The Fundamental Theorem of Algebra: From the Four Basic Operations

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Abstract. This paper presents an elementary and direct proof of the Fundamental Theorem of Algebra, via Weierstrass' Theorem on Minima, that avoids the following: every root extraction, angles, non-algebraic functions, differentiation, integration, series, and arguments by induction.

1. INTRODUCTION. This article aims, through combining an inequality proved by de Oliveira [**5**] and a lemma by Estermann [**6**], to show a very elementary proof of the Fundamental Theorem of Algebra that does not employ any root extraction. Following a suggestion given by Littlewood [**11**, p. 95], see also Remmert [**13**, pp. 113–114], the proof requires a minimum amount of "limit processes lying outside algebra proper." Hence, the proof avoids differentiation, integration, series, angles, and the transcendental functions (i.e., non-algebraic functions) $\cos\theta$, $\sin\theta$, and $e^{i\theta}$, for $\theta \in \mathbb{R}$. Another reason to avoid these functions is justified by the fact that the theory of transcendental functions is more profound than that of the Fundamental Theorem of Algebra (a polynomial result), see Burckel [**4**]. In particular, it is interesting to notice that the usual proof of the well known Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$, for $\theta \in \mathbb{R}$, requires series, differentiation, and the (transcendental) numbers e and π (see Rudin [**14**, pp. 167–169]). This proof also avoids arguments by induction as well as those of $\epsilon - \delta$ type and the asymptotic ones.

Many elementary proofs of the Fundamental Theorem of Algebra, implicitly assuming the modulus function $|z| = \sqrt{z\bar{z}}$ where $z \in \mathbb{C}$, assume either Weierstrass' Theorem on Minima or the Intermediate Value Theorem, plus polynomial continuity. In addition, further root extraction is used in the proof (see Aigner and Ziegler [**1**, pp. 127–129], Argand [**2**] and [**3**], de Oliveira [**5**], Estermann [**6**], Fefferman [**7**], Kochol [**9**], Körner [**10**], Littlewood [**11**], Redheffer [**12**], Remmert [**13**], Rudin [**14**, pp. 169–170], Searcöid [**16**, p. 110], Spivak [**17**, pp. 548–550], and Terkelsen [**18**]). See also Schep [**15**], Vaggione [**19**] and [**20**], and Výborný [**21**]. Beginning with Littlewood [**11**], some of these proofs include a proof by induction of the existence of every nth root, for $n \in \mathbb{N}$, of every complex number (see [**9**], [**13**], [**16**], and [**17**, p. 553]).

This proof has similarities with the indirect one given by Körner [**10**]. Both proofs rely on Weierstrass' Theorem, avoid differentiation, integration, and angles, and "elementarize" Estermann's proof. It is worth pointing out that Körner, at the end of his note—in which the square root function, asymptotic arguments, and the Euclidean norm are employed—indicates a simple artifice that further eliminates from his preceding proof "general results on the existence and behaviour of square roots." However, the proof in this article employs Weierstrass' Theorem just once. Furthermore, this proof of the Fundamental Theorem of Algebra is straightforward in two out of the three analyzed cases. In fact, we "elementarize" a result by Estermann with the sole purpose of proving the remaining case.

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2. NOTATIONS AND PRELIMINARIES. Given $z \in \mathbb{C}$ such that $z = a + ib$, with $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $i^2 = -1$, we write $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$. The complex conjugate of z is $\overline{z} = a - ib$.

In what follows, the norm $|z|_1 = |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$, for $z \in \mathbb{C}$, is employed in order to avoid the need for square roots! It is well known that all norms in $\mathbb{C}$ are equivalent (in particular, they are equivalent to the usual Euclidean one). Given four arbitrary real numbers $a, b, c,$ and d, a short calculation reveals that

$$\begin{aligned}(|a| + |b|)^2(|c| + |d|)^2 &\le 2(a^2 + b^2)2(c^2 + d^2)\\ &= 4[(ac - bd)^2 + (ad + bc)^2] \le 4[|ac - bd| + |ad + bc|]^2,\end{aligned}$$

and

$$|ac - bd| + |ad + bc| \le |a||c| + |b||d| + |a||d| + |b||c| = (|a| + |b|)(|c| + |d|).$$

Hence, putting $z = a + ib$ and $w = c + id$, we deduce the properties

$$|\overline{z}|_1 = |z|_1 \text{ and } \frac{|z|_1|w|_1}{2} \le |zw|_1 \le |z|_1|w|_1.$$

Moreover, we use the Binomial Formula $(z + w)^n = \sum_{j=0}^{n} \binom{n}{j} z^j w^{n-j}$, where $z \in \mathbb{C}$, $w \in \mathbb{C}$, $n \in \mathbb{N}$, $\binom{n}{j} = \frac{n!}{j!(n-j)!}$, and $0! = 1$. We only assume without proof the following:

- Polynomial continuity, and
- Weierstrass' Theorem: *Any continuous function $f : D \to \mathbb{R}$, with D a bounded and closed disk, has a minimum on D.*

3. THE FUNDAMENTAL THEOREM OF ALGEBRA. In what follows, k is an arbitrary nonzero natural number.

We start by proving a pair of inequalities for the case where k is even. These inequalities have been proved by Estermann [**6**] for every $k \in \mathbb{N} \setminus \{0\}$. Our proof uses the binomial formula and simplifies Estermann's argument, which is based on root extraction and induction. The case where k is odd can be proved similarly. It is appropriate to emphasize that, in this article, we employ the following Lemma 1 only when k is a multiple of 4.

Lemma 1 (Estermann). *For $\zeta = \left(1 + \frac{i}{k}\right)^2$ and k even, $k \ge 2$, we have that*

$$\mathrm{Re}[\zeta^k] < 0 < \mathrm{Im}[\zeta^k].$$

Proof. Since $k = 2m$ and $2k = 4m$, for some $m \in \mathbb{N}$, applying the formulas

$$\mathrm{Re}\left[\left(1+\frac{i}{k}\right)^{2k}\right] = 1 - \binom{2k}{2}\frac{1}{k^2} + \binom{2k}{4}\frac{1}{k^4} + \sum_{\text{odd } j,\, j=3}^{k-1}\left[-\binom{2k}{2j}\frac{1}{k^{2j}} + \binom{2k}{2j+2}\frac{1}{k^{2j+2}}\right]$$

and

$$\mathrm{Im}\left[\left(1 + \frac{i}{k}\right)^{2k}\right] = \sum_{\text{odd} j,\, j=1}^{k-1}\left[\binom{2k}{2j-1}\frac{1}{k^{2j-1}} - \binom{2k}{2j+1}\frac{1}{k^{2j+1}}\right],$$

we proceed by noticing that for every $j \in \mathbb{N}$, with $1 \leq j \leq k-1$, it follows that

$$\begin{aligned}
1-\binom{2k}{2}\frac{1}{k^2}+\binom{2k}{4}\frac{1}{k^4} &= 1-\left(2-\frac{1}{k}\right)\left(\frac{2}{3}+\frac{5}{6k}-\frac{1}{2k^2}\right) \leq 1-\frac{3}{2}\left(\frac{2}{3}+\frac{5k-3}{6k^2}\right) \\
&= -\frac{3}{2}\cdot\frac{5k-3}{6k^2} < 0, -\binom{2k}{2j}\frac{1}{k^{2j}}+\binom{2k}{2j+2}\frac{1}{k^{2j+2}} \\
&= -\frac{(2k)!}{(2j)!k^{2j}(2k-2j-2)!}\left[\frac{1}{(2k-2j)(2k-2j-1)}-\frac{1}{(2kj+2k)(2kj+k)}\right] \\
&< 0,
\end{aligned}$$

and

$$\begin{aligned}
&\binom{2k}{2j-1}\frac{1}{k^{2j-1}}-\binom{2k}{2j+1}\frac{1}{k^{2j+1}} \\
&\quad = \frac{(2k)!}{(2j-1)!(2k-2j-1)!}\frac{1}{k^{2j-1}}\left[\frac{1}{(2k-2j+1)(2k-2j)}-\frac{1}{(2kj+k)(2kj)}\right] \\
&\quad > 0 \qquad \blacksquare
\end{aligned}$$

Theorem 1. *If P is a non-constant complex polynomial, then P has a zero in $\mathbb{C}$.*

Proof. Putting $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, where $a_j \in \mathbb{C}$ for all j such that $0 \leq j \leq n$, $a_n \neq 0$, and $n \geq 1$, we define the nonnegative function

$$P(z)\overline{P(z)} = \sum_{j=0}^{n} a_j \overline{a_j} z^j \overline{z}^j + \sum_{0 \leq j < k \leq n} 2\mathrm{Re}[a_j \overline{a_k} z^j \overline{z}^k], \text{ for } z \in \mathbb{C}.$$

Applying the triangle inequality and the previously mentioned properties of $|\,.\,|_1$, we obtain

$$P(z)\overline{P(z)} \geq \frac{|a_n|_1^2 |z|_1^{2n}}{2^{2n+1}} - \sum_{0 \leq j < k \leq n} 2|a_j|_1 |a_k|_1 |z|_1^{j+k}, \text{ forall } z \in \mathbb{C}.$$

Hence, $P(z)\overline{P(z)} \to \infty$ as $|z|_1 \to \infty$ and thus, there is $R > 0$ such that $P(z)\overline{P(z)} > P(0)\overline{P(0)}$, if $|z|_1 > R$. By continuity and by Weierstrass' Theorem, the function $P\overline{P}$ restricted to the disk $D = \{z \in \mathbb{C} : |z|_1 \leq R\}$ has a minimum at some $z_0 \in D$. Since $0 \in D$, we obtain the inequality $P(0)\overline{P(0)} \geq P(z_0)\overline{P(z_0)}$. Thus, $P\overline{P}$ has a global minimum at z_0. By considering $P(z + z_0)$, which is a polynomial of degree n, with leading coefficient a_n and constant term $P(z_0)$, we may assume that $z_0 = 0$. Therefore,

$$P(z)\overline{P(z)} - P(0)\overline{P(0)} \geq 0, \text{ for all } z \in \mathbb{C}, \tag{1}$$

and $P(z) = P(0) + z^k Q(z)$, for some $k \in \{1, \ldots, n\}$, where Q is a polynomial and $Q(0) \neq 0$. Substituting this equation, evaluated at $z = r\zeta$, where $r \geq 0$ and ζ is arbitrary in $\mathbb{C}$, in inequality (1), we arrive at

$$2r^k \mathrm{Re}\left[\overline{P(0)}\zeta^k Q(r\zeta)\right] + r^{2k}\zeta^k Q(r\zeta)\overline{\zeta^k Q(r\zeta)} \geq 0, \text{ for all } r \geq 0 \text{ and all } \zeta \in \mathbb{C},$$

and, canceling $r^k > 0$, we deduce the inequality

$$2\mathrm{Re}\left[\overline{P(0)}\zeta^k Q(r\zeta)\right] + r^k \zeta^k Q(r\zeta)\overline{\zeta^k Q(r\zeta)} \geq 0, \text{ for all } r > 0 \text{ and all } \zeta \in \mathbb{C},$$

whose left-hand side is a continuous function of r, with $r \in [0, +\infty)$. Thus, taking the limit as $r \to 0$ we are led to consider

$$2\mathrm{Re}\left[\overline{P(0)}Q(0)\zeta^k\right] \geq 0, \text{ for all } \zeta \in \mathbb{C}. \tag{2}$$

Let $\alpha = \overline{P(0)}Q(0) = a + ib$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$. If k is odd, then substituting $\zeta = \pm 1$ and $\zeta = \pm i$ into (2), we deduce that $a = 0$ and $b = 0$. Hence, $\alpha = 0$ and then we conclude that $P(0) = 0$. Thus, the case where k is odd is proved. We proceed by splitting the case where k is even into two subcases. First, let us suppose that $k = 4j + 2$, for $j \in \mathbb{N}$. Taking $\zeta = 1$ in (2), we obtain $a \geq 0$. Taking $\zeta = i$ in (2), we obtain $-a \geq 0$. Hence, we deduce that $a = 0$. Taking $\zeta = 1 \pm i$ in (2), we arrive at $\mathrm{Re}[\pm ib(-4)^j 2i] \geq 0$. Thus, we deduce that $b = 0$. Hence, $\alpha = 0$ and then we conclude that $P(0) = 0$. Thus, the case where $k = 4j + 2$, for $j \in \mathbb{N}$, is proved. Finally, let us suppose that $k = 4j$, for $j \in \mathbb{N}$. Taking $\zeta = 1$ in (2), we obtain $a \geq 0$. Picking ζ as in Lemma 1, let us write $\zeta^k = x + iy$, with $x < 0$ and $y > 0$. Substituting ζ^k and $\overline{\zeta}^k = \overline{\zeta^k}$ into (2) we arrive at $\mathrm{Re}[\alpha(x \pm iy)] = ax \mp by \geq 0$. Hence, $ax \geq 0$ and then (since $x < 0$) we obtain $a \leq 0$. Thus, we deduce that $a = 0$. Therefore, we arrive at $\mp by \geq 0$. Hence, since $y \neq 0$, we deduce that $b = 0$. Hence, $\alpha = 0$ and then we conclude that $P(0) = 0$. Thus, the case where $k = 4j$, for $j \in \mathbb{N}$, is proved, and the proof is complete. ■

Having concluded the proof of the Fundamental Theorem of Algebra, and keeping our previous notation, we recognize that k is the algebraic multiplicity of $z = z_0$ as a zero of $P(z)$.

4. REMARKS.

Remark 1. By equipping $\mathbb{C}$ with the usual norm $|z| = \sqrt{z\overline{z}}$, for $z \in \mathbb{C}$, one can adapt the proof above to produce a "more familiar" and easier to follow proof of the Fundamental Theorem of Algebra, at the cost of the introduction of the square root function. One then has the inequality $|P(z)| \geq |a_n||z|^n - \sum_{j=0}^{n-1} |a_j||z|^j$, for $z \in \mathbb{C}$, which implies that the function $|P|$ has a global minimum at some $z_0 \in \mathbb{C}$. Thus, supposing without loss of generality that $z_0 = 0$, one can analyze the inequality $|P(z)|^2 - |P(0)|^2 \geq 0$ exactly as was done above.

Remark 2. The almost algebraic "Gauss' Second Proof" (see [**8**]) of the Fundamental Theorem of Algebra uses only that "every real polynomial of odd degree has a real zero" and the existence of a nonnegative square root of every nonnegative real number. Nevertheless, this proof by Gauss is not elementary.

Remark 3. It is possible to rewrite a small part of the given proof of the Fundamental Theorem of Algebra so that the polynomial continuity is employed only to guarantee the existence of z_0, a point of global minimum of the function $P\overline{P}$. In fact, to avoid extra use of polynomial continuity, let us keep the notation of the proof and set $Q(z) = Q(0) + zR(z)$, with R as a polynomial. Now substitute this expression for $Q(z)$, with $z = r\zeta$, only in the first term in the left-hand side of the inequality

$2\mathrm{Re}\left[\overline{P(0)}\zeta^k Q(r\zeta)\right] + r^k\zeta^k Q(r\zeta)\overline{\zeta^k Q(r\zeta)} \geq 0$, for all $r > 0$ and all $\zeta \in \mathbb{C}$, which appeared just above (2). Thus, we obtain

$$2\mathrm{Re}\left[\overline{P(0)}\zeta^k Q(0)\right] + 2r\mathrm{Re}\left[\overline{P(0)}\zeta^{k+1} R(r\zeta)\right] + r^k\zeta^k Q(r\zeta)\overline{\zeta^k Q(r\zeta)} \geq 0,$$

for all $r > 0$ and all $\zeta \in \mathbb{C}$. Fixing ζ arbitrary in $\mathbb{C}$, a calculation shows that there is a finite constant $M = M(\zeta)$ such that the following inequality holds:

$$\max\left(|P(0)\zeta^{k+1}R(r\zeta)|_1, |\zeta^k Q(r\zeta)|_1^2\right) \leq M,$$

for all $r \in (0, 1)$. Hence,

$$-2\mathrm{Re}\left[\overline{P(0)}\zeta^k Q(0)\right] \leq 2rM + r^k M \leq 3rM, \text{ for all } r \in (0, 1).$$

Therefore, we conclude that $-2\mathrm{Re}\left[\overline{P(0)}\zeta^k Q(0)\right] \leq 0$, with ζ arbitrary in $\mathbb{C}$. The rest of the proof continues as before.

Remark 4. It is worth pointing out that this proof of the Fundamental Theorem of Algebra easily implies an independent proof of the existence of a unique nonnegative nth root, for $n \geq 2$, of each nonnegative number c. To show this, let us fix $c \geq 0$. Considering $n = 2$, and applying the Fundamental Theorem of Algebra, we can pick $z = x + iy \in \mathbb{C}$, for $x \in \mathbb{R}$ and $y \in \mathbb{R}$, such that $c = z^2 = (x^2 - y^2) + 2xyi$. Hence, we have that $y = 0$ and $x^2 = c$. Thus, $(\pm x)^2 = c$. Let $\sqrt{c}$ be the unique nonnegative one of x and $-x$. Hence, the absolute value function $|z| = \sqrt{z\overline{z}}$, for $z \in \mathbb{C}$, thereby becomes well defined and nonnegative. Lastly, given an arbitrary $n \in \mathbb{N}$, where $n \geq 2$, let us pick $z \in \mathbb{C}$ such that $z^n = \sqrt{c}$. Therefore, we have that $z^{2n} = c$ and, by the well known properties of the absolute value function over $\mathbb{C}$, $(|z|^2)^n = |z^{2n}| = c$. The uniqueness of a nonnegative nth root of c is rather trivial.

Remark 5. We can find a motivation for Estermann's Lemma by freely employing the well known results about the complex exponential function. In fact, since $(1 + \frac{z}{k})^k \to e^z$ as $k \to +\infty$, for all $z \in \mathbb{C}$, we immediately deduce that $\zeta^k = (1 + \frac{i}{k})^{2k} \to e^{2i} = \cos 2 + i \sin 2$, as $k \to +\infty$. Yet, by geometric arguments we obtain the pair of inequalities $\cos 2 < 0 < \sin 2$.

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