P2 - A

1. a) $\lim_{(x,y)\to(0,0)} \frac{8x^3+y^3+x^2y}{4x^2+y^2} = \lim_{(x,y)\to(0,0)} 2x \frac{4x^2}{4x^2+y^2} +$

$+ \lim_{(x,y)\to(0,0)} y \frac{y^2}{4x^2+y^2} + \lim_{(x,y)\to(0,0)} y \frac{x^2}{4x^2+y^2} = 0$

pois $\frac{4x^2}{4x^2+y^2} \leq 1$, $\frac{y^2}{4x^2+y^2} \leq 1$, $\frac{x^2}{4x^2+y^2} \leq \frac{1}{4}$

e $x \to 0, y \to 0$

b) $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2}$. O limite não existe pois:

- $x = 0, y \to 0, y \neq 0 \Rightarrow \lim_{y\to 0} \frac{0}{y^2} = 0$
- $y = x^3, x \to 0, x \neq 0 \Rightarrow \lim_{x\to 0} \frac{x^6}{2x^6} = \frac{1}{2}$

2. Aproxime $\sqrt{1{,}01^3 + 2{,}98}$. Consideremos a função:
$f(x,y) = \sqrt{x^3+y}$. Temos que $f(1+0{,}01, 3-0{,}02) =$
$= \sqrt{1.01^3 + 2.98}$.

Por outro lado:

$f(1+0{,}01, 3-0{,}02) \approx f(1,3) + \frac{\partial f}{\partial x}(1,3)\cdot 0{,}01 + \frac{\partial f}{\partial y}(1,3)\cdot(-0{,}02)$.

Temos que $f(1,3) = \sqrt{4} = 2$,

$\frac{\partial f}{\partial x} = \frac{3x^2}{2\sqrt{x^3+y}}$, $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x^3+y}} \Rightarrow$

$$\frac{\partial f}{\partial x}(1,3) = \frac{3}{2\cdot 2} = \frac{3}{4}, \quad \frac{\partial f}{\partial y}(1,3) = \frac{1}{2\cdot 2} = \frac{1}{4}.$$ 2

Logo $\sqrt{1.01^3 + 2.98} \approx 2 + \frac{3}{4}\cdot\frac{1}{100} + \frac{1}{4}\cdot\frac{-2}{100} =$

$$= 2 + \frac{3-2}{4\cdot 100} = 2 + \frac{1}{400}$$

3. Ache o plano π que passa por $(1,0,-2)$, $(1,1,0)$ e é tangente ao gráfico de $f(x,y) = xy + x$.

Sol $\pi: z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$

Temos $\frac{\partial f}{\partial x} = y+1$, $\frac{\partial f}{\partial y} = x$, logo

$\pi: z = x_0y_0 + x_0 + (y_0+1)(x-x_0) + x_0(y-y_0)$

Como $(1,0,-2)$ e $(1,1,0) \in \pi$, obtemos:

$$\begin{cases} -2 = x_0y_0 + x_0 + (y_0+1)(1-x_0) + x_0(0-y_0) \\ 0 = x_0y_0 + x_0 + (y_0+1)(1-x_0) + x_0(1-y_0) \end{cases} \Rightarrow$$

$$\begin{cases} 2 = x_0 \\ 0 = 2y_0 + 2 + (y_0+1)(1-2) + 2(1-y_0) \end{cases} \Rightarrow$$

$0 = 2y_0 + 2 - y_0 - 1 + 2 - 2y_0 \Rightarrow y_0 = 3 \Rightarrow$

$\pi: z = 2\cdot 3 + 2 + (3+1)(x-2) + 2(y-3) \Rightarrow$

$z = 8 + 4x - 8 + 2y - 6 \Rightarrow z = 4x + 2y - 6.$

4. $f(x,y) = y \operatorname{sen}\left(\sqrt[3]{x^2+y^2}\right)$.

a) • $(x,y) \neq (0,0) \Rightarrow \frac{\partial f}{\partial x} = \frac{2xy\cos\left(\sqrt[3]{x^2+y^2}\right)}{3(x^2+y^2)^{\frac{2}{3}}}$

$$\frac{\partial f}{\partial y} = \operatorname{sen}\left(\sqrt[3]{x^2+y^2}\right) + \frac{2y^2\cos\left(\sqrt[3]{x^2+y^2}\right)}{3(x^2+y^2)^{\frac{2}{3}}}.$$

• $(x,y) = (0,0) \Rightarrow$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0} \frac{y \operatorname{sen}\left(\sqrt[3]{y^2}\right)}{y} = 0$$

$\Rightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ são definidas para todos $(x,y) \in \mathbb{R}^2$, e

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{2xy\cos\left(\sqrt[3]{x^2+y^2}\right)}{3(x^2+y^2)^{2/3}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \operatorname{sen}\left(\sqrt[3]{x^2+y^2}\right) + \frac{2y^2\cos\left(\sqrt[3]{x^2+y^2}\right)}{3(x^2+y^2)^{3/2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

b) Claramente as funções $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ são contínuas para todos $(x,y) \neq (0,0) \in \mathbb{R}^2$.

Consideremos o caso $(x,y) = (0,0)$.

Observe que $\lim_{(x,y)\to(0,0)} \frac{y^2}{(x^2+y^2)^{3/2}} = \lim_{(x,y)\to(0,0)} y^{\frac{2}{3}} \cdot \sqrt[3]{\frac{x^4}{(x^2+y^2)^2}} = 0$

pois função $\frac{x^4}{(x^2+y^2)^2}$ é limitada.

Da mesma maneira

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt[3]{x^2+y^2}} = 0, \quad \lim_{(x,y)\to(0,0)} \frac{y}{\sqrt[3]{x^2+y^2}} = 0$$

Assim, $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt[3]{(x^2+y^2)^2}} = \lim \frac{x}{\sqrt[3]{x^2+y^2}} \cdot \lim \frac{y}{\sqrt[3]{x^2+y^2}} = 0$

Assim, $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}(x,y) = \lim_{(x,y)\to(0,0)} \left(\operatorname{sen} \sqrt[3]{x^2+y^2} + \frac{2y^2 \cos(\sqrt[3]{x^2+y^2})}{3(x^2+y^2)^{\frac{2}{3}}} \right) =$

$= 0 + 0 = 0 = \frac{\partial f}{\partial y}(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = \lim_{(x,y)\to(0,0)} \left[\frac{2xy\cos(\sqrt[3]{x^2+y^2})}{3(x^2+y^2)^{\frac{2}{3}}} \right] = 0 = \frac{\partial f}{\partial x}(0,0)$$

$\Rightarrow \frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são contínuas para todos $(x,y) \in \mathbb{R}^2$

c) usando b) obtemos que $f(x,y)$ é derivável em $\mathbb{R}^2$

5. Seja $f: \mathbb{R}^2 \to \mathbb{R}$ derivável. Suponha que $f(3t^2+1, 2t^2-1) = t^3$, $t \in \mathbb{R}$ e $\frac{\partial f}{\partial y}(4,1) = 1$. Ache $\frac{\partial f}{\partial x}(4,1)$.

<u>Sol:</u> $f(\gamma(t)) = g(t)$, onde $\gamma(t) = (3t^3+1, 2t^2-1)$, $g(t) = t^3$

Pela regra da Cadeia $g'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t) =$

$= \frac{\partial f}{\partial x}(\gamma(t)) \cdot 9t^2 + \frac{\partial f}{\partial y}(\gamma(t)) \cdot 4t = 3t^2$. Seja $t = 1$

$\Rightarrow \frac{\partial f}{\partial x}(4,1) \cdot 9 + \frac{\partial f}{\partial y}(4,1) \cdot 4 = 3 \Rightarrow$

$\frac{\partial f}{\partial x}(4,1) = \frac{1}{9}\left(3 - 4\frac{\partial f}{\partial y}(4,1)\right) = \frac{1}{9}(3-4) = -\frac{1}{9}$