij}$ satisfy the orthogonal matrix unit relations. That is, for any two partitions $\lambda$ and $\mu$ of $n$, we have*

$$U^\lambda_{ij}\, U^\mu_{k\ell} = \delta_{\lambda\mu}\, \delta_{jk}\, U^\lambda_{i\ell}.$$

*Furthermore, the isomorphism*

$$\phi^{-1}\colon \bigoplus_{\lambda} M_{d_\lambda}(\mathbb{F}) \to \mathbb{F}\,S_n$$

*is defined on basis elements by*

$$\phi^{-1}(E^{\lambda}_{ij}) = U^{\lambda}_{ij}.$$

**Example 1.6.** For $n = 3$, $\phi^{-1}\colon \mathbb{F} \oplus M_2(\mathbb{F}) \oplus \mathbb{F} \to \mathbb{F}S_3$ is given by

$$\begin{aligned}
\phi^{-1}\big(E^{(3)}_{11}\big) &= \frac{1}{6}(123 + 132 + 213 + 231 + 312 + 321),\\
\phi^{-1}\big(E^{(21)}_{11}\big) &= \frac{1}{3}(123 + 213 - 312 - 321),\\
\phi^{-1}\big(E^{(21)}_{12}\big) &= \frac{1}{3}(132 + 231 - 312 - 321),\\
\phi^{-1}\big(E^{(21)}_{21}\big) &= \frac{1}{3}(132 - 213 - 231 + 312),\\
\phi^{-1}\big(E^{(21)}_{22}\big) &= \frac{1}{3}(123 - 213 - 231 + 321),\\
\phi^{-1}\big(E^{(111)}_{11}\big) &= \frac{1}{6}(123 - 132 - 213 + 231 + 312 - 321).
\end{aligned}$$

## 2. Polynomial identities and representation theory

The application of the representation theory of the symmetric group to polynomial identities was initiated independently by Malcev [45] and Specht [56] in 1950. Starting in the 1970's the method was further developed by Regev (see [50]). The implementation of this theory in computer algebra was initiated by Hentzel [25, 26] in 1977.

Let $X = \{x_1, x_2, \dots\}$ be a set of variables. We construct the set $M[X]$ of (noncommutative and nonassociative) *monomials* inductively as follows: $X \subset M[X]$; if $x_i,\ x_j \in X$ then $x_i x_j \in M[X]$; if $u, v \in M[X]{-}X$ then $x_i(u),\ (u)x_i,\ (u)(v) \in M[X]$. Let

$$\mathbb{F}[X] = \Big\{ \sum_{i=1}^{r} \alpha_i\ u_i \mid r \in \mathbb{N},\ \alpha_i \in \mathbb{F},\ u_i \in M[X] \Big\}$$

be the vector space over $\mathbb{F}$ spanned by $M[X]$. The elements of $\mathbb{F}[X]$ are called *nonassociative polynomials* in the variables $x_i$. We define in $\mathbb{F}[X]$ a multiplication by the following rules:

$$\begin{gathered}
x_i.x_j = x_i x_j,\ x_i.u = x_i(u),\ u.x_i = (u)x_i,\ u.v = (u)(v),\\
(\sum_{i=1}^{r} \alpha_i\ u_i)\ .\ (\sum_{j=1}^{s} \beta_j\ v_j) = \sum_{i,j=1}^{r,s} \alpha_i \beta_j\ u_i.v_j,
\end{gathered}$$

where $x_i, x_j \in X$, $u, u_i, v_j \in M[X] - X$. We obtain then an algebra called *free nonassociative algebra generated by* $X$. We denote this algebra by $\mathbb{F}[X]$ or $\mathbb{F}[x_1, x_2, \dots]$.

**2.1. Polynomial identities of an algebra.** Let $A$ be a nonassociative algebra over $\mathbb{F}$. A polynomial $f = f(x_1, x_2, \ldots, x_n) \in \mathbb{F}[X]$ is an *identity* of $A$ if $f(a_1, a_2, \ldots, a_n) = 0$ for all $a_1, a_2, \ldots, a_n \in A$. When $f$ is an identity of $A$ we say also that $A$ *satisfies* $f = 0$. Any identity of degree $\leq n$ over a field $\mathbb{F}$ of characteristic 0 or $p > n$ is equivalent to a finite set of multilinear identities. If $f = \sum_{i=1}^{r} \alpha_i \, u_i$ is a *multilinear polynomial* of degree $n$ then each term $\alpha_i \, u_i$ has a coefficient $\alpha_i \in \mathbb{F}$ together with a monomial $u_i$ which is a permutation of $x_1 \cdots x_n$ with an association type.

An *association type* is a placement of parentheses. For $n = 3$ we have two association types, $(xx)x$ and $x(xx)$. For $n = 4$ we have five association types:

| $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ |
|---|---|---|---|---|
| $((xx)x)x$ | $(x(xx))x$ | $(xx)(xx)$ | $x((xx)x)$ | $x(x(xx))$ |

Here $x$ is just a place holder. For the general case of a nonassociative algebra, the number of distinct association types in degree $n$ equals the *Catalan number*,

$$c(n) = \frac{1}{n}\binom{2n-2}{n-1}.$$

The numbers $c(n)$ grow very rapidly; here is a short table:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c(n)$ | 1 | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1430 | 4862 | 16796 | 58786 |

We denote the association types of degree $n$ by $T_1, \ldots, T_{c(n)}$.

If $f = f(x_1, \ldots, x_n)$ is a multilinear polynomial then

$$f = f_1 + \cdots + f_{c(n)}.$$

Within each component $f_k$, all the terms have association type $k$ and differ only in the permutation of the variables. So each component $f_k$ can be regarded as an element of the group algebra $\mathbb{F}S_n$:

$$f_k = \sum_{\pi \in S_n} \alpha_{k\pi} \, (x_1 x_2 \ldots x_n)_\pi \ \in \mathbb{F}S_n.$$

The permutation $\pi$ applies to the position, not the subscript. Thus $(x_1 x_2 \ldots x_n)_\pi = x_{\pi^{-1}(1)} x_{\pi^{-1}(2)} \ldots x_{\pi^{-1}(n)}$. For example, $(x_1 x_2 x_3)_{(231)} = x_3 x_1 x_2$. We can therefore regard $f$ as an element of the direct sum of $c(n)$ copies of $\mathbb{F}S_n$:

$$f = f_1 + \cdots + f_{c(n)} \ \in \ \mathbb{F}S_n \oplus \cdots \oplus \mathbb{F}S_n.$$

Let $\phi_\lambda \colon \mathbb{F}\, S_n \to M_{d_\lambda}(\mathbb{F})$ be the $\lambda$-component of the Wedderburn decomposition $\phi \colon \mathbb{F}\, S_n \to \bigoplus_\lambda M_{d_\lambda}(\mathbb{F})$. Applying $\phi_\lambda$ to $f$ we obtain the representation matrix of $f$ in partition $\lambda$ given by Table 1.

$$\begin{array}{ccccc} T_1 & T_2 & \cdots & T_{c(n)-1} & T_{c(n)} \\ \end{array}$$
$$\left[\ \phi_\lambda(f_1) \mid \phi_\lambda(f_2) \mid \cdots \mid \phi_\lambda(f_{c(n)-1}) \mid \phi_\lambda(f_{c(n)})\ \right]$$

TABLE 1. Representation matrix of $f$ in partition $\lambda$

The basic problem concerning identities of an algebra is the following:

**Problem 2.1.** Let $f^1, \ldots, f^k, f$ be a set of multilinear identities of degre $n$. Is $f$ a consequence of $f^1, \ldots, f^k$?

For $i = 1, \ldots, k$ we consider

$$f^i = f^i_1 + f^i_2 + \cdots + f^i_{c(n)} \ \in \ \mathbb{F}S_n \oplus \cdots \oplus \mathbb{F}S_n$$

and also

$$f = f_1 + f_2 + \cdots + f_{c(n)} \ \in \ \mathbb{F}S_n \oplus \cdots \oplus \mathbb{F}S_n.$$

We then apply $\phi_\lambda$ to $f^1, \ldots, f^k, f$ to obtain the representation matrix

$$M_\lambda = \begin{bmatrix} \phi_\lambda(f^1_1) & \phi_\lambda(f^1_2) & \cdots & \phi_\lambda(f^1_{c(n)-1}) & \phi_\lambda(f^1_{c(n)}) \\ \phi_\lambda(f^2_1) & \phi_\lambda(f^2_2) & \cdots & \phi_\lambda(f^2_{c(n)-1}) & \phi_\lambda(f^2_{c(n)}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_\lambda(f^k_1) & \phi_\lambda(f^k_2) & \cdots & \phi_\lambda(f^k_{c(n)-1}) & \phi_\lambda(f^k_{c(n)}) \\ \phi_\lambda(f_1) & \phi_\lambda(f_2) & \cdots & \phi_\lambda(f_{c(n)-1}) & \phi_\lambda(f_{c(n)}) \end{bmatrix}$$

Let $N_\lambda$ be the matrix consisting of the first $k$ rows of blocks of $M_\lambda$. We denote by $\mathrm{RCF}(M_\lambda)$ the row canonical form of $M_\lambda$ and by $\mathrm{RCF}(N_\lambda)$ the row canonical form of $N_\lambda$. Let $\mathrm{rank}(M_\lambda)$ be the number of nonzero rows in $\mathrm{RCF}(M_\lambda)$ and $\mathrm{rank}(N_\lambda)$ the number of nonzero rows in $\mathrm{RCF}(N_\lambda)$.

**Proposition 2.2. (Solution of Problem 2.1)** *Let $f^1, \ldots, f^k, f$ be a set of multilinear identities of degree $n$. Then $f$ a consequence of $f^1, \ldots, f^k$ if and only if $\mathrm{rank}(M_\lambda) = \mathrm{rank}(N_\lambda)$ for all partition $\lambda$ of $n$.*

**Rational arithmetic and modular arithmetic.** We prefer to use rational arithmetic ($\mathbb{F} = \mathbb{Q}$). However, even for a sparse matrix with integer entries, very large rational numerators and denominators can appear during the computation of the row canonical form. Therefore, when it is necessary, we use modular arithmetic ($\mathbb{F} = \mathbb{Z}_p$ where $p$ is a prime and $p > n$). The ranks are the same over $\mathbb{Q}$ and over $\mathbb{Z}_p$ (see Bremner and Peresi [12], Lemma 8).

**2.2. An example in alternative algebras.** In any algebra the *associator* $(a, b, c)$ and the *commutator* $[a, b]$ are defined by $(a, b, c) = (ab)c - a(bc)$ and $[a, b] = ab - ba$. An algebra $A$ is called an *alternative algebra* if it satisfies the *left* and *right alternative laws* $(a, a, c) = 0$ and $(a, b, b) = 0$. The most familiar example of an alternative algebra is the octonions which appear in the chain

$$\text{Reals} \subset \text{Complexes} \subset \text{Quaternions} \subset \text{Octonions}.$$

The Cayley-Dickson algebras are generalizations of the octonions (see §2.4). The classification of simple alternative algebras is given by

**Theorem 2.3. (Kleinfeld 1953)** [42] *A simple alternative algebra (which is not associative) is a Cayley-Dickson algebra over its center.*

A well-known identity satisfied by alternative algebras is given by

**Proposition 2.4.** *Let $A$ be any alternative algebra. Then $A$ satisfies the identity*

$$f(a, b, c, d) = (ab, c, d) + (a, b, [c, d]) - a(b, c, d) - (a, c, d)b = 0.$$

To prove that $f(a,b,c,d)=0$ is an identity in any alternative algebra we need to verify that it is a consequence of the alternative laws. We do this using Proposition 2.2.The linerized form of $(a,a,b)=0$ is $(a,b,c)+(b,a,c)=0$. The linerized form of $(a,b,b)=0$ is $(a,b,c)+(a,c,b)=0$. From this we obtain the following *lifting identities* to degree 4:

$$\begin{array}{ll} (a,b,c)d+(b,a,c)d=0, & (a,b,c)d+(a,c,b)d=0, \\ a(b,c,d)+a(c,b,d)=0, & a(b,c,d)+a(b,d,c)=0, \\ (ab,c,d)+(c,ab,d)=0, & (ab,c,d)+(ab,d,c)=0, \\ (a,bc,d)+(a,d,bc)=0, & (a,b,cd)+(b,a,cd)=0. \end{array}$$

For each partition $\lambda$ we calculate the matrix $M_\lambda$ representing the lifting identities and $f(a,b,c,d)$, and the matrix $N_\lambda$ representing only the lifting identities. The ranks of the row canonical forms of these matrices are given in Table 2. The ranks of $N_\lambda$ are given in the column labeled Lift and those of $M_\lambda$ are given in the column labeled Lift and f. The ranks are the same in both columns for all partitions. Therefore, $f(a,b,c,d)$ is a consequence of the lifting identities to degree 4 of the linearized forms of the alternative laws.

| Partition $\lambda$ | $d_\lambda$ | Lift | Lift and $f$ |
|---|---|---|---|
| 4 | 1 | 4 | 4 |
| 31 | 3 | 12 | 12 |
| 22 | 2 | 8 | 8 |
| 211 | 3 | 10 | 10 |
| 1111 | 1 | 2 | 2 |

Table 2. Ranks of liftings identities and $f(a,b,c,d)$

**2.3. Computing polynomial identities.** If $A$ is an algebra over $\mathbb{F}$ we denote by $T(A)$ the set of polynomial identities of $A$. The set $T(A)$ is an *ideal of identities* (or *T-ideal*). That is, $T(A)$ is an ideal of $\mathbb{F}[X]$ and $\psi(T(A))\subset T(A)$ for any homomorphism $\psi:\mathbb{F}[X]\to\mathbb{F}[X]$.

**Problem 2.5. (Specht 1950)** Given a class of algebras, is it true that any algebra $A$ in this class has a finite basis of identities (i.e., the ideal of identities $T(A)$ is generated by a finite number of identities)?

Specht posed this problem for associative algebras over a field of characteristic zero. The complete solution was given by Kemer [41].

**Theorem 2.6. (Kemmer 1987)** *Any associative algebra over a field of characteristic zero has a finite basis of identities.*

Similar results were obtained by Vais and Zelmanov [57] in 1989 for finitely generated Jordan algebras, and by Iltyakov [36, 37] for finetely generated alternative algebras (1991) and Lie algebras (1992).

In this subsection we consider the problem of finding polynomial identities of a fixed degree.

**Problem 2.7.** Let $A$ be an algebra over a field $\mathbb{F}$ of finite dimension $d$. Let $\{v_1, \ldots, v_d\}$ be a basis of $A$ and multiplication table given by

$$v_i v_j = \sum_{k=1}^{d} c_{ij}^k v_k \qquad (c_{ij}^k \in \mathbb{F}).$$

Determine all the multilinear polynomial identities of degree $n$ satisfied by $A$. What is the mininal degree of a polynomial identity?

For the matrix algebra $M_n(\mathbb{F})$ the polynomial identities of minimal degree is given by

**Theorem 2.8. (Amitsur and Levitzki 1950)** [2] *The minimal degree of a polynomial identity of $M_n(\mathbb{F})$ is $2n$. Furthermore, any multilinear polynomial identity of degree $2n$ is a multiple of the standard polynomial*

$$s_{2n}(x_1, \ldots, x_{2n}) = \sum_{\sigma \in S_{2n}} \epsilon(\sigma)\ x_{\sigma(1)} \ldots x_{\sigma(2n)}.$$

In 1973 Leron [44] proved that if $\mathrm{char}(\mathbb{F}) = 0$ and $n > 2$ then all polynomial identities of degree $2n+1$ of $M_n(\mathbb{F})$ are consequences of $s_{2n}(x_1, \ldots, x_{2n})$. In particular, the identities of degree 7 of $M_3(\mathbb{F})$ are consequences of $s_6$. When $\mathrm{char}(\mathbb{F}) = 0$, Drensky and Kasparian [16] found in 1983 all identities of degree 8 of $M_3(\mathbb{F})$ and showed that they are consequences of the standard identity $s_6$.

The computational methods used to study polynomial identities of matrices are surveyed by Benanti et. al. [3]. We remark also that there are in the literature many results about the ideal of identities $T(M_2(\mathbb{F}))$.

In 1997 Bondari [6] introduced an algorithm for computing an independent generating set for the multilinear identifies and central identities of $M_n(\mathbb{F})$ over a field $\mathbb{F}$ of characteristic zero or large enough prime. Constructed all the multilinear identifies and all the multilinear central identities of degree $\leq 8$ for $M_3(\mathbb{F})$. Checked the existing results in the literature and found a new central identity in degree 8 for $M_3(\mathbb{F})$.

The algorithm of Bondari can be used to find identities of any finite dimensional algebra, giving a solution for Problem 2.7.

**Algorithm 2.9. (Bondari 1997)** Let $f(x_1, \ldots, x_n)$ be a multilinear polynomial of degree $n$ and write

$$f = f_1 + \cdots + f_{c(n)} \ \in \ \mathbb{F}S_n \oplus \cdots \oplus \mathbb{F}S_n.$$

For $k = 1, \ldots, c(n)$, assume that the $d_\lambda \times d_\lambda$ matrix $\phi_\lambda(f_k)$ is given by $\phi_\lambda(f_k) = a_{11}^k E_{11}^\lambda + \cdots + a_{1d_\lambda}^k E_{1d_\lambda}^\lambda$ $(a_{1j}^k \in \mathbb{F})$. In other words, assume that $\phi_\lambda(f_k)$ is a matrix such that the only nonzero row is the first row $a_{11}^k a_{12}^k \ldots a_{1d_\lambda}^k$. (Any nonzero row can be moved to the first row.) In this case we say that $f$ is an *irreducible polynomial for partition $\lambda$*.

We consider $\phi^{-1}(E_{1j}^\lambda) = U_{1j}^\lambda \in \mathbb{F}S_n$ as a multilinear associative polynomial of degree $n$. For $k = 1, \ldots, c(n)$ we denote by $[U_{1j}^\lambda]_k\,(x_1, \ldots, x_n)$ the multilinear nonassociative polynomial obtained by applying association type $T_k$ to every term of $U_{1j}^\lambda$. Therefore

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{c(n)} \sum_{j=1}^{d_\lambda} a_{1j}^k\ [U_{1j}^\lambda]_k\,(x_1, \ldots, x_n).$$

Assume that $f$ is an identity of $A$. Then $f(a_1, \ldots, a_n) = 0$ for any choice of $a_1, \ldots, a_n \in A$. Given $n$ arbitrary elements $a_1, \ldots, a_n \in A$, we can evaluate

$$[U^{\lambda}_{1j}]_k\,(a_1, \ldots, a_n) = \alpha^1_{jk} v_1 + \cdots + \alpha^d_{jk} v_d := (\alpha^1_{jk}, \ldots, \alpha^d_{jk}) \in \mathbb{F}^d.$$

Therefore

$$\sum_{k=1}^{c(n)} \sum_{j=1}^{d_\lambda} a^k_{1j}\ (\alpha^1_{jk}, \ldots, \alpha^d_{jk}) = 0.$$

This gives the following linear system with $d$ equations and $c(n)d_\lambda$ unknowns $a^k_{1j}$:

$$\sum_{k=1}^{c(n)} \sum_{j=1}^{d_\lambda} \alpha^i_{jk}\ a^k_{1j} = 0 \qquad (i = 1, \ldots, d).$$

Solving this linear system we find the coeficients $a^k_{1j}$ of the *irreducible polynomial identity* $f(x_1, \ldots, x_n) = 0$ *of* $A$ *for partition* $\lambda$.

**2.4. Polynomial identities of Cayley-Dickson algebras.** A *Cayley-Dickson algebra* is an eight-dimensional alternative algebra $C = C(\alpha, \beta, \gamma)$ over a field $\mathbb{F}$ with three non-zero parameters $\alpha,\ \beta,\ \gamma$ in $\mathbb{F}$. Over the reals $C(-1, -1, -1)$ is the algebra of octonions. The algebras $C$ can be constructed by the Cayley-Dickson process (see Zhevlakov et. al. [59], §2.2). Over a field $\mathbb{F}$ of characteristic $\neq 2$ it is possible to choose a basis $1, e_1, \ldots, e_7$ with the multiplication table given below.

| $C$ | $1$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
| $e_1$ | $e_1$ | $\alpha$ | $e_3$ | $\alpha e_2$ | $e_5$ | $\alpha e_4$ | $-e_7$ | $-\alpha e_6$ |
| $e_2$ | $e_2$ | $-e_3$ | $\beta$ | $-\beta e_1$ | $e_6$ | $e_7$ | $\beta e_4$ | $\beta e_5$ |
| $e_3$ | $e_3$ | $-\alpha e_2$ | $\beta e_1$ | $-\alpha\beta$ | $e_7$ | $\alpha e_6$ | $-\beta e_5$ | $-\alpha\beta e_4$ |
| $e_4$ | $e_4$ | $-e_5$ | $-e_6$ | $-e_7$ | $\gamma$ | $-\gamma e_1$ | $-\gamma e_2$ | $-\gamma e_3$ |
| $e_5$ | $e_5$ | $-\alpha e_4$ | $-e_7$ | $-\alpha e_6$ | $\gamma e_1$ | $-\alpha\gamma$ | $\gamma e_3$ | $\alpha\gamma e_2$ |
| $e_6$ | $e_6$ | $e_7$ | $-\beta e_4$ | $\beta e_5$ | $\gamma e_2$ | $-\gamma e_3$ | $-\beta\gamma$ | $-\beta\gamma e_1$ |
| $e_7$ | $e_7$ | $\alpha e_6$ | $-\beta e_5$ | $\alpha\beta e_4$ | $\gamma e_3$ | $-\alpha\gamma e_2$ | $\beta\gamma e_1$ | $\alpha\beta\gamma$ |

**Problem 2.10.** Find a basis for the ideal of identities $T(C)$.

Isaev [38] found in 1984 a finite basis of $T(C)$ when $\mathbb{F}$ is a finite field. When $\mathrm{char}(\mathbb{F}) = 0$, Iltyakov [35] proved in 1985 that $T(C)$ is finitely generated without giving a set of generators. For fields of characteristic $\neq 2, 3, 5$, Racine [49] found in 1988 the identities of degree $\leq 5$ of $C$.

Let $A$ be an algebra over $\mathbb{F}$ with identity element 1. Then $A$ is a *quadratic algebra* if each element $x \in A$ satisfies $x^2 - t(x)\ x + n(x)\ 1 = 0$, where the *trace* $t : A \to \mathbb{F}$ is a linear map and the *norm* $n : A \to \mathbb{F}$ is a quadratic form. The trace and the norm are uniquely defined.

The Cayley-Dickson algebra $C$ is a quadratic algebra. Let $x = a\ 1 + \sum_{i=1}^{7}\ a_i\ e_i \in C$ and $\overline{x} = a\ 1 - \sum_{i=1}^{7}\ a_i\ e_i$. The trace and norm of $x$ is given by $t(x) = x + \overline{x} = 2a$ and $n(x) = x\ \overline{x} = a^2 - \alpha a_1^2 - \beta a_2^2 + \alpha\beta a_3^2 - \gamma a_4^2 + \alpha\gamma a_5^2 + \beta\gamma a_6^2 - \alpha\beta\gamma a_7^2$.

The *Jordan product* $x \circ y$ is defined by $x \circ y = xy + yx$. Let $V_x$ and $V$ given by $V_x(y) = x \circ y$ and $V = V_a V_b V_c + V_c V_a V_b + V_b V_c V_a - V_b V_a V_c - V_a V_c V_b - V_c V_b V_a$.

**Proposition 2.11. (Racine 1985)** [48] *Any quadratic algebra satisfies the identity*

$$V(d^2) - V(d) \circ d = 0.$$

**Corollary 2.12.** *The Cayley-Dickson algebra $C$ satisfies the identity*

$$V(d^2) - V(d) \circ d = 0.$$

The linearized form of $x^2 - t(x)\, x + n(x)\, 1 = 0$ is $x \circ y - t(x)\, y - t(y)\, x + q(x, y)\, 1 = 0$, where $q(x, y) = n(x + y) - n(x) - n(y)$. For every $a, b, c, d \in C$, if $x = [a, b]$ and $y = [c, d]$ then $t(x) = 0$, $t(y) = 0$. Thus $[a, b] \circ [c, d] = -q([a, b], [c, d])\, 1 \in \mathbb{F}$. It follows that for every $e \in C$ we have $[[a, b] \circ [c, d], e] = 0$. Therefore $C$ satisfies the identity $[[a, b] \circ [c, d], e] = 0$.

Using the Algorithm 2.9 we can obtain the identities of degree $\leq 7$ satisfied by $C$. For each degree we list only the new ones, that is, those that are not implied by identities of lower degree.

**Theorem 2.13. (Hentzel and Peresi 1997)** [29] *The identities of degree $\leq 6$ of the Cayley-Dickson algebras $C$ are given by (char$(\mathbb{F}) = 0$ or $>$ degree of the identity):*

*degree 1 and 2: None.*

*degree 3: $(a, a, b) = 0$ and $(a, b, b) = 0$ (alternative laws).*

*degree 4: No new ones (just the lifted alternative identities).*

*degree 5: $V(d^2) - V(d) \circ d = 0$ and $[[a, b] \circ [c, d], e] = 0$.*

*degree 6:*

$[\ ALTSUM_{\{a,b,c,d,e\}}\ \{\ 24\ a(b(c(de))) + 8\ a((b, c, d)e) - 11\ (a, b, (c, d, e))\ \}\ , f\ ] = 0,$

*where $ALTSUM_{\{a,b,c,d,e\}}$ denotes the alternating sum over all the permutations of the variables $\{a, b, c, d, e\}$.*

We remark that although all the calculations done by Racine in [49] are correct, to obtain all degree 5 identities of $C$, we need the identity $[[a, b] \circ [c, d], e] = 0$. The weaker form $[[a, b] \circ [a, b], e] = 0$ used by Racine is not enough.

**Theorem 2.14. (Bremner and Peresi 2012)** *The identities of degree 7 of the Cayley-Dickson algebra $C$ are consequences of identities of degree $\leq 6$.*

In 2002, Bremner and Hentzel [8] studied identities for alternative algebras which are restricted in the sense that the terms in the identities must be built out of associators. These authors discovered two new identities in degree 7 satisfied by the associator in every alternative algebra and five new identities in degree 7 satisfied by the associator in the algebra of octonions. For fields of characteristic zero, Shestakov and Zhukavets [55] found in 2009 a basis consisting of 3 identities (one of degree 5 and two of degree 6) for the skew-symmetric identities of $C$. One of this identity of degree 6 is two times the one given in Theorem 2.13. In 2011, Shestakov [53] found a basis of identities for a Cayley-Dickson algebra modulo an associator ideal of a free alternative algebra. More precisely, denote by $T_{as}(C)$ a homomorphic image of the ideal of identities $T(C)$ of the split Cayley-Dickson algebra $C$, in a free associative algebra. Over a field F of characteristic not 2, 3, or 5, Shestakov found a basis for the ideal $T_{as}(C)$. Using the relations that appear in the Cayley-Dickson process, Henry [24] found in 2012 a basis for the $\mathbb{Z}_2^2$ and $\mathbb{Z}_2^3$-graded identities for $C$. The $\mathbb{Z}_2^3$-grading requires characteristic $\neq 2$.

## 3. Nuclear elements in the free alternative algebra

The *nucleus* of an algebra $A$ is the set

$$N(A) = \{p \in A \mid (p, x, y) = (x, p, y) = (x, y, p) = 0, \ \forall x, y \in A\}.$$

The *center* of $A$ is the set

$$C(A) = \{p \in N(A) \mid [p, x] = 0, \ \forall x \in A\}.$$

Let $\mathbb{F}[X]$ be the free nonassociative algebra over the field $\mathbb{F}$ in generators $X = \{x_1, x_2, \ldots, x_n\}$. Let $Alt[X]$ denote the ideal of $\mathbb{F}[X]$ generated by the elements $(f_1, f_1, f_2)$, $(f_2, f_1, f_1)$ $(f_1, f_2 \in F[X])$. We denote by $Alt_n[X]$ the subspace of multilinear elements of degree $n$ in $Alt[X]$. The *free alternative algebra* in generators $X$ is the quotient algebra $ALT[X] = F[X]/Alt[X]$.

In 1953 Kleinfeld [42] showed that for any $x$ and $y$ in an alternative algebra, the element $[x, y]^4$ is in the nucleus and this element is nonzero in the free alternative algebra on two or more generators. This result was used by Kleinfeld to prove Theorem 2.3. Subsequently, other authors have found elements of larger and smaller degree in the nucleus, as well as elements in the center.

In the *Dniester Notebook* published in 1993, Shestakov posed the folowing problem (see [20], Problem 2.121):

**Problem 3.1.** Describe the center and the associative center (i. e., the nucleus in our terminology) of a free alternative algebra as completely characteristic subalgebras. Are they finitely generated?

A subalgebra $\mathcal{S}$ of an algebra $\mathcal{A}$ is *completely characteristic* (or *$T$-subalgebra*) if $\psi(\mathcal{S}) \subset \mathcal{S}$ for all homomorphisms $\psi : \mathcal{A} \to \mathcal{A}$.

When the free alternative algebra has more then five free generators, Filippov [19] found in 1999 an element of degree 7 in the center. And Filippov conjectured that the minimal degree of nonzero elements in the center of the free alternative algebra is 7. In 2003 (see [30]) we presented a new central element of degree 7. In 2006 (see [31]) we proved that Fillippov conjecture is true and found all the central elements of degree 7 in the free alternative algebra over the field $\mathbb{Z}_{103}$. An infinite set $\{a_n \mid n = 4k \ (k > 1) \text{ or } n = 4k + 1 \ (k > 0)\}$ of central elements was constructed in 2006 by Shestakov and Zhukavets [54].

In this section we consider in more details the problem of finding nuclear elements of degree $n$ in the free alternative algebra. We give the calculations for degree 5 and 6. Let $p$ be an element of the free nonassociative algebra $\mathbb{F}[X]$ over the field $\mathbb{F}$ in generators $X = \{x_1, x_2, \ldots, x_n\}$. We say that $p$ is an *element of the nucleus of the free alternative algebra in generators $X$* if in the free alternative algebra on generators $X \cup \{x_{n+1}, x_{n+2}\}$ one has that $(p, x_{n+1}, x_{n+2}) = 0$.

**3.1. Nuclear elements of degree $n$.** A multilinear element $p \in F[X]$ is in the nucleus if $(p, x_{n+1}, x_{n+2})$ is in $Alt_{n+2}$ $[X \cup \{x_{n+1}, x_{n+2}\}]$ that is the subspace of multilinear elements of degree $n + 2$ in $Alt[X \cup \{x_{n+1}, x_{n+2}\}]$. This subspace is generated by the liftings identities to degree $n + 2$ of the alternative laws.

We consider only permutations of the variables $x_1, x_2, \ldots, x_n$. We do this by using $c(n)$ association types of degree $n$, $T_1, \ldots, T_{c(n)}$, and $k(n) = c(n+2)\binom{n+2}{2}$ association types of degree $n+2$, $T'_1, \ldots, T'_{k(n)}$. In each type $T'_j$ we consider only the permutation of $x_1, \ldots, x_n$ and the positions of $x_{n+1}$ and $x_{n+2}$. Also we assume that $x_{n+1}$ and $x_{n+2}$ are skew-symmetric since $(p, x_{n+1}, x_{n+2}) = -\,(p, x_{n+2}, x_{n+1})$.

Let $w_k$ be the monomial $x_1 x_2 \ldots x_n$ associated as in type $T_k$. We expand the associator $(w_k, x_{n+1}, x_{n+2})$ as

$$-\,(w_k x_{n+1}) x_{n+2} \;+\; w_k (x_{n+1} x_{n+2}) \;+\; (w_k, x_{n+1}, x_{n+2}) = 0$$

and call

$$-\,(w_k x_{n+1}) x_{n+2} \;+\; w_k (x_{n+1} x_{n+2})$$

the *expansion of the associator* $(w_k, x_{n+1}, x_{n+2})$. This expansion of associator can be represented as in Table 3, where $\iota_n$ is the identity permutation in $S_n$.

| $T'_{i_k}$ <br> $-\,(w_k\ x_{n+1})\ x_{n+2}$ | $T'_{j_k}$ <br> $w_k\ (x_{n+1}\ x_{n+2})$ | $T_k$ <br> $(w_k, x_{n+1}, x_{n+2})$ |
|---|---|---|
| $-\,\iota_n$ | $\iota_n$ | $\iota_n$ |

Table 3. Expansion of associator

Let $\lambda$ be a partition of $n$. Using $\phi_\lambda \colon \mathbb{F}\,S_n \to M_{d_\lambda}(\mathbb{F})$ we obtain the representation matrix in Table 4, where $\phi_\lambda(\iota_n) = I_{d_\lambda}$ is the $d_\lambda \times d_\lambda$ identity matrix.

$$\begin{array}{ccc} T'_{i_k} & T'_{j_k} & T_k \\[1ex] \multicolumn{3}{c}{\left[\; \phi_\lambda(-\,\iota_n) \;\mid\; \phi_\lambda(\iota_n) \;\mid\; \phi_\lambda(\iota_n) \;\right]} \end{array}$$

Table 4. Representation matrix of expansion of associator

We apply $\phi_\lambda$ to the lifted alternative identities of degree $n+2$ obtaining the representation matrix of these identities. We append to the bottom of this matrix the representation matrix given in Table 4 for all the associators. We obtain the representation matrix $M_\lambda$ given in Table 5. We calculate the row canonical form $RCF(M_\lambda)$ of $M_\lambda$. The nonzero rows of $RCF(M_\lambda)$ that involve only the types $(w_k, x_{n+1}, x_{n+2})$ $(k = 1, \ldots, c(n))$ represent the identities of the form $(p, x_{n+1}, x_{n+2}) = 0$ with $p \in F[X]$. Therefore $p$ is an element of degree $n$ in the the nucleus of the free alternative algebra $ALT[X]$. Some of these nuclear elements $p$ are not new in the sense that they are consequence of the lifted alternative identities of degree $n$ and the lifted nuclear elements of degree $n$ of possible nuclear elements of degree $< n$. We calculate the matrix $N_\lambda$ representing these lifted alternative identities and possible lifted nuclear elements. The nonzero rows that appear in the right-hand portion of $RCF(M_\lambda)$ but do not appear in $RCF(N_\lambda)$ represent the new nuclear elements of degree $n$.

| $T'_1 \quad T'_2 \quad \cdots \quad T'_{k(n)}$ | $T_1 \quad T_2 \quad \cdots \quad T_{c(n)}$ |
|---|---|
| representation matrix of lifted alternative identities of degree $n+2$ | zero matrix |
| representation matrix of expantion of associators | $\begin{matrix} I_{d_\lambda} & 0 & \cdots & 0 \\ 0 & I_{d_\lambda} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{d_\lambda} \end{matrix}$ |

TABLE 5. Representation matrix for nuclear elements of degree $n$

**3.2. Nuclear elements of degree 5.** Let $A$ be any alternative algebra. Then by Proposition 2.4 $A$ satisfies the identity

$$(ab, c, d) + (a, b, [c, d]) = a(b, c, d) + (a, c, d)b.$$

Therefore, if $(x, c, d) = 0$ for all $x \in A$, then $[c, d]$ is in the nucleus of $A$.

Since $(x, [a, b][a, b], a) = 0$ is an identity in $A$ ([59], §13.4, Lemma 15) it follows that

$$[\ [a, b][a, b],\ a\ ] \tag{1}$$

is in the nucleus of $A$. Filippov [17] proved that

$$(x, [[[b, a], c], a] - [[[b, a], a], c] + 2\ [[b, a], [c, a]], a) = 0$$

is an identity of $A$. Therefore

$$[[[[b, a], c], a], a] - [[[[b, a], a], c], a] + 2\ [[[b, a], [c, a]],\ a] \tag{2}$$

is in the nucleus of $A$. These nuclear elements have degree 5. In the next theorem we obtain that 5 is the minimal degree and obtain all the nuclear elements of degree 5 in the free alternative algebra.

**Theorem 3.2. (Hentzel and Peresi 2006)** [32] *In the free alternative algebra over $\mathbb{Z}_{103}$ on generators $\{a, b, c,\ d, e\}$ we have:*

*(i) There are no nonzero nuclear elements of degree $< 5$.*

*(ii) All the nuclear elements of degree 5 are consequences of the lifted alternative identities of degree 5 and*

$$([a, b][a, c])a - (a[a, b])[a, c]. \tag{3}$$

As described in §3.1 we compare the right-hand portion of $RCF(M_\lambda)$ with the $RCF(N_\lambda)$. For degree $< 5$ they are the same. The ranks for degree 5 are given in Table 6. The nozero rows containing the new nuclear elements of degree 5 are given in Table 7. Modulo the alternative laws the nuclear element in partition 32 is (1) and the one in partition 311 is (2). Further work shows that these two nuclear elements are equivalent (modulo the alternative laws) to the nuclear element (3).

| Partition $\lambda$ | $d_\lambda$ | Alternative laws | Nuclear elements |
|---|---|---|---|
| 5 | 1 | 13 | 13 |
| 41 | 4 | 52 | 52 |
| 32 | 5 | 65 | 66 • |
| 311 | 6 | 75 | 76 • |
| 221 | 5 | 63 | 63 |
| 2111 | 4 | 46 | 46 |
| 11111 | 1 | 10 | 10 |

TABLE 6. Nuclear elements of degree 5: rank of matrices

| Partition | Type 14 $x(x(x(xx)))$ |
|---|---|
| 32 | 1 0 1 1 1 |
| 311 | 1 0 −1 −3 −1 2 |

TABLE 7. New nuclear elements of degree 5

**3.3. Nuclear elements of degree 6.** The degree 6 nuclear elements implied by nuclear element (3) are the following *lifted nuclear elements of degree 6*:

$$([a,bd][a,c])a-(a[a,bd])[a,c], \tag{4}$$

$$([a,b][a,cd])a-(a[a,b])[a,cd], \tag{5}$$

$$\begin{aligned}&([de,b][a,c])a+([a,b][de,c])a+([a,b][a,c])(de)\\&-((de)[a,b])[a,c]-(a[de,b])[a,c]-(a[a,b])[de,c].\end{aligned} \tag{6}$$

For each partition $\lambda$, we compare the right-hand portion of $RCF(M_\lambda)$ with the $RCF(N_\lambda)$ (see §3.1). The matrix $N_\lambda$ represents the lifted alternative identities of degree 6 and the lifted nuclear elements (4), (5) and (6). We also compare $N_\lambda$ with the matrix representing just the lifted alternative identities of degree 6. The ranks are given in Table 8. The new nuclear elementes of degree 6 are the ones that are not consequences of the alternative laws, (4), (5) and (6). The new nuclear elements occur in partitions 33, 222, 2211 and 21111. The new nuclear element in partition 33 is

$$[[a,b][a,b],b]a \tag{7}$$

and was obtained by a different method not the representation of $S_n$ (see [33], §2 and 4.3). For the other partitions we give the nonzero rows and the corresponding new nuclear element (modulo the alternative laws, (4), (5) and (6)).

| Partition $\lambda$ | $d_\lambda$ | Alternative laws | Alternative laws and nuclear elements (4)-(6) | Nuclear elements |
|---|---|---|---|---|
| 6 | 1 | 41 | 41 | 41 |
| 51 | 5 | 205 | 205 | 205 |
| 42 | 9 | 369 | 372 | 372 |
| 411 | 10 | 406 | 409 | 409 |
| 33 | 5 | 205 | 206 | 207 $\bullet$ |
| 321 | 16 | 652 | 658 | 658 |
| 3111 | 10 | 400 | 403 | 403 |
| 222 | 5 | 202 | 202 | 204 $\bullet$ |
| 2211 | 9 | 360 | 362 | 363 $\bullet$ |
| 21111 | 5 | 194 | 195 | 196 $\bullet$ |
| 111111 | 1 | 36 | 36 | 36 |

TABLE 8. Nuclear elements of degree 6: rank of matrices

**Partition 222.** Nuclear elemenets (8) and (9).

$$x(x(x(x(xx))))$$

1 1 1 0 -1

$$\begin{aligned} & 2\ [[a,b],a] \circ [[b,c],c] \ - \ 2\ [[a,b],b] \circ [[a,c],c] \\ - & \ 2\ [[a,c],a] \circ [[b,c],b] \ + \ 2\ [[a,c],b] \circ [[a,c],b] \\ + & \ 6\ [[b,c],a] \circ [[b,c],a] \ - \ 6\ [[a,c],b] \circ [[b,c],a] \\ + & \ 4\ [[a,b],c] \circ [[b,c],a] \ + \ 3\ [a,b] \circ [[a,c],[b,c]] \\ - & \ 3\ [a,c] \circ [[a,b],[b,c]] \ - \ 3\ [b,c] \circ [[a,c],[a,b]] \end{aligned} \tag{8}$$

$$(xx)(x((xx)x)) \qquad x(x((xx)(xx))) \qquad x(x(x((xx)x))) \qquad x(x(x(x(xx))))$$

0 0 0 0 1     0 -1 0 0 0     0 0 0 0 1     0 1 0 -1 -2

$$\begin{aligned} & \langle a,a,b\rangle\langle c,c,b\rangle \ + \ \langle a,a,c\rangle\langle b,b,c\rangle \\ - & \ \langle a,b,c\rangle\langle a,b,c\rangle \ + \ \langle b,b,a\rangle\langle c,c,a\rangle \\ + & \ \langle b,b,c\rangle\langle a,a,c\rangle \ - \ \langle b,c,a\rangle\langle b,c,a\rangle \\ - & \ \langle c,a,b\rangle\langle c,a,b\rangle \ + \ \langle c,c,a\rangle\langle b,b,a\rangle \\ + & \ \langle c,c,b\rangle\langle a,a,b\rangle \ - \ 72\ (a,b,c)(a,b,c), \end{aligned} \tag{9}$$

where $\langle x,y,z\rangle$ is the *Jordan associator* $\langle x,y,z\rangle = (x \circ y) \circ z - x \circ (y \circ z)$.

**Partition 2211.** Nuclear elemenet (10).

$$x(x(x(x(xx))))$$

0 2 3 1 -1 1 -5 1 4

$$[V(d^2) - V(d) \circ d, a] \tag{10}$$

**Partition 21111.** Nuclear elemenet (11).

| $(xx)(x(x(xx)))$ | $x((xx)(x(xx)))$ | $x(x((xx)(xx)))$ | $x(x(x((xx)x)))$ | $x(x(x(x(xx))))$ |
|---|---|---|---|---|
| 0 0 0 0 1 | 0 0 0 0 -1 | 1 0 0 1 0 | 0 -2 0 0 -1 | -1 1 1 0 0 |

$$ALTSUM_{\{b,c,d,e\}} \Big\{ \begin{array}{rl} 3 & [[[a,b],a],c]\ [d,e] \ +\ [[[a,b],c],a]\ [d,e] \\ -2 & [[[a,b],c],d]\ [a,e] \ + 2\ [[[b,c],a],a]\ [d,e] \\ -4 & [[[b,c],a],d]\ [a,e] \ \Big\} \end{array} \tag{11}$$

where $ALTSUM_{\{b,c,d,e\}}$ denotes the alternating sum over all the permutations of the variables $\{b,c,d,e\}$.

Therefore we have:

**Theorem 3.3. (Hentzel and Peresi 2008)** [33] *In the free alternative algebra over $\mathbb{Z}_{103}$ on generators $\{a,b,c,d,e,f\}$ all the nuclear elements of degree 6 are consequences of the lifted alternative identities of degree 6, the lifted nuclear elements (4), (5),(6) and the nuclear elements (7) - (11).*

## 4. Special identities for Bol algebras

The Bol loops were introduced by Bol [5] in 1937. In the 1980's Sabinin and Mikheev constructed the theory of smooth Bol loops (see Sabinin [51]). In particular, they introduced the notion of Bol algebra in 1982, and proved that any finite-dimensional Bol algebra over $\mathbb{R}$ can be realized as the tangent algebra of a local analytic Bol loop [52].

**4.1. Bol algebras.** A (left) *Bol algebra* is a vector space over $\mathbb{F}$ equipped with a binary operation $[a,b]$ and a ternary operation $\{a,b,c\}$ satisfying these identities:

$$[a,b]+[b,a]=0, \tag{12}$$

$$\{a,b,c\}+\{b,a,c\}=0, \tag{13}$$

$$\{a,b,c\}+\{b,c,a\}+\{c,a,b\}=0, \tag{14}$$

$$\begin{aligned}[\{a,b,c\},d]-[\{a,b,d\},c]+\{c,d,[a,b]\}\\ -\{a,b,[c,d]\}+[[a,b],[c,d]]=0,\end{aligned} \tag{15}$$

$$\begin{aligned}\{a,b,\{c,d,e\}\}-\{\{a,b,c\},d,e\}\\ -\{c,\{a,b,d\},e\}-\{c,d,\{a,b,e\}\}=0.\end{aligned} \tag{16}$$

In 1994 Filippov [18] gave the classification of homogeneous Bol algebras, i.e, the Bol algebras satisfying $\{a,b,c\} = \alpha[[a,b],c] + \beta[[b,c],a] + \gamma[[c,a],b]$ $(\alpha,\beta,\gamma \in \mathbb{F})$. Kuzmin and Zaĭdi [43] investigated the solvability and semisimplicity of Bol algebras in 1993. An envelope for Bol algebras was constructed by Pérez-Izquierdo [47] in 2005, and it was proved that any Bol algebra is located inside the generalized left alternative nucleus of the envelope.

From the classification of all two-dimensional right Bol algebras over $\mathbb{R}$ given in [43] we obtain two examples of (left) Bol algebras.

**Example 4.1.** The Bol algebra $\mathcal{B}_1$ has basis $\{e_1,\ e_2\}$, $[a,b]=0$ and the nonzero ternary products are given by

$$\{e_1,e_2,e_1\}=e_2,\ \{e_1,e_2,e_2\}=-e_1,\ \{e_2,e_1,e_1\}=-e_2,\ \{e_2,e_1,e_2\}=e_1.$$

**Example 4.2.** The Bol algebra $\mathcal{B}_2$ has basis $\{e_1,\ e_2\}$ and the nonzero products are given by

$$[e_1,e_2]=e_1,\ [e_2,e_1]=-e_1,$$
$$\{e_1,e_2,e_1\}=-e_1,\ \{e_1,e_2,e_2\}=e_2,\ \{e_2,e_1,e_1\}=e_1,\ \{e_2,e_1,e_2\}=-e_2.$$

Starting with a *left altenartive algebra* (i.e., an algebra satisfying $(x,x,y)=0$) or a *right altenartive algebra* (i.e., an algebra satisfying $(x,y,y)=0$) $A$, we construct a Bol algebra $A^b$.

**The Bol algebra $A^b$.** Let $A$ be any algebra with multiplication $ab$. We denote by $A^b$ the algebra having the same underlying vector space as $A$, the binary operation $[a,b]:=ab-ba$ and the ternary operation $\{a,b,c\}:=\langle b,c,a\rangle$. Then $A^b$ satisfies (12), (13) and (14). If $A$ is a left or right alternative algebra then $A^b$ satisfies also (15) and (16), and therefore $A^b$ is a Bol algebra.

As an example we prove that if $A$ is a right alternative algebra then $A^b$ satisfies (16). The algebra $A^{\#}=\mathbb{F}.1\oplus A$ is also a right alternative as proved by Albert ([1], Lemma 3). Therefore, we may assume that $A$ has an identity element 1. Let $R_b: A\to A$ be the map given by $R_b(a)=ab$. We denote by $[R_a,R_b]$ the operator $R_aR_b-R_bR_a$. We remind that $[a,b]:=ab-ba$ and $\{a,b,c\}:=\langle b,c,a\rangle$. For any $a,b,c\in A$ we can prove that $[[R_a,R_b],R_c]=R_{\{a,b,c\}}$. Therefore we have:

$$\begin{aligned}
&\{a,b,\{c,d,e\}\}-\{\{a,b,c\},d,e\}-\{c,\{a,b,d\},e\}-\{c,d,\{a,b,e\}\}\\
&=R_{\{a,b,\{c,d,e\}\}}(1)-R_{\{\{a,b,c\},d,e\}}(1)-R_{\{c,\{a,b,d\},e\}}(1)-R_{\{c,d,\{a,b,e\}\}}(1)\\
&=[[R_a,R_b],R_{\{c,d,e\}}](1)-[[R_{\{a,b,c\}},R_d],R_e](1)\\
&\quad-[[R_c,R_{\{a,b,d\}}],R_e](1)-[[R_c,R_d],R_{\{a,b,e\}}](1)\\
&=[[R_a,R_b],[[R_c,R_d],R_e]](1)-[[[[R_a,R_b],R_c],R_d],R_e](1)\\
&\quad-[[R_c,[[R_a,R_b],R_d]],R_e](1)-[[R_c,R_d],[[R_a,R_b],R_e]](1)=0.
\end{aligned}$$

**The free Bol algebra $\boldsymbol{Bol[X]}$.** Let $X=\{x_1,x_2,x_3,\dots\}$ be a set of variables. We construct the set of *binary-ternary monomials* $BTM[X]$ inductively as follows: $X\subset BTM[X]$ and we assume that $BTM[X]$ is closed under $[a,b]$ and $\{a,b,c\}$. Let

$$BT[X]=\Big\{\ \textstyle\sum_{i=1}^n \alpha_i\ u_i \mid n\in\mathbb{N},\ \alpha_i\in\mathbb{F},\ u_i\in BTM[X]\Big\}$$

be the vector space over $\mathbb{F}$ spanned by $BTM[X]$. The elements of $BT[X]$ are called *binary-ternary polynomials* in the variables $x_i$. We define in $BT[X]$ a multiplication by the following rules: if $f=\sum_{i=1}^l \alpha_i\ u_i,\ g=\sum_{j=1}^m \beta_j\ v_j$ and $h=\sum_{k=1}^n \gamma_k\ w_k$ are polynomials in $BT[X]$ then

$$[f,g]=\textstyle\sum_{i,j=1}^{l,m}\alpha_i\beta_j\ [u_i,v_j],\qquad \{f,g,h\}=\sum_{i,j,k=1}^{l,m,n}\alpha_i\beta_j\gamma_k\ \{u_i,v_j,w_k\}.$$

The algebra $BT[X]$ is called the *free binary-ternary algebra*. The usual notion of ideal of identities (T-ideal) of $\mathbb{F}[X]$ can be extended to $BT[X]$. The *free Bol algebra* $Bol[X]$ is $BT[X]$ modulo the T-ideal of $BT[X]$ generated by the (binary-ternary polynomial) identities (12) - (16).

The *Bol types* of degree $n$ are ways to construct a product of degree $n$ in $BT[X]$. Identities (12) and (13) reduce the number of these types. The Bol types of degree

$\leq 3$ are $x,\ [x,x],\ [[x,x],x],\ \{x,x,x\}$. The Bol types of degree $\leq 4$ are

$$\{x,x,[x,x]\},\ \{[x,x],x,x\},\ [[x,x],[x,x]],[\{x,x,x\},x],\ [[[x,x],x],x].$$

Again $x$ is just a place holder. The number of Bol types we use is denoted by $b(n)$, and for $n \leq 8$ this number is given by

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $b(n)$ | 1 | 1 | 2 | 5 | 13 | 38 | 113 | 354 |

We denote the Bol types of degree $n$ by $B_1,\dots,B_{b(n)}$.

Let $B$ be a Bol algebra. An element $f = f(x_1,\dots,x_n) \in BT[X]$ is an *identity* of $B$ if $f(a_1,\dots,a_n) = 0$ for all $a_1,\dots,a_n \in B$. Any multilinear identity $f$ of degree $n$ can be written as a linear combination of multilinear monomials, where each monomial has one of the Bol types of degree $n$. Collecting the monomials which have the same Bol type,

$$f = f_1 + \cdots + f_{b(n)},$$

where $f_k$ is a linear combination of monomials having Bol type $k$. We can therefore (as in §2.1) identify each $f_k$ with an element of the group algebra $\mathbb{F}S_n$ of the symmetric group $S_n$. Therefore, we can regard $f$ as an element of the direct sum of $b(n)$ copies of $\mathbb{F}S_n$. Applying $\phi_\lambda\colon \mathbb{F}\,S_n \to M_{d_\lambda}(\mathbb{F})$ to $f$ we obtain the representation matrix of $f$ in partition $\lambda$:

$$\begin{array}{ccccc} B_1 & B_2 & \cdots & B_{b(n)-1} & B_{b(n)} \\ \end{array}$$
$$\left[\begin{array}{c|c|c|c|c} \phi_\lambda(f_1) & \phi_\lambda(f_2) & \cdots & \phi_\lambda(f_{b(n)-1}) & \phi_\lambda(f_{b(n)}) \end{array}\right]$$

**4.2. Special identities.** An algebra $J$ is called a *Jordan algebra* if it satisfies $xy = yx$ and $(x^2,y,x) = 0$. For any algebra $A$ with multiplication $ab$, we denote by $A^+$ the algebra having the same underlying vector space as $A$ and the Jordan product $a \circ b = ab + ba$ as the multiplication. If $A$ is associative then $A^+$ is a Jordan algebra. A Jordan algebra $J$ is *special* if there is an associative algebra $A$ such that $J$ is isomorphic to a subalgebra of $A^+$. Otherwise, $J$ is called *exceptional*. The *Albert algebra*, consisting of the matrices $M \in M_3(C(\alpha,\beta,\gamma))$ such that $\overline{M}^t = M$, has dimension 27 and is an exemple of a exceptional Jordan algebra. In the 1960's Glennie [21, 22, 23] discovered a special identity for Jordan algebras, i.e., an identity satisfied by special Jordan algebras that is not satisfied by all Jordan algebras (see also Hentzel [27]).

Velásquez and Felipe [58] introduced in 2008 a new class of algebras of Jordan type, the quasi-Jordan algebras, defined by two identities. Two years later, Bremner [7] proved that it is more natural to consider the subclass consisting of the algebras that satisfy a third identity. We say that an algebra is a *quasi-Jordan algebra* if it satisfies the identities

$$x(yz) = x(zy),\ (yx)x^2 = (yx^2)x,\ (y,x^2,z) = 2(y,x,z)x.$$

If $(D,+,\dashv,\vdash)$ is an associative dialgebra and we define the *quasi-Jordan product* $a \bullet b = a \dashv b + b \vdash a$ we obatin that $D^+ = (D,+,\bullet)$ is a quasi-Jordan algebra. A quasi-Jordan algebra is *special* if it is isomorphic to a subalgebra of $D^+$ for some associative dialgebra $D$. Bremner and Peresi [13] found in 2011 special identities for quasi-Jordan algebras, i.e., identities satisfied by all special quasi-Jordan algebras but not satisfied by all quasi-Jordan algebras. One of these special identities for quasi-Jordan algebras is a noncommutative preimage of the Glennie identity. In both cases (Jordan and quasi-Jordan) the minimal degree of a special identity is 8.

We consider the analogous problem of finding special identities for Bol algebras. We denote by $LALT[X]$ ($RALT[X]$) the free left (right) alternative algebra. Since a homogeneous element in $BT[X]$ is an identity of $LALT[X]^b$ if and only if it is an identity of $RALT[X]^b$, we consider only right alternative algebras. A *special identity* is an identity satisfied by $A^b$ for every right alternative algebra $A$, but not satisfied by the free Bol algebra.

**Example 4.3.** In the free binary-ternary algebra $BT[a,b,c]$ let

$$d = \{a,b,a\} + [[b,a],a].$$

Then

$$H(a,b,c) := \{\{\{\{c,d,d\},d,d\},d,d\},d,d\}$$

is a special identity. The operations in $RALT[a,b,c]^b$ are $[x,y] := xy - yx$ and $\{x,y,z\} := \langle y,z,x\rangle$. We have that $d = \{a,b,a\} + [[b,a],a] = -2(a,a,b)$. In every right alternative algebra the Mikheev's identity $(x,x,y)^4 = 0$ is valid (see Zhevlakov [59], §16.1, Theorem 2). Therefore $d^4 = 0$. We obtain

$$\begin{aligned} H(a,b,c) &= \{\{\{\{c,d,d\},d,d\},d,d\},d,d\} = R_{\{\{\{\{c,d,d\},d,d\},d,d\},d,d\}}(1) \\ &= [[[[[[[[R_c,R_d],R_d],R_d],R_d],R_d],R_d],R_d],R_d](1) = 0, \end{aligned}$$

since in each term of the expansion of the commutators of operators appears $R_{d^4} = 0$. Therefore $H(a,b,c) = 0$ is an identity of $RALT[a,b,c]^b$. Now, we prove that $H(a,b,c)$ is not zero in the free Bol algebra $Bol[a,b,c]$. It is enough to give an example of a Bol algebra where $H(a,b,c)$ does not hold. We consider the algebra $\mathcal{B}_1$ (see Example 4.1). For $a = e_2$, $b = e_1$ and $c = e_2$, we have $d = \{e_2,e_1,e_2\} + [[e_1,e_2],e_2] = e_1$ and $\{c,d,d\} = \{e_2,e_1,e_1\} = -e_2$. Therefore $H(e_2,e_1,e_2) = e_2 \neq 0$.

The special identity $H(a,b,c,d)$ has degree 25 which is far from the minimal degree. In §4.3 we obtain that the minimal degree for special identities is 8.

**Remark 4.4.** Not every Bol algebra is isomorphic to a subalgebra of $A^b$ for some right alternative algebra $A$. For instance, consider the Bol algebra $\mathcal{B}_1$ (see Example 4.1). Assume that $\mathcal{B}_1$ can be imbedded into $A^b$ for some right alternative algebra $A$. Since $A^b$ satisfies the identity $H(a,b,c) = 0$ then $H(a,b,c)$ evaluates to zero in $\mathcal{B}_1$, a contradiction since $H(e_2,e_1,e_2) = e_2 \neq 0$.

**4.3. Degree $n$ identities of $RALT[X]^b$.** Let $RAlt_n[X]$ be the subspace of multilinear elements of degree $n$ in $RALT[X]$ and $\{\mathcal{I}_1,\dots,\mathcal{I}_{g(n)}\}$ a generating set of $RAlt_n[X]$. We use a generating set with $g(n)$ elements where $g(n)$ (for $n \le 8$) is given by

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| $g(n)$ | 1 | 4 | 16 | 61 | 234 | 895 |

Each $\mathcal{I}_i$ can be written as

$$\mathcal{I}_i = \sum_{j=1}^{c(n)} f_j^i$$

where all monomials in $f_j^i$ have type $T_j$.

| $T_1$ | $T_2$ | $\cdots$ | $T_{c(n)}$ | $B_1$ | $B_2$ | $\cdots$ | $B_{b(n)-1}$ | $B_{b(n)}$ |
|---|---|---|---|---|---|---|---|---|
| $\phi_\lambda(f_1^1)$ | $\phi_\lambda(f_2^1)$ | $\cdots$ | $\phi_\lambda(f_{c(n)}^1)$ | $O$ | $O$ | $\cdots$ | $O$ | $O$ |
| $\phi_\lambda(f_1^2)$ | $\phi_\lambda(f_2^2)$ | $\cdots$ | $\phi_\lambda(f_{c(n)}^2)$ | $O$ | $O$ | $\cdots$ | $O$ | $O$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ |
| $\phi_\lambda(f_1^{g(n)-1})$ | $\phi_\lambda(f_2^{g(n)-1})$ | $\cdots$ | $\phi_\lambda(f_{c(n)}^{g(n)-1})$ | $O$ | $O$ | $\cdots$ | $O$ | $O$ |
| $\phi_\lambda(f_1^{g(n)})$ | $\phi_\lambda(f_2^{g(n)})$ | $\cdots$ | $\phi_\lambda(f_{c(n)}^{g(n)})$ | $O$ | $O$ | $\cdots$ | $O$ | $O$ |
| $\phi_\lambda(g_1^1)$ | $\phi_\lambda(g_2^1)$ | $\cdots$ | $\phi_\lambda(g_{c(n)}^1)$ | $-I_{d_\lambda}$ | $O$ | $\cdots$ | $O$ | $O$ |
| $\phi_\lambda(g_1^2)$ | $\phi_\lambda(g_2^2)$ | $\cdots$ | $\phi_\lambda(g_{c(n)}^2)$ | $O$ | $-I_{d_\lambda}$ | $\cdots$ | $O$ | $O$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ |
| $\phi_\lambda(g_1^{b(n)-1})$ | $\phi_\lambda(g_2^{b(n)-1})$ | $\cdots$ | $\phi_\lambda(g_{c(n)}^{b(n)-1})$ | $O$ | $O$ | $\cdots$ | $-I_{d_\lambda}$ | $O$ |
| $\phi_\lambda(g_1^{b(n)})$ | $\phi_\lambda(g_2^{b(n)})$ | $\cdots$ | $\phi_\lambda(g_{c(n)}^{b(n)})$ | $O$ | $O$ | $\cdots$ | $O$ | $-I_{d_\lambda}$ |

TABLE 9. Representation matrix $M_\lambda$: Bol types of degree $n$

Let $b_i = b_i(x_1, \dots, x_n)$ be the monomial $x_1 \dots x_n$ written using Bol type $B_i$. Using $[a,b] := ab - ba$ and $\{a,b,c\} := \langle b,c,a \rangle$ we obtain

$$\sum_{j=1}^{c(n)} g_j^i - b_i = 0$$

where all monomials in $g_j^i$ have type $T_j$.

For each partition $\lambda$, we apply $\phi_\lambda : \mathbb{F}S_n \to M_{d_\lambda}(\mathbb{F})$ to the expansion of $b_i$ to obtain the (direct sum of matrices)

$$\sum_{j=1}^{c(n)} \phi_\lambda(g_j^i) - I_{d_\lambda} = 0,$$

where $I_{d_\lambda}$ is the $d_\lambda \times d_\lambda$ identity matrix. Also applying $\phi_\lambda$ to each $\mathcal{I}_i$ we obtain the (direct sum of matrices)

$$\sum_{j=1}^{c(n)} \phi_\lambda(f_j^i).$$

Putting all these $d_\lambda \times d_\lambda$ blocks together we obtain the representation matrix $M_\lambda$ in Table 9. We compute the row canonical form $RCF(M_\lambda)$ of $M_\lambda$. The identities of degree $n$ for $RALT[X]^b$ are represented in the lower right block of $RCF(M_\lambda)$, under Bol types $B_1, \dots, B_{b(n)}$.

| $B_1$ | $B_2$ | $\cdots$ | $B_{b(n)}$ |
|---|---|---|---|
| $\phi_\lambda(h_1^1)$ | $\phi_\lambda(h_2^1)$ | $\cdots$ | $\phi_\lambda(h_{b(n)}^1)$ |
| $\phi_\lambda(h_1^2)$ | $\phi_\lambda(h_2^2)$ | $\cdots$ | $\phi_\lambda(h_{b(n)}^2)$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $\phi_\lambda(h_1^{k(n)-1})$ | $\phi_\lambda(h_2^{k(n)-1})$ | $\cdots$ | $\phi_\lambda(h_{b(n)}^{k(n)-1})$ |
| $\phi_\lambda(h_1^{k(n)})$ | $\phi_\lambda(h_2^{k(n)})$ | $\cdots$ | $\phi_\lambda(f_{b(n)}^{k(n)})$ |

TABLE 10. Representation matrix $K_\lambda$: known identities of degree $n$

Some of these identities of degree $n$ satisfied by $RALT[X]^b$ are known, i.e., they are consequences of identities (12) - (16). Therefore we need to find the nonzero rows in $RCF(M_\lambda)$ that come from the known identities. To do this we lift to degree $n$ all the known identities of degree $< n$. This gives a generating set $\{\mathcal{K}_1, \ldots, \mathcal{K}_{k(n)}\}$ of the subspace $Bol_n[X]$ consisting of multilinear elements of degree $n$ in $Bol[X]$. We use a generating set with $k(n)$ elements where $k(n)$ (for $4 \leq n \leq 8$) is given by

| | | | | | |
|---|---|---|---|---|---|
| $n$ | 4 | 5 | 6 | 7 | 8 |
| $k(n)$ | 9 | 39 | 148 | 516 | 1885 |

Each $\mathcal{K}_i$ can be written as

$$\mathcal{K}_i = \sum_{j=1}^{b(n)} h_j^i$$

where all monomials in $h_j^i$ have Bol type $B_j$. Applying $\phi_\lambda$ to each $\mathcal{K}_i$ we obtain the representation matrix $K_\lambda$ given in Table 10. The known identities of degree $n$ for $RALT[X]^b$ are represented by the nonzero rows of the row canonical form $RCF(K_\lambda)$ of $K_\lambda$.

We denote by *all* the rank of the lower right block of $RCF(M_\lambda)$ and by *known* the rank of $RCF(K_\lambda)$. The nonzero rows that appear in the lower right block of $RCF(M_\lambda)$ but not in $RCF(K_\lambda)$ are the irreducible new identities for $RALT[X]^b$. The number of new identities is *new* = *all* - *known*.

| Partition $\lambda$ | $d_\lambda$ | *known* | *all* | *new* |
|---|---|---|---|---|
| 4 | 1 | 5 | 5 | |
| 31 | 3 | 12 | 12 | |
| 22 | 2 | 7 | 8 | 1 |
| 211 | 3 | 9 | 10 | 1 |
| 1111 | 1 | 3 | 3 | |

TABLE 11. Degree 4: ranks of matrix representations

For $n = 4$ and $n = 5$ the ranks are given in Tables 11 and 12. There are 2 new identities in degree 4 and 3 in degree 5. For $n = 4$ the matrix $RCF(K_\lambda)$ are

| Partition $\lambda$ | $d_\lambda$ | *known* | *all* | *new* |
|---|---|---|---|---|
| 5 | 1 | 13 | 13 | |
| 41 | 4 | 47 | 47 | |
| 32 | 5 | 55 | 56 | 1 |
| 311 | 6 | 64 | 65 | 1 |
| 221 | 5 | 51 | 52 | 1 |
| 2111 | 4 | 39 | 39 | |
| 11111 | 1 | 9 | 9 | |

TABLE 12. Degree 5: ranks of matrix representations

obtained using (12) - (14). Since we know that (15) is an identity of $RALT[X]^b$, we append the matrix $\phi_\lambda((15))$ to the bottom of $K_\lambda$. Since the row canonical form of this new matrix coincides with the lower right block of $RCF(M_\lambda)$, we have that the new identities of degree 4 of $RALT[X]^b$ are consequences of (15). For $n = 5$ we obtain $RCF(K_\lambda)$ using (12) - (15), and by a similar procedure we verify that the new identities of degree 5 of $RALT[X]^b$ are consequences of (16). For $n = 6, 7$, *known* $=$ *all* for all partitions and there are no new identities. Therefore we have:

**Theorem 4.5. (Hentzel and Peresi 2012)** [34]
**(i)** *A degree 4 element in $BT[X]$ is an identity of $RALT[X]^b$ if and only if it is a consequence of identities (12)-(15).*
**(ii)** *A degree 5, 6 or 7 element in $BT[X]$ is an identity of $RALT[X]^b$ if and only if it is a consequence of identities (12)-(16).*

As a consequence of Theorem 4.5 we obtain that there are no special identity of degree $< 8$. For $n = 8$ the ranks are given in Table 13. Therefore we have:

**Theorem 4.6. (Hentzel and Peresi 2012)** [34] *There are 13 irreducible special identities of degree 8. They are distributed as follows:*

| *Partition $\lambda$* | *62* | *53* | *521* | *44* | *431* | *3311* |
|---|---|---|---|---|---|---|
| *new* | *1* | *4* | *1* | *2* | *4* | *1* |

**4.4. Special identities in partition 62.** Given a partition $\lambda = \lambda_1\lambda_2 \dots \lambda_k$ of $n$, we give a general method to construct special identities of degree $n \geq 8$. We then apply this method to $n = 8$ and $\lambda = 62$. The method is based on the algorithm described by Hentzel and Jacobs [28] in 1991. The computer implementation `Albert` of this algorithm is given by Jacobs [39].

**The algebra $\mathcal{A}_\lambda[x_1, \dots, x_k]$.** The free right alternative algebra $RALT[x_1, \dots, x_k]$ is the quotient algebra $\mathbb{F}[x_1, \dots, x_k]/I$, where $I$ is the $T$-ideal of $\mathbb{F}[x_1, \dots, x_k]$ generated by the right alternative law $(a, b, b) = 0$. Let $L$ be the subspace of $\mathbb{F}[x_1, \dots, x_k]$ spanned by the words $w$, where for some $i$ in $\{1, \dots, k\}$, the degree of $x_i$ in $w$ is $> \lambda_i$. Then $L$ is an ideal of $\mathbb{F}[x_1, \dots, x_k]$. The algebra $\mathcal{A}_\lambda[x_1, \dots, x_k]$ is the quotient algebra $\mathbb{F}[x_1, \dots, x_k]/(I + L)$. We denote by $\psi : \mathbb{F}[x_1, \dots, x_k] \to \mathcal{A}_\lambda[x_1, \dots, x_k]$ the homomorphism extending the map $\{x_1, \dots, x_k\} \to \mathcal{A}_\lambda[x_1, \dots, x_k]$ that sends $x_i$ to $x_i+(I+L)$ for $i = 1, \dots, k$. We denote by $r(n, \lambda)$ the dimension of $\mathcal{A}_\lambda[x_1, \dots, x_k]$. A basis and multiplication table for $\mathcal{A}_\lambda[x_1, \dots, x_k]$ can be constructed using `Albert`.

| Partition $\lambda$ | $d_\lambda$ | *known* | *all* | *new* |
|---|---|---|---|---|
| 8 | 1 | 354 | 354 | |
| 71 | 7 | 2457 | 2457 | |
| 62 | 20 | 6959 | 6960 | 1 |
| 611 | 21 | 7288 | 7288 | |
| 53 | 28 | 9665 | 9669 | 4 |
| 521 | 64 | 22000 | 22001 | 1 |
| 5111 | 35 | 11981 | 11981 | |
| 44 | 14 | 4815 | 4817 | 2 |
| 431 | 70 | 23896 | 23900 | 4 |
| 422 | 56 | 19052 | 19052 | |
| 4211 | 90 | 30503 | 30503 | |
| 41111 | 35 | 11780 | 11780 | |
| 332 | 42 | 14209 | 14209 | |
| 3311 | 56 | 18884 | 18885 | 1 |
| 3221 | 70 | 23488 | 23488 | |
| 32111 | 64 | 21344 | 21344 | |
| 311111 | 21 | 6921 | 6921 | |
| 2222 | 14 | 4653 | 4653 | |
| 22211 | 28 | 9251 | 9251 | |
| 221111 | 20 | 6549 | 6549 | |
| 2111111 | 7 | 2250 | 2250 | |
| 11111111 | 1 | 313 | 313 | |

Table 13. Degree 8: ranks of matrix representations

| $\lambda$ | new | $b(n,\lambda)$ | $s(n)$ | $r(n,\lambda)$ | $t(n)$ | null$(N_\lambda)$ | fewest |
|---|---|---|---|---|---|---|---|
| 62 | 1 | 297 | 142 | 659 | 309 | 1 | 81 |
| 53 | 4 | 740 | 389 | 1510 | 766 | 5 | 187 |
| 521 | 1 | 2519 | 1333 | 5166 | 2677 | 7 | 409 |
| 44 | 2 | 984 | 530 | 1966 | 1020 | 7 | 186 |
| 431 | 4 | 4712 | 2605 | 9298 | 5003 | 17 | 359 |
| 3311 | 1 | 21996 | 12552 | 42730 | 23820 | 45 | 502 |

Table 14. Degree 8: data from creating the special identities

**The algebra $\mathcal{B}_\lambda[x_1,\dots,x_k]$.** The free Bol algebra $Bol[x_1,\dots,x_k]$ is the quotient algebra $BT[x_1,\dots,x_k]/I$, where $I$ is the $T$-ideal of $BT[x_1,\dots,x_k]$ generated by (12)-(16). Let $L$ be the subspace of $BT[x_1,\dots,x_k]$ spanned by the words $w$, where for some $i$ in $\{1,\dots,k\}$, the degree of $x_i$ in $w$ is $> \lambda_i$. Then $L$ is an ideal of $BT[x_1,\dots,x_k]$. The algebra $\mathcal{B}_\lambda[x_1,\dots,x_k]$ is the quotient algebra $BT[x_1,\dots,x_k]/(I+L)$. We denote by $b(n,\lambda)$ the dimension of $\mathcal{B}_\lambda[x_1,\dots,x_k]$. A basis and multiplication table for $\mathcal{B}_\lambda[x_1,\dots,x_k]$ can be constructed using a method similar to the one used in `Albert` (see [34], p. 2327).

Using the operations $[a,b] := ab - ba$ and $\{a,b,c\} := \langle b,c,a\rangle$, we expand the degree $n$ basis elements of $\mathcal{B}_\lambda[x_1,\dots,x_k]$ in terms of the degree $n$ basis elements of $\mathcal{A}_\lambda[x_1, \ldots, x_k]$. This gives us a matrix $N_\lambda$ with $s(\lambda)$ rows and $t(\lambda)$ columns. The $(i, j)$ entry of $N_\lambda$ is the $j$-th coordinate of the expansion of the $i$-th degree $n$ basis element of $B_\lambda[x_1, \ldots, x_k]$. Any dependence relation between these expanded basis elements becomes a special identity, since it is not an identity in the free Bol algebra because it is a linear combination of basis elements of $\mathcal{B}_\lambda[x_1, \ldots, x_k]$. It is an identity in any algebra $A^b$, where $A$ is a right alternative algebra, because it expands to zero in the free right alternative algebra $RALT[x_1, \ldots, x_k]$. We compute the left null space of $N_\lambda$, and denote its dimension by null$(N_\lambda)$. Also, we use lattice basis reduction to attempt to find a special identity with *fewest* terms. The number of terms is not necessarily the smallest. For $n = 8$ the data is given in Table 14.

For $n = 8$ and $\lambda = 62$ we obtain a special identity with 81 terms, each term having 6 a's and 2 b's. Using the identity (14), we can express this identity using only 32 terms. This simpler form is called $S(a, b)$ and it is given below.

We know that $S(a, b)$ is not zero in the free Bol algebra $Bol[a, b]$. We can also verify this by giving an example of a Bol algebra where $S(a, b)$ does not hold. We consider the Bol algebra $\mathcal{B}_2$ over a field of characteristic not 2 (§4.1, Example 4.2) and verify that $S(e_2, e_1) = -2e_2 \neq 0$.

We know that $S(a, b)$ is an identity of $RALT[a, b]^b$ and it follows that $S(a, b)$ is an identity in $A^b$ for any right alternative algebra $A$. We can also verify that $S(a, b)$ is an identity of $RALT[a, b]^b$ as follows. We construct the algebra $\mathcal{A}_{62}[a, b]$ of dimension 659 and the homomorphism $\psi : \mathbb{F}[a, b] \to \mathcal{A}_{62}[a, b]$. Using the operations $[a, b] := ab - ba$ and $\{a, b, c\} := \langle b, c, a\rangle$, the identity $S(a, b)$ becomes $T(a, b)$. Expanding the commutators and Jordan associators, and simplifying, we obtain that $\psi(T(a, b)) = 0$. Then, by Hentzel and Jacobs [28] (Corollary of Theorem), we have that $T(a, b)$ is an identity of $RALT[a, b]$. Therefore, $S(a, b)$ is an identity $RALT[a, b]^b$.

Finally, using $\phi_{62} \colon \mathbb{F}S_8 \to M_{20}(\mathbb{F})$, we verify that $S(a, b)$ is a special identity, and that all special identities in partition 62 are consequences of $S(a, b)$. We construct the matrix $K_{62}$ representing the 1885 multilinear identities that are a generating set for the lifted identities of (12)-(16) to degree 8. The rank of the row canonical form $RCF(K_{62})$ is 6959. Denote by $LS(a, b)$ the linearized form of $S(a, b)$. The binary-ternary polynomial $LS(a, b)$ is a linear combination of 46080 multilinear binary-ternary monomials. We append to the bottom of the matrix $K_{62}$ the $20 \times 20$ blocks that come from the representation matrix $\phi_{62}(LS(a, b))$. We calculate the row canonical form of this new matrix; the rank increases by 1 and is now 6960. Therefore, $S(a, b)$ is not an identity of the free Bol algebra. Also the row canonical form of this new matrix is the same matrix as the lower right block of $RCF(M_{62})$. Therefore $S(a, b)$ is a special identity and all the special identities for partition 62 are consequences of $S(a, b)$.

**Theorem 4.7. (Hentzel and Peresi 2012)** [34] *The binary-ternary polynomial $S(a,b)$ is a special identity. All special identities in partition six-two are consequences of the special identity $S(a,b)$, where*

$$\begin{aligned}
S(a,b) := \quad & 2\{[\{a,b,a\},a],a,[[b,a],a]\} - \{[\{a,b,a\},a],[[b,a],a],a\} \\
& +2\{[[[b,a],a],a],a,[[b,a],a]\} - \{[[[b,a],a],a],[[b,a],a],a\} \\
& -\{\{[[b,a],a],a,a\},a,[b,a]\} + 2\{\{[[b,a],a],a,a\},[b,a],a\} \\
& -\{\{[[b,a],a],a,[b,a]\},a,a\} + 3\{\{\{[b,a],a,a\},a,b\},a,a\} \\
& -\{\{\{[b,a],a,a\},b,a\},a,a\} + \{\{\{[b,a],a,a\},a,a\},a,b\} \\
& -3\{\{\{[b,a],a,a\},a,a\},b,a\} - \{\{[\{a,b,a\},a],a,b\},a,a\} \\
& -\{\{[\{a,b,a\},a],a,a\},a,b\} + 2\{\{[\{a,b,a\},a],a,a\},b,a\} \\
& -\{\{[[[b,a],a],a],a,b\},a,a\} + 2\{\{[[[b,a],a],a],a,a\},b,a\} \\
& -\{\{[[[b,a],a],a],a,a\},a,b\} + [\{[\{a,b,a\},a],a,a\},[b,a]] \\
& +[\{[[[b,a],a],a],a,a\},[b,a]] - [\{[[b,a],a],a,[[b,a],a]\},a] \\
& -2[\{[\{a,b,a\},a],a,[b,a]\},a] + [\{[\{a,b,a\},a],[b,a],a\},a] \\
& -2[\{[[[b,a],a],a],a,[b,a]\},a] + [\{[[[b,a],a],a],[b,a],a\},a] \\
& +[\{\{\{a,b,a\},a,a\},a,b\},a] - 3[\{\{\{a,b,a\},a,b\},a,a\},a] \\
& +2[\{\{\{a,b,b\},a,a\},a,a\},a] - [\{\{[[b,a],a],a,a\},a,a\},b] \\
& -8[\{\{[[b,a],a],a,a\},b,a\},a] + 5[\{\{[[b,a],a],a,a\},a,b\},a] \\
& -[\{\{[[b,a],a],b,a\},a,a\},a] + 5[\{\{[[b,a],a],a,b\},a,a\},a].
\end{aligned}$$

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