The Change of Variable for the Riemann Integral on the Real Line

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The simplest version of this theorem supposes that all the functions and derivatives involved are continuous on closed intervals, under the additional condition that the inverse function of the change of variable is differentiable.

A comment based on T. M. Apostol Análisis Matemático, pp. 199–200. The most general version of the Change of Variable Theorem (for the unidimensional Riemann integral) does not require the continuity of the function involved and does not require that the substitution is invertible. Such version has the form

$$\int_{G(\alpha)}^{G(\beta)} f(x)dx = \int_{\alpha}^{\beta} f(G(t))g(t)dt,$$

with \(f\) integrable on the interval \(G([\alpha, \beta])\) and the substitution

\[G(t) = G(\alpha) + \int_{\alpha}^{t} g(\tau)d\tau, \text{ where } t \in [\alpha, \beta],\]

for some function \(g\) integrable on \([\alpha, \beta]\). It is worth noting that the identity \(G'(t) = g(t)\) is true on the points of continuity of \(g\) but is not assured on other points. The first proof of this general version is due to H. Kestelman (1961). In the same journal, same number, R. Davies simplifies the proof of Kestelman. Since then, many articles have been published about this general version.

Apostol’s book does not prove the general version. Kestelman’s proof employs the concept measure zero, from Lebesgue measure theory.

The version proven in this note presents the following features.

- The enunciate is very simple and the proof is very elementary.
- It applies in some cases where the “general version” does not.
- It is stronger than the versions in Basic Real Analysis, A. W. Knapp, pp. 37–38 (the proof in this note has some similarities and many differences with respect to the proof in such book) and in Principles of Mathematical Analysis, W. Rudin, p. 133.
- It does not appear in the material cited in References.
Along this proof we adopt the following type of partitions

\[ \mathcal{X} = \{ a = x_0 \leq \cdots \leq x_n = b \}, \]

with repeated points. Many authors (Knapp, Lang, Rudin, Spivak, etc.) adopt it. One advantage is that the integral of a function over a single point is automatically zero. Regarding the investigation of the existence and the value of the integral of a given function, it does not matter if the partition has repeated points or not.

**Theorem (Generalized Change of Variable Theorem).** Let us consider

\[ f : [a, b] \rightarrow \mathbb{R} \text{ integrable and } \varphi : [\alpha, \beta] \rightarrow [a, b] \]

surjective, increasing (not necessarily strictly increasing) and continuous. Suppose that \( \varphi \) is differentiable on the open interval \((\alpha, \beta)\). The following are true.

- If \( \varphi' \) is integrable on \([\alpha, \beta]\), then the map \((f \circ \varphi)\varphi' \) is integrable on \([\alpha, \beta]\).
- If the product \((f \circ \varphi)\varphi' \) is integrable on \([\alpha, \beta]\), then we have the formula

\[
\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.
\]

**Proof.**

\(\diamond\) The hypothesis that \( \varphi \) is continuous is superfluous. Given \( t \in [\alpha, \beta] \), we put \( x = \varphi(t) \in [a, b] \). Let \( x' \) and \( x'' \) be such that \( x \in [x', x''] \subset [a, b] \). Since \( \varphi \) is surjective and increasing, there exist \( t' \) and \( t'' \), both in \([\alpha, \beta]\) with \( t' \leq t'' \), such that \( \varphi(t') = x' \) and \( \varphi(t'') = x'' \). Since \( \varphi \) is increasing, we have \( \varphi([t', t'']) \subset [x', x''] \). Hence, \( \varphi \) is continuous [we notice that if \( x' < x \) then \( t' < t \) and, analogously, if \( x < x'' \) then \( t < t'' \)].

\(\diamond\) The function \( \varphi \) is uniformly continuous. It is trivial.

\(\diamond\) We have \( \varphi' \geq 0 \) on the open interval \((\alpha, \beta)\). It is obvious, since \( \varphi \) is increasing.

\(\diamond\) To integrate \( \varphi' \), it is clear that we may define \( \varphi'(\alpha) \) and \( \varphi'(\beta) \) arbitrarily.
Let $\mathcal{T} = \{\alpha = t_0 \leq \cdots \leq t_n = \beta\}$ be an arbitrary partition of the interval $[\alpha, \beta]$ and $\mathcal{X} = \{a = x_0 \leq \cdots \leq x_n = b\}$ be the partition of the interval $[a, b]$ given by $\mathcal{X} = \varphi(\mathcal{T})$. That is, suppose that $x_i = \varphi(t_i)$ for each $i = 0, \ldots, n$.

The mean-value theorem yields a point $\bar{t}_i \in [t_{i-1}, t_i]$ satisfying the condition $\Delta x_i = \varphi(t_i) - \varphi(t_{i-1}) = \varphi'(\bar{t}_i) \Delta t_i$.

If $|\mathcal{T}| \to 0$, then $|\mathcal{X}| \to 0$. It follows from the uniform continuity of $\varphi$, since given $\delta_1 > 0$ there exists $\delta_2 > 0$ such that $|\varphi(t) - \varphi(\tau)| \leq \delta_1$ if $|t - \tau| \leq \delta_2$.

If $\varphi'$ is integrable, then $(f \circ \varphi) \varphi'$ is integrable. Let $\tau_i$ be arbitrary in $[t_{i-1}, t_i]$. We notice that $\varphi(\tau_i) \in (x_{i-1}, x_i]$. Let us investigate the Riemann sum

$$
\sum f(\varphi(\tau_i)) \varphi'(\tau_i) \Delta t_i = \sum f(\varphi(\tau_i)) \Delta x_i + \sum f(\varphi(\tau_i)) [\varphi'(\tau_i)] - \varphi'(\bar{t}_i)] \Delta t_i.
$$

If $|\mathcal{T}| \to 0$, then $|\mathcal{X}| \to 0$ and the first sum on the right side goes to $\int_a^b f dx$. Let $M$ be a constant such that $|f| \leq M$ (obviously, $f$ is bounded). We have

$$
\frac{\left|\sum f(\varphi(\tau_i)) [\varphi'(\tau_i)] - \varphi'(\bar{t}_i)\right| \Delta t_i}{|\mathcal{T}| \to 0} \leq M \left[ S(\varphi', \mathcal{T}) - s(\varphi', \mathcal{T}) \right] 
$$

Thus, $(f \circ \varphi) \varphi'$ is integrable (the value of its integral equals the one of $f$).

If $(f \circ \varphi) \varphi'$ is integrable, then the value of its integral equals the one of $f$.

With the above notation, we choose $\tau_i = \bar{t}_i$ and write $x_i = \varphi(\bar{t}_i)$.

Hence, we have

$$
\sum f(\varphi(\bar{t}_i)) \varphi'(\bar{t}_i) \Delta t_i = \sum f(x_i) \Delta x_i.
$$

If $|\mathcal{T}| \to 0$, by definition the left hand side goes to the integral of $(f \circ \varphi) \varphi'$. If $|\mathcal{T}| \to 0$, then $|\mathcal{X}| \to 0$ and the right hand side goes to the integral of $f$. 

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Corollary. Keeping the other hypotheses of the theorem, let us suppose that $\varphi$ is monotonous (i.e., either increasing or decreasing). Then we have

$$\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_{a}^{b} f(\varphi(t))\varphi'(t)dt.$$ 

Proof.

- The case $\varphi$ increasing is already proven. Let us suppose $\varphi$ decreasing. Then

$$\psi(s) = \varphi(a + b - s),$$

where $t \in [a, b]$, is increasing. From the theorem we obtain

$$\int_{\psi(a)}^{\psi(b)} f(x)dx = \int_{a}^{b} f(\psi(s))\psi'(s)ds = -\int_{a}^{b} f(\varphi(a + b - s))\varphi'(a + b - s)ds.$$ 

Therefore, with the change of variable $t = a + b - s$ we may conclude that

$$\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_{a}^{b} f(\varphi(t))\varphi'(t)dt.$$ 

Next, we have an example telling apart this version from the “general version”.

Example (An example for which the version in this note applies but “the general version” does not). Let us consider the pair of functions

$$f(x) = x \text{ for all } x \in [0, 1], \text{ and } \varphi(t) = \sqrt{t} \text{ for all } t \in [0, 1].$$

Evidently $f$ is integrable. Moreover, $\varphi : [0, 1] \to [0, 1]$ is surjective, increasing, and continuous. Yet, the derivative $\varphi'$ is defined on the open interval $(0, 1)$ and

$$\varphi'(t) = \frac{1}{2\sqrt{t}}.$$ 

It is not difficult to see that $\varphi'$ is not bounded and thus not integrable on $(0, 1)$. Clearly, the function

$$f(\varphi(t))\varphi'(t) = \frac{\sqrt{t}}{2\sqrt{t}} = \frac{1}{2}$$

is integrable. Hence, from the theorem proven in this note, we obtain

$$\int_{0}^{1} x dx = \int_{0}^{1} \frac{1}{2} dt.$$ 

On the other hand, $\varphi'$ is not integrable on $(0, 1)$ and thus we cannot write

$$\sqrt{t} = \int_{0}^{t} \frac{1}{2\sqrt{\tau}} d\tau, \text{ for all } t \in [0, 1].$$ 

Therefore, the “general version” (with $g = \varphi'$) does not apply in this case.
References.


