A New Proof of Darboux's Theorem

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In this short note we present a (new?) proof of Darboux's theorem, which states that any derivative has the intermediate value property. Recall that a real-valued function $f : I \to \mathbb{R}$ defined on an interval I is said to have the *intermediate value property* if for all a and b in I with $a \leq b$ and for each number y between $f(a)$ and $f(b)$ there exists a number x in $[a, b]$ such that $f(x) = y$.

It is well known, and proved in any course in real analysis, that a continuous function defined on an interval has the intermediate value property. It was widely believed by many nineteenth-century mathematicians that the intermediate value property was, in fact, equivalent to continuity. However, in 1875 the French mathematician Jean Gaston Darboux (1842–1917) [**4**] proved that this is not the case. Darboux showed that any derivative has the intermediate value property and gave examples of differentiable functions with discontinuous derivatives. Because of Darboux's work, the fact that any derivative has the intermediate value property is now known as Darboux's theorem.

Darboux's Theorem. *Let I be an open interval, and let $f : I \to \mathbb{R}$ be a differentiable function. If a and b are points of I with $a \leq b$ and if y lies between $f'(a)$ and $f'(b)$, then there exists a number x in $[a, b]$ such that $f'(x) = y$.*

Darboux's theorem is sometimes proved in courses in real analysis as an example of a nontrivial application of the fact that a continuous function defined on a compact interval has a maximum. All textbooks, including classical texts such as those by Bartle and Sherbert [**1**, Theorem 6.2.12], Boas [**2**, p. 122], Hardy [**5**, sec. 129], Rudin [**9**, Theorem 5.12], Spivak [**10**, Exercise 39, p. 187] and Stromberg [**13**, Exercise 17, p. 186], in addition to the other text-books ([**3**, Theorem 3.10], [**6**, Theorem 7.6], [**7**, Theorem 26.9], [**8**, Theorem 29.8], [**11**, Theorem 5.2.13], [**12**, p. 158], [**14**, Theorem 7.31]) in real analysis that I have on the bookshelf next to my desk, present the following proof of Darboux's theorem: We may clearly assume that y lies strictly between $f'(a)$ and $f'(b)$. Define $\varphi : I \to \mathbb{R}$ by

$$\varphi(t) = f(t) - yt.$$

Then $\varphi'(a) = f'(a) - y$ and $\varphi'(b) = f'(b) - y$. We therefore conclude that either $\varphi'(a) > 0$ and $\varphi'(b) < 0$, or $\varphi'(a) < 0$ and $\varphi'(b) > 0$. Without loss of generality we may assume that $\varphi'(a) > 0$ and $\varphi'(b) < 0$. This implies that neither a nor b can be a point where φ attains even a local maximum. Since φ is continuous, it must therefore attain its maximum on $[a, b]$ at an interior point x of $[a, b]$. From this we conclude that $0 = \varphi'(x) = f'(x) - y$, whence $f'(x) = y$.

However, it is my experience that this proof is somewhat unconvincing to many beginning undergraduate students in real analysis. In particular, the fact that $\varphi'(a) > 0$ and $\varphi'(b) < 0$ implies that neither a nor b can be a maximum point of φ seems to cause most problems. Most students typically either think that this is obvious (and that the lecturer is being overly pedantic by insisting on a proof), or they see the need for a proof but find the ε-δ-gymnastics in the proof less than convincing. The reader is referred to Lemma 6.2.11 in Bartle and Sherbert's textbook [**1**] for a typical ε-δ proof of this fact.

The proof of Darboux's theorem that follows is based only on the mean value theorem for differentiable functions and the intermediate value theorem for continuous functions. It is my experience that this proof is more convincing than the "standard" one to beginning undergraduate students in real analysis.

Proof of Darboux's theorem. We may clearly assume that y lies strictly between $f'(a)$ and $f'(b)$. Define continuous functions $f_a, f_b : I \to \mathbb{R}$ by

$$f_a(t) = \begin{cases} f'(a) & \text{for } t = a, \\ \dfrac{f(a) - f(t)}{a - t} & \text{for } t \neq a, \end{cases}$$

and

$$f_b(t) = \begin{cases} f'(b) & \text{for } t = b, \\ \dfrac{f(t) - f(b)}{t - b} & \text{for } t \neq b. \end{cases}$$

It follows that $f_a(a) = f'(a)$, $f_a(b) = f_b(a)$, and $f_b(b) = f'(b)$. Hence, y lies between $f_a(a)$ and $f_a(b)$, or y lies between $f_b(a)$ and $f_b(b)$.

If y lies between $f_a(a)$ and $f_a(b)$, then (by the continuity of f_a) there exists s in $(a, b]$ with

$$y = f_a(s) = \frac{f(s) - f(a)}{s - a}.$$

The mean value theorem ensures that there exists x in $[a, s]$ such that

$$y = \frac{f(s) - f(a)}{s - a} = f'(x).$$

If y lies between $f_b(a)$ and $f_b(b)$, then an analogous argument (exploiting the continuity of f_b) shows that there exist s in $[a, b)$ and x in $[s, b]$ such that

$$y = \frac{f(b) - f(s)}{b - s} = f'(x).$$

This completes the proof. ∎

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Tiling Our Universe

In our infinite universe
I could talk forever,
About the tilt of your head
The shape of your smile—
Endlessly shifting.

Were we merely a fiction?
I used to be amazed
At how we had found one another,
From the thousands of others,
I chose you and you me.

Yes, I have loved before, of course,
But we were not a repetition,
This love was larger
Of this I am certain,
A truth of unimaginable scale.

We fit, our love was concrete,
As we came together in endless patterns,
There was the proof—
Nothing abstract in the way
We covered our universe.

And yet, with predictable ease,
You committed secret treason,
Shifted, changing shape,
'Just part of nature'
You repeated to me, again . . . and again.

We reached an impasse, as I feared we would,
No rhyme nor reason,
But even knowing you will repeat,
Different but the same,
Still I backtrack
So we can create our love again . . . and again.

——Submitted by Jennifer Granville, Ohio University School of Film, Athens, Ohio, and inspired by the writers and mathematicians at the BIRS Workshop on Mathematics and Creative Writing, April 2004.