Sibuya´s Measure of Local Dependence

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ABSTRACT

Common measures of association quantify the global association between two variables, however, the data may have different behaviours of association or dependence if we consider subsets of the data. The aim of this paper is to study the function of local dependence of Sibuya (1960) for two continuous random variables. We rewrite this function in terms of copula. We propose three nonparametric estimators, deriving their respective properties. Simulations and applications to real data are also given.

RESUMEN

Medidas de asociación comunes cuantifican la asociación geral entre dos variables, sin embargo, los datos pueden tener diferentes comportamientos de asociación o dependencia si se tiene en cuenta subconjuntos de los datos. El objetivo de este trabajo es estudiar la función de la dependencia local de Sibuya (1960) para dos variables aleatorias continuas. Estamos reescribir esta función en términos de la cópula. Proponemos tres estimadores no paramétrico, que se derivan de sus respectivas propiedades. Simulaciones y aplicaciones a datos reales también se dan.

Key words: local dependence, copulas, nonparametric estimation

1 Introduction

Some measures like Pearson´s correlation coefficient, Kendall´s tau and Spearman´s rho, measure the global association between two variables. However, the data may have different behaviours of association if we consider subsets of this data. In order to detect local behaviours in the data, local measures were proposed for two variables, such as the function of local dependence of Holland and Wang (1987), the curve of correlation of Bjerve and Doksum (1993) and the measure of local dependence of Bairamov et al. (2003). Also, other proposals to measure the local dependence have been suggested by Nelsen (1999) and Sibuya (1960), for which Kolev et al. (2007) developed additional properties.

The measure of local dependence between two variables X e Y proposed by Bairamov et al. (2003) is given by the expression

$$H(x,y)=\frac{E\left[(X-E[X/Y=y])(Y-E[Y/X=x])\right]}{\sqrt{E[(X-E[X/Y=y])^2]}\sqrt{E[(Y-E[Y/X=x])^2]}},\quad \forall(x,y)\in S,$$

which refers to the known Pearson correlation coefficient where $E[X]$ and $E[Y]$ was replaced by $E[X/Y=y]$ and $E[Y/X=x]$, respectivelly. Thus, this measure can be interpreted as a local dependence between X and Y at the point (x,y). Here, S denote the support of (X,Y). An alternative representation of the above expression is given by

$$H(x,y)=\frac{\rho_{XY}+\varphi_X(y)\varphi_Y(x)}{\sqrt{1+\varphi_X^2(y)}\sqrt{1+\varphi_Y^2(x)}},$$

where $\varphi_X(y)=\frac{E[X]-E[X/Y=y]}{\sqrt{Var[X]}}$, $\varphi_Y(x)=\frac{E[Y]-E[Y/X=x]}{\sqrt{Var[Y]}}$ and $\rho_{XY}=\frac{Cov(X,Y)}{\sqrt{Var[X]}\sqrt{Var[Y]}}$.

Bjerve and Doksum (1993) introduced the correlation curve between X and Y which is given by

$$\rho(x)=\frac{\beta(x)\sigma_X}{\sqrt{\beta^2(x)\,\sigma_X^2+\sigma_\varepsilon^2(x)}}\ ,\quad \forall x\in S,$$

where $\beta(x)=\frac{\partial}{\partial x}E[Y/X=x]$ and $\sigma_\varepsilon^2(x)=Var[Y/X=x]$, assuming that σ_X^2 and $\sigma_\varepsilon^2(x)$ exist, $\forall x\in S$, where S is the domain of X. This local measure is a generalization of the Pearson correlation coefficient for usual linear models which can be written by $\rho=\beta\sigma_X \Big/ \{\beta^2\sigma_X^2+\sigma_\varepsilon^2\}^{1/2}$. Thus, this local measure is appropriate for nonlinear models because it considers nonconstant mean and variance. However, it is not symmetric in X and Y.

The expression of the function of local dependence proposed by Holland and Wang (1987) is

$$\gamma(x,y)=\frac{\partial^2}{\partial x\partial y}\log f(x,y)=\frac{1}{f(x,y)}\left(f^{11}(x,y)-\frac{f^{10}(x,y)f^{01}(x,y)}{f(x,y)}\right)$$

where f is the bivariate density function of (X,Y) such that the mixed partial derivatives $f^{ij}(x,y)$ exist and f is defined in a Cartesian product set. The motivation of these authors was the extension of an $r\times c$ contingence table for two discrete variables to the case of two continuous variables, taking it simply partitions in a thin rectangular grid. Then, this measure are motivated from considering continuous density functions as contingency tables. These authors observe that this measure not considers the marginal distributions.

The aim of this paper is to study the function of Sibuya (1960) for two continuous random variables. To this end, we rewrite this function in terms of copula, discussing their properties. Also, we propose one empirical estimator and two kernel smoothed estimators, checking their convergence in probability and in distribution. The behaviour of this local measure and their smoothed estimators were evaluated through simulations and applications to real data (in this case, we made a comparison with copula and copula density).

Aiming to extend the notion of extreme statistics from the univariate case to the bivariate case, Sibuya (1960) proposed a function of dependence between two continuous random variables, which relates the joint distribution function with their marginal distribution functions.

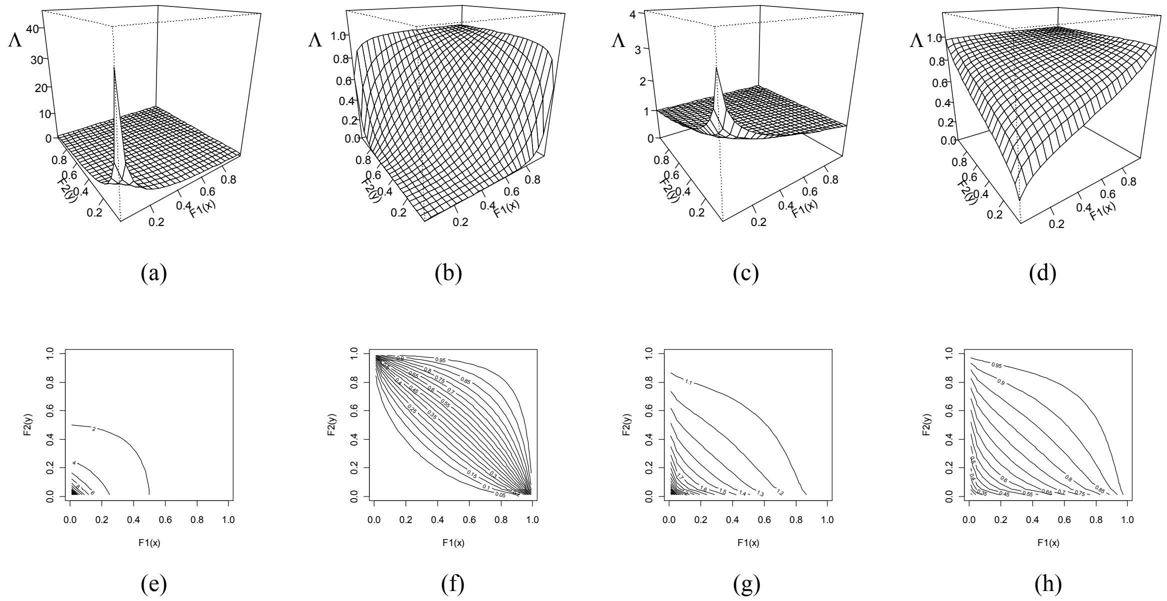
Let X and Y be continuous random variables defined in a probability space $(\Omega,\mathcal{A},\mathcal{P})$. Let $F(x,y)=P[X\le x,Y\le y]$ be a joint distribution of X and Y, and $F_1(x)=P[X\le x]$ and $F_2(y)=P[Y\le y]$ their respectives marginal distributions. Then, the dependence function of Sibuya, $\Lambda=\Lambda(F_1(x),F_2(y))$, is given by

$$\Lambda(F_1(x),F_2(y))=\frac{F(x,y)}{F_1(x)F_2(y)}\ ,\ \forall(x,y)\in\Re^2, \tag{1}$$

where $F_1(x) > 0$ and $F_2(y) > 0$. For $F_1(x) = 0$ or $F_2(y) = 0$, then $\Lambda(F_1(x), F_2(y))$ may be defined if the limit $\lim_{F_1(x)\to 0} F(x,y)/(F_1(x)F_2(y))$ or $\lim_{F_2(y)\to 0} (F(x,y)/F_1(x)F_2(y))$ exists, respectively.

The behaviour of this measure of local dependence for a bivariate random vector with standardized normal distribution and correlation coefficient +0,80 , -0,80 , +0,20 and –0,20 are shown in Figure 1. The graphics of this figure, whose scales are not always the same, have bivariate grid with 99% of the central distribution of each marginal.

Figure 1 – Graphics of perspective and corresponding graphics of level curves of the function of local dependence of Sibuya given by (1) for (*X*,*Y*) with standardized normal distribution and correlation coefficient ρ = +0,80 in (a) and (e), ρ = -0,80 in (b) and (f), ρ = +0,20 in (c) and (g), and ρ = -0,20 in (d) and (h).



The properties of Λ defined by Sibuya (1960) are given below. See also Kolev et al. (2007).

(i) $máx\left(0, \dfrac{F_1(x)+F_2(y)-1}{F_1(x)F_2(y)}\right) \le \Lambda(F_1(x), F_2(y)) \le mín\left(\dfrac{1}{F_1(x)}, \dfrac{1}{F_2(y)}\right), \ \forall (x,y) \in \Re^2$;

(ii) $\Lambda(F_1(x), F_2(y)) = 1, \ \forall (x,y) \in \Re^2$, if and only if X and Y are independent;

(iii) If X and Y are *PQD* (*NQD*) [(1)], then $\Lambda(F_1(X), F_2(Y)) \ge 1$ (≤ 1) almost shure;

(iv) Consider the arbitrary function $\varphi(.)$ and $\psi(.)$ of X and Y, respectively, such that $\varphi^{-1}(.)$ and $\psi^{-1}(.)$ exist. Under the notation $\overline{F}_1(x) = 1 - F_1(x)$, $\overline{F}_2(y) = 1 - F_2(y)$, $F_{\varphi(X)}(x) = P[\varphi(X) \le x]$, $F_{\psi(Y)}(y) = P[\psi(Y) \le y]$, $\Lambda_{\varphi(X)\psi(Y)}(x,y) = \Lambda(F_{\varphi(X)}(x), F_{\psi(Y)}(y))$ and $\Lambda_{XY}(\varphi^{-1}(x), \psi^{-1}(y)) = \Lambda(F_1(\varphi^{-1}(x)), F_2(\psi^{-1}(y)))$, and S being the support of (X,Y), then:

(a) If $\varphi(.)$ and $\psi(.)$ are increasing functions on the support of X and Y, respectively, then

$$\Lambda_{\varphi(X)\psi(Y)}(x,y) = \Lambda_{XY}(\varphi^{-1}(x), \psi^{-1}(y)), \ \forall (x,y) \in S ;$$

(b) If $\varphi(.)$ is a decreasing function on the support of X and $\psi(.)$ is an increasing function on the support of Y, then

$$\Lambda_{\varphi(X)\psi(Y)}(x,y)=\frac{1}{\overline{F}_1(\varphi^{-1}(x))}-\frac{F_1(\varphi^{-1}(x))}{\overline{F}_1(\varphi^{-1}(x))}\Lambda_{XY}(\varphi^{-1}(x),\psi^{-1}(y)),\ \forall(x,y)\in S\ ;$$

(c) If $\varphi(.)$ is an increasing function on the support of X, and $\psi(.)$ is a decreasing function on the support of Y, then

$$\Lambda_{\varphi(X)\psi(Y)}(x,y)=\frac{1}{\overline{F}_2(\psi^{-1}(y))}-\frac{F_2(\psi^{-1}(y))}{\overline{F}_2(\psi^{-1}(y))}\Lambda_{XY}(\varphi^{-1}(x),\psi^{-1}(y)),\ \forall(x,y)\in S\ ;$$

(d) If $\varphi(.)$ and $\psi(.)$ are decreasing functions on the support of X and Y, respectively, then

$$\Lambda_{\varphi(X)\psi(Y)}(x,y)=\frac{1-F_1(\varphi^{-1}(x))-F_2(\psi^{-1}(y))}{\overline{F}_1(\varphi^{-1}(x))\overline{F}_2(\psi^{-1}(y))}+\frac{F_1(\varphi^{-1}(x))F_2(\psi^{-1}(y))}{\overline{F}_1(\varphi^{-1}(x))\overline{F}_2(\psi^{-1}(y))}\Lambda_{XY}(\varphi^{-1}(x),\psi^{-1}(y)),$$

$\forall(x,y)\in S$;

(v) If $\rho_{XY}=0\ (>0,\ <0)$ then $\Lambda(F_1(X),F_2(Y))=1\ (>1,\ <1)$, in bivariate normal case.

By property (iv), we see that at the point (x,y), Λ is not invariant under strictly monotone transformations since it depend of the functions $\varphi^{-1}(.)$ and $\psi^{-1}(.)$, among possible others.

Also, Kolev et al. (2007) established to Λ :

- exact form of the cumulative distribution of Λ for the cases where X and Y are countermonotonic, independent or comonotonic, and also derived the relation between these distributions;
- lower bounds for expectation of Λ.

We observe that the Sibuya´s dependence function Λ given by (1) can also be write through copula.

Copulas are functions that couple the joint distributions with their respectives univariate marginal distributions, or copulas are joint distributions whose univariate marginal distributions are uniform in the interval $[0,1]$. From now on let us consider the bivariate case. Then a copula denoted by C is a function $C:[0,1]^2\to[0,1]$ that satisfy some properties (see Nelsen, 1999).

By Sklar´s theorem (1959), there exist a copula C such that $F(x,y)=C(F_1(x),F_2(y))$, $\forall(x,y)\in\Re^2$, and this is unique if F_1 and F_2 are continuous. An immediate corollary of this theorem shows that $C(u,v)=F(F_1^{-1}(u),F_2^{-1}(v))$, $\forall(u,v)\in[0,1]^2$, where $F_1^{-1}(u)=\inf\{x\in\Re:F_1(x)\geq u\}\equiv\varsigma_x$ and $F_2^{-1}(v)=\inf\{y\in\Re:F_2(y)\geq v\}\equiv\varsigma_y$ are the generalized inverse of F_1 and F_2, respectively. If F_i is strictly increasing, then its generalized inverse is the ordinary inverse, $i=1,2$. We assume that $F_i,i=1,2$, are such that $F_1(x)=u$ and $F_2(y)=v$, $u,v\in[0,1]$, admit a unique solution denoted by ς_x and ς_y, respectively.

Then, rewriting (1) using copula, we have

$$\Lambda(u,v)=\frac{C(u,v)}{uv},\ \forall(u,v)\in(0,1]^2, \quad (2)$$

whose behaviour is equivalent of that shown in Figure 1.

The properties (i) to (v) of Λ given by (1), remain valid for Λ given by (2), with necessary adjustments. Considering (2), we observe that $\Lambda_{\varphi(X)\psi(Y)}(u,v)=\Lambda_{XY}(u,v)$ if $\varphi(.)$ and $\psi(.)$ are strictly increasing.

2 Estimators and their properties

To estimate Λ given by (1), Kolev et al. (2007) suggest the empirical estimator Λ_n, which is built by plug-in method using empirical estimators of the joint and marginal distributions. To get a better estimator for small samples, we suggest a smoothed estimator $\hat{\Lambda}$ for (1), which consider the kernel smoothed estimators for the joint and marginal distributions. Also, we derive an empirical estimator and a kernel smoothed estimator for Λ given by (2).

Let $\{(X_i,Y_i),\ i=1,\ldots,n\}$ be a random sample from (X,Y). To build our estimator $\hat{\Lambda}$, we will consider the bivariate kernel of simmetrical type (or product) and bandwidths $\mathbf{H}=diag(h_1^2,h_2^2)$.

Then, let $k:\Re^2\to\Re$ be a bivariate kernel function such that $\int\int k(u,v)dudv=1$ and $k(x,y;h_1,h_2)=\frac{1}{h_1h_2}k\left(\frac{x}{h_1},\frac{y}{h_2}\right)$, with $h_j>0$, $j=1,2$, being functions of n such that $h_j\to 0$ as $n\to\infty$. Then, the estimator of joint density $f(x,y)$ is given by $\hat{f}(x,y)=\frac{1}{n}\sum_{i=1}^{n}k(x-X_i,y-Y_i;h_1,h_2)=\frac{1}{nh_1h_2}\sum_{i=1}^{n}k\left(\frac{x-X_i}{h_1},\frac{y-Y_i}{h_2}\right)$. Consider the primitive function of $k(x,y)$ given by $K(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y}k(u,v)dudv$ with $K(x,y;h_1,h_2)=K\left(\frac{x}{h_1},\frac{y}{h_2}\right)$. Then, the smoothed estimator of the joint distribution $F(x,y)$ is given by $\hat{F}(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y}\hat{f}(u,v)dudv=\frac{1}{n}\sum_{i=1}^{n}K(x-X_i,y-Y_i;h_1,h_2)=\frac{1}{n}\sum_{i=1}^{n}K\left(\frac{x-X_i}{h_1},\frac{y-Y_i}{h_2}\right)$.

Let $k_j(.)$, $j=1,2$, be real bounded symmetric functions, such that $\int k_j(u)du=1$ and $k_j(u;h_j)=\frac{1}{h_j}k_j\left(\frac{u}{h_j}\right)$. Then, the estimators of the marginal densities $f_1(x)$ and $f_2(y)$ are given respectivelly by $\hat{f}_1(x)=\frac{1}{n}\sum_{i=1}^{n}k_1(x-X_i;h_1)=\frac{1}{nh_1}\sum_{i=1}^{n}k_1\left(\frac{x-X_i}{h_1}\right)$ and $\hat{f}_2(y)=\frac{1}{nh_2}\sum_{i=1}^{n}k_2\left(\frac{y-Y_i}{h_2}\right)$. Now, considering the primitive functions $K_j(.)$ of $k_j(.)$, $j=1,2$, given by $K_j(w)=\int_{-\infty}^{w}k_j(u)du$ with $K_j(w;h_j)=K_j\left(\frac{w}{h_j}\right)$, then the smoothed estimators of marginal distributions $F_1(x)$ and $F_2(y)$ are given respectivelly by $\hat{F}_1(x) = \int_{-\infty}^{x} \hat{f}_1(u)du = \frac{1}{n}\sum_{i=1}^{n} K_1(x - X_i; h_1) = \frac{1}{n}\sum_{i=1}^{n} K_1\left(\frac{x - X_i}{h_1}\right)$ and $\hat{F}_2(y) = \frac{1}{n}\sum_{i=1}^{n} K_2\left(\frac{y - Y_i}{h_2}\right)$.

Therefore, a smoothed estimator $\hat{\Lambda}$ of Λ given by (1) is

$$\hat{\Lambda}(\hat{F}_1(x), \hat{F}_2(y)) = \frac{\hat{F}(x, y)}{\hat{F}_1(x)\hat{F}_2(y)}, \ \forall (x, y) \in \Re^2, \text{ with } \hat{F}_1(x), \hat{F}_2(y) > 0, \quad (3)$$

where $\hat{F}(.,.)$, $\hat{F}_1(.)$ and $\hat{F}_2(.)$ are as before.

Now, we derive two nonparametric estimators for Λ given by (2).

Let the empirical estimators of the joint and marginal distributions of X and Y be $F_n(x, y) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{I}(X_i \le x, Y_i \le y)$, $F_{1n}(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x)$ (or $F_{1n} = F_n(x, +\infty)$) and $F_{2n}(y) = \frac{1}{n}\sum_{i=1}^{n} I(Y_i \le y)$ (or $F_{2n} = F_n(+\infty, y)$), respectively. Then according to Fermanian et al. (2004), using the plug-in method, we have that $C_n(u, v) = F_n(F_{1n}^{-1}(u), F_{2n}^{-1}(v))$, $\forall (u, v) \in [0,1]^2$, is the empirical copula. Then, an empirical estimator for (2) is given by

$$\Lambda_n(u, v) = \frac{C_n(u, v)}{uv}, \ \forall (u, v) \in (0,1]^2, \quad (4)$$

where $F_{1n}^{-1}(u)$ and $F_{2n}^{-1}(v)$ are the empirical quantiles.

But, to obtain better results, we propose the use of a kernel smoothed estimator which is built by plug-in method. Let $\hat{F}$, $\hat{F}_1$ and $\hat{F}_2$ as before. Then, according to Fermanian et al. (2004), the smoothed estimator for the copula is given by $\hat{C}(u, v) = \hat{F}(\hat{F}_1^{-1}(u), \hat{F}_2^{-1}(v))$ where $\hat{F}_1^{-1}(u) = \inf\{x : \hat{F}_1(x) \ge u\}$ and $\hat{F}_2^{-1}(v) = \inf\{y : \hat{F}_2(y) \ge v\}$. As a consequence, the corresponding smoothed estimator for (2) results in

$$\hat{\Lambda}(u, v) = \frac{\hat{C}(u, v)}{uv}, \ \forall (u, v) \in (0,1]^2. \quad (5)$$

In what follows, we will get the properties of estimators (3), (4) and (5).

Consider the following regularity conditions.

(C1) F has a bounded qth derivative;
(C2) F_1 and F_2 are Lipschitz;
(C3) $h \to 0$, as $n \to \infty$;
(C4) $\lim_{n \to \infty} \sqrt{n} h^q = 0$;
(C5) $\int_{\Re}\int_{\Re} x^j y^l k(x, y) dxdy = 0$, $1 \le j + l < q$;
(C6) $\int_{\Re}\int_{\Re} |x|^j |y|^l |k(x, y)| dxdy < \infty$, $1 \le j + l \le q$;

(C7) $\int_{\Re}\int_{\Re}(|x|+|y|)dK(x,y)<\infty$.

Theorem 1. Let (X,Y) be a continuous random vector. Under assumptions (C1) to (C7), then

$$\hat{\Lambda}(\hat{F}_1(x),\hat{F}_2(y)) \xrightarrow[n\to\infty]{P} \Lambda(F_1(x),F_2(y)), \text{ for each fixed } (x,y)\in\Re^2,$$

with $\hat{F}_1(x),\hat{F}_2(y)>0$ and $F_1(x),F_2(y)>0$.

Proof. Assuming the conditions (C1) and (C4) to (C6) valid, then the Lemma 8 of Fermanian et al. (2004) applies, which results in one of the conditions of Lemma 7 of these authors. Now, also consider valid the conditions (C2), (C3) and (C7), then the referred Lemma 7 is valid, ie, the smoothed process $\{\sqrt{n}(\hat{F}(x,y)-F(x,y)),\ x,y\in\Re\}$ converges weakly to a tight Brownian bridge in $D(\Re^2)$, as $n\to\infty$, where $D(\Re^2)$ is the space of functions in $\Re^2$ that are right continuous and have left limits. Therefore, by Theorem 2.3.4. of Lehmann (1999), $\hat{F}(x,y)\xrightarrow{P}F(x,y)$ as $n\to\infty$. Set $\hat{F}_1(x)=\hat{F}(x,\infty)$, $\hat{F}_2(y)=\hat{F}(\infty,y)$, $F_1(x)=F(x,\infty)$ and $F_2(y)=F(\infty,y)$, then by Theorem 5.1.4 of Fuller, we have that $\hat{F}_1(x)\xrightarrow{P}F_1(x)$ and $\hat{F}_2(y)\xrightarrow{P}F(y)$, as $n\to\infty$.
Now, consider $\hat{\boldsymbol{\theta}}=(\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3)'=(\hat{F}(x,y),\hat{F}_1(x),\hat{F}_2(y))'$, $\boldsymbol{\theta}=(\theta_1,\theta_2,\theta_3)'=(F(x,y),F_1(x),F_2(y))'$ and $g(\mathbf{w}):\Re^3\to\Re$ a continuous function defined by $g(\mathbf{w})=w_1/(w_2w_3)$ with $\mathbf{w}=(w_1,w_2,w_3)'$ and $w_2,w_3>0$. We saw that $\hat{\theta}_i\xrightarrow{P}\theta_i$, $i=1,2,3$, as $n\to\infty$. Then, by Lemma 5.1.3 of Fuller (1976), $\hat{\boldsymbol{\theta}}\xrightarrow{P}\boldsymbol{\theta}$, and by Theorem 5.1.4 of Fuller, $g(\hat{\boldsymbol{\theta}})\xrightarrow{P}g(\boldsymbol{\theta})$, ie, $\hat{F}(x,y)/(\hat{F}_1(x)\hat{F}_2(y))\xrightarrow{P}F(x,y)/(F_1(x)F_2(y))$. Therefore, as $n\to\infty$, we have that $\hat{\Lambda}(\hat{F}_1(x),\hat{F}_2(y))\xrightarrow{P}\Lambda(F_1(x),F_2(y))$, for each fixed $(x,y)\in\Re^2$.

Theorem 2. Let (X,Y) be a continuous random vector. Suppose that F has continuous marginal distribution functions and that the copula function $C(u,v)$ has continuous partial derivatives. Then

$$\Lambda_n(u,v)\xrightarrow[n\to\infty]{P}\Lambda(u,v), \text{ for each fixed } (u,v)\in(0,1]^2 .$$

Proof. Assuming that F has continuous marginal distribution functions and that the copula function $C(u,v)$ has continuous partial derivatives, then by Theorem 3 of Fermanian et al. (2004), the empirical copula process $\{Z_n(u,v)\equiv\sqrt{n}(C_n(u,v)-C(u,v)),\ 0<u,v\le1\}$ converges weakly to the Gaussian process $\{G_C(u,v),\ 0<u,v\le1\}$ in $l^{\infty}((0,1]^2)$, which is the space of almost shure bounded functions in $(0,1]^2$ with the supremum norm. Then, by Theorem 2.3.4 of Lehmann (1999), if $n\to\infty$ then $C_n(u,v)\xrightarrow{P}C(u,v)$, for each fixed $(u,v)\in(0,1]^2$. Because a function of this estimator also converge in probability, the result follows.

Theorem 3. Let (X,Y) be a continuous random vector. Suppose that F has continuous marginal distribution functions and that the copula function $C(u,v)$ has continuous partial derivatives. Then

$$\{W_n(u,v) \equiv \sqrt{n}(\Lambda_n(u,v) - \Lambda(u,v)), 0 < u, v \le 1\}$$

converges weakly to the centered Gaussian process $\{G_\Lambda(u,v), 0 < u, v \le 1\}$ with variance $\{C(u,v) - C^2(u,v) + \{\partial_1 C(u,v)\}^2 u(1-u) + \{\partial_2 C(u,v)\}^2 v(1-v) - 2\{\partial_1 C(u,v)\}C(u,v)(1-u) - 2\{\partial_2 C(u,v)\}C(u,v)(1-v) + 2\{\partial_1 C(u,v)\}\{\partial_2 C(u,v)\}\{C(u,v) - uv\}\} \times \{\partial C(u,v)/uv\}^2$, in $l^\infty((0,1]^2)$, for each fixed $(u,v) \in (0,1]^2$.

Note: ∂, ∂_1 and ∂_2 represent the derivative with respect to C, u and v, respectively.

Proof. The function $\Lambda(u,v) = C(u,v)/uv$ can be represented by the mapping $C \mapsto (C, C_1, C_2) \mapsto C/(C_1 C_2)$ where $C_1 = C(u,1) = u$ and $C_2 = C(1,v) = v$. The first map is linear and continuous and then Hadamard differenciable (see the proof of Lemma 3.9.28 in van der Vaart and Wellner, 1996), and the second map is Hadamard differenciable on the domain of functions that are away from zero (see section 3.9.4.3 in van der Vaart and Wellner). Then, by the chain rule (Lemma 3.9.3 in van der Vaart and Wellner), the map $\phi: D((0,1]^2) \to l^\infty((0,1]^2)$ where $\phi(C)(u,v) = C(u,v)/\{C_1(u,v)C_2(u,v)\} = C(u,v)/uv$ (or $C \mapsto C/C_1 C_2$) is Hadamard differenciable at C tangentially to $C((0,1]^2)$.

Assuming that F has continuous marginal distribution functions and that the copula function $C(u,v)$ has continuous partial derivatives, then the Theorem 3 of Fermanian et al. is valid (see the proof of theorem 2 above), where G_C takes its values in $C([0,1]^2)$. Also, these authors (page 851) show that the limited Gaussian process can be written as $G_C(u,v) = B_C(u,v) - \partial_1 C(u,v) B_C(u,1) - \partial_2 C(u,v) B_C(1,v)$, where B_C is a Brownian bridge in $[0,1]^2$ with covariance function $E[B_C(u,v)B_C(u',v')] = C(u \wedge u', v \wedge v') - C(u,v)C(u',v')$, for each $0 \le u, u', v, v' \le 1$. Then, $E[G_C(u,v)] = 0$ and $Var[G_C(u,v)] = C(u,v) - C^2(u,v) + \{\partial_1 C(u,v)\}^2 u(1-u) + \{\partial_2 C(u,v)\}^2 v(1-v) - 2\{\partial_1 C(u,v)\}C(u,v)(1-u) - 2\{\partial_2 C(u,v)\}C(u,v)(1-v) + 2\{\partial_1 C(u,v)\}\{\partial_2 C(u,v)\}\{C(u,v) - uv\}$.

By the delta method (Theorem 3.9.4 of van der Vaart and Wellner) the empirical process $\{W_n(u,v), 0 < u, v \le 1\}$ of Λ converges weakly to the centered Gaussian process G_Λ with variance given above.

Theorem 4. Let (X,Y) be a continuous random vector. Under assumptions (C1) to (C7), and supposing that C has continuous partial derivatives, then

$$\hat{\Lambda}(u,v) \xrightarrow[n\to\infty]{P} \Lambda(u,v)$$, for each fixed $(u,v) \in (0,1]^2$.

Proof. Assuming the conditions (C1) to (C7) satisfied, with C having continuous partial derivatives, then the Theorem 10 of Fermanian et al. (2004) is valid, ie, the smoothed copula process $\{\hat{Z}(u,v) \equiv \sqrt{n}(\hat{C}(u,v) - C(u,v)),\ 0 \le u, v \le 1\}$ converges weakly to a Gaussian process $\{G_C(u,v),\ 0 \le u, v \le 1\}$ in $l^\infty([0,1]^2)$. Because $\hat{C}(u,v) \xrightarrow{P} C(u,v)$, as $n \to \infty$, then by Theorem 5.1.4 of Fuller we have that $\hat{\Lambda}(u,v) \xrightarrow{P} \Lambda(u,v)$, for each fixed $(u,v) \in (0,1]^2$.

Theorem 5. Let (X,Y) be a continuous random vector. Under assumptions (C1) to (C7), and supposing that C has continuous partial derivatives, then

$$\{\hat{W}(u,v) \equiv \sqrt{n}(\hat{\Lambda}(u,v) - \Lambda(u,v)),\ 0 < u, v \le 1\}$$

converges weakly to the centered Gaussian process $\{G_{\Lambda}(u,v),\ 0 < u,v \le 1\}$ with variance $\{C(u,v) - C^2(u,v) + \{\partial_1 C(u,v)\}^2 u(1-u) + \{\partial_2 C(u,v) v(1-v)\} - 2\{\partial_1 C(u,v)\} C(u,v)(1-u) - 2\{\partial_2 C(u,v)\}$ $C(u,v)(1-v) + 2\{\partial_1 C(u,v)\}\{\partial_2 C(u,v)\}\{C(u,v) - uv\}\} \times \{\partial C(u,v)/uv\}^2$ in $l^{\infty}([0,1]^2)$, for each fixed $(u,v) \in (0,1]^2$.

Proof. Assuming (C1) to (C7) valid, with C having continuous partial derivatives, then the Theorem 10 of Fermanian et al. is valid (see proof of Theorem 4 above). Also, in the proof of Theorem 3, we derived $E[G_C(u,v)]$ and $Var[G_C(u,v)]$. Because Λ is differentiable at C, then by the delta method, $\{\hat{W}(u,v) \equiv \sqrt{n}\ (\hat{\Lambda}(u,v) - \Lambda(u,v)),\ 0 < u,v \le 1\}$ converges weakly to a centered Gaussian process $\{G_{\Lambda}(u,v),\ 0 < u,v \le 1\}$ with variance like above, in $l^{\infty}((0,1]^2)$.

3 Simulations

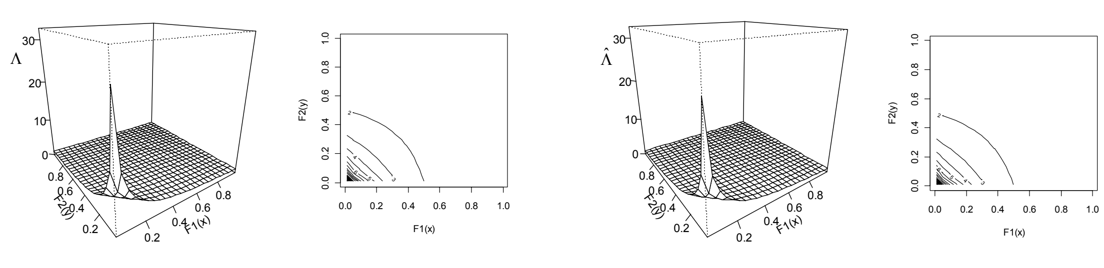
In this section we wish to investigate the sample properties of our kernel estimators.

Initially we consider $\hat{\Lambda}$ given by (3) and then given by (5). For each estimator, we consider 1,000 samples (with different sizes) observed from a bivariate normal random vector with correlation 0.7. We calculated the bias, the mean squared error and the *p*-value of normality test of Jarque Bera in some points of the bivariate grid (25×25 points). In these simulations, we use product of two Gaussian kernel, optimal bandwidth of Hansen (2004) and 99% of the central data.

Let us firstly consider the local measure written in terms of distribution functions.

Considering a random vector (X,Y) with normal distribution with mean $(3.05;6.44)'$, $vec(\mathbf{\Sigma}) = (1.13;1.49;1.49;3.99)'$ and correlation 0.70, we calculate Λ and we estimate $\hat{\Lambda}$ using 1,000 samples of size 2,000 observed from this random vector. See their graphs in Figure 2. The resulting values of usual statistics of the estimator are shown in Table 1 to Table 3.

Figure 2 – (a) Theoretical function of dependence of Sibuya given through distribution functions, and their (b) graphs of level curves, considering a random vector (X,Y) with normal distribution with mean (3.05;6.44)´ and vec(Σ)=(1.13;1.49;1.49;3.99)´ (correlation 0.7). Also, we present the (c) estimated function of dependence of Sibuya and their (d) graph of level curves , using 1,000 random samples of size 2,000.



(a) (b) (c) (d)

This simulation was repeated for samples of sizes 500 and 1,000, whose results are also shown in these tables. Comparing the three simulations, we conclude that the bias, the mean squared error and the rejection of normality decreases with increasing the size of the sample.

Table 1 – Bias for some points of the bivariate grid for the estimated function of dependence of Sibuya obtained through 1,000 random samples of size 500, 1,000 and 2,000, observed from the random vector (X,Y) with normal distribution with mean (3.05;6.44)´ and vec(Σ)=(1.13;1.49;1.49;3.99)´ (correlation 0.7).

n	**Grid**							
		0,01	**0,05**	**0,25**	**0,50**	**0,75**	**0,95**	**0,99**
500	**0,01**	-4,418	-1,382	-0,129	-0,010	0,006	0,004	0,001
	0,05	-1,384	-0,778	-0,159	-0,026	0,004	0,004	0,001
	0,25	-0,126	-0,161	-0,097	-0,035	-0,003	0,003	0,001
	0,50	-0,009	-0,023	-0,035	-0,024	-0,007	0,002	0,001
	0,75	0,006	0,004	-0,004	-0,008	-0,005	0,001	0,001
	0,95	0,004	0,004	0,003	0,002	0,001	0,000	0,000
	0,99	0,001	0,001	0,001	0,001	0,001	0,000	0,000
1,000	**0,01**	-3,112	-0,831	-0,069	-0,001	0,005	0,002	0,001
	0,05	-0,823	-0,411	-0,089	-0,013	0,004	0,002	0,001
	0,25	-0,086	-0,088	-0,060	-0,022	-0,001	0,002	0,001
	0,50	-0,007	-0,013	-0,020	-0,014	-0,004	0,002	0,001
	0,75	0,005	0,004	0,000	-0,004	-0,003	0,001	0,001
	0,95	0,002	0,002	0,002	0,001	0,000	0,000	0,000
	0,99	0,001	0,001	0,001	0,001	0,001	0,000	0,000
2,000	**0,01**	-1,815	-0,526	-0,044	-0,002	0,002	0,002	0,000
	0,05	-0,586	-0,329	-0,055	-0,007	0,001	0,002	0,000
	0,25	-0,038	-0,056	-0,035	-0,014	-0,002	0,001	0,000
	0,50	-0,001	-0,005	-0,013	-0,009	-0,003	0,001	0,000
	0,75	0,003	0,002	-0,001	-0,003	-0,002	0,000	0,000
	0,95	0,001	0,001	0,001	0,001	0,000	0,000	0,000
	0,99	0,000	0,000	0,000	0,000	0,000	0,000	0,000

Table 2 – Mean squared error for some points of the bivariate grid for the estimated function of dependence of Sibuya obtained through 1,000 random samples of size 500, 1,000 and 2,000 observed from the random vector (X,Y) with normal distribution with mean (3.05;6.44)´ and vec(Σ)=(1.13;1.49;1.49;3.99)´ (correlation 0.7).

n	**Grid**							
		0,01	**0,05**	**0,25**	**0,50**	**0,75**	**0,95**	**0,99**
500	**0,01**	207,707	12,681	0,139	0,008	0,001	0,000	0,000
	0,05	11,148	2,152	0,087	0,007	0,001	0,000	0,000
	0,25	0,148	0,095	0,026	0,006	0,001	0,000	0,000
	0,50	0,008	0,007	0,006	0,003	0,001	0,000	0,000
	0,75	0,001	0,001	0,001	0,001	0,000	0,000	0,000
	0,95	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	0,99	0,000	0,000	0,000	0,000	0,000	0,000	0,000
1,000	**0,01**	114,974	6,644	0,069	0,004	0,001	0,000	0,000
	0,05	6,148	1,128	0,046	0,004	0,001	0,000	0,000
	0,25	0,075	0,044	0,013	0,003	0,000	0,000	0,000
	0,50	0,005	0,004	0,003	0,001	0,000	0,000	0,000
	0,75	0,001	0,001	0,001	0,000	0,000	0,000	0,000
	0,95	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	0,99	0,000	0,000	0,000	0,000	0,000	0,000	0,000
2,000	**0,01**	57,217	3,250	0,040	0,002	0,000	0,000	0,000
	0,05	3,587	0,610	0,024	0,002	0,000	0,000	0,000
	0,25	0,039	0,024	0,007	0,002	0,000	0,000	0,000
	0,50	0,002	0,002	0,001	0,001	0,000	0,000	0,000
	0,75	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	0,95	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	0,99	0,000	0,000	0,000	0,000	0,000	0,000	0,000

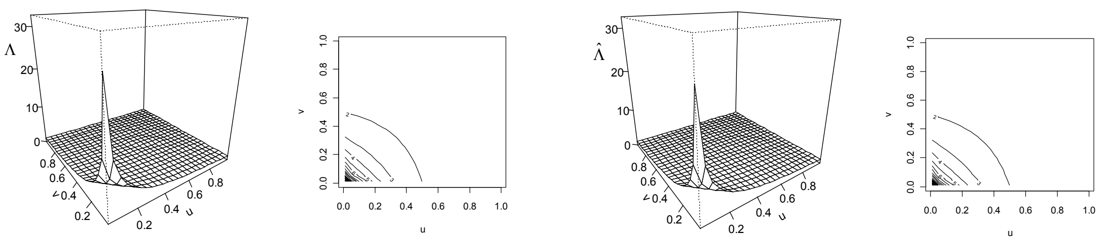
Table 3 – P-value of normality test for some points of the bivariate grid for the estimated function of dependence of Sibuya obtained through 1,000 random samples of size 500, 1,000 and 2,000, observed from the random vector (X,Y) with normal distribution with mean (3.05;6.44)´ and vec(Σ)=(1.13;1.49;1.49;3.99)´ (correlation 0.7).

n		**Grid**						
		0,01	**0,05**	**0,25**	**0,50**	**0,75**	**0,95**	**0,99**
500	**0,01**	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	0,05	0,000	0,000	0,001	0,114	0,165	0,000	0,000
	0,25	0,000	0,000	0,007	0,001	0,002	0,000	0,000
	0,50	0,000	0,006	0,003	0,002	0,001	0,000	0,000
	0,75	0,009	0,653	0,579	0,015	0,001	0,000	0,000
	0,95	0,028	0,021	0,015	0,026	0,001	0,000	0,000
	0,99	0,000	0,000	0,000	0,000	0,000	0,000	0,000
1,000	**0,01**	0,000	0,002	0,000	0,001	0,000	0,022	0,000
	0,05	0,000	0,000	0,857	0,978	0,001	0,024	0,000
	0,25	0,403	0,003	0,098	0,266	0,042	0,038	0,000
	0,50	0,000	0,180	0,046	0,501	0,440	0,019	0,000
	0,75	0,000	0,039	0,200	0,225	0,140	0,001	0,000
	0,95	0,012	0,011	0,021	0,034	0,217	0,004	0,000
	0,99	0,000	0,000	0,000	0,000	0,000	0,000	0,000
2,000	**0,01**	0,000	0,032	0,037	0,000	0,051	0,720	0,269
	0,05	0,000	0,310	0,687	0,027	0,210	0,686	0,262
	0,25	0,002	0,021	0,026	0,168	0,968	0,788	0,246
	0,50	0,000	0,251	0,030	0,876	0,890	0,848	0,291
	0,75	0,000	0,023	0,065	0,896	0,923	0,928	0,171
	0,95	0,671	0,639	0,656	0,701	0,399	0,087	0,130
	0,99	0,094	0,094	0,086	0,113	0,046	0,088	0,000

Next, we consider the measure of local dependence written in terms of copula.

From the random vector $(X,Y) \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with correlation 0.7 considered above, we observed 1,000 samples of size 2,000. Then, we obtain the graphs of the theoretical and estimated functions (Figure 3), and the usual statistics which are presented in Table 4 to Table 6. In these tables we see also the results of simulations with samples of sizes 500 and 1,000, showing that as n increases the bias, mean squared error and the rejection of normality generally decreases, but the last very slowly.

Figure 3 – (a) Theoretical function of dependence of Sibuya given through copula and their (b) graph of level curves, considering a random vector (X,Y) with normal distribution with mean (3.05;6.44)´ and vec(Σ)=(1.13;1.49;1.49;3.99)´ (correlation 0.7). Also, we present the (c) estimated function of dependence of Sibuya and their (d) graph of level curves, considering 1,000 random samples of size 2,000.



(a) (b) (c) (d)

Comparing the behaviour of the two estimators, we see that the estimator using copulas has lower bias, mean squared error and less rejection of normality.

Table 4 – Bias for some points of the bivariate grid for the estimated function of dependence of Sibuya obtained through 1,000 random samples of size 500, 1,000 and 2,000, observed from the random vector (X,Y) with normal distribution with mean (3.05;6.44)´ and vec(Σ)=(1.13;1.49;1.49;3.99)´ (correlation 0.7).

n		Grid						
		0,01	0,05	0,25	0,50	0,75	0,95	0,99
500	0,01	-3,156	-1,003	-0,081	-0,011	-0,003	-0,002	-0,002
	0,05	-0,928	-0,526	-0,106	-0,023	-0,004	-0,001	-0,001
	0,25	-0,079	-0,116	-0,073	-0,031	-0,008	-0,001	0,000
	0,50	-0,011	-0,021	-0,031	-0,023	-0,011	-0,001	0,000
	0,75	-0,003	-0,003	-0,009	-0,011	-0,008	-0,002	0,000
	0,95	-0,002	-0,001	-0,001	-0,001	-0,002	-0,002	-0,001
	0,99	-0,002	-0,001	0,000	0,000	0,000	0,000	0,000
1,000	0,01	-2,444	-0,599	-0,054	-0,007	-0,003	-0,002	-0,002
	0,05	-0,581	-0,267	-0,066	-0,014	-0,002	-0,001	-0,001
	0,25	-0,063	-0,063	-0,048	-0,021	-0,005	0,000	0,000
	0,50	-0,011	-0,015	-0,019	-0,015	-0,007	-0,001	0,000
	0,75	-0,003	-0,003	-0,005	-0,007	-0,006	-0,001	0,000
	0,95	-0,002	-0,001	0,000	-0,001	-0,001	-0,001	0,000
	0,99	-0,002	-0,001	0,000	0,000	0,000	0,000	0,000
2,000	0,01	-1,442	-0,334	-0,035	-0,007	-0,003	-0,002	-0,002
	0,05	-0,389	-0,200	-0,039	-0,009	-0,002	-0,001	-0,001
	0,25	-0,026	-0,038	-0,027	-0,014	-0,004	0,000	0,000
	0,50	-0,006	-0,008	-0,013	-0,010	-0,004	0,000	0,000
	0,75	-0,003	-0,002	-0,004	-0,005	-0,003	-0,001	0,000
	0,95	-0,002	-0,001	0,000	0,000	-0,001	-0,001	0,000
	0,99	-0,002	-0,001	0,000	0,000	0,000	0,000	0,000

Table 5 – Mean squared error for some points of the bivariate grid for the estimated function of dependence of Sibuya obtained through 1,000 random samples of size 500, 1,000 and 2,000 observed from the random vector (X,Y) with normal distribution with mean (3.05;6.44)´ and vec(Σ)=(1.13;1.49;1.49;3.99)´ (correlation 0.7).

n		Grid						
		0,01	0,05	0,25	0,50	0,75	0,95	0,99
500	0,01	180,596	10,795	0,087	0,002	0,000	0,000	0,000
	0,05	9,735	1,748	0,054	0,003	0,000	0,000	0,000
	0,25	0,080	0,057	0,015	0,003	0,000	0,000	0,000
	0,50	0,002	0,002	0,003	0,001	0,000	0,000	0,000
	0,75	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	0,95	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	0,99	0,000	0,000	0,000	0,000	0,000	0,000	0,000
1,000	0,01	105,282	6,085	0,046	0,001	0,000	0,000	0,000
	0,05	5,563	0,929	0,029	0,001	0,000	0,000	0,000
	0,25	0,048	0,028	0,008	0,002	0,000	0,000	0,000
	0,50	0,002	0,001	0,001	0,001	0,000	0,000	0,000
	0,75	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	0,95	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	0,99	0,000	0,000	0,000	0,000	0,000	0,000	0,000
2,000	0,01	54,800	3,000	0,025	0,001	0,000	0,000	0,000
	0,05	3,261	0,515	0,014	0,001	0,000	0,000	0,000
	0,25	0,024	0,015	0,004	0,001	0,000	0,000	0,000
	0,50	0,001	0,001	0,001	0,000	0,000	0,000	0,000
	0,75	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	0,95	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	0,99	0,000	0,000	0,000	0,000	0,000	0,000	0,000

Table 6 – P-value of normality test of Jarque Bera for some points of the bivariate grid for the estimated function of dependence of Sibuya obtained through 1,000 random samples of size 500, 1,000 and 2,000, observed from the random vector (X,Y) with normal distribution with mean (3.05;6.44)´ and vec(Σ)=(1.13;1.49;1.49;3.99)´ (correlation 0.7).

n	**Grid**							
		0,01	**0,05**	**0,25**	**0,50**	**0,75**	**0,95**	**0,99**
500	**0,01**	0,000	0,002	0,000	0,000	0,000	0,000	0,000
	0,05	0,109	0,591	0,002	0,000	0,000	0,000	0,000
	0,25	0,000	0,000	0,086	0,030	0,000	0,000	0,000
	0,50	0,000	0,000	0,414	0,565	0,154	0,000	0,000
	0,75	0,000	0,000	0,000	0,319	0,291	0,002	0,000
	0,95	0,000	0,000	0,000	0,000	0,000	0,258	0,003
	0,99	0,000	0,000	0,000	0,000	0,000	0,002	0,000
1,000	**0,01**	0,001	0,521	0,000	0,000	0,000	0,000	0,000
	0,05	0,055	0,188	0,000	0,000	0,000	0,000	0,000
	0,25	0,000	0,000	0,173	0,046	0,000	0,000	0,000
	0,50	0,000	0,000	0,116	0,587	0,324	0,000	0,000
	0,75	0,000	0,000	0,000	0,059	0,859	0,001	0,000
	0,95	0,000	0,000	0,000	0,000	0,000	0,362	0,299
	0,99	0,000	0,000	0,000	0,000	0,000	0,047	0,001
2,000	**0,01**	0,077	0,167	0,000	0,000	0,000	0,000	0,000
	0,05	0,674	0,689	0,009	0,000	0,000	0,000	0,000
	0,25	0,000	0,000	0,431	0,025	0,000	0,000	0,000
	0,50	0,000	0,000	0,201	0,401	0,458	0,000	0,000
	0,75	0,000	0,000	0,005	0,243	0,022	0,232	0,000
	0,95	0,000	0,000	0,000	0,000	0,531	0,351	0,002
	0,99	0,000	0,000	0,000	0,000	0,000	0,237	0,109

4 Applications to real data

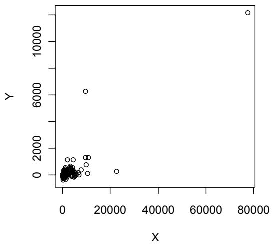
This section illustrates the implementation of the procedure described in Section 3.

Considering the largest companies in Brazil in 2006, according to the criterion of the brazilian magazine Exame, we select all private and state companies (except banks and insurers) which result in 1,018 companies (see http://app.exame.abril.com.br/servicos/melhoresemaiores/). Then, we analyzed some variables concerning the rates of annual performance. Firstly, we use the estimator (3) and then the estimator (5). In this estimation we used bivariate grid with 30×30 points, 90% of the central data and bandwidht also according to Azzaline (1981).

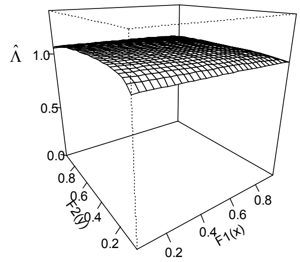
Sales ($ million) and net profit ($ million) for the 884 companies that contains this information, present Spearman´s correlation equal to 0.455, indicating global positive association. In Figure 4 we see the scatter plot of these two variables in (a), the graph of $\hat{\Lambda}$ given by (3) in (b) and (c) which show mild positive dependence, the scatter plot of normalized ranks and the kernel smoothed copula density in (d), (e) and (f) clearly show positive dependence, and the kernel smooted copula in (g) and (h) indicate positive dependence. In Figure 5 we see the graphs of $\hat{\Lambda}$ given by (5), which result in a similar behaviour of the graph (b) and (c) in Figure 4.

Spearman´s correlation between net profit ($ million) and sales margin (%) for the 864 companies considered, results in 0.848. In Figure 6, we see the scatter plot of the original variables in (a) and the remaining graphs indicate strong positive dependence. For these data, $\hat{\Lambda}$ given by (5) are shown in Figure 7.

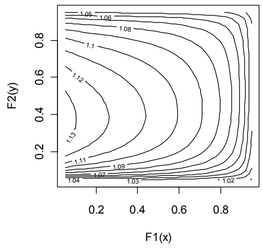
Figure 4 – Considering sales (X, in \$ million) and net profit (Y, in \$ million) of the companies in 2006, we have (a) scatter plot, (b) estimated function of dependence of Sibuya given through distribution functions and their (c) graph of level curves. For the corresponding normalized ranks, we have the (d) scatter plot, the (e) kernel smoothed copula density and their (f) level curves, the (g), kernel smoothed copula and their (h) level curves.



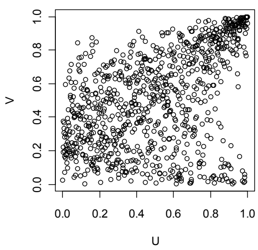
(a)



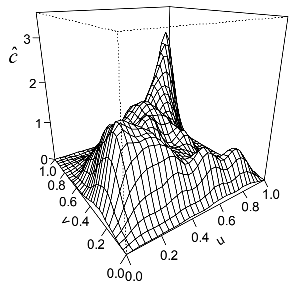
(b)



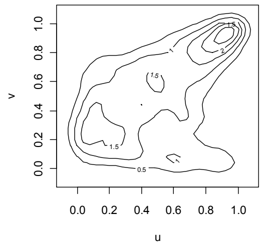
(c)



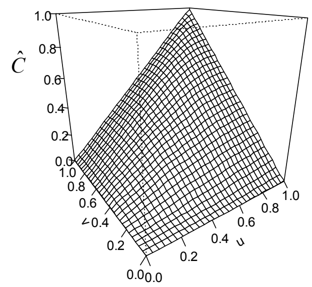
(d)



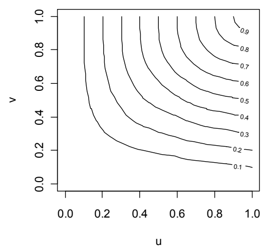
(e)



(f)

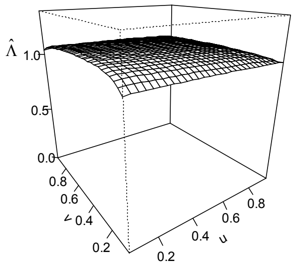


(g)

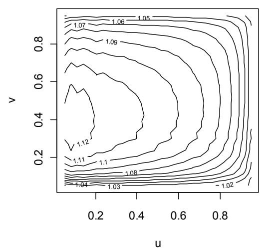


(h)

Figure 5 – Considering sales (X, in \$ million) and net profit (Y, in \$ million) of the companies in 2006, we have the estimated function of dependence of Sibuya given through copula in (a) and their graph of level curves in (b).

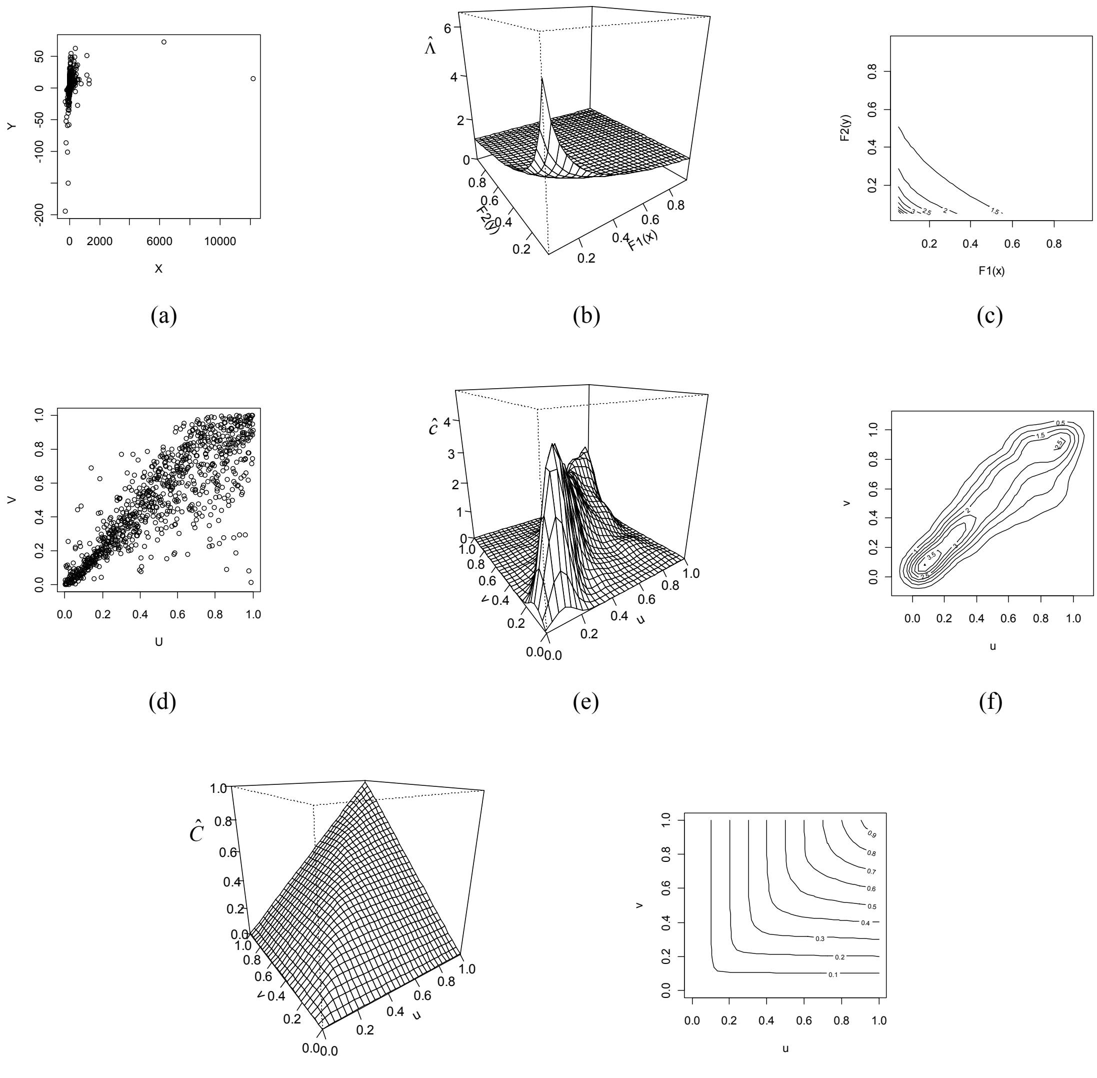


(a)



(b)

Figure 6 – Considering net profit (X, \$ million) and sales margin (Y, %) of the companies in 2006, we have (a) scatter plot, (b) estimated function of dependence of Sibuya given through distribution functions and their (c) graph of level curves. For the corresponding normalized ranks, we have the (d) scatter plot, the (e) kernel smoothed copula density and their (f) level curves, the (g) kernel smooted copula and their (h) level curves.

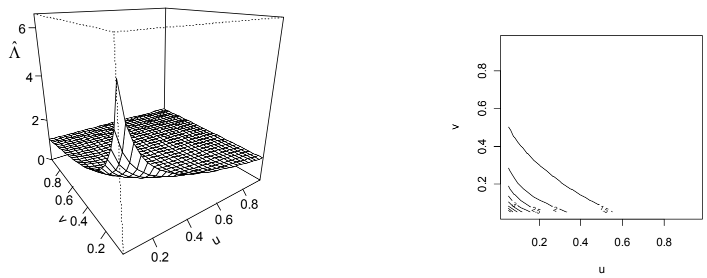


(a) (b) (c)

(d) (e) (f)

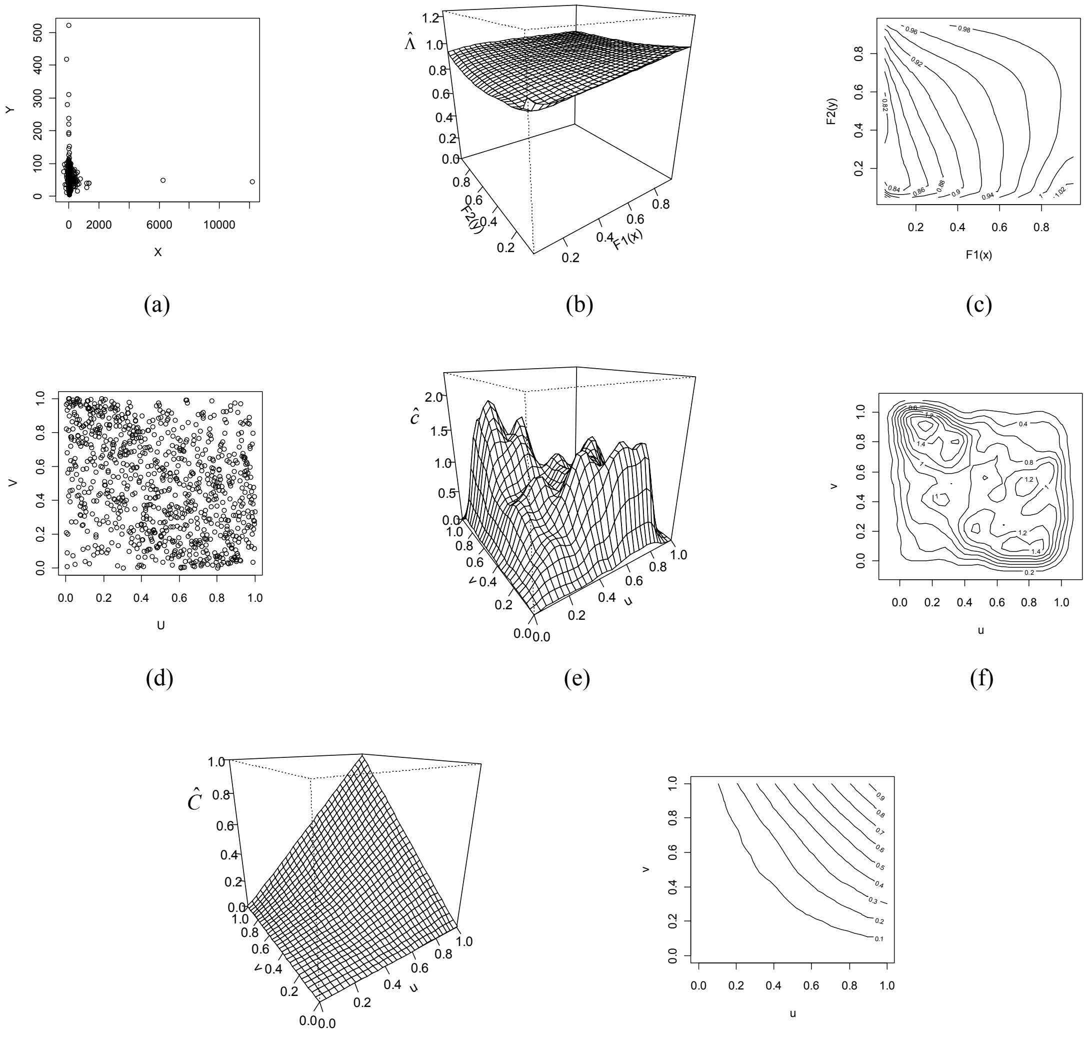
(g) (h)

Figure 7 – Considering net profit (X, \$ million) and sales margin (Y, %) of the companies in 2006, we have the estimated function of dependence of Sibuya given through copula in (a) and their graph of level curves in (b).



(a) (b)

Figure 8 – Considering net profit (X, \$ million) and general indebtedness (Y, %) of the companies in 2006, we have (a) scatter plot, (b) estimated function of dependence of Sibuya given through distribution functions and their (c) graph of level curves. For the their normalized ranks, we have the (d) scatter plot, the (e) kernel smoothed copula density and their (f) level curves, the (g) kernel smoothed copula and their (h) level curves.

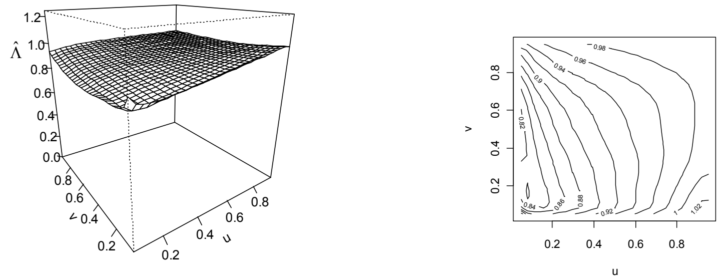


(a) (b) (c)

(d) (e) (f)

(g) (h)

Figure 9 – Considering net profit (X, \$ million) and general indebtedness (Y, %) of the companies in 2006, we have the estimated function of dependence of Sibuya given through copula in (a) and their graph of level curves in (b).



(a) (b)

Figure 10 – Considering growth in sales (X, %) and net profit (Y, \$ million) of the companies in 2006, we have (a) scatter plot, (b) estimated function of dependence of Sibuya given through distribution functions and their (c) graph of level curves. For the corresponding normalized ranks, we have the (d) scatter plot, the (e) kernel smoothed copula density and their (f) level curves, the (g) kernel smoothed copula and their (h) level curves.

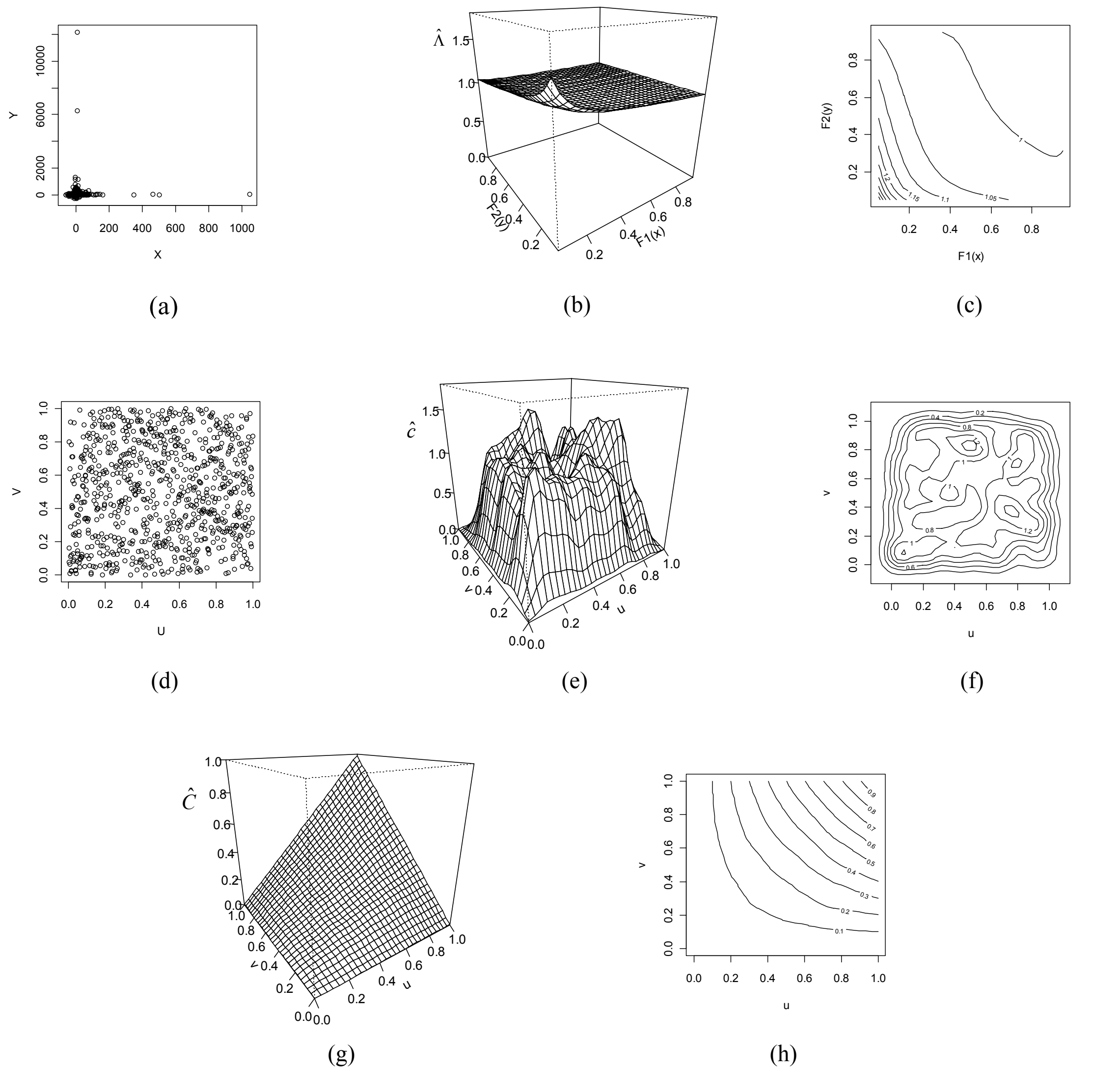
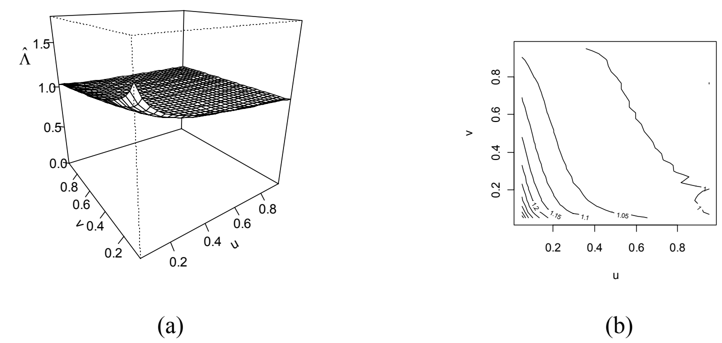


Figure 11 – Considering growth in sales (X, %) and net profit (Y, \$ million) of the companies in 2006, we have the estimated function of dependence of Sibuya given through copula in (a) and their graph of level curves in (b).



The net profit ($ million) and general indebtedness (%) for 884 companies, result in -0.333 for the Spearman´s correlation. In Figure 8 and Figure 9 we see the graphs of local dependence which indicate negative dependence.

The correlation of Spearman between growth in sales (%) and net profit ($ million) for 808 companies result in 0.047, and the local graphs indicate independence between these variables.

Acknowledgements.

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Note:

(1) X and Y are PQD - positively quadrant dependent (NQD - negatively quadrant dependent), if for all fixed $(x,y) \in \Re^2$, $F(x,y) \geq (\leq)\ F_1(x)F_2(y)$.