Finite Volume schemes for global dynamical core development

Cubed sphere and icosahedral grid approaches

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July 2019
CPTEC

Summary

Summary

1. 1D FV
   - 1D Advection
   - Reconstruct-Evolve-Average (REA)

1D FV conservation law

$$\frac{\partial q}{\partial t} + \frac{\partial (uq)}{\partial x} = 0$$

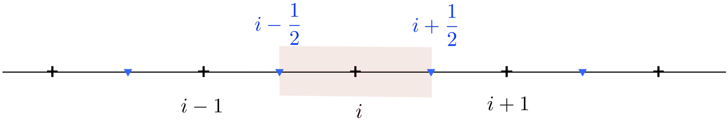
In FV notation:

$$\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = 0, \qquad \text{where } f(q(t,x)) = u(t,x)q(t,x) \quad \text{(Flux function)}$$

Finite Volume: Integrate over cells!

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial q}{\partial t} dx = - \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial (uq)}{\partial x} dx$$

$$\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} q(t,x)dx = - \underbrace{u(t,x_{i+1/2})q(t,x_{i+1/2})}_{\text{Right Flux } f(q(t,x_{i+1/2}))} + \underbrace{u(t,x_{i-1/2})q(t,x_{i-1/2})}_{\text{Left Flux } f(q(t,x_{i-1/2}))}$$



LeVeque - FV for Hyperbolic Problems
Transgeinstein - Num Sol Hyperbolic PDES

Integrated quantities

Finite Volume (mean quantity):

$$Q_i(t) = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(t,x)dx$$

$$\frac{dQ_i(t)}{dt} = \underbrace{-\frac{1}{\Delta x}\delta_x \left(u(t,x_i)q(t,x_i)\right),}_{\text{Mean Fluxes}} \qquad \delta_x q(x) = q(x+\Delta x/2) - q(x-\Delta x/2),$$

Integrate over time

$$\int_{t_n}^{t_{n+1}} \frac{dQ_i(t)}{dt} dt = -\frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} \delta_x \left(u(t,x_i)q(t,x_i)\right) dt$$

$$Q_i(t_{n+1}) = Q_i(t_n) - \frac{1}{\Delta x}\delta_x \int_{t_n}^{t_{n+1}} u(t,x_i)q(t,x_i)dt$$

$$Q_i(t_{n+1}) = Q_i(t_n) \underbrace{-\frac{1}{\Delta x}\delta_x \int_{t_n}^{t_{n+1}} f(q(t,x_i))dt}_{F}$$

General FV Scheme

$$Q_i^{n+1} = Q_i^n + F, \qquad F = -\frac{\Delta t}{\Delta x}\frac{1}{\Delta t}\delta_x \int_{t_n}^{t_{n+1}} \underbrace{f(q(t,x_i))}_{u(t,x_i)q(t,x_i)} dt$$

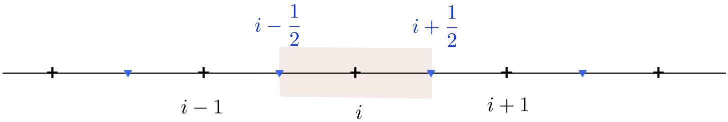
General scheme 3-point scheme:

$$Q_i^{n+1} = Q_i^n + \left(F_{i-1/2} - F_{i+1/2}\right)$$

Possible numerical fluxes:

$$F_{i-1/2} = \mathcal{F}(Q_{i-1}^n, Q_i^n) \approx \frac{\Delta t}{\Delta x}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(t,x_{i-1/2}))dt$$

$$F_{i+1/2} = \mathcal{F}(Q_i^n, Q_{i+1}^n) \approx \frac{\Delta t}{\Delta x}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(t,x_{i+1/2}))dt$$



Summary

1. 1D FV
   - 1D Advection
   - Reconstruct-Evolve-Average (REA)

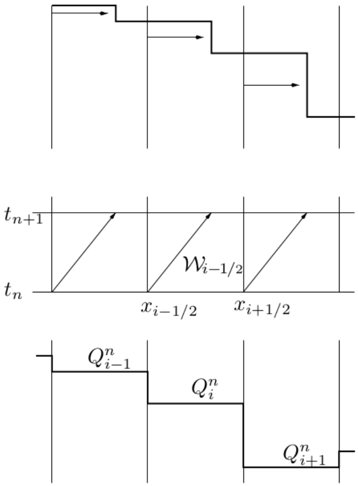
Godunov's Scheme

REA algorithm (reconstruct-evolve-average)

Assume Q is piecewise constant:

- Given initial condition $q(t_0, x)$
- Integrate over each cell exactly to obtain $Q_i(t_0)$
- Solve equation (advection) for Δt (but with small Δt, so that advection stays within 1 cell)
- Integrate solution at Δt (exactly) to obtain $Q_i(t_0 + \Delta t)$

The exact advection with discontinuity is called the Riemman problem



(Fig from LeVeque's book)

Godunov and upwind

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = 0, \quad u > 0, \quad q(t_0, x) = q_0(x), \quad q(t, 0) = q(t, 1)$$

- Integrate $q_0(x)$ in each interval to obtain Q_i^0 (constant for each cell)
- Solve the advection problem to obtain $\widetilde{Q}^{n+1}$.
- The edge point $x_{i-1/2}$ moves to $x_{i-1/2} + u\Delta t$
- Integrate advected quantity, which is

$$\begin{aligned} Q_i^{n+1} &= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \widetilde{Q}_i^{n+1} \\ &= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i-1/2}+u\Delta t} Q_{i-1}^n + \frac{1}{\Delta x}\int_{x_{i-1/2}+u\Delta t}^{x_{i-1/2}+\Delta x} Q_i^n \\ &= \frac{u\Delta t}{\Delta x} Q_{i-1}^n + \frac{\Delta x - u\Delta t}{\Delta x} Q_i^n \end{aligned}$$

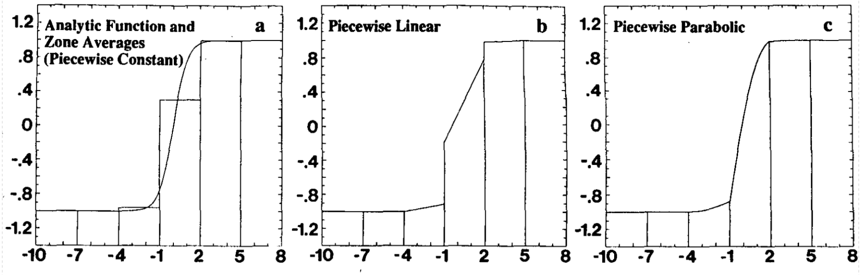
And *voilà*: Upwind!

$$Q_i^{n+1} = Q_i^n + \frac{u\Delta t}{\Delta x}(Q_{i-1}^n - Q_i^n)$$

Higher order reconstructions

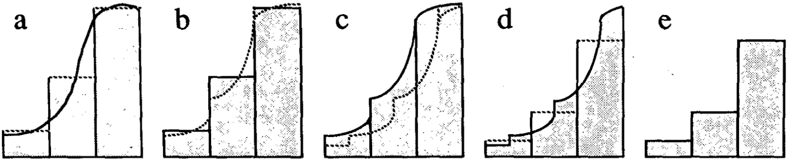
REA algorithm:

- Godunov (1959) - upwind - Piecewise constant
- van Leer (1979) - Piecewise linear (Monotonic Upwind Scheme for Conservation Laws - MUSCL)
- Woodward & Colella (1984) - Piecewise Parabolic Method (PPM)



(Carpenter et al (1990))

PPM

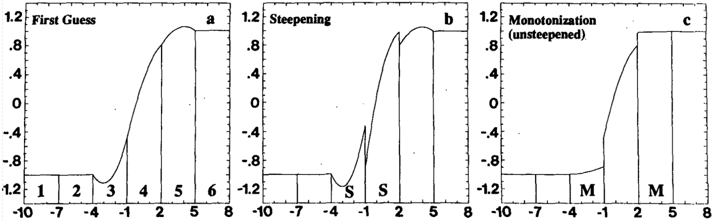


(a) Compute initial mean value (analytically)
(b) Construct parabolas from mean values
(c) Advect parabolas with constant velocity
(d) Integrate advected parabolas
(e) New mean quantities are obtained

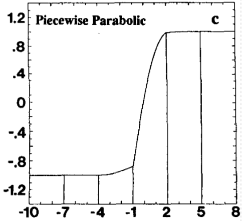
(Carpenter et al (1990), van Leer (1977))

Monotonicity and steepening

Parabolas are adjusted to capture discontinuities and avoid under/overshoots.



d) Steeping + Monotonization



(Carpenter et al (1990))

Summary

2. 2D FV
   - Dimension splitting
   - FV3 advection scheme

2D advection

$$\frac{\partial q}{\partial t} + \nabla \cdot (\vec{v} q) = 0$$

$$\vec{v} = (u, v)$$

$$\nabla = (\partial_x, \partial_y)$$

- 2D problems use dimension splitting to 2 $\times$ 1D problems
- FV3 uses PPM method with dimension splitting

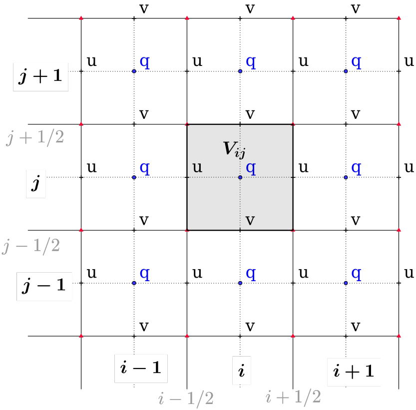
FV on C-grid - Advection

$$\frac{\partial q}{\partial t} + \nabla \cdot (\vec{v}q) = 0$$

$$\frac{1}{|V_{ij}|}\int_{V_{ij}} \frac{\partial q}{\partial t} dV = -\frac{1}{|V_{ij}|}\int_{V_{ij}} \nabla\cdot(\vec{v}q)$$

$$Q_{ij} = \frac{1}{|V_{ij}|}\int_{V_{ij}} q\, dV$$

$$\begin{aligned}
\frac{d\,Q_{ij}}{dt} &= -\frac{1}{|V_{ij}|}\int_{V_{ij}} \nabla \cdot (\vec{v}q)\, dV \\
\text{(Div Thm)} \quad &= -\frac{1}{|V_{ij}|}\int_{\partial V_{ij}} q\vec{v}\cdot\vec{n} dV \\
&= -\frac{1}{\Delta x \Delta y}\left(\int_E qu - \int_W qu\right) \\
&\quad -\frac{1}{\Delta x \Delta y}\left(\int_N qv - \int_S qv\right)
\end{aligned}$$



W=West, E=East, N=North, S=South

FV on C-grid - Dimension Splitting

$$\frac{d\,Q_{ij}}{dt} = -\underbrace{\frac{1}{\Delta x \Delta y}\left(\int_E qu - \int_W qu\right)}_{\text{Mean flux in x direction}} - \underbrace{\frac{1}{\Delta x \Delta y}\left(\int_N qv - \int_S qv\right)}_{\text{Mean flux in y direction}}$$

Integrating from t_n to t_{n+1} leads to:

$$Q_{ij}(t_{n+1}) = Q_{ij}(t_n) - \underbrace{\frac{\Delta t}{\Delta x \Delta y}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(\int_E qu - \int_W qu\right)}_{\text{Mean flux in x direction over time}} - \underbrace{\frac{\Delta t}{\Delta x \Delta y}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(\int_N qv - \int_S qv\right)}_{\text{Mean flux in y direction over time}}$$

FV method with dimension splitting (used in FV3, for example):

$$Q_{ij}^{n+1} = Q_{ij}^{n} + F + G$$

where

$$F \approx -\frac{\Delta t}{\Delta x \Delta y}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(\int_E qu - \int_W qu\right)$$

$$G \approx -\frac{\Delta t}{\Delta x \Delta y}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(\int_N qv - \int_S qv\right)$$

W=West, E=East, N=North, S=South

FV on C-grid - 1st try - Not FV3!

$$Q_{ij}^{n+1} = Q_{ij}^n + F + G$$

$$F \approx -\frac{\Delta t}{\Delta x \Delta y} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left(\int_E qu - \int_W qu \right)$$

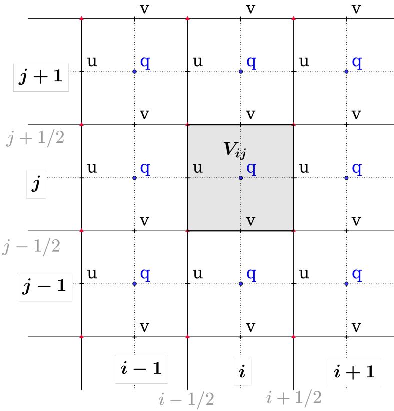
Midpoint rule integration in space, explicit in time

$$\begin{aligned} F &= -\frac{\Delta t}{\Delta x \Delta y} \left(\Delta y (qu)_{i+1/2,j}^n - \Delta y (qu)_{i-1/2,j}^n \right) \\ &= -\frac{\Delta t}{\Delta x} \left((qu)_{i+1/2,j}^n - (qu)_{i-1/2,j}^n \right) \end{aligned}$$

How to obtain $(qu)_{i\pm 1/2,j}^n$ given values of Q_{ij} and u on C-Grid?

$$(q)_{i+1/2,j}^n \approx \left(Q_{i+1,j}^n + Q_{i,j}^n \right) / 2$$

W=West, E=East, N=North, S=South



FV on C-grid - 1st try - Not FV3!

$$Q_{ij}^{n+1} = Q_{ij}^{n} + F + G$$

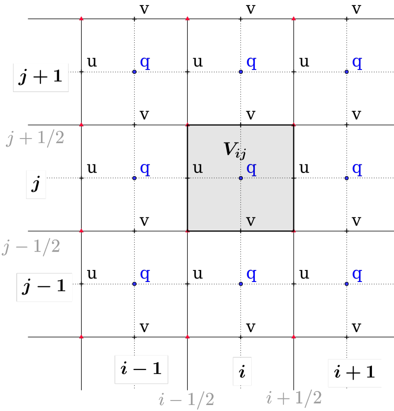
$$\begin{aligned}
F &= -\frac{\Delta t}{\Delta x \Delta y}\left(\Delta y (qu)_{i+1/2,j} - \Delta y (qu)_{i-1/2,j}\right) \\
&= -\frac{\Delta t}{\Delta x}\left((qu)^n_{i+1/2,j} - (qu)^n_{i-1/2,j}\right) \\
&\approx -\frac{\Delta t}{\Delta x}\delta_x(\overline{Q^n_{ij}}^x u^n_{ij})
\end{aligned}$$

$$\delta_x q_{ij} = (q_{i+1/2,j} - q_{i-1/2,j})$$

$$\overline{q_{ij}}^x = (q_{i+1/2,j} + q_{i-1/2,j})/2$$

Compactly:

$$Q^{n+1} = Q^n - \Delta t(\frac{1}{\Delta x}\delta_x(\overline{Q^n}^x u^n)) - \Delta t(\frac{1}{\Delta y}\delta_y(\overline{Q^n}^y v^n))$$



Hey! This looks like FD!

We could try this out assuming constant (u, v) known on a C-grid on a bi-periodic plane. Will it work? No, it is unstable :-(

Summary

2 2D FV
- Dimension splitting
- FV3 advection scheme

FV on C-grid - FV3 scheme

$$Q^{n+1} = Q^n + F + G$$

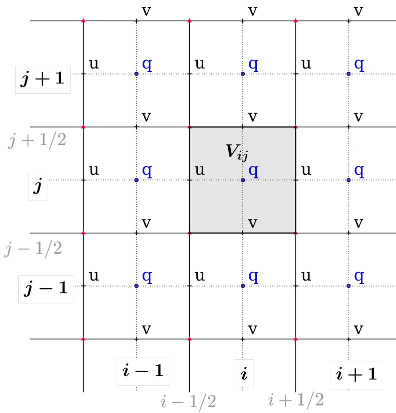
where

$$F \approx -\frac{\Delta t}{\Delta x \Delta y}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(\int_E qu - \int_W qu\right)$$

$$G \approx -\frac{\Delta t}{\Delta x \Delta y}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(\int_N qv - \int_S qv\right)$$

Assume (u, v) known, obtain F and G using neighbour values of Q that:

- Conserves mass exactly
- Consistent (preserve constant field)
- Stable



Lin & Rood (1996)

FV3 Advection - Dimension splitting

$$Q^{n+1} = Q^n + F(Q^n, u) + G(Q^n, v)$$

$$F(Q^n, u) \approx -\frac{\Delta t}{\Delta x \Delta y} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left(\int_E qu - \int_W qu \right)$$

$$G(Q^n, v) \approx -\frac{\Delta t}{\Delta x \Delta y} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left(\int_N qv - \int_S qv \right)$$

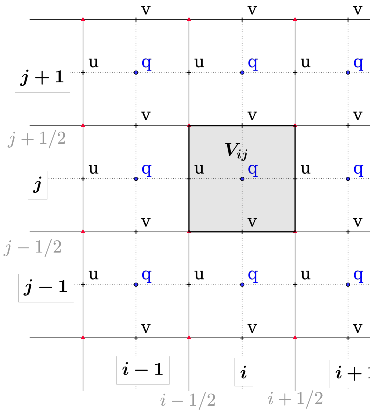
- Splitting (1st x then y):

$$Q^x = Q^n + F(Q^n, u)$$
$$Q^{yx} = Q^x + G(Q^x, v)$$

- Splitting (1st y then x):

$$Q^y = Q^n + G(Q^n, v)$$
$$Q^{xy} = Q^y + F(Q^y, u)$$

- Directional bias avoided with (assuming linearity of F and G on Q)



$$Q^{n+1} = \frac{1}{2}(Q^{xy} + Q^{yx}) = Q^n + F(Q^n, u) + G(Q^n, v) + \frac{1}{2}(G(F(Q^n, u), v) + F(G(Q^n, v), u))$$

FV3 Advection - Dimension splitting

Splitting without directional bias (omitting u, v):

$$Q^{n+1} = Q^n + F(Q^n) + G(Q^n) + \frac{1}{2}(GF(Q^n) + FG(Q^n))$$

If $Q^n = \alpha$ constant, $Q^{n+1} = \alpha$ for non-divergent flows ($\nabla \cdot \vec{v} = 0$).
Example:

$$F(\alpha, u) = -\frac{\alpha \Delta t}{\Delta x \Delta y} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left(\int_E u - \int_W u \right) = \alpha \overline{U}$$

Thus

$$Q^{n+1} = \alpha + \alpha \overline{U} + \alpha \overline{V} + \alpha \overline{UV}$$

with $\overline{U} + \overline{V} = 0$.

Conclusion

This method does not conserve constant quantities on simple flows, with $\alpha \overline{UV}$ error !!!

FV3 Advection - Dimension splitting

Splitting without directional bias (omitting u, v):

$$Q^{n+1} = Q^n + F(Q^n + \frac{1}{2}G(Q^n)) + G(Q^n + \frac{1}{2}F(Q^n))$$

Eliminate the error using inner advective fluxes:

$$Q^{n+1} = Q^n + F(Q^n + \frac{1}{2}g(Q^n)) + G(Q^n + \frac{1}{2}f(Q^n))$$

where

$$f(Q^n, u) \approx -\Delta t \overline{u^*}^x \frac{\partial Q}{\partial x},$$

$$g(Q^n, v) \approx -\Delta t \overline{v^*}^y \frac{\partial Q}{\partial y},$$

$$F(Q^n, u) \approx -\frac{\Delta t}{\Delta x \Delta y} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left(\int_{\mathrm{E}} qu - \int_{\mathrm{W}} qu \right)$$

$$G(Q^n, v) \approx -\frac{\Delta t}{\Delta x \Delta y} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left(\int_{\mathrm{N}} qv - \int_{\mathrm{S}} qv \right)$$

Lin & Rood (1996)

FV3 Advection - 2D Example

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = 0, \quad u > 0, v > 0 \quad q(t_0, x, y) = q_0(x, y)$$

with doubly periodic boundary conditions.

1D Upwinding with splitting:

$$Q^x_{ij} = Q^n_{ij} + F(Q^n, u), \quad F(Q^n, u) = \frac{u\Delta t}{\Delta x}(Q^n_{i-1,j} - Q^n_{ij})$$

$$Q^y_{ij} = Q^n_{ij} + G(Q^n, v), \quad G(Q^n, v) = \frac{v\Delta t}{\Delta y}(Q^n_{i,j-1} - Q^n_{ij})$$

Full scheme:

$$Q^{n+1} = Q^n + F(Q^n + \frac{1}{2}g(Q^n)) + G(Q^n + \frac{1}{2}f(Q^n))$$

where

$$f(Q, u) = -\frac{\Delta t\, u}{\Delta x}\overline{\delta_x Q}^x, \quad g(Q, u) = -\frac{\Delta t\, v}{\Delta y}\overline{\delta_y Q}^y,$$

FV3 Advection - 2D Example

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = 0, \quad u > 0, v > 0 \quad q(t_0, x, y) = q_0(x, y)$$

with doubly periodic boundary conditions.

Full scheme:

$$Q^{n+1} = Q^n + F\left(Q^n - \frac{1}{2}\frac{\Delta t\, v}{\Delta y}\overline{\delta_y Q^n}^y\right) + G\left(Q^n - \frac{1}{2}\frac{\Delta t\, u}{\Delta x}\overline{\delta_x Q^n}^x\right)$$

$$F(Q, u) = \frac{u\Delta t}{\Delta x}(Q_{i-1,j} - Q_{ij}), \qquad G(Q, v) = \frac{v\Delta t}{\Delta y}(Q_{i,j-1} - Q_{ij})$$

Remember that:

$$\delta_x\phi(x) = \phi(x + \Delta x/2) - \phi(x - \Delta x/2)$$

$$\overline{\phi(x)}^x = \frac{1}{2}(\phi(x + \Delta x/2) + \phi(x - \Delta x/2))$$

FV3 Advection - 2D Example

Advective fluxes:

$$Q^y = Q^n - \frac{1}{2}\frac{\Delta t\, v}{\Delta y}\overline{\delta_y Q^n}^y$$

$$Q^x = Q^n - \frac{1}{2}\frac{\Delta t\, u}{\Delta x}\overline{\delta_x Q^n}^x$$

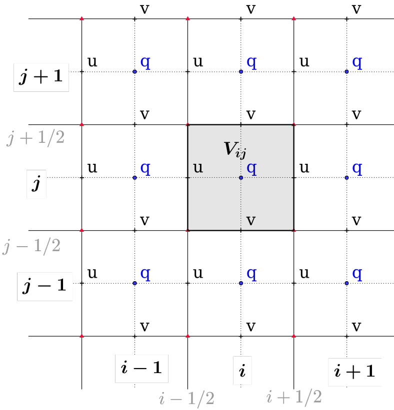
Integrated fluxes

$$F^y = F(Q^y, u) = \frac{u\Delta t}{\Delta x}(Q^y_{i-1,j} - Q^y_{ij})$$

$$G^x = G(Q^x, v) = \frac{v\Delta t}{\Delta y}(Q^x_{i,j-1} - Q^x_{ij})$$

Advance in time

$$Q^{n+1} = Q^n + F^y + G^x$$



Task of the day

Implement:

- Linear advection with velocity $(u, v) = (1, 1)$ and bi-periodic boundary conditions on $[0, 1] \times [0, 1]$
- Use dimension splitting with upwinding
- Use as initial condition $q_0(x, y)$ as a square in the middle of the domain with height 1 and width $1/4$.
- Experiment different Δt and Δx, Δy.

What is still missing:

- MUSCL and PPM (as in FV3)
- Imposition of monotonicity and steepness control.

This is too deep for this course, but now you know the tools and path. Feel feel to try, if you find the initial task easy!

Summary

3 Dynamics
- 3D equations
- Shallow Water Equations

Governing equations - Dynamics

Compressible Euler equations for atmosphere (ideal gas) in vector form:

$$
\begin{aligned}
\frac{D\mathbf{u}}{Dt} &= -2\,\boldsymbol{\Omega} \times \mathbf{u} - \frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{F}_r \;\; \text{(Momentum)} \\
\frac{D\rho}{Dt} &= -\rho\nabla \cdot \mathbf{u} \;\; \text{(Continuity)} \\
c_v \frac{DT}{Dt} &= -\frac{p}{\rho}\nabla \cdot \mathbf{u} \, \text{(Thermodynamics)}
\end{aligned}
$$

- $\mathbf{u} = (u, v, w)$: wind velocity
- p: pressure
- ρ: density
- T: temperature
- $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$: Material derivative

Governing equations

- Compressible Euler
  - Hydrostatic vs Non-hydrostatic
  - Shallow atmosphere vs Deep atmosphere

- Primitive equations: hydrostatic and shallow atmosphere
  - Shallow water equations
  - Quasigeostrophic equations
  - Barotropic vorticity equations
  - Passive transport equation

Summary

3 Dynamics
- 3D equations
- Shallow Water Equations

Recall

3D

$\vec{v} = (u, v, w)$

$\nabla = (\partial_x, \partial_y, \partial_z)$: Gradient

$\nabla \cdot \vec{v} = \partial_x u + \partial_y v + \partial_z w$: Divergence

2D

$\vec{v} = (u, v)$

$\nabla = (\partial_x, \partial_y)$: Gradient

$\nabla \cdot \vec{v} = \partial_x u + \partial_y v$: Divergence

$\vec{k} \times \vec{v}$: Cross product between $\vec{k}$ and $\vec{v}$ (rotate $\vec{v}$ ccw 90 degrees).

$\vec{k} \times (u, v) = (-v, u)$

3D to 2D dynamics

3D Incompressibility (constant density):

$$\nabla \cdot \vec{v} = 0, \quad (\rho = \rho_0)$$

Hydrostatic balance:

$$\frac{\partial p}{\partial z} = -\rho_0 g,$$

Free surface $h_T(x, y, t)$ where $h_T = h_0 + h$, with $h_0(x, y)$ topography, $h(x, y, t)$ fluid depth.

$$\int_z^{h_T} \frac{\partial p}{\partial z} dz = -\int_z^{h_T} \rho_0 g dz$$

$$p(z) = \rho_0 g(h_T - z) + \underbrace{p(h_T)}_{\text{Constant}}$$

Pressure gradient:

$$\nabla p = \rho_0 g \nabla h_T$$

Horizontal Momentum Equations ($\vec{v} = (u, v)$):

$$\frac{\partial \vec{v}}{\partial t} + \underbrace{(\vec{v} \cdot \nabla)\vec{v}}_{\text{nonlinear advection}} = \underbrace{-f\vec{k} \times \vec{v}}_{\text{Coriolis}} \underbrace{-g\nabla(h + h_0)}_{\text{Pressure}}$$

(f-plane approximation)

3D to 2D dynamics

3D Incompressibility :

$$\nabla \cdot \vec{v} = \partial_x u + \partial_y v + \partial_z w = 0,$$

$$\partial_z w = -\partial_x u - \partial_y v$$

Free surfave must have w velocity:

$$\frac{Dh_T}{Dt} = w(x, y, h_T, t)$$

Topography height must have w_0 velocity:

$$\frac{Dh_0}{Dt} = w(x, y, h_0, t) \Rightarrow w(x, y, h_0, t) = \vec{v} \cdot \nabla h_0$$

Integrate 3D incompressibility:

$$\int_{h_0}^{h} \partial_z w dz = -\int_{h_0}^{h} (\partial_x u + \partial_y v) dz$$

$$w_h - w_0 = -(h - h_0) \nabla \cdot \vec{v}$$

2D Continuity equation:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{v}) = 0$$

Shallow Water Equations - Advective form

$$\vec{v}(x,y,t) = (u(x,y,t), v(x,y,t))$$
$$h(x,y,t) = h_T(x,y,t) - h_0(x,y)$$

2D Continuity equation:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{v}) = 0$$
$$\frac{\partial h}{\partial t} + \underbrace{\vec{v}\cdot\nabla h}_{\text{transport}} = \underbrace{-h\nabla\cdot\vec{v}}_{\text{flow divergence}}$$

2D Horizontal Momentum Equations:

$$\frac{\partial \vec{v}}{\partial t} + \underbrace{(\vec{v}\cdot\nabla)\vec{v}}_{\text{nonlinear advection}} = \underbrace{-f\vec{k}\times\vec{v}}_{\text{Coriolis}}\underbrace{-g\nabla(h+h_0)}_{\text{Pressure}}$$

(f-plane approximation)

Shallow Water Equations - Vector invariant form

Split nonlinear advection ($\vec{v} = (u, v)$):

$$(\vec{v} \cdot \nabla)\vec{v} = (\nabla \times \vec{v}) \times \vec{v} + \frac{1}{2}\nabla\left(\vec{v} \cdot \vec{v}\right)$$

Relative and absolute vorticities:

$$\zeta = \vec{k} \cdot (\nabla \times \vec{v}), \qquad \eta = \zeta + f$$

Kinetic energy:

$$K = \frac{1}{2}\left(\vec{v} \cdot \vec{v}\right)$$

Bernoulli potential:

$$B = K + g(h + h_0)$$

Vector invariant momentum equations :

$$\frac{\partial \vec{v}}{\partial t} + \eta \vec{k} \times \vec{u} = -\nabla B$$

Exercise: Use expanded equation to show the equation transformation

Shallow water equations on the sphere

Vector invariant form:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{v}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \eta\vec{k} \times \vec{v} = -\nabla K - g\nabla(h + h_0)$$

- $\vec{v}$ is the 3D velocity vector tangent to the sphere
- $f = 2\Omega \sin(\theta)$
- ∇ gradient on tangent plane
- $\vec{k}$ unit vector point out of the sphere

Many properties: Conserves mass, energy, enstrophy, Coriolis neutral in energy budget, normal modes - inertia-gravity waves, Rossby waves, etc...

Shallow water equations with PV

Explicitly include the potential vorticity:

$$q = \eta / h$$

into SWE:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{v}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + q\vec{k} \times (h\vec{v}) = -\nabla K - g\nabla(h + h_0)$$

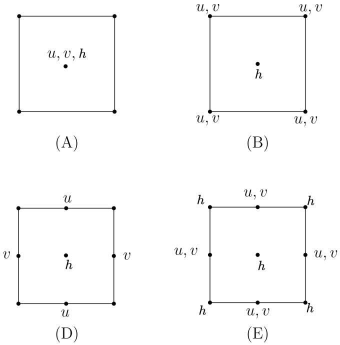
- PV conservation

Summary

4 Logically rectangular schemes
- FV Continuity equation - C-Grid
- FV Momentum Equations - D-Grid
- FV Full C-Grid

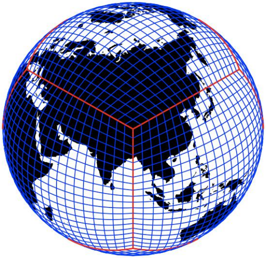
Plane vs Quasi uniform grids

Plane



Arakawa, A., & Lamb, V. R. (1977)

Cubed sphere



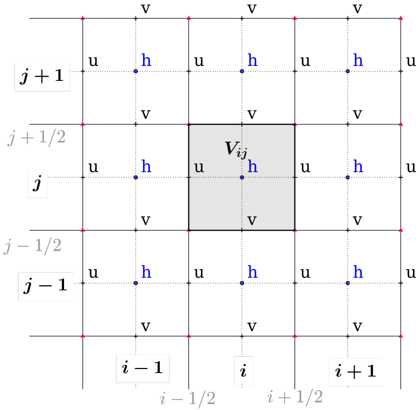
FV on C-grid - Continuity

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{v}) = 0$$

$$\frac{1}{|V_{ij}|}\int_{V_{ij}} \frac{\partial h}{\partial t} dV = -\frac{1}{|V_{ij}|}\int_{V_{ij}} \nabla \cdot (h\vec{v}) dV$$

$$H_{ij} = \frac{1}{|V_{ij}|}\int_{V_{ij}} h\, dV$$

$$\begin{aligned}
\frac{dH_{ij}}{dt} &= -\frac{1}{|V_{ij}|}\int_{V_{ij}} \nabla \cdot (h\vec{v}) dV \\
\text{(Div Thm)} \quad &= -\frac{1}{|V_{ij}|}\int_{\partial V_{ij}} h\vec{v} \cdot \vec{n} dV \\
&= -\frac{1}{\Delta x \Delta y}\left(\int_E hu - \int_W hu\right) \\
&\quad -\frac{1}{\Delta x \Delta y}\left(\int_N hv - \int_S hv\right)
\end{aligned}$$



W=West, E=East, N=North, S=South

FV on C-grid - Dimension Splitting

$$\frac{d\,H_{ij}}{dt} = -\underbrace{\frac{1}{\Delta x \Delta y}\left(\int_{E} hu - \int_{W} hu\right)}_{\text{Mean flux in x direction}} - \underbrace{\frac{1}{\Delta x \Delta y}\left(\int_{N} hv - \int_{S} hv\right)}_{\text{Mean flux in y direction}}$$

Integrating from t_n to t_{n+1} and dividing by Δt, leads to:

$$H_{ij}(t_{n+1}) = H_{ij}(t_n) - \underbrace{\frac{\Delta t}{\Delta x \Delta y}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(\int_{E} hu - \int_{W} hu\right)}_{\text{Mean flux in x direction over time}} - \underbrace{\frac{\Delta t}{\Delta x \Delta y}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(\int_{N} hv - \int_{S} hv\right)}_{\text{Mean flux in y direction over time}}$$

FV method with dimension splitting (used in FV3, for example):

$$H_{ij}^{n+1} = H_{ij}^{n} + F + G$$

where

$$F \approx -\frac{\Delta t}{\Delta x \Delta y}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(\int_{E} hu - \int_{W} hu\right)$$

$$G \approx -\frac{\Delta t}{\Delta x \Delta y}\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(\int_{N} hv - \int_{S} hv\right)$$

W=West, E=East, N=North, S=South

Method

Apply 2D PPM advection method on dimensionally split F and G

FV - FV3 scheme for continuity

Dimension split Fluxes

$$H^{n+1} = H^n + F(H^n + \frac{1}{2}g(H^n)) + G(H^n + \frac{1}{2}f(H^n))$$

where

$$f(H^n, u^*) \approx -\Delta t \overline{u^*}^x \frac{\partial H}{\partial x},$$

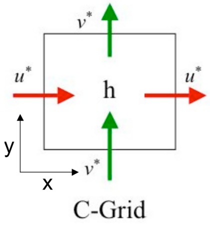
$$g(H^n, v^*) \approx -\Delta t \overline{v^*}^y \frac{\partial H}{\partial y},$$

$$F(H^n, u^*) \approx -\frac{\Delta t}{\Delta x \Delta y} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left(\int_{\mathrm{E}} hu - \int_{\mathrm{W}} hu \right)$$

$$G(H^n, v^*) \approx -\frac{\Delta t}{\Delta x \Delta y} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left(\int_{\mathrm{N}} hv - \int_{\mathrm{S}} hv \right)$$

- (u^*, v^*) are time averaged normal velocities
- PPM or MUSCL for F and G

Lin & Rood (1996), Putman (2007)



Summary

4 Logically rectangular schemes
- FV Continuity equation - C-Grid
- FV Momentum Equations - D-Grid
- FV Full C-Grid

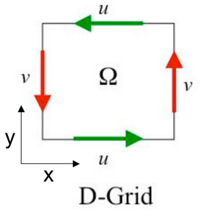
FV on D-grid - Momentum Equation

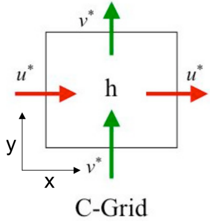
$$\frac{\partial \vec{v}}{\partial t} + \eta \vec{k} \times \vec{v} = -\nabla B, \qquad B = K + g(h + h_0)$$

$$\frac{\partial u}{\partial t} = +\eta v - \partial_x B$$
$$\frac{\partial v}{\partial t} = -\eta u - \partial_y B$$

FV3 uses a D-grid: Prognostic variables are

- H: Cell averaged depth in centre of cell
- (U, V): Edge integrated winds tangent to cell edges





C-Grid

Advection is done on C-grid. How are (U, V) converted to C-Grid (u^*, v^*)?

FV on D-grid - Momentum Equation

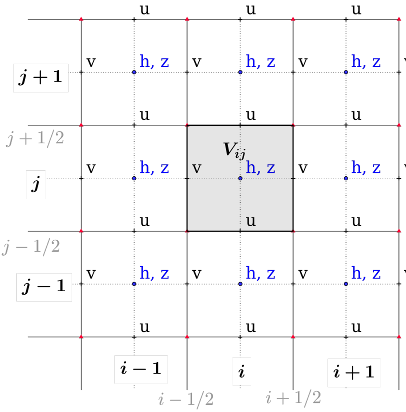
D-Grid favours calculation of vorticity, but we need C-grid quantities for FV-continuity

- Interpolate tangent winds to normal winds

$$u_c = \overline{u}^{xy}$$
$$v_c = \overline{v}^{xy}$$

- Advance equations in half step ($\Delta t/2$) to obtain h^*, u^*, v^* time/space averaged diagnostics on C-Grid (Details in next slides)
- Use C-grid quantities to advance full step to t_{n+1}

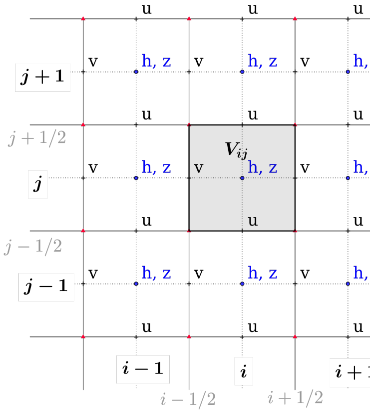
D-GRID



Vorticity on D-Grid

$$\zeta = \vec{k} \cdot (\nabla \times \vec{v}), \qquad \eta = \zeta + f$$

D-GRID



FV vorticity (Div/Vort Theorem)

$$\begin{aligned} Z &= \frac{1}{|V|}\int_V \zeta + f\, dS \\ &= \frac{1}{|V|}\int_V \vec{k}\cdot(\nabla\times\vec{v}) + f\, dS \\ &= \frac{1}{|V|}\int_{\partial V(\text{CCW})} \vec{v}\cdot\vec{t} + \frac{1}{|V|}\int_V f \\ &= \frac{1}{\Delta x \Delta y}\left(\int_E v - \int_W v - \int_N u + \int_S u\right) + \bar{f} \\ &= \frac{1}{\Delta x}(V_E - V_W) + \frac{1}{\Delta y}(-U_N + U_S) + \bar{f} \\ &= \frac{1}{\Delta x}\delta_x(V) - \frac{1}{\Delta y}\delta_y(U) + \bar{f} \end{aligned}$$

This is exact for U and V given! (No approximation)

Half step on D-Grid

All time operators (F, G, f, g) with step of $\Delta t/2$:

- Interpolations:
  - $\cdot_c$: Average to edges (C-grid)
  - $\cdot_a$: Average to cell centers (A-grid)
  - $\cdot_b$: Average to corners (B-grid)
- Depth at half step (Dimension split fluxes)

$$H^* = H^n + F(H^n + \frac{1}{2} g(H^n, v_a), u_c) + G(H^n + \frac{1}{2} f(H^n, u_a), v_c)$$

- Advected vorticity

$$Z_c^{x/2} = Z_c^n + \frac{1}{2} f(Z_c^n, u_b^n), \quad Z_c^{y/2} = Z_c^n + \frac{1}{2} g(Z_c^n, v_b^n)$$

- C-grid winds at half step

$$u^* = u_c^n + \frac{\Delta t}{2} G_x(Z_c^{x/2}, v^n) - \frac{\Delta t}{2} \frac{1}{\Delta x} \delta_x(\kappa^{**} + g(H^* - H_0))$$

$$v^* = v_c^n - \frac{\Delta t}{2} F_y(Z_c^{y/2}, u^n) - \frac{\Delta t}{2} \frac{1}{\Delta y} \delta_y(\kappa^{**} + g(H^* - H_0))$$

Half step on D-Grid

- C-grid winds at half step

$$u^* = u_c^n + \frac{\Delta t}{2}\mathcal{Y}(Z_c^{x/2}, v^n) - \frac{\Delta t}{2}\frac{1}{\Delta x}\delta_x(\kappa^{**} + g(H^* + H_0))$$

$$v^* = v_c^n - \frac{\Delta t}{2}\mathcal{X}(Z_c^{y/2}, u^n) - \frac{\Delta t}{2}\frac{1}{\Delta y}\delta_y(\kappa^{**} + g(H^* + H_0))$$

- Time averaged fluxes

$$\mathcal{Y}(Q, v) \approx \frac{2}{\Delta t}\int_{t_n}^{t_n+\Delta t/2} vQ, \quad \mathcal{X}(Q, u) \approx \frac{2}{\Delta t}\int_{t_n}^{t_n+\Delta t/2} uQ$$

- Upwind biased kinetic energy

$$\kappa^{**} = \frac{1}{2}(\mathcal{X}(u_a, u_c) + \mathcal{Y}(v_a, v_c))$$

Full step on C-grid

All operators $(F, G, f, g, \mathcal{X}, \mathcal{Y})$ with Δt step:

$$H^{n+1} = H^n + F(H^n + \frac{1}{2}g(H^n, \overline{v^*}^y), u^*) + G(H^n + \frac{1}{2}f(H^n, \overline{u^*}^x), v^*)$$

$$U^{n+1} = U^n + \Delta t \mathcal{Y}(Z^x, v^*) - \frac{\Delta t}{\Delta x}\delta_x(\kappa^* + g(\overline{H^{n+1} + H_0}^{xy}))$$

$$V^{n+1} = V^n - \Delta t \mathcal{X}(Z^y, u^*) - \frac{\Delta t}{\Delta y}\delta_y(\kappa^* + g(\overline{H^{n+1} + H_0}^{xy}))$$

$$\kappa^{**} = \frac{1}{2}(\mathcal{X}(\overline{u^*}^y, U^n) + \mathcal{Y}(\overline{v^*}^x, V^n))$$

Details in Lin & Rood (1997)

FV3 horizontal dynamics

Cubed sphere ou Lat-lon grid

Same idea but include coordinate metric terms!

Details on documentation of FV3 and references therein (Lin et al papers)

Summary

4. Logically rectangular schemes
   - FV Continuity equation - C-Grid
   - FV Momentum Equations - D-Grid
   - FV Full C-Grid

FV on C-grid - Momentum Equation

$$\frac{\partial \vec{v}}{\partial t} + \eta \vec{k} \times \vec{v} = -\nabla B, \qquad B = K + g(h + h_0)$$

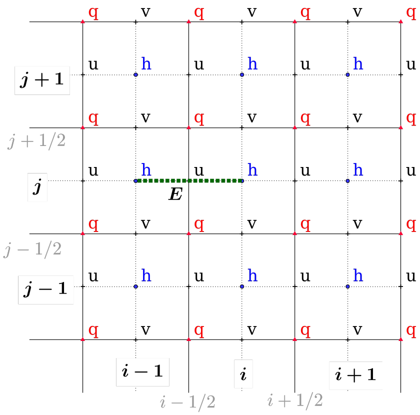
$$\frac{\partial u}{\partial t} = +\eta v - \partial_x B$$
$$\frac{\partial v}{\partial t} = -\eta u - \partial_y B$$

Integrate over dual edges E:

$$U = \frac{1}{\Delta x} \int_E u$$

Integrating u equation leads to:

$$\frac{dU}{dt} = \frac{1}{\Delta x} \int_E qhv - \frac{1}{\Delta x} \int_E \partial_x B$$



This is the circulation approach, but a flux approach exists too.

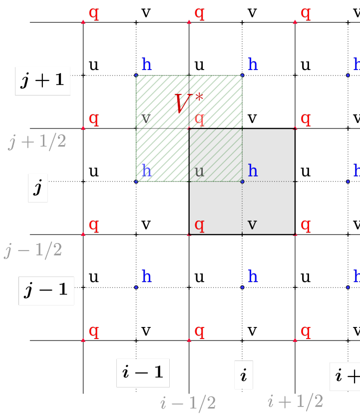
FV on C-grid - The dual grid

The dual grid is formed by cells around corners of primal cells.
It forms a D-grid!

$$\zeta = \vec{k}\cdot(\nabla\times\vec{v}), \qquad \eta = \zeta + f$$

FV vorticity (Div/Vort Theorem)

$$\begin{aligned}
Z &= \frac{1}{|V*|}\int_{V*}\zeta + f\,dS \\
&= \frac{1}{|V*|}\int_{V*}\vec{k}\cdot(\nabla\times\vec{v}) + f\,dS \\
&= \frac{1}{|V|}\int_{\partial V*\,(\mathrm{CCW})}\vec{v}\cdot\vec{t} + \frac{1}{|V*|}\int_{V*} f \\
&= \frac{1}{\Delta x\Delta y}\left(\int_{E*} v - \int_{W*} v - \int_{N*} u + \int_{S*} u\right) + \bar{f} \\
&= \frac{1}{\Delta x}(V_E - V_W) + \frac{1}{\Delta y}(-U_N + U_S) + \bar{f} \\
&= \frac{1}{\Delta x}\delta_x(V) - \frac{1}{\Delta y}\delta_y(U) + \bar{f}
\end{aligned}$$



FV on C-grid - Gradient

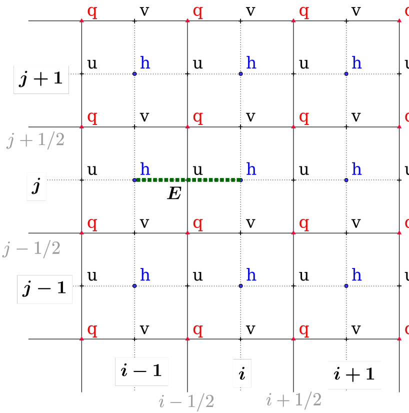
$$\nabla B$$

x-derivative (∂_x)

$$\frac{1}{\Delta x}\int_{x-edge}\frac{\partial B}{\partial x}dx = \frac{1}{\Delta y}(B_E - B_W) = \frac{1}{\Delta x}\delta_x(B)$$

y-derivative (∂_y)

$$\frac{1}{\Delta y}\int_{y-edge}\frac{\partial B}{\partial y}dy = \frac{1}{\Delta y}(B_N - B_S) = \frac{1}{\Delta y}\delta_y(B)$$



FV on C-grid - Momentum Equation

$$\frac{\partial \vec{v}}{\partial t} + \eta \vec{k} \times \vec{v} = -\nabla B, \qquad B = K + g(h + h_0)$$

Prognostic Variables:

$$H = \frac{1}{\Delta x \Delta y} \int_{\text{Primal Cell}} h$$

$$U = \frac{1}{\Delta x} \int_{\text{x-edge}} u, \quad V = \frac{1}{\Delta y} \int_{\text{y-edge}} v$$

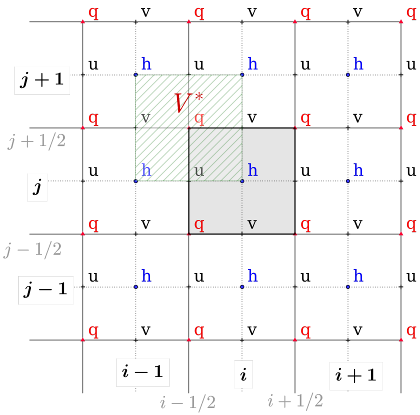
Diagostics:

$$\eta \approx \frac{1}{\Delta x} \delta_x(V) - \frac{1}{\Delta y} \delta_y(U) + \bar{f}^*$$

$$\vec{k} \times (U, V) = (-V, U)$$

$$\nabla B = \left(\frac{1}{\Delta x} \delta_x B, \frac{1}{\Delta y} \delta_y B\right)$$

$$K = \frac{1}{2}\left(\overline{U^2}^x + \overline{V^2}^y\right)$$



Sadourny's energy conserving scheme

$$\frac{\partial \vec{v}}{\partial t} + q\vec{k} \times (h\vec{v}) = -\nabla B, \qquad B = K + g(h + h_0)$$

Integrated discrete equations

$$\frac{dU}{dt} = +\overline{q\overline{\overline{H}^y V}^x}^y - \frac{1}{\Delta x}\delta_x B$$

$$\frac{dV}{dt} = -\overline{q\overline{\overline{H}^x U}^y}^x - \frac{1}{\Delta y}\delta_y B$$

$$\frac{dH}{dt} = -\frac{1}{\Delta x}\delta_x(\overline{H}^x U) - \frac{1}{\Delta y}\delta_y(\overline{H}^y V)$$

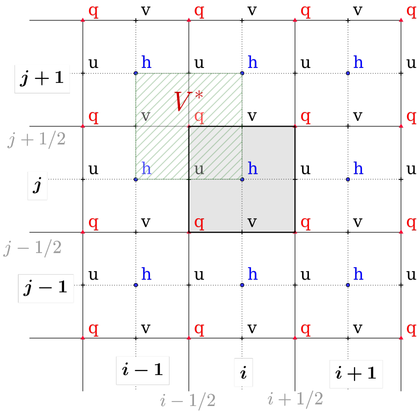
Diagostics:

$$\eta \approx \frac{1}{\Delta x}\delta_x(V) - \frac{1}{\Delta y}\delta_y(U) + \overline{f}$$

$$q = \eta / \overline{H}^{xy}$$

$$B = K + g(H + H_0)$$

$$K = \frac{1}{2}\left(\overline{U^2}^x + \overline{V^2}^y\right)$$



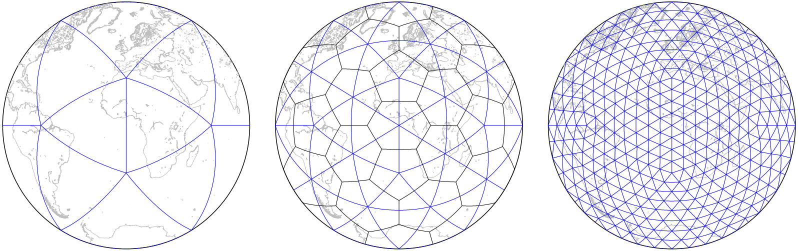
Task of the day

Implement FV-C-grid Sadourny's energy conserving scheme!

Summary

5 Arbitrary Polygonal grids
- Quasi uniform grids
- 2D FV

Icosahedral grids



- May be used as triangular or Hexagonal/Pentagonal grid
- May be optimized (Spring Dynamics, Centroidal Voronoi, HR95)
- May be locally refined (Hierarchically or with Centroidal optimizations)
- But are not perfectly isotropic ...

Summary

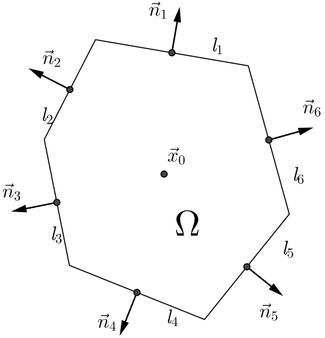
5 Arbitrary Polygonal grids
- Quasi uniform grids
- 2D FV

Continuity equation

Horizontal continuity equation (Shallow water model)

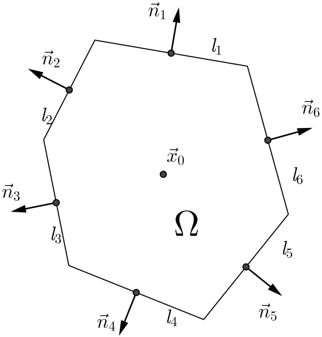
$$\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{u}) = 0$$

- h is the fluid depth
- $\vec{u} = (u, v)$ is the fluid horizontal velocity



Divergence theorem

$$\begin{aligned}
\nabla\cdot(h\vec{u})|_{\vec{x}_0} &\approx \frac{1}{|\Omega|}\int_{\Omega}\nabla\cdot(h\vec{u})\,d\Omega \\
(DivThm)\quad &= \frac{1}{|\Omega|}\int_{\partial\Omega} h\vec{u}\cdot\vec{n}\,d\partial\Omega \\
&\approx \frac{1}{|\Omega|}\sum_{i=1}^{n} h_i\vec{u}_i\cdot\vec{n}_i\,l_i.
\end{aligned}$$



$$\frac{\partial h}{\partial t} = -\frac{1}{|\Omega|}\sum_{i=1}^{n} h_i\vec{u}_i\cdot\vec{n}_i\,l_i$$

Interpolations required to obtain h_i and $\vec{u}_i$ depending on the staggering (A,C,...)

Problems...

- Can we get all the nice properties obtained in finite difference models and also scalability?

Desired:

1. Accurate
2. Stable
3. Conservative (mass, energy, PV, axial-angular momentum)
4. Mimetic Properties (spurious modes)

And also:

- Scalable on supercomputers
- Arbitrary spherical grids

Is it possible?

Shallow Water Model

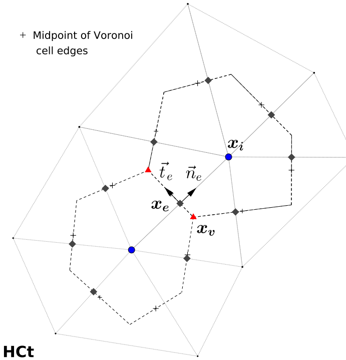
$$\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{u}) = 0,$$

$$\frac{\partial \vec{u}}{\partial t} + qh\vec{u}^{\perp} = -g\nabla(h + h_0) - \nabla K,$$

Discrete model

$$\begin{aligned} \frac{\partial h_i}{\partial t} &= -D_i, \\ \frac{\partial u_e}{\partial t} &= -Q_e^{\perp} - G_e, \end{aligned}$$

- $h_i = \frac{1}{|\Omega|} \int_\Omega h$
- $u_e = \frac{1}{|e|} \int_{\text{tr-edge}} \vec{v} \cdot \vec{n}_e$
- D_i discrete version of $\nabla \cdot (h\vec{u})$
- G_e discrete version of $\nabla(g(h + b) + K)$
- $Q_e^{\perp}$ discrete version of $qh\vec{u}^{\perp}$



See TRSK
Thuburn et al (2009)
Ringler et al (2010) Used in MPAS

Discrete operators

- Divergence Theorem:

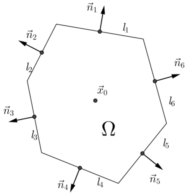
$$D_i = \frac{1}{A_i} \sum_e h_e u_e n_{ei} l_e,$$

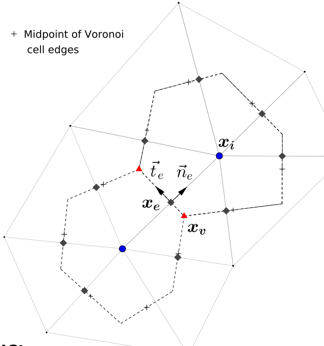
- "Divergence Theorem" for Vorticity:

$$\zeta_v = \frac{1}{A_v} \sum_e u_e t_{ev} \, d_e,$$

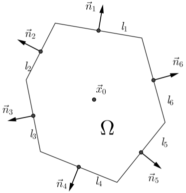
- Potential Vorticity: Interpolate h to triangle centers
- Kinetic Energy - Energy conserving (TRSK) - in general inconsistent:

$$K_i = \frac{1}{4A_i} \sum_e l_e \, d_e \, u_e^2$$





Discrete operators



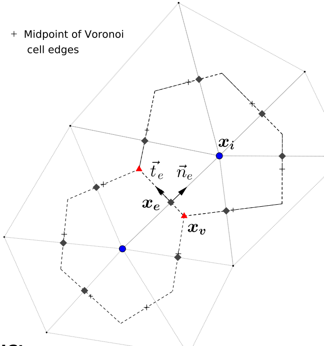
- Perpendicular term (TRSK):

$$Q_e^{\perp} = \sum_{e'} w_{ee'} h_{e'} u_{e'} q_{ee'} ,$$

$$w_{ee'} = c_{ee'} \frac{l_{e'}}{d_e} \left(\frac{1}{2} - \sum_{v} \frac{A_{iv}}{Ai} \right) n_{e' i},$$

- Gradient - Fund Thm of Calculus

$$G_e = (B_i - B_j)/d_e$$



Accuracy Summary

	Order of Accuracy		
Operator \Method	TRSK SCVT	TRSK HR95	MODF HR95
Divergence	0	1	1
Vorticity	1	1	1
Kinetic energy	0	1	2*
Gradient	-1	0	1
Perp	1	0	1
Overall	0*	0	1

See Peixoto (2016)

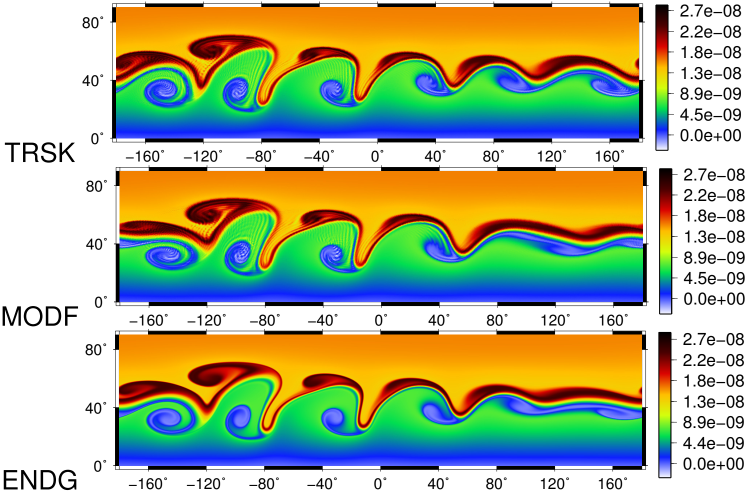
Properties of TRSK

- Mass conservation
- Accurate representation of geostrophic balance (C stag)
- Curl-free pressure gradient
- Energy conservation of pressure terms
- Energy conserving Coriolis term
- Conservation of total energy
- Steady geostrophic modes (f-sphere)
- Compatible discretization of PV
- Local operators
- Arbitrary orthogonal grids

Issue: Very low order (Inconsistent)

Obs: On rectangular grids, this reproduces a scheme similar to Sadourny's (1975) energy conserving scheme! (2nd order accurate)

Barotropicaly Unstable Jet - with perturbation - PV



Used HR95 grid level 7 (≈ 60km) and ENDG with ≈ 20km

Task of the day

Play with iModel!

`https://github.com/pedrospeixoto/iModel`

Summary

Conclusion

"All models are wrong but some are useful"

— George Box

Thank you!

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