The Fundamental Theorem of Algebra: an Elementary and Direct Proof

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Abstract. We present a simple, differentiation-free, integration- free, trigonometry-free, direct and elementary proof of the Fundamental Theorem of Algebra.

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1. Introduction

One can ask, as soon as the complex numbers have been defined, whether every polynomial has a zero in the complex numbers. In this article we consider how early in the development of complex analysis this question can be answered.

As pointed out by Remmert [11], Burckel [2] and others, two of the best proofs of the FTA are the easy and short but not elementary one given by Argand [1] (see also Burckel [3], Chrystal [4], Fefferman [6], Fine and Rosenberger [7], Redheffer [10], Remmert [11], Rudin [12], Stillwell 14]), and the elementary but not so easy or short one given by Littlewood [9] (see also Estermann [5], Körner [8], [11], Searcóid [13]). All these works, except [2], use or prove d'Alembert's Lemma [14] or Argand's Inequality [11]:

"If P is a nonconstant complex polynomial and $P(z_0) \neq 0$, where $z_0 \in \mathbb{C}$, then any neighborhood of z_0 contains a point w such that $|P(w)| < |P(z_0)|$ ".

The proof of the FTA we now present does not apply d'Alembert's Lemma or Argand's Inequality. Instead we assume without proof only the continuity of complex polynomials and the following consequences of the completeness of $\mathbb{R}$:

- *any continuous function $f : D \to \mathbb{R}$, with D a bounded and closed disk, has a minimum on D.*
- *Every positive real number has a positive square root.*

Square Roots. It is well known that $z^2 = a + ib$, where $a, b \in \mathbb{R}$, is solvable on $\mathbb{C}$. We have

$$\pm z = \sqrt{\frac{a}{2} + \frac{\sqrt{a^2+b^2}}{2}} + i\mathrm{sgn}(b)\sqrt{-\frac{a}{2} + \frac{\sqrt{a^2+b^2}}{2}},$$

with $\mathrm{sgn}(b) = \frac{b}{|b|}$, if $b \neq 0$, and $\mathrm{sgn}(0) = 1$. Applying this formula repeatedly we find all the 2^j-roots, where $j \in \mathbb{N}$, of $z = \pm 1$ and $z = \pm i$.

2. Fundamental Theorem of Algebra

Theorem. *Let P be a complex polynomial, with degree$(P) = n \geq 1$. Then there exists $z_0 \in \mathbb{C}$ satisfying $P(z_0) = 0$.*

Proof. Writing $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, with $a_j \in \mathbb{C}$, $0 \leq j \leq n$, $a_n \neq 0$, we have $|P(z)| \geq |a_n|\,|z|^n - |a_0| - ... - |a_{n-1}|\,|z|^{n-1}$ from which follows that $\lim_{|z|\to\infty} |P(z)| = \infty$. By continuity, it is easy to see that $|P|$ has an absolute minimum at some $z_0 \in \mathbb{C}$. Suppose without loss of generality that $z_0 = 0$. Hence, putting $S^1 = \{\omega \in \mathbb{C} : |\omega| = 1\}$,

$$(1) \qquad |P(r\omega)|^2 - |P(0)|^2 \geq 0, \ \ \forall r \geq 0, \forall \omega \in S^1,$$

and $P(z) = P(0) + z^k Q(z)$, for some $k \in \{1, \ldots, n\}$, where Q is a polynomial and $Q(0) \neq 0$. Combining this equation, at $z = r\omega$, with inequality (1) we get

$$|P(0)+r^k\omega^k Q(r\omega)|^2 - |P(0)|^2 = 2r^k \mathrm{Re}[\overline{P(0)}\omega^k Q(r\omega)] + r^{2k}|Q(r\omega)|^2 \geq 0, \forall r \geq 0, \ \forall \omega \in S^1,$$

and, dividing by $r^k > 0$,

$$2\mathrm{Re}\left[\overline{P(0)}\omega^k Q(r\omega)\right] + r^k\left|Q(r\omega)\right|^2 \geq 0\,, \ \forall\, r > 0, \ \forall\, \omega \in S^1,$$

whose left side is a continuous function of r, where $r \in [0, +\infty)$. Thus, letting $r \to 0$, we have

$$(2) \qquad 2\mathrm{Re}[\,\overline{P(0)}Q(0)\,\omega^k\,] \geq 0, \ \ \forall\, \omega \in S^1.$$

Let $\alpha = \overline{P(0)}Q(0)$. Factoring out powers of 2 we can write $k = 2^j m$ where m is odd. Taking $\omega = 1$ in (2) we conclude $\mathrm{Re}[\alpha] \geq 0$. Choosing ω so that $\omega^{2^j} = -1$ we conclude that $\mathrm{Re}[\alpha] \leq 0$, and hence $\mathrm{Re}[\alpha] = 0$. Choosing ω so that $\omega^{2^j} = i$ we conclude that $\omega^k = \pm i$ and $\overline{\omega}^k = \mp i$. Substituting ω and $\overline{\omega}$ into (2) we conclude that $\mathrm{Im}[\alpha] = 0$. So, $\alpha = 0$, and $P(0) = 0$.

3. Final Remark

The last paragraph of the proof was a trick to avoid appealing to trigonometry. An easy but not elementary proof that (2) implies $P(0) = 0$ can be done with the help of De Moivre's Formula $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, n natural and θ real, as follows. Putting $\omega = \cos\theta + i\sin\theta$, with $\omega \in S^1$ and θ a real number, we choose values of θ so that $\omega^k = \cos k\theta + i\sin k\theta$, with k as in the above proof of the FTA, assumes the values ± 1 and $\pm i$. Hence, we get $\mathrm{Re}[\pm\overline{P(0)}Q(0)] \geq 0$ and $\mathrm{Re}[\pm\overline{P(0)}Q(0)i] \geq 0$ and conclude that $P(0) = 0$.

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