SCHWARZ THEOREM (mixed partial derivatives)
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Let us write \( \mathbb{R}^2 = \{(x, y) : x \in \mathbb{R} \text{ e } y \in \mathbb{R}\} \).

In this note we employ the following lemma.

Lemma (Limit X Iterated Limit). Let us consider \((a, b) \in \mathbb{R}^2\) and a function \(g : \mathbb{R}^2 \setminus \{(a, b)\} \to \mathbb{R}\). Let us suppose that the following limits exist,
\[
\begin{align*}
\lim_{(x,y)\to(a,b)} g(x, y) &= L \in \mathbb{R} \\
\lim_{x\to a} g(x, y) &= G(y) \in \mathbb{R}, \text{ for all } y \text{ in an open neighborhood of } b.
\end{align*}
\]
Then, the following iterated limit exists and satisfies
\[
\lim_{y\to b} \lim_{x\to a} g(x, y) = L.
\]

Proof. See https://www.ime.usp.br/~oliveira/ELE-IteratedLimits.pdf

Given a real function \(F = F(x, y)\), we also write
\[
F_x = \frac{\partial F}{\partial x}, \quad F_y = \frac{\partial F}{\partial y}, \quad F_{xy} = \frac{\partial^2 F}{\partial y \partial x} \quad \text{and} \quad F_{yx} = \frac{\partial^2 F}{\partial x \partial y}.
\]

Theorem (Schwarz). Let \(F : \mathbb{R}^2 \to \mathbb{R}\) be such that \(F_x, F_y\) and \(F_{xy}\) exist on a neighborhood of \((0,0)\), with \(F_{xy}\) continuous at \((0,0)\). Then, \(F_{yx}(0,0)\) exists and
\[
F_{yx}(0,0) = F_{xy}(0,0).
\]

Proof.

\( \ast \) Let us consider \(h \in \mathbb{R} \setminus \{0\}\) and \(k \in \mathbb{R} \setminus \{0\}\), both small enough. We have
\[
F_{xy}(0,0) \approx F_x(0,k) - F_x(0,0) \approx \frac{F(h,k) - F(0,k)}{h} - \frac{F(h,0) - F(0,0)}{k}.
\]
Without logical rigor, this points out to
\[
F_{xy}(0,0) \approx \frac{F(h,k) - F(0,k) - F(h,0) + F(0,0)}{hk}.
\]
\[ F(h, k) - F(0, k) - F(h, 0) + F(0, 0) = [F(h, k) - F(h, 0)] - [F(0, k) - F(0, 0)]. \]

The function \( x \mapsto F(x, k) - F(x, 0) \) is differentiable near \( x = 0 \). The mean-value theorem gives a point \( \overline{h} \), between 0 and \( h \), such that
\[ [F(h, k) - F(h, 0)] - [F(0, k) - F(0, 0)] = [F_x(\overline{h}, k) - F_x(\overline{h}, 0)]h. \]

The function \( y \mapsto F_x(\overline{h}, y) \) is differentiable near \( y = 0 \). The mean-value theorem gives a point \( \overline{k} \), between 0 and \( k \), such that
\[ F_x(\overline{h}, k) - F_x(\overline{h}, 0) = F_{xy}(\overline{h}, \overline{k})k. \]

The last two highlighted identities show that
\[ F_{xy}(\overline{h}, \overline{k}) = \frac{1}{h} \left[ \frac{F(h, k) - F(h, 0)}{k} - \frac{F(0, k) - F(0, 0)}{k} \right]. \]

By the continuity of \( F_{xy} \) at the origin we know that
\[ \lim_{(h, k) \to (0, 0)} F_{xy}(\overline{h}, \overline{k}) = F_{xy}(0, 0). \]

However, fixing \( h \), it also exists the limit
\[ \lim_{k \to 0} \frac{1}{h} \left[ \frac{F(h, k) - F(h, 0)}{k} - \frac{F(0, k) - F(0, 0)}{k} \right] = \frac{F_y(h, 0) - F_y(0, 0)}{h}. \]

By the lemma we conclude that
\[ F_{xy}(0, 0) = \lim_{h \to 0} \lim_{k \to 0} \frac{1}{h} \left[ \frac{F(h, k) - F(h, 0)}{k} - \frac{F(0, k) - F(0, 0)}{k} \right] = \lim_{h \to 0} \frac{F_y(h, 0) - F_y(0, 0)}{h} = F_{yx}(0, 0). \]

REFERENCE