Revisão (ano 2019)

Ex1 Dados 2 sist. de coord-s $\Sigma_1 = (O_1, E)$, $\Sigma_2 = (O_2, F)$ e, matriz de mudança

$$M_{EF} = \begin{pmatrix} a & 1 & -1 \\ b & 0 & 1 \\ c & -1 & 0 \end{pmatrix}, \quad O_2 = (1,-1,0)_{\Sigma_1}$$

a) Ache a, b, c, sabendo que $(2,3,1)_{\Sigma_1} = (1,0,-1)_{\Sigma_2}$

b) Ache eq-o vetorial da reta

$r: [X = (1,2,-1) + t(1,0,1)]_{\Sigma_1}$ no sist. Σ_2

c) —"— geral do plano $[x - y = 1]_{\Sigma_1}$ no sist Σ_2.

Sol Tem-se $$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\Sigma_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}_{\Sigma_1} + \begin{pmatrix} a & 1 & -1 \\ b & 0 & 1 \\ c & -1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{\Sigma_2}$$

a) $$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}_{\Sigma_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}_{\Sigma_1} + \begin{pmatrix} a & 1 & -1 \\ b & 0 & 1 \\ c & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}_{\Sigma_2} \Rightarrow$$

$$\begin{cases} 2 = 1 + a + 1 \\ 3 = -1 + b - 1 \\ 1 = 0 + c \end{cases} \Rightarrow \boxed{\begin{cases} a = 0 \\ b = 5 \\ c = 1 \end{cases}}$$

b) Assim, $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 \\ 5 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \Rightarrow$$

$$\begin{cases} x = 1 + y' - z' \\ y = -1 + 5x' + z' \\ z = x' - y' \end{cases} \Rightarrow \begin{cases} x + y = y' + 5x' \\ x + z = 1 + x' - z' \\ z = x' - y' \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x+y+z = 6x' \\ x+z = 1+x'-z' \\ z = x'-y' \end{cases} \Rightarrow \begin{cases} x' = \dfrac{x+y+z}{6} \\ y' = x'-z = \dfrac{x+y+z}{6} - z \\ z' = 1+x'-x-z = \\ \quad = 1 + \dfrac{x+y+z}{6} - x - z \end{cases}$$

$$\Rightarrow \begin{cases} x' = \dfrac{x+y+z}{6} \\ y' = \dfrac{x+y-5z}{6} \\ z' = 1 + \dfrac{-5x+y-5z}{6} \end{cases}$$

Tem-se $r: \begin{cases} x = 1+t \\ y = 2 \\ z = -1+t \end{cases} \Rightarrow \begin{cases} x' = \dfrac{1+t+2-1+t}{6} \\ y' = \dfrac{1+t+2+5-5t}{6} \\ z' = 1 + \dfrac{-5-5t+2+5-5t}{6} \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} x' = \frac{1}{3} + \frac{1}{3}t \\ y' = \frac{4}{3} - \frac{2}{3}t \\ z' = \frac{4}{3} - \frac{5}{3}t \end{cases} \Rightarrow$$

$$\boxed{r: \left[X = \left(\tfrac{1}{3}, \tfrac{4}{3}, \tfrac{4}{3}\right) + t\left(\tfrac{1}{3}, -\tfrac{2}{3}, -\tfrac{5}{3}\right)\right]_{\Sigma_2}}$$

c) $[\pi: x-y=1]_{\Sigma_1} \Rightarrow 1+y'-z'-(-1+5x'+z') = 1$

$\Rightarrow \boxed{-5x'+y'-2z' = -1}$ eq. geral no sist Σ_2.

<u>Ex 2</u> Seja sistema Σ_2 obtido pela translação do sist. cartesiano Σ_1 para p-to O_2 da reta $r: [X = (1,0,1) + t(1,4,2)]_{\Sigma_1}$. Ache O_2 em Σ_1 se a) $d(P, [x'=0]) = 3$, $P = (-1,-2,0)_{\Sigma_1}$

b) $d(Q, [y=0]) = 3$, $Q = (2,1,1)_{\Sigma_2}$

$O_2 \in r \Rightarrow O_2 = (1,0,1) + t_0(1,4,2) =$ ③
$= (1+t_0, 4t_0, 1+2t_0) \Rightarrow$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\Sigma_1} = \begin{pmatrix} 1+t_0 \\ 4t_0 \\ 1+2t_0 \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{\Sigma_2}$$

a) $[x'=0]_{\Sigma_2}$ ou $[x-1-t_0=0]_{\Sigma_1}$ ou $[x = \underbrace{1+t_0}_{d}]_{\Sigma_1}$

Seja $\pi: ax+by+cz=d$ e $P=(x_P, y_P, z_P) \Rightarrow$

$$d(P,\pi) = \frac{|ax_P + by_P + cz_P - d|}{\sqrt{a^2+b^2+c^2}}$$

Logo, $d(P, [x'=0]) = d(P, [x=1+t_0]) = \frac{|1\cdot(-1)+0\cdot(-2)+0\cdot 0 - 1 - t_0|}{1} = 3$

$\Rightarrow |1-t_0| = 3 \Rightarrow \begin{cases} 1-t_0 = 3 \\ 1-t_0 = -3 \end{cases} \Rightarrow \begin{cases} t_0 = -2 \\ t_0 = 4 \end{cases}$

$\Rightarrow \boxed{\begin{cases} O_2 = (-1,-8,-3) \\ O_2 = (5,16,9) \end{cases}}$

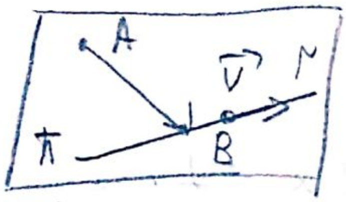
b) $[y=0]_{\Sigma_1}$ ou $[y'+4t_0=0]_{\Sigma_2}$ ou $[y'=-4t_0]_{\Sigma_2}$

$d(Q,[y=0]) = d(Q,[y'=-4t_0]) =$
$= \frac{|2\cdot 0 + 1\cdot 1 + 1\cdot 0 - (-4t_0)|}{1} = 3 \Rightarrow |1+4t_0| = 3$

$\Rightarrow \begin{cases} 1+4t_0 = 3 \\ 1+4t_0 = -3 \end{cases} \Rightarrow \begin{cases} t_0 = \frac{1}{2} \\ t_0 = -1 \end{cases} \Rightarrow \boxed{\begin{cases} O_2 = (\frac{3}{2}, 2, 2) \\ O_2 = (0,-4,-1) \end{cases}}$

Ex3 Ache a equação geral do plano π passando pelo ponto $A=(0,1,1)$ e contendo a reta $r: \frac{x-3}{1} = \frac{y-1}{3} = \frac{-z}{2}$

Sol $\pi: ax+by+cz=d$, onde $\vec{n}=(a,b,c)$ é vetor normal.



Seja $B \in r$ e $\vec{v}$ vetor diretor de r.

$\Rightarrow \vec{n} = \vec{AB} \times \vec{v}$.

$B=(3,1,0) \in r \Rightarrow \vec{AB} = (3,0,-1)$. Como $\vec{v}=(1,3,-2)$

obtemos $\vec{n} = \vec{AB} \times \vec{v} = \det \left(\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 3 & 0 & -1 & 3 & 0 \\ 1 & 3 & -2 & 1 & 3 \end{array}\right) =$

$= -\vec{j} + 9\vec{k} + 3\vec{i} + 6\vec{j} =$

$= (3,5,9)$.

Portanto, $\pi: 3x+5y+9z=d$.

Como $A=(0,1,1) \in \pi$, obtemos $3\cdot 0 + 5\cdot 1 + 9\cdot 1 = 14 = d$

$\Rightarrow \boxed{\pi: 3x+5y+9z=14.}$

Ex 4 Ache a posição relativa das r_1 e r_2

$r_1: \begin{cases} x-y-z=2 \\ x+y-z=0 \end{cases} \quad r_2: \begin{cases} 2x-3y+z=5 \\ x+y-2z=0 \end{cases}$

Sol ~~E~~ Procuremos equações vet-s das r_1 e r_2

Vetor diretor da ~~E1~~ r_1:

Seja $x=0 \Rightarrow \begin{cases} -y-z=2 \\ y-z=0 \end{cases} \Rightarrow \begin{cases} z=-1 \\ y=-1 \end{cases} \Rightarrow$

$A_1=(0,-1,-1) \in r_1$

Seja $z=0 \Rightarrow \begin{cases} x-y=2 \\ x+y=0 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=-1 \end{cases} \Rightarrow$

$B_1=(1,-1,0) \in r_1$

$\Rightarrow \vec{v_1} = \overrightarrow{A_1B_1} = (1,0,1)$ é vetor diretor de r_1.

$\vec{v_2}$: Seja $y=0 \Rightarrow \begin{cases} 2x + z = 5 \\ x - 2z = 0 \end{cases} \Rightarrow \begin{cases} z = 1 \\ x = 2 \end{cases}$

$\Rightarrow A_2 = (2,0,1)$

Seja $x=0 \Rightarrow \begin{cases} \cancel{0} -3y + z = 5 \\ y - 2z = 0 \end{cases} \Rightarrow \begin{cases} z = -1 \\ y = -2 \end{cases}$

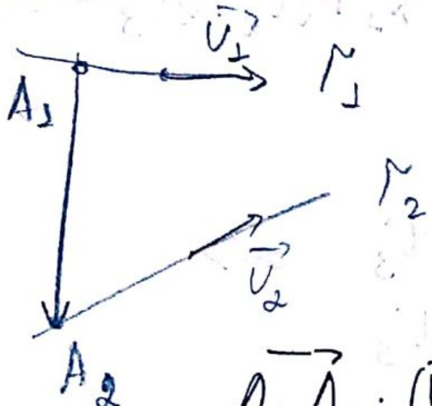
$\Rightarrow B_2 = (0,-2,-1)$

Logo, $\vec{v_2} = \overrightarrow{A_2B_2} = (-2,-2,-2)$

Portanto, $r_1: X = (0,-1,-1) + t(1,0,1)$

$r_2: X = (2,0,1) + t(-2,-2,-2)$

Como $\vec{v_1} \nparallel \vec{v_2}$, r_1 e r_2 são ou concorrentes ou reversas.



r_1 e r_2 são reversas $\Leftrightarrow$

$\overrightarrow{A_1A_2} \cdot (\vec{v_1} \times \vec{v_2}) \neq 0$

$\overrightarrow{A_1A_2} = (2,1,2) \Rightarrow$

$\overrightarrow{A_1A_2} \cdot (\vec{v_1} \times \vec{v_2}) = \det \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ -2 & -2 & -2 \end{pmatrix} \begin{matrix} 2 & 1 \\ 1 & 0 \\ -2 & -2 \end{matrix} =$

$= -2 -2 + 4 + 2 = 2 \neq 0$

$\Rightarrow \boxed{r_1 \text{ e } r_2 \text{ são reversas}}$.

<u>Ex 5</u> Ache uma eq-o vet-l da reta r que contém ponto $P=(0,2,1)$ e forma ângulos iguais com as retas!

$r_1: X = (1,2,3) + t\underbrace{(0,3,0)}_{\vec{v_1}}$, $r_2: X = (1,2,0) + t\underbrace{(0,0,3)}_{\vec{v_2}}$ ⑥

$r_3: X = (0,0,0) + t\underbrace{(1,2,2)}_{\vec{v_3}}$

Sol Temos, $(\widehat{r,r_1}) = (\widehat{r,r_2}) = (\widehat{r,r_3})$ em

$\cos(\widehat{r,r_1}) = \cos(\widehat{r,r_2}) = \cos(\widehat{r,r_3}) \iff$

$$\frac{|\vec{v}\cdot\vec{v_1}|}{\|\vec{v}\|\cdot\|\vec{v_1}\|} = \frac{|\vec{v}\cdot\vec{v_2}|}{\|\vec{v}\|\cdot\|\vec{v_2}\|} = \frac{|\vec{v}\cdot\vec{v_3}|}{\|\vec{v}\|\cdot\|\vec{v_3}\|}$$

Como $\vec{v_1} = (0,3,0)$, $\vec{v_2} = (0,0,3)$, $\vec{v_3} = (1,2,2)$, $\vec{v} = (v_1, v_2, v_3)$

$$\Rightarrow \frac{|3\cdot v_2|}{\sqrt{v_1^2+v_2^2+v_3^2}\cdot 3} = \frac{|3\cdot v_3|}{\sqrt{v_1^2+v_2^2+v_3^2}\cdot 3} = \frac{|v_1+2v_2+2v_3|}{\sqrt{v_1^2+v_2^2+v_3^2}\cdot 3}$$

$$\Rightarrow \begin{cases} |v_2| = |v_3| \\ |v_1+2v_2+2v_3| = |3v_2| \end{cases} \iff \begin{cases} v_2 = \pm v_3 \\ v_1+2v_2+2v_3 = \pm 3v_2 \end{cases}$$

Temos 4 casos:

1) $\begin{cases} v_2 = v_3 \\ v_1+2v_2+2v_3 = 3v_2 \end{cases} \Rightarrow \begin{cases} v_1 = -v_3 \\ v_2 = v_3 \end{cases} \Rightarrow$

$\vec{v} = (-v_3, v_3, v_3) = v_3(-1,1,1) \Rightarrow$

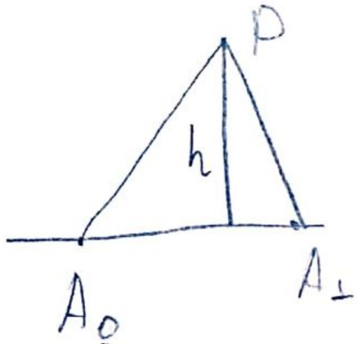
$\boxed{X = (0,2,1) + t(-1,1,1)}$

Considere 2)-4).

<u>Ex 6</u> Sejam $r: \begin{cases} x = 4 \\ y + z = 3 \end{cases}$ e $P = (m, 1, 1)$

Ache m t.q. $d(P, r) = 2$.

<u>Sol</u>



$$d(P,r) = h = \frac{\|\vec{A_0P} \times \vec{A_0A_1}\|}{\|\vec{A_0A_1}\|}$$

Temos: $A_0 = (4, 0, 3) \in r$
$A_1 = (4, 1, 2) \in r$

$\Rightarrow \vec{A_0A_1} = (0, 1, -1)$, $\|\vec{A_0A_1}\| = \sqrt{2}$, $\vec{A_0P} = (m-4, 1, -2)$

$$\Rightarrow \vec{A_0P} \times \vec{A_0A_1} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ m-4 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix} \begin{matrix} \vec{i} & \vec{j} \\ m-4 & 1 \\ 0 & 1 \end{matrix} =$$

$$= -\vec{i} + (m-4)\vec{k} + 2\vec{i} + (m+4)\vec{j} = \vec{i} + (m-4)\vec{j} + (m-4)\vec{k}$$

$$= (1, (m-4), (m-4)) \Rightarrow \|\vec{A_0P} \times \vec{A_0A_1}\| =$$

$$= \sqrt{1 + 2(m+4)^2}$$

Portanto $d(P,r) = \dfrac{\sqrt{1 + 2(m-4)^2}}{\sqrt{2}} = 2 \Rightarrow$

$$1 + 2(m-4)^2 = 8 \Rightarrow (m+4)^2 = \frac{7}{2} \Rightarrow$$

$$m - 4 = \pm\sqrt{\frac{7}{2}} \Rightarrow m = \pm\sqrt{\frac{7}{2}} + 4 \Rightarrow$$

$$\boxed{\left[\begin{array}{l} P = \left(\sqrt{\frac{7}{2}} + 4, 1, 1\right) \\ P = \left(-\sqrt{\frac{7}{2}} + 4, 1, 1\right) \end{array}\right.}$$