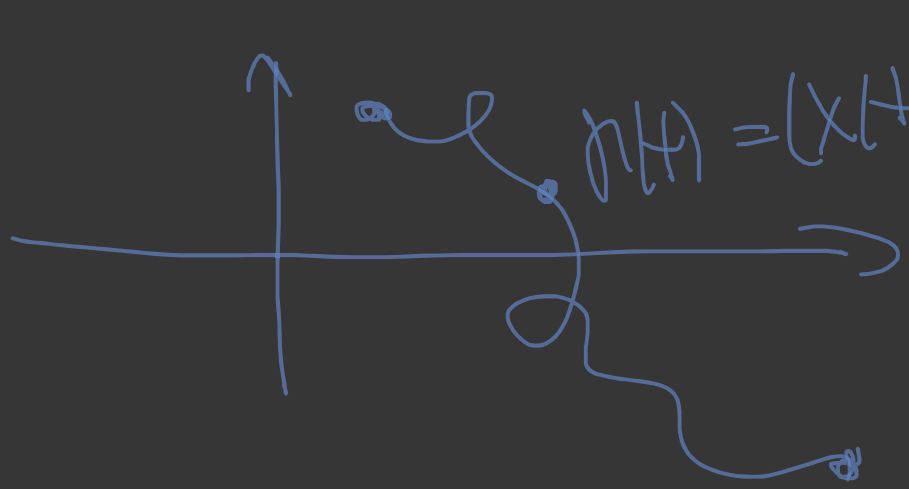
Integrais de linha

Curvas em $\mathbb{R}^2$ e $\mathbb{R}^3$

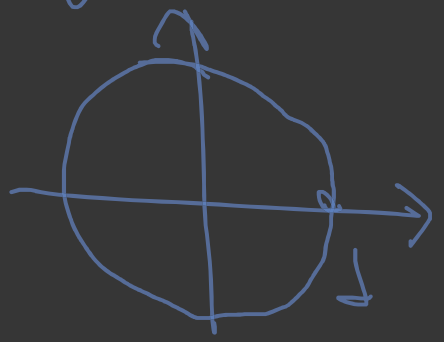
- Parametrização de curvas

Seja $I \subseteq \mathbb{R}$ intervalo. Seja γ uma aplicação de I em $\mathbb{R}^2$ ou $\mathbb{R}^3$, $\gamma(t) = (x(t), y(t))$ $(\gamma(t) = (x(t), y(t), z(t))$

Def. A imagem de γ é dita **curva**. As funções $x(t), y(t), z(t)$ são ditas parametrizações de γ.

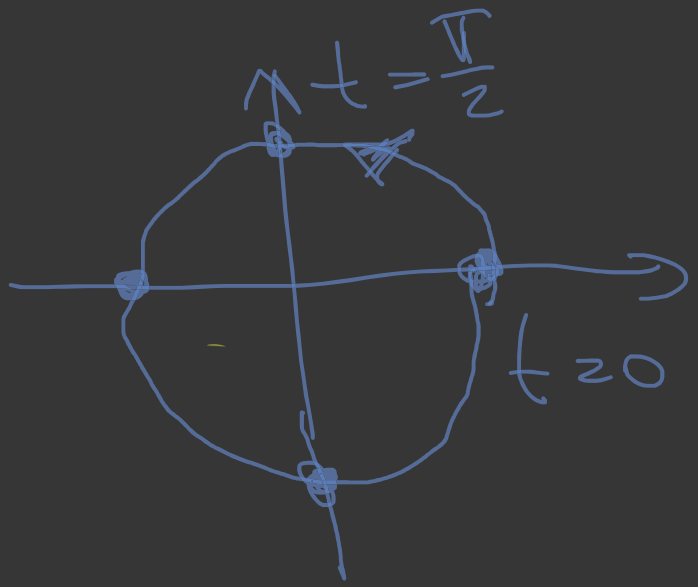


Ex. Obtenha uma parametrização da curva $x^2+y^2=1$



1) $\gamma(t) = (\underbrace{\cos t}_{x(t)}, \underbrace{\operatorname{sen} t}_{y(t)})$, $0 \le t \le 2\pi$

$\cos^2 t + \operatorname{sen}^2 t = 1 = x^2(t) + y^2(t)$

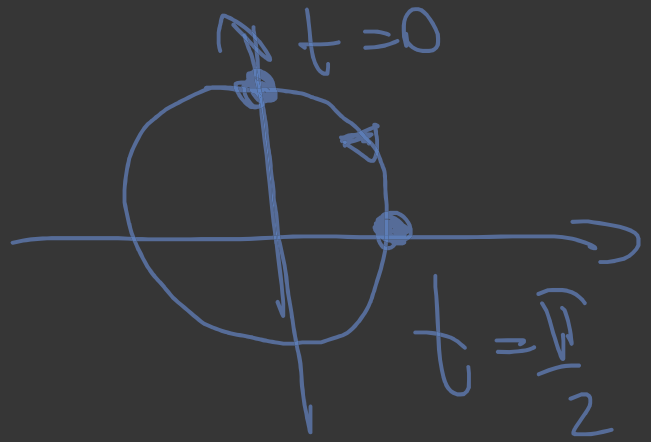


$t = 0 \Rightarrow$

$\gamma(0) = (1, 0)$

$t = \frac{\pi}{2} \Rightarrow \gamma\left(\frac{\pi}{2}\right) = (0, 1)$

2) $\gamma(t) = (\underbrace{\operatorname{sen} t}_{x(t)}, \underbrace{\cos t}_{y(t)})$, $0 \leq t \leq 2\pi$



$t = 0 \Rightarrow$

$\gamma(0) = (0, 1)$

$\gamma\left(\frac{\pi}{2}\right) = (1, 0)$

3) $\gamma(t) = (\underbrace{\cos 2t}_{x(t)}, \underbrace{\operatorname{sen} 2t}_{y(t)})$, $0 \leq t \leq \pi$

Ex2 Esboce a curva

1) $\gamma(t) = (\underbrace{\cos t}_{x(t)}, \underbrace{\cos^2 t}_{y(t)})$, $t \in [0, 2\pi]$

$y(t) = x^2(t)$. Observe que $-1 \leq \underbrace{\cos t}_{x(t)} \leq 1$

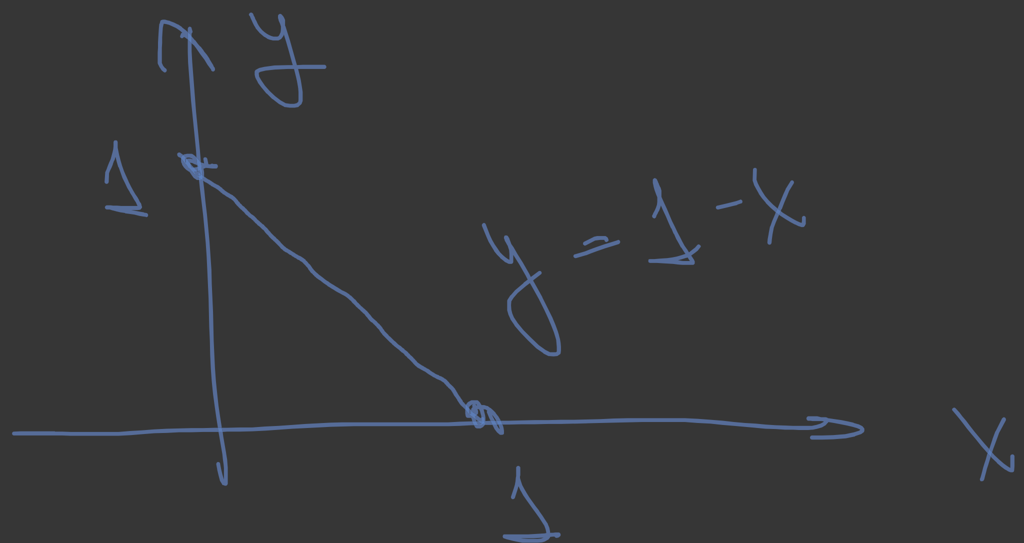
$0 \leq \underbrace{\cos^2 t}_{y(t)} \leq 1$

2) $\gamma(t) = (\underbrace{\cos^2 t}_{x(t)}, \underbrace{\operatorname{sen}^2 t}_{y(t)}), \; t \in [0, \pi]$

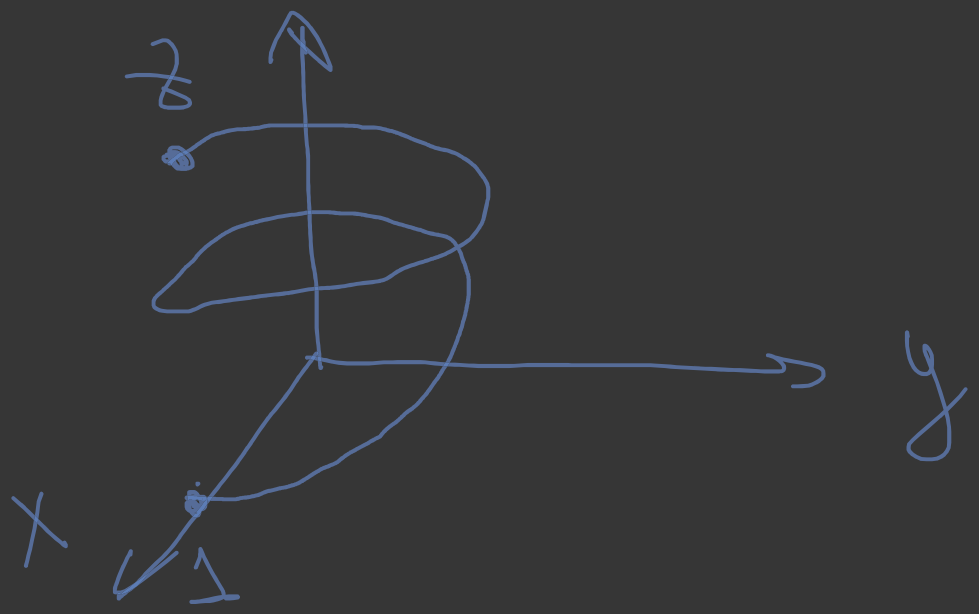
Temos $x(t) + y(t) = 1 \implies y(t) = 1 - x(t)$

Observe que $0 \leq x(t) \leq 1$
$ y(t)$

Então



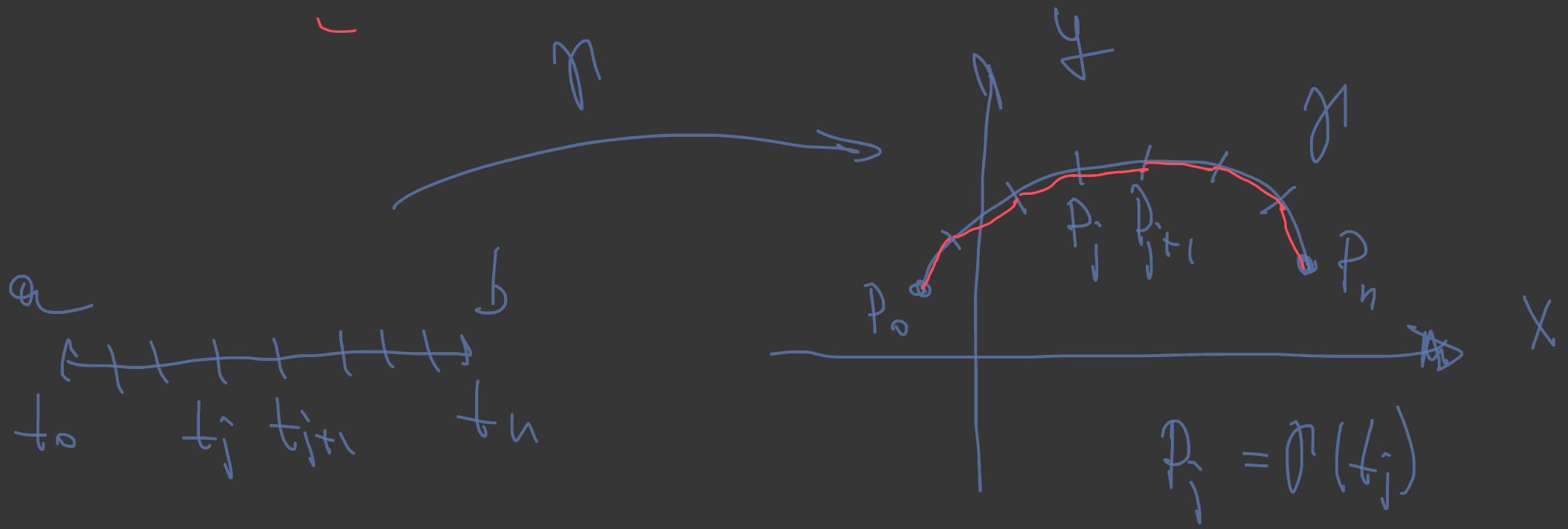
3) $\vec{r}(t) = (\underbrace{\cos t}_{x(t)}, \underbrace{\sin t}_{y(t)}, \underbrace{t}_{z(t)}), \quad t \in [0, 4\pi]$



Integral de linha

- Comprimento de arco da curva

Temos $\gamma : [a,b] \to \mathbb{R}^2$



Seja P uma poligonal que consiste das cordas $P_j P_{j+1}$

Observe que $\ell(\mathcal{P}) \approx \ell(P)$ e

$$\ell(P) = \sum_{j=0}^{n-1} \|\overrightarrow{P_j P_{j+1}}\| \approx \ell(\mathcal{P}) = \sum_{j=0}^{n-1} \|\overset{\frown}{P_j P_{j+1}}\|$$

Temos que $\|\overrightarrow{P_j P_{j+1}}\| = \|(x(t_{j+1}) - x(t_j), y(t_{j+1}) - y(t_j))\|$

$$= \|(x'(\tilde{t}_j)\Delta t_j, y'(\tilde{\tilde{t}}_j)\Delta t_j)\|, \quad \tilde{t}_j, \tilde{\tilde{t}}_j \in [t_j, t_{j+1}]$$

T. do valor médio

$$\approx \|(x'(\tilde{t}_j), y'(\tilde{t}_j))\| \Delta t_j = \underline{\|\gamma'(\tilde{t}_j)\| \Delta t_j}$$

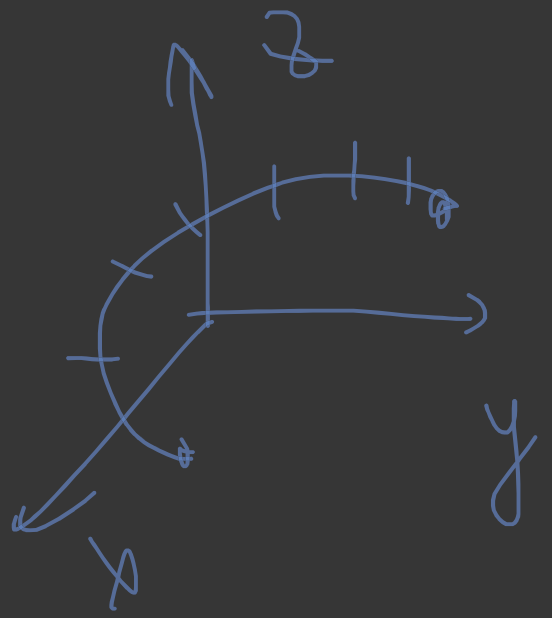
Então, $\ell(\Gamma) \approx \ell(P) \approx \sum_{j=0}^{n-1} \|\gamma'(\tilde{t}_j)\| \Delta t_j \Rightarrow$

$$\ell(\Gamma) = \lim_{n \to \infty} \sum_{j=0}^{n-1} \|\gamma'(\tilde{t}_j)\| \Delta t_j = \int_a^b \underbrace{\|\gamma'(t)\|}_{(x'(t), y'(t))} dt =$$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Obs Se tivesse $r(t) = (x(t), y(t), z(t)) \Rightarrow$

$$\ell(\Gamma) = \int_a^b \underbrace{\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}}_{\|r'(t)\|} \, dt$$



- Integral de linha de

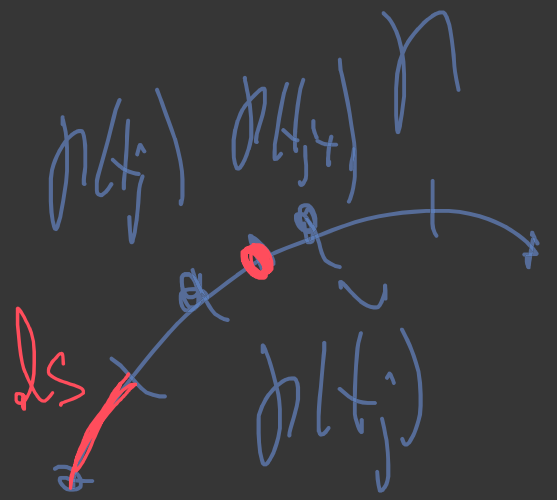
$f : D \subseteq \mathbb{R}^2 \text{ (ou } \mathbb{R}^3) \longrightarrow \mathbb{R}$ e $\gamma \subset D$

Somas de Riemann:

$$\sum_{i=0}^{n-1} f(\gamma(\tilde{t}_j)) \widehat{\|P_j P_{j+1}\|} \approx$$

$$\sum_{i=0}^{n-1} f(\gamma(\tilde{t}_j)) \overrightarrow{\|P_j P_{j+1}\|} \approx$$

$$\sum_{i=0}^{n-1} f(\gamma(\tilde{t}_j)) \|\gamma'(\tilde{t}_j)\| \Delta t_j$$



Def 2 A integral de linha de $f : D \to \mathbb{R}$ ao longo de $\gamma : [a,b] \to \mathbb{R}^2 (\mathbb{R}^3)$ é:

$$\int_\gamma f(s)\,ds = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\gamma(\tilde{t}_i))\,\|\widehat{P_i P_{i+1}}\| =$$

$$= \int_a^b f(\gamma(t))\,\|\gamma'(t)\|\,dt = \int_a^b f(x(t),y(t))\cdot\sqrt{(x'(t))^2+(y'(t))^2}\,dt$$

Ex 3 Ache $\ell(\gamma)$, onde

$$\gamma(t) = (\underbrace{\cos t}_{x(t)}, \underbrace{\operatorname{sen} t}_{y(t)}, \underbrace{2t}_{z(t)}), \quad 0 \le t \le 2\pi$$

Temos $\gamma'(t) = (-\operatorname{sen} t, \cos t, 2) \Rightarrow$

$$\|\gamma'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{\underbrace{\operatorname{sen}^2 t + \cos^2 t}_{1} + 4}$$

$$= \sqrt{5}$$

Lembrando que $\ell(\gamma) = \int_0^{2\pi} \|\gamma'(t)\|\, dt$,

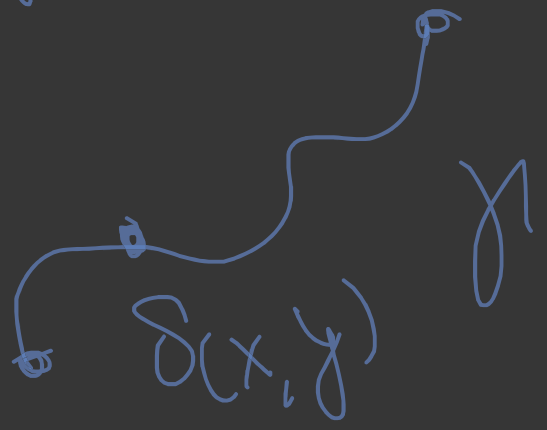
obtemos $\ell(\gamma) = \int_0^{2\pi} \sqrt{5}\, dt = \sqrt{5}\, t \Big|_0^{2\pi} = \underline{2\pi\sqrt{5}}$

Obs Mais uma aplicação da integral de linha. Suponha que tem um arame $\gamma(t)$ com densidade $\delta(x, y)$

$\Rightarrow$ a massa de arame é

$$M(\alpha) = \int_{\alpha} \delta(s)\,ds = \int_a^b \delta(\alpha(t))\,\|\alpha'(t)\|\,dt =$$

$$= \int_a^b \delta(x(t), y(t))\,\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$



Ex 4 Ache a massa da curva de interseção de $x^2+z^2=4$ com

$x + y = 1$ com densidade $\delta(x,y,z) = \sqrt{4+z^2}$

Temos: $x(t) = 2\cos t$, $z(t) = 2\,\text{sen}\,t$,

$t \in [0, 2\pi] \Rightarrow y(t) = 1 - x(t) = 1 - 2\cos t$

Logo, $r(t) = (2\cos t, 1 - 2\cos t, 2\,\text{sen}\,t) \Rightarrow$

$r'(t) = (-2\,\text{sen}\,t, 2\,\text{sen}\,t, 2\cos t) \Rightarrow$

$$\|r'(t)\| = \sqrt{\underbrace{4\,\text{sen}^2 t}_{x'(t)^2} + \underbrace{4\,\text{sen}^2 t}_{y'(t)^2} + \underbrace{4\cos^2 t}_{z'(t)^2}} = \sqrt{4\,\text{sen}^2 t + 4}$$

Temos $\delta(\gamma(t)) = \sqrt{4 + \underbrace{4\operatorname{sen}^2 t}_{(\gamma_2(t))^2}}$

Finalmente, $M(\gamma) = \int_0^{2\pi} \delta(\gamma(t)) \, \|\gamma'(t)\| \, dt =$

$= \int_0^{2\pi} \sqrt{4 + 4\operatorname{sen}^2 t} \cdot \sqrt{4\operatorname{sen}^2 t + 4} \, dt =$

$= \int_0^{2\pi} \left(4 + 4\operatorname{sen}^2 t\right) dt = \int_0^{2\pi} \left(4 + 2 - 2\cos 2t\right) dt$

$\operatorname{sen}^2 t = \frac{1 - \cos 2t}{2}$

$$= \left(6t - \frac{2\operatorname{sen} 4t}{2}\right)\Big|_0^{2\pi} = \underline{12\pi}$$

Ex 5 Ache $l(\gamma)$, onde γ é o gráfico da função $y = x^2 - \frac{1}{8}\ln x$, $x \in [1, 2]$

Parametrização de γ :

$$x(t) = t \;,\; y(t) = t^2 - \frac{1}{8}\ln t \;,\; t \in [1,2]$$

ou seja $\gamma(t) = \left(t,\ t^2 - \frac{1}{8}\ln t\right) \Rightarrow$

$$\gamma'(t) = \left(1,\ 2t - \frac{1}{8}\frac{1}{t}\right) \Rightarrow \|\gamma'(t)\| =$$

$$= \sqrt{1 + 4t^2 - \frac{1}{2} + \frac{1}{64t^2}} = \sqrt{\frac{1}{2} + 4t^2 + \frac{1}{64t^2}}$$

$$\left(2t - \frac{1}{8}\frac{1}{t}\right)^2$$

$$= \sqrt{\left(2t + \frac{1}{8t}\right)^2} = 2t + \frac{1}{8t} \Rightarrow$$

$$\ell(\gamma) = \int_1^2 \|\gamma'(t)\| dt = \int_1^2 \left(2t + \frac{1}{8t}\right) dt =$$

$$= \left(t^2 + \frac{1}{8}\ln t\right)\Big|_1^2 = 4 - 1 + \frac{1}{8}(\ln 2 - \ln 1)$$

$$= 3 + \frac{1}{8}\ln 2$$

<u>Obs</u> Se γ for um grafico de

$y = f(x)$, $x \in [a,b]$ $\Rightarrow$

a parametri grafo è

$$\alpha(t) = (t, f(t)), \quad t \in [a,b] \longrightarrow$$

$$\ell(\alpha) = \int_a^b \|\alpha'(t)\| dt = \int_a^b \sqrt{1 + (f'(t))^2}\, dt$$

$(x'(t))^2 \quad (y'(t))^2$