On scale-mixture Birnbaum-Saunders distributions

Alexandre G. Patriota

Departamento de Estatística, Universidade de São Paulo, São Paulo/SP, 05508-090, Brazil

Abstract

We present for the first time a justification on the basis of central limit theorems for the family of life distributions generated from scale-mixture of normals. This family was proposed by Balakrishnan et al. (2009) and can be used to accommodate unexpected observations for the usual Birnbaum-Saunders distribution generated from the normal one. The class of scale-mixture of normals includes normal, slash, Student-t, logistic, double-exponential, exponential power and many other distributions. We present a model for the crack extensions where the limiting distribution of total crack extensions is in the class of scale-mixture of normals. Moreover, simple Monte Carlo simulations are reported in order to illustrate the results.

Key words: Birmbaum-Saunders distribution, central limit theorem, life distribution, scale-mixture of normals.

1 Introduction

All materials are subject to structural damages when exposed to fluctuating stresses and tensions, namely, computer devices, mobile phones, airplanes, beam bridges, raceways, human cells and so forth. Usually, the ultimate failure of the specimen is assumed to be due to the growth of a dominant crack in the material. The Birnbaum-Saunders distribution (BS distribution for short) has been successfully used in life studies and in material-fatigue life studies. This distribution was firstly proposed by Birnbaum and Saunders (1969) and extended versions called scale-mixture Birnbaum-Saunders (SBS) distribution and generalized Birnbaum-Saunders (GBS) distribution were proposed by Balakrishnan et al. (2009) and Días-Garcia and Leiva-Sánchez (2005), respectively. The SBS distribution is a special case of the GBS distribution. Theoretical developments, inference and diagnostics methods, goodness-of-fit tests and random generation algorithms for the GBS distribution are described in Leiva-Sánchez et al. (2008) and Sanhueza et al. (2008).

The construction of the standard BS distribution is based on the central limit theorem (CLT) for independent and identically distributed (iid) random variables. Desmond (1985) derived the BS distribution under a biological context, but he also relied on the standard CLT (the idea presented here can also be applied in this context). Here, for simplicity, we focus only on the original idea of Birnbaum and Saunders (1969). Since the derivation of the BS model is the main aspect for this article, in what follows we shall summarize the steps given in Birnbaum and Saunders (1969).

Suppose that a specific material is subject to fluctuating stresses caused by a cycling stress source. Birnbaum and Saunders (1969) considered that, in each cycle, the material is subject to m microscopic incremental cracks related to loading oscillations, say $Y_{i,1}, \ldots, Y_{i,m}$ for cycle i. Let X_i be the material crack extension in the ith cycle of the stress source, it is assumed that $X_i = Y_{i,1} + \ldots + Y_{i,m}$, where $Y_{i,j}$ and $Y_{k,l}$ are independent for all $i \neq k$ and also that $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2 < \infty$ for all $i \geq 1$. The total crack extension is defined then by $W_n = \sum_{i=1}^{n} X_i$, where n is the number of cycles required to crack. As $X_1, \ldots, X_n$ are iid random variables with finite second moment, by the standard CLT,

$$\frac{W_n - n\mu}{\sigma\sqrt{n}} = \frac{W_n}{\sigma\sqrt{n}} - \frac{\sqrt{n}\mu}{\sigma}$$

is approximately normally distributed. In Desmond (1985), it is considered dependent crack extensions, but the magnitude of the loadings ("impulses") are regarded as iid random variables. In the core of these theories, they made similar assumptions (i.e., the standard CLT holds).

Let C be the number of cycles until failure and ω the material critical value for cracking, then we have $P(C \leq n) = 1 - P(W_n \leq \omega)$. Let T be the continuous extension of the discrete variable C and define $\alpha = \frac{\sigma}{\sqrt{\mu\omega}}$ and $\beta = \frac{\omega}{\mu}$, then, following Birnbaum and Saunders (1969), we obtain that

$$\frac{1}{\alpha}\left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \stackrel{d}{=} \mathcal{N}(0,1), \tag{1}$$

where "$\stackrel{d}{=}$" means "equally distributed as" and $\mathcal{N}(0,1)$ represents the standard normal distribution. The distribution of T is the one that satisfies the relation (1). Naturally, relation (1) must be seen as an approximation that is reasonably justified through standard CLT.

Balakrishnan et al. (2009) generalized this distribution by relaxing the relation (1) to the scale-mixture of normals. Below we define a multivariate scale mixture of normals which will be useful in next section.

DEFINITION **1.1.** *(Eltoft et al., 2006) We say that a random vector $\boldsymbol{X}$ has a n-variate scale mixture of normals, if $\boldsymbol{X} \stackrel{d}{=} \boldsymbol{\mu} + V\boldsymbol{U}$, where $\boldsymbol{\mu} \in \mathbb{R}^n$ is a fixed vector, $\boldsymbol{U}$ has an n-variate normal distribution with zero mean and variance matrix $\boldsymbol{\Sigma}$ and V is a univariate positive random variable, independent* *of $\boldsymbol{U}$, having distribution function H. In notation: $\boldsymbol{X} \sim SMN_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, H)$. When $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix, we say that $\boldsymbol{X}$ follows a standard n-variate scale mixture of normals.*

Backing to the life distribution of T, Balakrishnan et al. (2009) considered the following relation

$$\frac{1}{\alpha}\left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \stackrel{d}{=} SMN_1(0, 1, H) \tag{2}$$

instead of (1). Evidently, relation (2) includes (1) and therefore, the class of distributions of T that satisfies relation (2) includes the usual BS distribution. Balakrishnan et al. (2009) offered the following reasons for using their generalization:

> *"The three main reasons for developing this class of distributions are the following: (i) the use of the SBS specification to model observable data enables us to make robust estimation of parameters in a similar way to that of the SMS specification, which is not possible with the BS distribution or any other well-known compatible model such as the lognormal distribution, (ii) the theoretical arguments established in the genesis of the BS distribution can be transferred to the SBS one and thus it is an appropriate model for describing different phenomena that present accumulation of some type under stress, and (iii) SBS distributions allow us to efficiently compute the maximum likelihood (ML) estimates of the model parameters by using the EM-algorithm, which is not possible with the classical BS distribution; "* (Balakrishnan et al., 2009)

Basically, the robustness justification is given for any other situation when a normal distribution is extended to the scale-mixture of normals (or to the elliptical class of distributions). In this paper, we point out a theoretical justification for the use of scale-mixture of normal distributions based on a CLT. Section 2 presents a general CLT which legitimates relation (2). Section 3 offers some examples when relation (2) may occur. Results of simple Monte Carlo simulations are reported in Section 4. Finally, Section 5 concludes the paper.

2 A general central limit theorem

It is well known that normal distributions are attractors in CLTs for sequences of iid random variables with some finite moments (see, for instance Athreya and Lahiri, 2006). If the iid condition is replaced by exchangeability for a sequence $\{X_i\}_{i\geq 1}$ with $E(X_1) = 0$ and $E(X_1^2) = 1$, the normal distribution is still the attractor in central limit theorems (CLTs), if and only if $\text{Cov}(X_i, X_j) = \text{Cov}(X_i^2, X_j^2) = 0$ for all $i \neq j$ (Bum et al., 1958, pg. 225). Based on these results, sums of random variables with finite moments are often claimed to be approximately normally distributed. However it is not always true, below we present a simple result where asymptotic normality does not hold even for sequences with finite second moments.

LEMMA **2.1.** *Let* $\{X_i\}_{i\geq 1}$ *be a sequence of random variables such that the first* n *variables* $(X_1, \ldots, X_n) \sim SMN_n(\mathbf{0}, \boldsymbol{I}, H)$. *Then,*

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \xrightarrow{d} SMN_1(0, 1, H). \tag{3}$$

Proof. By the stochastic representation, the proof is straightforward. Define $\boldsymbol{X}_n = (X_1, \ldots, X_n)^\top$, then $\boldsymbol{X}_n \stackrel{d}{=} V\boldsymbol{U}_n$, where $\boldsymbol{U}_n = (U_1, \ldots, U_n)$ follows a standard n-variate normal distribution. Then, for all $n \geq 1$,

$$Z_n \stackrel{d}{=} V \frac{\sum_{i=1}^{n} U_i}{\sqrt{n}} \stackrel{d}{=} VZ,$$

where $Z \sim \mathcal{N}(0, 1)$. Thus, $Z_n \sim SMN_1(0, 1, H)$ for all $n \geq 1$. □

Non-standard CLTs are well known when the second moment is infinite for the class of α-stable distributions (see, for instance, Gnedenko and Kolmogorov, 1954; Meerschaert and Scheffler, 2001). That is, the sum of a number of random variables with power law tail distributions decreasing proportional to $|x|^{-\alpha}$ (where $0 < \alpha < 2$, which implies infinite variance) will converge to a stable distribution as the number of variables grows. By using this result, we can only justify stable distributions as attractors, but in the context of BS distribution, we must also justify other than normal limit distributions when the involved random variables have finite variances (e.g., Student distribution with $k > 2$ degrees of freedom). Lemma 2.1 shows that scale mixture of normals can also be attractors in CLTs when the involved random variables have a specific structure of dependence. Under complicated conditions, Jiang and Hahn (2003) generalized Lemma 2.1 when the sequence $\{X_i\}_{i\geq 1}$ is **exchangeable**.

In what follows, we provide (1) simple conditions (which is readily applied to the fatigue-life context) and a new Fourier-analytic demonstration that sums of sequences of exchangeable random variables with finite second moments converge to scale-mixture of normals (a generalization of Lemma 2.1, but a specialized version of Jiang and Hahn, 2003); (2) the rate of convergence in this general CLT when the absolute third moment is finite; (3) a standardized version which converges to the standard normal distribution. To the best of our knowledge, all these results are new. First, we introduce some important results and definitions.

A sequence of random variables $\{X_i\}_{i\geq 1}$ defined in a probability space $(\Omega, \mathcal{F}, P)$ is said to be exchangeable if for each n,

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) = P(X_{\pi_1} \leq x_{\pi_1}, \ldots, X_{\pi_n} \leq x_{\pi_n})$$

for any permutation $\pi = (\pi_1, \ldots, \pi_n)$ of $\{1, \ldots, n\}$. Let $\mathcal{G}$ be the tail σ-field of $\{X_i\}_{i\geq 1}$, i.e., $\mathcal{G} = \bigcap_{n\geq 1} \sigma\langle X_n, X_{n+1}, \ldots\rangle$, where $\sigma(X)$ is the smallest σ-field generated by X. An important result which will be used in the proof of the below theorem is that an exchangeable sequence of random variables is conditionally iid given $\mathcal{G}$ (de Finetti, 1937; Hewitt and Savage, 1955). Now, we are able to enunciate the theorem of this section.

THEOREM **2.1.** *Let $\{Y_{i,j},\ 1 \leq j \leq m\}_{i\geq 1}$ be an infinity sequence of random vectors such that $Y_{i,j} = \frac{\mu}{m} + \frac{\sigma}{m} r_{i,j}$, with $E(r_{i,j}) = 0$ and $E(r_i^2) = 1$, where $r_i = m^{-1}\sum_{j=1}^m r_{i,j}$. Define $X_i = \sum_{j=1}^m Y_{i,j} = \mu + \sigma r_i$, then $E(X_i) = \mu$ and Var$(X_i) = \sigma^2$. Consider that $\{r_i\}_{i\geq 1}$ is a sequence of exchangeable and identically distributed random variables and let $\mathcal{G}$ be its tail σ-field.*

(i) If $\gamma_2 = E(r_1^2|\mathcal{G}) < \infty$ and $E(r_1|\mathcal{G}) = 0$ almost surely, then

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{X_i - \mu}{\sigma} \xrightarrow{d} \sqrt{\gamma_2} Z. \tag{4}$$

(ii) In addition to (i), if $\gamma_3 = E(|r_1|^3|\mathcal{G}) < \infty$ almost surely and $\kappa = E(\gamma_3\gamma_2^{-3/2}) < \infty$, then

$$\sup_{x\in\mathbb{R}} \left|P(Z_n < x) - P(\sqrt{\gamma_2} Z < x)\right| \leq O\left(\frac{\kappa}{\sqrt{n}}\right). \tag{5}$$

(iii) In addition to (ii), if $\gamma_4 = E(r_1^4|\mathcal{G}) < \infty$ almost surely and $E(\gamma_2^2) < \infty$, then

$$\frac{Z_n}{s_n} \xrightarrow{d} Z, \tag{6}$$

where $s_n^2 = \frac{1}{n}\sum_{i=1}^n \frac{(X_i-\mu)^2}{\sigma^2}$ and $Z \sim N(0,1)$.

As the ultimate cracks are often the balanced nonstandardized sums of many unobserved random events, this theorem provides a partial explanation for the appearance of the scale-mixture of normal distributions. Equation (5) gives a more quantitative form of the central limit theorem (4) and equation (6) shows that standardized sums that satisfy the theorem's conditions are approximately normally distributed.

Note that, here $\{Y_{i,j}\}_{1\leq j\leq m, i\geq 1}$ does not need to be an exchangeable sequence of random variables, provided that $\{X_i\}_{i\geq 1}$ be. Accordingly, we may assume that the events related to the material cracks are exchangeable just in the cycle levels not in the microscopic ones. From a physical point of view, we highlight that exchangeable assumption for the crack extensions seems to be much more reasonable than independency.

From now on, we shall consider just the sequence of crack extensions $\{X_i\}_{i\geq 1}$. Since conditionally to $\mathcal{G}$, X_1 and X_2 are independent, we have that Cov$(X_1, X_2) = E($Cov$(X_1, X_2|\mathcal{G})) +$

$\mathrm{Cov}(E(X_1|\mathcal{G}), E(X_2|\mathcal{G})) = \mathrm{Cov}(E(X_1|\mathcal{G}), E(X_2|\mathcal{G}))$. In Theorem 2.1, it is required that $E(X_1|\mathcal{G}) = \mu$ almost surely, therefore, it is equivalent to require uncorrelated random variables, i.e., $\mathrm{Cov}(X_1, X_2) = 0$. If $E(X_1|\mathcal{G}) = \mu + \sigma E(r_1|\mathcal{G})$, with $E(r_1|\mathcal{G})$ being a non degenerate random variable, then the random variables $X_1, X_2, \ldots$ will be correlated and Theorem 2.1 will not apply.

Depending on the dependence structure of the crack extensions, γ_2 is indeed a random variable. However, on the one hand, if the sequence $\{X_i\}_{i\geq 1}$ is also independent, then the tail σ-field $\mathcal{G}$ is trivial (just contains events of probability zero or one). In this context, $\gamma_2 = \mathrm{E}(r_1^2|\mathcal{G}) = \mathrm{E}(r_1^2) = 1$ and the standard CLT applies. On the other hand, if $\{X_i\}_{i\geq 1}$ is not an independent sequence of random variables, the tail σ-field $\mathcal{G}$ may not be trivial and then γ_2 is a non degenerate random variable. Suppose that $\mathcal{G}$ is generated by a random variable s, then we can just write $\gamma_2 = \mathrm{E}(r_1^2|s)$.

In actual problems, we hardly know the dependence structure of the random variables, then it is almost impossible to derive the distribution of $\gamma_2 = \mathrm{E}(r_1^2|\mathcal{G})$. However, statistical tools may be employed to test possible distributions based on the observed data (see Section 3.4 of Sanhueza et al., 2008). If, after a statistical analysis, one chooses the function H for relation (2) and normality is not tenable (i.e., γ_2 is not constant), then possibly the crack extensions $X_1, X_2, \ldots$ have some type of persistent dependence structure. Notice that, they are uncorrelated $\mathrm{Cov}(X_i, X_k) = 0$ with a persistent dependence structure, since $\mathrm{Cov}(X_i^2, X_k^2) = \mathrm{Cov}(E(X_1^2|\mathcal{G}), E(X_2^2|\mathcal{G})) = \sigma^4\mathrm{Var}(\gamma_2) \geq 0$ for all $i \neq k$. In general, $\mathrm{Cov}(\tilde{f}(X_i), \tilde{g}(X_j)) = \mathrm{Cov}(E(\tilde{f}(X_1)|\mathcal{G}), E(\tilde{g}(X_2)|\mathcal{G}))$ for all $i \neq j$, where $\tilde{f}$ and $\tilde{g}$ are measurable functions. That is, the dependence structure between two crack extensions are always the same independently of how distant they are from each other. This means that the incremental microscopic cracks $\{Y_{i,j},\ 1 \leq j \leq m\}$ and $\{Y_{k,j},\ 1 \leq j \leq m\}$ for $i \neq k$ are not independent as assumed by Birnbaum and Saunders (1969). According to Owen (2006), long dependence structures for the crack extensions are quite realistic assumptions. The source of this long dependence can be very complex and hard to establish precisely. It can be mixtures from the material type, environment, temperature, direction of loading, size and distribution of internal defects, surface quality, geometry and many others.

3 Examples

As aforementioned, in actual problems, we may employ statistical tests to find the function H in relation (2), this is equivalent to assign a limiting distribution for the sequence of crack extensions $\{X_i\}_{i\geq 1}$. In this section we present a simple but illuminating instance of a model for the crack extensions which allows interpretations regarding their dependence structure. By using this model, we can predict the value of $\mathrm{Cov}(X_1^2, X_2^2)$ based only on the chosen limiting distribution. We apply this model when the limiting distribution is Student, Laplace, contaminated normal and slash distributions.

EXAMPLE **3.1.** *(Model for the crack extensions) Let* $Y_{i,j} = \frac{\mu}{m} + \frac{\tilde{\sigma}}{m} r_{i,j}$*, where* $r_{i,j} = a_j B_i$*, in which* $a_1, \ldots, a_m$ *are (possible dependent) random variables and* $\{B_i\}_{i\geq 1}$ *is an iid sequence of random variables independent of* $a_1, \ldots, a_m$ *with* $E(B_1) = 0$ *and* $E(B_1^2), E(a^2) < \infty$*, where* $a = m^{-1} \sum_{j=1}^m a_j$*. The random variables* $a_1, \ldots, a_m$ *may be interpreted as* m *random effects related to the material characteristic, while the random variables* $B_1, B_2, \ldots$ *are related to the cycles. Assuming that the events related to the cracks are independent among the cycles, then independence of the sequence* $\{B_i\}_{i\geq 1}$ *is reasonably justified. In this model, the variation of microscopic cracks are driven by multiplicative effects from two sources: one related to the cycles and other related to the material properties.*

Notice that, $X_i = \mu + \tilde{\sigma} r_i$ *for all* $i \geq 1$*, where* $r_i = aB_i$*. On the one hand, as* $B_1, B_2, \ldots$ *are iid random variables, by the standard CLT, we have* $n^{-1/2} \sum_{i=1}^n (X_i - \mu)/\tilde{\sigma} \xrightarrow{d} \sqrt{E(B_1^2)} aZ$*, i.e., it converges to a scale-mixture of normals. On the other hand, by Theorem 2.1,* $n^{-1/2} \sum_{i=1}^n (X_i - \mu)/\sigma \xrightarrow{d} \sqrt{\gamma_2} Z$*, where* $\sigma^2 = \tilde{\sigma}^2 E(a^2) E(B_1^2)$ *and then, by equating the limiting distribution, we conclude that* $\gamma_2 = \frac{a^2}{E(a^2)}$ *almost surely.*

Observe that, in this simple model, $\{X_i\}_{i\geq 1}$ is a sequence of uncorrelated random variables $\text{Cov}(X_1, X_2) = 0$, but with long dependence. After a straightforward calculation we obtain

$$\text{Cov}(X_i^2, X_j^2) = \sigma^4 \frac{\text{Var}(a^2)}{E(a^2)^2} > 0$$

for any $i \neq j$ (i.e., the dependence of X_j and X_i does not decrease when $|i - j|$ increases), where $\sigma^2 = \tilde{\sigma}^2 E(B_1^2) E(a^2)$. Based on Example 3.1, we compute the covariance $\text{Cov}(X_i, X_j)$ for Student, Laplace, contaminated normal and slash limiting distributions.

EXAMPLE **3.2.** *(Student distribution) Consider Example 3.1 and suppose that a Student distribution with* $\nu > 4$ *is the limiting distribution. Then,* $a^{-2} \stackrel{d}{=} \frac{\chi^2(\nu)}{\nu}$*,* $E(a^2) = \frac{\nu}{\nu - 2}$*, Var*$(a^2) = \frac{2\nu^2}{(\nu-2)^2(\nu-4)}$ *and*

$$\textit{Cov}(X_i^2, X_j^2) = \frac{2\sigma^4}{\nu - 4} = O(\nu^{-1})$$

as $\nu \to \infty$*.*

EXAMPLE **3.3.** *(Laplace distribution) Consider Example 3.1 and suppose that a Laplace distribution is the limiting distribution. Then,* $a^2 \stackrel{d}{=} Exponential(\nu)$*,* $E(a^2) = \nu^{-1}$*, Var*$(a^2) = \nu^{-2}$ *and*

$$\textit{Cov}(X_i^2, X_j^2) = \sigma^4.$$

EXAMPLE **3.4.** *(Contaminated normal distribution) Consider Example 3.1 and suppose that a contaminated normal distribution is the limiting distribution. Then,* a^{-2} *has discrete distribution, where*

$P(a^{-2}=\gamma)=\nu$ and $P(a^{-2}=1)=1-\nu$ for $\gamma\in(0,1]$ and $\nu\in[0,1]$. Notice that, $E(a^2)=\frac{\gamma+\nu(1-\gamma)}{\gamma}$, Var$(a^2)=\frac{\nu(1-\gamma)^2(\nu-1)}{\gamma^2}$ and

$$\textit{Cov}(X_i^2,X_j^2)=\frac{\sigma^4\nu(\nu-1)(1-\gamma)^2}{(\gamma+\nu(1-\gamma))^2}=O(\nu^2)$$

as $\nu\to 0$.

EXAMPLE **3.5.** *(Slash distribution) Consider Example 3.1 and suppose that a slash distribution with parameter $\nu>4$ is the limiting distribution. Then, $a^{-1}\overset{d}{=}Beta(\nu,1)$, $E(a^2)=\frac{\nu}{\nu-2}$, Var$(a^2)=\frac{4\nu}{(\nu-4)(\nu-2)^2}$ and*

$$\textit{Cov}(X_i^2,X_j^2)=\frac{4\sigma^4}{(\nu-4)\nu}=O(\nu^{-2})$$

as $\nu\to\infty$.

In summary, we may assume the model proposed in Example 3.1 to describe the dependence structure of the crack extensions, therefore, based on the limit distribution one can find an expression for Cov(X_i^2,X_j^2) for any i,j.

4 A simple simulation study

The generation of uncorrelated exchangeable random variables is a hard task. In this section, we consider a multivariate Student distribution, which components are known to be exchangeable and uncorrelated. Let $\{X_i\}_{i\geq 1}$ be an exchangeable sequence of random variables where the first n random variables $\tilde{X}_n=(X_1,X_2,\ldots,X_n)$ has standard n-variate Student distribution with $\nu>2$ degrees of freedom. Its density is

$$f_{\tilde{X}_n}(x)\propto\left(\frac{1}{1+\frac{1}{\nu}x^\top x}\right)^{\frac{n+\nu}{2}}.$$

We know that $X_1,X_2,\ldots,X_n$ are exchangeable and uncorrelated but not independent random variables. Here, we will study the limit distributions of the sequences

$$S_{1n}=\sqrt{\frac{\nu-2}{\nu}}Z_n\qquad\text{and}\qquad S_{2n}=\frac{Z_n}{s_n},$$

where $Z_n=n^{-1/2}\sum_{i=1}^n X_i$ and $s_n^2=n^{-1}\sum_{i=1}^n X_i^2$. By Bum et al. (1958), S_{1n} cannot converge to a normal distribution. By Lemma 2.1 and Theorem 2.1, we have that S_{1n} converges to a univariate Student distribution with ν degrees of freedom and S_{2n} converges to a standard normal distribution.

Now, assume that $W_1, \ldots, W_n$ are iid random variables having Student-t distribution with $\nu > 2$ degrees of freedom. By the standard CLT

$$S_n^{(I)} = \sqrt{\frac{\nu - 2}{\nu}} \sum_{i=1}^{n} \frac{W_i}{\sqrt{n}}$$

converges to the standard normal distribution.

We generate 5000 Monte Carlo simulations from multivariate and univariate Student-t with $\nu = 4$ degrees of freedom. The sample sizes are $n = 100, 500$ and 1000 and for each simulation we compute S_{1n}, S_{2n} and $S_n^{(I)}$. Figure 4 displays normal qqplots of the quantities S_{1n}, S_{2n} and $S_n^{(I)}$, the closer the points are to the 45% line, the more normally distributed they are. Notice that, as expected, S_{1n} does not converge to the normal distribution while S_{2n} and $S_n^{(I)}$ do.

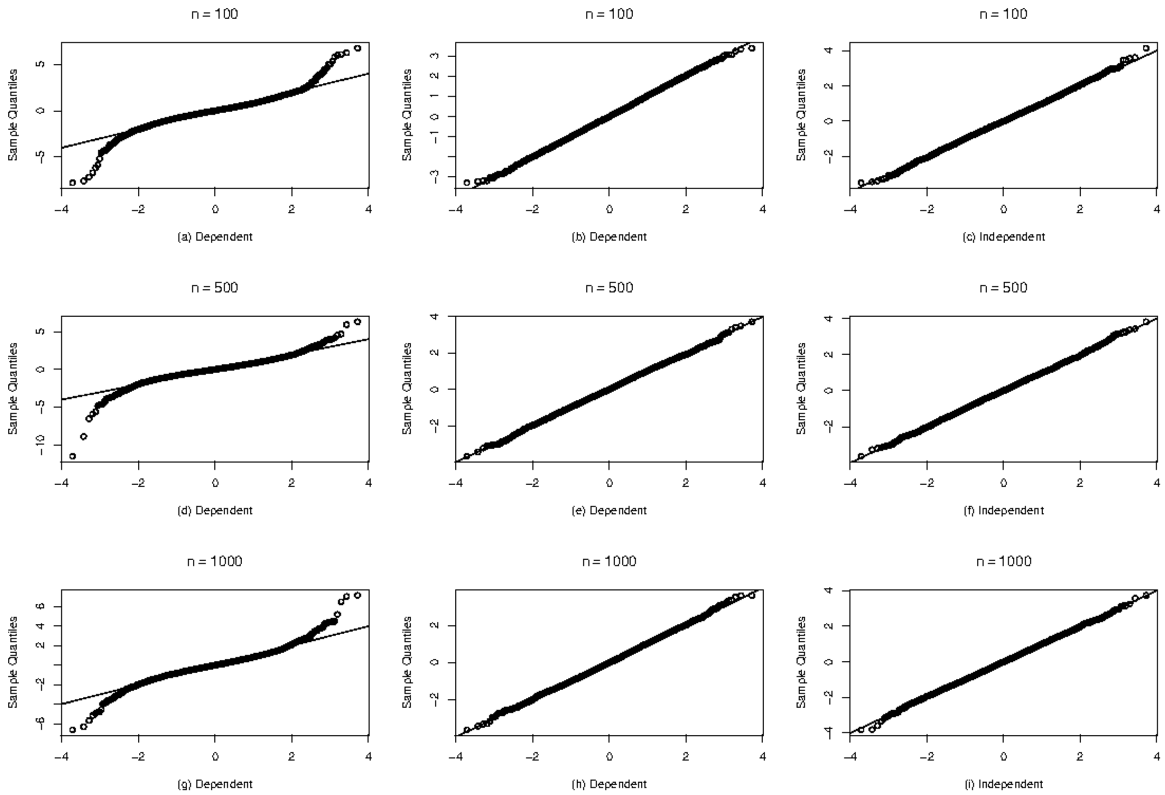

Figure 1: Normal qqplots of the Monte Carlo samples for: (a) S_{1n} and $n = 100$; (b) S_{2n} and $n = 100$; (c) $S_n^{(I)}$ and $n = 100$; (d) S_{1n} and $n = 500$; (e) S_{2n} and $n = 500$; (f) $S_n^{(I)}$ and $n = 500$; (a) S_{1n} and $n = 1000$; (b) S_{2n} and $n = 1000$; (c) $S_n^{(I)}$ and $n = 1000$.

5 Concluding remarks

In this note, we presented a justification for the use of scale-mixture of normals as underlying distributions for generating families of life distributions. When there exists a persevering dependence structure for the crack extensions, then the usual Birnbaum-Saunders distribution generated from the normal one may not be tenable. Instead, life distributions generated from scale-mixture of normals must be adopted with justification on the basis of central limit theorems for exchangeable random variables with persistent dependence structure. Moreover, if the variation source of microscopical cracks is defined as multiplicative effects from a source related to the material and another related to the stress cycles, we can also interpret the generalized Birnbaum-Saunders distribution within a physical perspective.

Acknowledgments

This work was supported by FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo) – Brazil (10/50014-0).

A Proof of Theorem 2.1

Without lost of generality, take $\mu = 0$ and $\sigma = 1$. From Taylor's expansion, we have

$$\exp(itX_1) = 1 + itX_1 - \frac{t^2}{2}X_1^2 + o(|t|^2X_1^2)$$

and

$$\exp(itX_1) = 1 + itX_1 - \frac{t^2}{2}X_1^2 + O(|t|^3|X_1|^3).$$

Taking conditional expectations, if $\gamma_2 < \infty$ we obtain

$$E(\exp(itX_1)|\mathcal{G}) = 1 - \frac{t^2}{2}\gamma_2 + o(|t|^2\gamma_2).$$

By de Finnetti's Theorem, given the tail σ-field $\mathcal{G}$, $X_1, \ldots, X_n$ are iid random variables, then

$$E(\exp(itZ_n)|\mathcal{G}) = \left[E(\exp(itX_1/\sqrt{n})|\mathcal{G})\right]^n = \left[1 - \frac{t^2}{2n}\gamma_2 + o\left(\frac{|t|^2}{n}\gamma_2\right)\right]^n$$

and therefore

$$E(\exp(itZ_n)|\mathcal{G}) \to \exp\left(-\frac{t^2}{2}\gamma_2\right)$$

almost surely. Notice that, for any $t \in \mathbb{R}$, we have

$$\xi_n(t) = E(\exp(itZ_n)) = E(E(\exp(itZ_n)|\mathcal{G})) \to E\left(\exp\left(-\frac{t^2}{2}\gamma_2\right)\right).$$

This implies that $Z_n \xrightarrow{d} \sqrt{\gamma_2}Z$, where $Z \sim \mathcal{N}(0,1)$ and $\gamma_2 = E(X_1^2|\mathcal{G})$. This proves (4).

In order to prove (5), we consider that $\gamma_3 < \infty$. Then,

$$E(\exp(itX_1)|\mathcal{G}) = 1 - \frac{t^2}{2}\gamma_2 + O(|t|^3\gamma_3)$$

and

$$E(\exp(itZ_n)|\mathcal{G}) = \exp\left(-\frac{t^2}{2}\gamma_2 + O\left(\frac{|t|^3}{\sqrt{n}}\gamma_3\right)\right).$$

By Taylor's expansion we arrive at

$$\xi_n(t) = E\left(\exp\left(-\frac{t^2}{2}\gamma_2\right)\left(1 + O\left(\frac{|t|^3}{\sqrt{n}}\gamma_3\right)\right)\right).$$

Then,

$$\xi_n(t) = \xi(t) + E\left(O\left(\frac{|t|^3}{\sqrt{n}}\gamma_3 \exp(-t^2\gamma_2/2)\right)\right),$$

where $\xi(t) = E(\exp(it\gamma_2 Z))$.

By Lemma 11.4.2 of Athreya and Lahiri (2006), we have, for some constant $C_0 \in (0,\infty)$, that

$$\sup_{x\in\mathbb{R}} \left|P(Z_n < x) - P(\sqrt{\gamma_2}Z < x)\right| \leq \frac{1}{\pi}\int_{-n}^{n} \frac{|\xi_n(t) - \xi(t)|}{|t|}dt + \frac{24C_0}{\pi n}. \tag{7}$$

Notice that

$$\frac{\gamma_3}{\pi}\int_{-n}^{n} t^2 \exp(-t^2\gamma_2/2)dt \leq \sqrt{\frac{1}{2\pi}}\left(\frac{\gamma_3}{\gamma_2^{3/2}} - \frac{2n\gamma_3}{\gamma_2}\exp\left(-\frac{n^2}{2}\gamma_2\right)\right).$$

Then, plugging this result into (7), by Fubini's Theorem, we have that

$$\sup_{x\in\mathbb{R}} \left|P(Z_n < x) - P(\sqrt{\gamma_2}Z < x)\right| \leq O\left(\frac{\kappa}{\sqrt{n}}\right),$$

where $\kappa = E(\gamma_3\gamma_2^{-3/2})$.

If $\gamma_4 < \infty$, by Chebychev's inequality, we have that

$$P(|s_n^2 - \gamma_2| > \epsilon) < \frac{E[(s_n^2 - \gamma_2)^2]}{\epsilon^2},$$

where

$$E[(s_n^2-\gamma_2)^2] = E\Bigg(E[(s_n^2-\gamma_2)^2\big|\mathcal{G}]\Bigg) = E\Bigg(E(s_n^4\big|\mathcal{G}) - 2\gamma_2 E(s_n^2\big|\mathcal{G}) + \gamma_2^2\Bigg),$$

with $E(s_n^4\big|\mathcal{G}) = \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n E(X_i^2 X_j^2\big|\mathcal{G}) = \frac{1}{n^2}\Bigg(n\gamma_4 + n(n-1)\gamma_2^2\Bigg)$ and $E(s_n^2\big|\mathcal{G}) = \gamma_2$. Then,

$$E[(s_n^2-\gamma_2)^2] = \frac{E(\gamma_4)}{n} - \frac{1}{n}E(\gamma_2^2)$$

and, as $E(\gamma_2^2) < \infty$,

$$\lim_{n\to\infty} P(|s_n^2-\gamma_2| > \epsilon) \to 0.$$

We conclude that $s_n^2 \xrightarrow{P} \gamma_2$. By Slutsky's device, we have that $\frac{Z_n}{s_n} \xrightarrow{d} Z$, where $Z \sim \mathcal{N}(0,1)$. Hence, (6) is demonstrated.

References

Athreya, KB, Lahiri, SN (2006). *Measure Theory and Probability Theory*, Springer Verlag.

Balakrishnan, N, Leiva-Sánchez, V, Sanhueza, A, Filidor-Vilca, L (2009). Estimation in the Birnbaum-Saunders distribution based on scale-mixture of normals and the EM-algorithm, *Test*, **33**, 171–192.

Birnbaum, ZW, Saunders, SC (1969). A new family of life distributions, *Journal of Applied probability*, **6**, 319–327.

Blum, JR, Chernoff, H, Rosenblatt, M, Teicher, H (1958). Central limit theorems for interchangeable processes, *Canadian Journal of Mathematics*, **10**, 222–229.

De Finetti, B. (1937). La Prévision, ses lois logiques, ses sources subjectives. *Ann. Inst. H. Poincaré*, **7**, 1–68.

Desmond, A (1985). Stochastic models of failure in random environments, *The Canadian Journal of Statistics*, **13**, 171–183.

Días-Garcia, JA, Leiva-Sánchez, V (2005). A new family of life distributions based on the elliptically contoured distributions, *Journal of Statistical Planning and Inference*, **128**, 445–457.

Eltoft, T, Kim, T, Lee T-W (2006). Multivariate scale mixture of gaussians modeling, *Lecture Notes in Computer Science*, **3889**, 799–806.

Gnedenko, BV, Kolmogorov, AN (1954). *Limit distributions for sums of independent random variables*, Addison-Wesley, Reading.

Hewitt, E, Savage, LJ (1955) Symmetric measures on cartesian products, *Transactions of the American Mathematical Society*, **80**, 470–501.

Jiang, X, Hahn, MG (2003). Central limit theorems for exchangeable random variables when limits are scale mixtures of normals, *Journal of Theoretical Probability*, **76**, 543–571.

Leiva-Sánchez, Riquelme, M, V, Balakrishnan, N, Sanhueza, A (2008). Lifetime analysis based on the generalized Birnbaum-Saunders distribution, *Computational Statistics & Data Analysis*, **52**, 2079–2097.

Meerschaert, MM, Scheffler, HP (2001). *Limit distributions for sums of independent random vectors, heavy tails in theory and practice*, John Wiley and Sons, Inc.

Owen, WJ (2006). A new three-parameter extension to the Birnbaum-Saunders distribution, *IEEE Transactions on Reliability*, **55**, 475–479.

Sanhueza, A, Leiva-Sánchez, V, Balakrishnan, N (2008). The generalized Birnbaum-Saunders distribution and its theory, methodology, and application, *Communications in Statistics - Theory and Methods*, **37**, 645–670.