Bochner-Kähler Structures (after R. Bryant)

Rui Loja Fernandes

Department of Mathematics
University of Illinois at Urbana-Champaign, USA

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This talk is based on:

- RLF & I. Stuchiner, The Classifying Algebroid of a G-structure I & II, (see `arXive`).
- R. Bryant, Bochner-Kähler metrics. *J. of Amer. Math. Soc.*, **14** (2001), 623–715.

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Plan:

1. Bochner-Kähler metrics
2. Classification problems and Lie algebroids
3. Lie algebroids and Lie groupoids
4. Main results

Notations

(M, g, J, ω) – Kähler manifold with curvature tensor $R : (TM)^4 \to \mathbb{R}$

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Ricci tensor:

$$S(X, Y) = \sum_{i=1}^{n} R(X, JY, e_i, Je_i), \qquad (\{e_i, Je_i\}_{i=1,\dots,n}, \text{ orthonormal basis})$$

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Traceless Ricci tensor:

$$S_0(X, Y) = S(X, Y) - \frac{s}{2n}$$

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Symmetries of the curvature tensor $R : (TM)^4 \to \mathbb{R}$:

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$$R(X,Y,Z,W) + R(X,Z,W,Y) + R(X,W,Y,Z) = 0$$
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$$R = R_0 + R_1 + R_2$$

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- Given Kähler metric g and scalar s, the tensor:

$$R_0(X,Y,Z,W) := \frac{s}{4}\left\{\begin{array}{c} g(X,Z)g(Y,W) - g(X,W)g(Y,Z) + \\ +g(JX,Z)g(JY,W) - g(JX,W)g(JY,Z) + \\ +2g(X,Y)g(Z,W) \end{array}\right.$$

satisfies all symmetries. $R_0 = 0$ if Kähler metric is scalar-flat and $R = R_0$ iff it has constant holomorphic scalar curvature.

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Decomposing the action of $U(n)$ on such tensors into irreducible factors:

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- Given traceless symmetric 2-tensor S_0, the tensor:

$$R_1(X,Y,Z,W) := \frac{1}{4}\left\{ \begin{array}{c} g(X,Z)S_0(Y,W) - g(X,W)S_0(Y,Z) - g(Y,Z)S_0(X,W) + g(Y,W)S_0(X,Z) \\ g(JX,Z)S_0(JY,W) - g(JX,W)S_0(JY,Z) - g(JY,Z)S_0(JX,W) + g(JY,W)S_0(JX,Z \\ +2g(X,Y)S_0(Z,W) + 2g(Z,W)S_0(X,Y) \end{array}\right.$$

satisfies all symmetries. The trace of R_1 is S_0, and $R_1 = 0$ iff metric is Kähler-Einstein.

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Decomposing the action of $U(n)$ on such tensors into irreducible factors:

$$R = R_0 + R_1 + R_2$$

- R_0 represents scalar curvature;
- R_1 represents traceless Ricci;
- $R_2 := R - R_0 - R_1$ is called the Bochner tensor.

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- $\mathbb{CP}(p)_c \times \mathbb{CP}(n-p)_{-c}$ – also with constant holomorphic sectional curvature;

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- $\mathbb{C}^n$ – with symplectic form

$$\omega = \frac{i}{2}\partial\bar{\partial}f(|z|), \quad f''(t) = (a\, f'(t)\, t + k)f'(t)^2.$$

Up to scalar multiples, there is exactly one complete such example which is not locally symmetric (Tachibana & Liu).

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- Bochner-Kähler orbifolds, e.g., weighted projective spaces.

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The coframe (η, θ) on $\mathrm{F}_{U(n)}$ satisfies the structure equations:

$$\begin{cases} \mathrm{d}\theta = -\eta \wedge \theta \\ \mathrm{d}\eta = -\eta \wedge \eta + R(\theta \wedge \theta) \end{cases}$$

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Using the Bochner-Kähler condition, the curvature can be written:

$$.R(\theta \wedge \theta) = (S\theta^*) \wedge \theta - (S\theta) \wedge \theta^* - (\theta \wedge \theta^*)S + (\theta^* \wedge S\theta)I,$$

where $S : \mathrm{F}_{U(n)} \to i\,\mathfrak{u}(n)$ takes values in hermitian symmetric matrices.

Invariants

Differentiating the structure equations and using $\mathrm{d}^2 = 0$, we find functions $T \in C^\infty(\mathrm{F}_{U(n)}, \mathbb{C}^n)$ and $U \in C^\infty(\mathrm{F}_{U(n)}, \mathbb{R})$ such that:

$$\begin{cases} \mathrm{d}S = -\eta S + S\eta + T\theta^* + \theta T^* + \frac{1}{2}(T^*\theta + \theta^* T)I_n \\ \mathrm{d}T = -\eta T + (UI_n + S^2)\theta \\ \mathrm{d}U = T^* S\theta + \theta^* ST \end{cases}$$

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The functions $(S, T, U) : \mathrm{F}_{U(n)} \to i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ provide a set of invariants.

(Local) Classification Problem

Find all (germs of) manifolds P carrying:

1. a free action of $U(n)$,
2. a coframe $(\eta, \theta) \in \Omega^1(P, \mathfrak{u}(n) \oplus \mathbb{C}^n)$ and
3. functions $(S, T, U) : P \to i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$,

such that the following equations are satisfied:

$$\begin{cases} \mathrm{d}\theta = -\eta \wedge \theta \\ \mathrm{d}\eta = -\eta \wedge \eta + S\theta^* \wedge \theta - S\theta \wedge \theta^* - \theta \wedge \theta^* S + (\theta^* \wedge S\theta) I_n \\ \mathrm{d}S = -\eta S + S\eta + T\theta^* + \theta T^* + \frac{1}{2}(T^*\theta + \theta^* T) I_n \\ \mathrm{d}T = -\eta T + (U I_n + S^2)\theta \\ \mathrm{d}U = T^* S\theta + \theta^* S T \end{cases}$$

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Then $M = P/U(n)$ is Bochner-Kähler and $P = \mathrm{F}_{U(n)}$ is its unitary frame bundle.

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such that the following equations are satisfied:

$$\begin{cases} \mathrm{d}\eta = k\theta \wedge \theta \\ \mathrm{d}\theta = -\eta \wedge \theta \\ \mathrm{d}k = k_1\theta_1 + k_2\theta_2 \\ \mathrm{d}k_1 = \frac{1}{2}(1 - k^2)\theta_1 - k_2\eta \\ \mathrm{d}k_2 = \frac{1}{2}(1 - k^2)\theta_2 + k_1\eta. \end{cases}$$

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Then $\Sigma = P/SO(2)$ is a surface with a metric g of Hessian type, i.e., its Gaussian curvature satisfies

$$\mathrm{Hess}_g(k) = \frac{1}{2}(1 - k^2)g,$$

and $P = \mathrm{F}_{SO(2)}$ is its orthogonal frame bundle.

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One is given **Cartan Data**:

- a closed Lie subgroup $G \subset \mathrm{GL}_n$
- a G-manifold X
- equivariant maps $c : X \to \mathrm{Hom}(\wedge^2\mathbb{R}^n, \mathbb{R}^n)$, and $R : X \to \mathrm{Hom}(\wedge^2\mathbb{R}^n, \mathfrak{g})$
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and asks for the existence of **realizations**:

- a principal G-bundle P with a coframe $(\eta, \theta) \in \Omega^1(P, \mathfrak{g} \oplus \mathbb{R}^n)$ and an equivariant map $h : P \to X$

satisfying the structure equations:

$$\left\{ \begin{array}{l} \mathrm{d}\theta = c(h)(\theta \wedge \theta) - \eta \wedge \theta \\ \mathrm{d}\eta = R(h)(\theta \wedge \theta) - \eta \wedge \eta \\ \mathrm{d}h = F(h, \theta) + \psi(h, \eta) \end{array} \right. \tag{1}$$

($\psi : X \times \mathfrak{g} \to TX$ is the infinitesimal $\mathfrak{g}$-action determined by the G action)

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($\psi : X \times \mathfrak{g} \to TX$ is the infinitesimal $\mathfrak{g}$-action determined by the G action)

$\Rightarrow M = P/G$ and $P = F_G(M)$ is a G-structure with connection η and tautological 1-form θ, satisfying the structure equations (1)

Example : : Bochner-Kähler

- $X = i\,\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ with global coordinates (S, T, U)
- $G = U(n)$ acts diagonally on X by
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- $R : X \to \mathrm{Hom}(\wedge^2\mathbb{C}^n, \mathfrak{u}(n))$:

  $$z \wedge w \mapsto (z^* S w - w^* S z) I_n - (z w^* - w z^*) S - S(w z^* - z w^*) + (\mathrm{tr}\, S)(z^* w - w^* z) I_n$$
- $c : X \to \mathrm{Hom}(\wedge^2\mathbb{C}^n, \mathbb{C}^n)$ identically zero (no torsion)
- $F : X \times \mathbb{C}^n \to TX$,

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Classification of Bochner-Kähler metrics	$\Leftrightarrow$	Cartan's Realization Problem

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- $R : X \to \mathrm{Hom}(\wedge^2\mathbb{R}^2, \mathfrak{so}(2)) \simeq \mathbb{R}$, $(k, k_1, k_2) \mapsto k$;
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Classification of surfaces (Σ, g) of Hessian type	$\Leftrightarrow$	Cartan's Realization Problem

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- vector bundle $A \to X$: trivial bundle with fiber $\mathbb{R}^n \oplus \mathfrak{g}$;
- anchor $\rho : A \to TX$: bundle map $\rho(u, \alpha) = F(u) + \psi(\alpha)$;
- bracket $[\ ,\ ] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$: skew-symmetric bracket defined on constant sections by

$$[(u, \alpha), (v, \beta)] = (\alpha \cdot v - \beta \cdot u - c(u, v), [\alpha, \beta]_{\mathfrak{g}} - R(u, v)).$$

and extended to any sections so that Leibniz holds:

$$[s_1, fs_2] = f[s_1, s_2] + (\mathcal{L}_{\rho(s_1)} f)s_2.$$

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- anchor $\rho : A \to TX$: bundle map $\rho(u, \alpha) = F(u) + \psi(\alpha)$;
- bracket $[\ ,\ ] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$: skew-symmetric bracket defined on constant sections by

$$[(u, \alpha), (v, \beta)] = (\alpha \cdot v - \beta \cdot u - c(u, v), [\alpha, \beta]_{\mathfrak{g}} - R(u, v)).$$

and extended to any sections so that Leibniz holds:

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Cartan's Data and Lie algebroids

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How to encode **realizations**?

- Each realization gives a bundle map:

$$\begin{array}{ccc} TP & \xrightarrow{(\theta,\eta)} & A \\ \downarrow & & \downarrow \\ P & \xrightarrow[h]{} & X \end{array}$$

Cartan's Data and Lie algebroids

Proposition

If there is a solution to Cartan's realization problem for every $x \in X$ then the bracket satisfies the Jacobi identity:

$$[[s_1, s_2], s_3] + [[s_2, s_3], s_1] + [[s_3, s_1], s_2] = 0.$$

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- How can one solve the classification problem using $(A, \rho, [\ ,\ ])$?
- What does $(A, \rho, [\ ,\ ])$ say about symmetries? Moduli space of solutions? *etc.*

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Examples:

- Tangent bundle: $A = TX \to X$, $[\ ,\ ]$ usual Lie bracket of vector fields and $\rho =$id;
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Basic concepts:

- **Orbits** $\mathcal{O}$**:** $\rho([s_1, s_2]) = [\rho(s_1), \rho(s_2)] \Rightarrow \mathrm{Im}\,\rho$ is integrable (singular) distribution.
- **Isotropy Lie algebras** $\mathfrak{g}_x$**:** For $x \in X$, $[\ ,\ ]$ restricts to Lie bracket on $\mathfrak{g}_x := \ker \rho$.

Example : : Bochner-Kähler classifying algebroid

$$X = i\mathfrak{u}_n \oplus \mathbb{C}^n \oplus \mathbb{R}, \qquad A = X \times (\mathbb{C}^n \oplus \mathfrak{u}_n) \to X,$$

- Lie bracket of constant sections $(u, \alpha), (v, \beta) \in \mathbb{C}^n \oplus \mathfrak{u}_n$:

$$[(u, \alpha), (v, \beta)]|_{(S,T,U)} = (\alpha \cdot v - \beta \cdot u, [\alpha, \beta]_{\mathfrak{u}_n} - (uv^* - vu^*)S - S(vu^* - uv^*) + \cdots)$$

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Each Bochner-Kähler manifold, has an associated U_n-structure $P = F_{\mathrm{U}_n}(M)$ yielding a Lie algebroid map:

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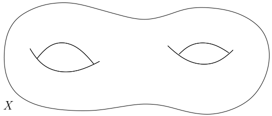
Groupoids

X – topological space; look at paths $\gamma : [0, 1] \to X$



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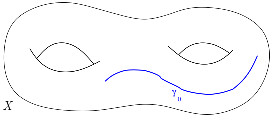

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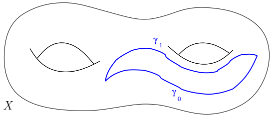

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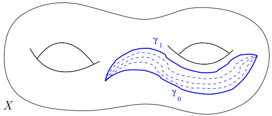

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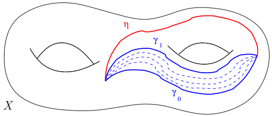
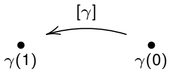


$$\Pi_1(X) = \{[\gamma] \mid \gamma : [0,1] \to X\}$$
$$\mathbf{t} \downarrow\downarrow \mathbf{s}$$
$$X$$



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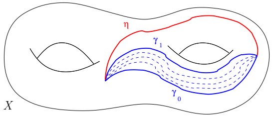

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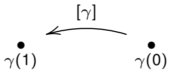
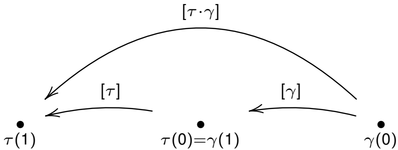


- **product**:



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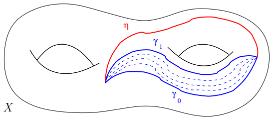

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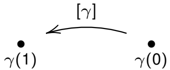

- **identity**:

$$u : X \hookrightarrow \Pi_1(X)$$

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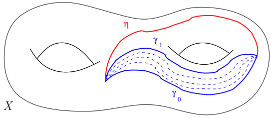

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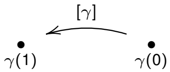
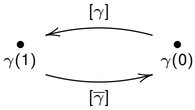


- **inverse**:

$$\iota : G \longrightarrow G$$



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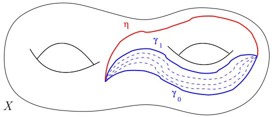

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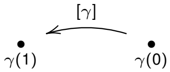

- The space $\Pi_1(X)$ has a natural topology and the source, target, multiplication and inverse are all continous maps: $\Pi_1(X) \rightrightarrows X$ is an example of a topological groupoid.

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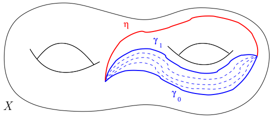

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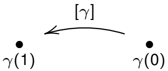
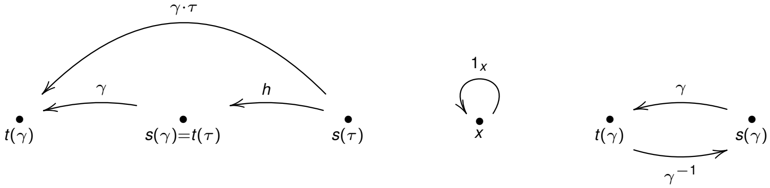


- The space $\Pi_1(X)$ has a natural topology and the source, target, multiplication and inverse are all continous maps: $\Pi_1(X) \rightrightarrows X$ is an example of a topological groupoid.
- If X is a manifold, the space $\Pi_1(X)$ is a manifold and the source, target, multiplication and inverse are all smooth maps: then $\Pi_1(X) \rightrightarrows X$ is an example of a Lie groupoid.

Lie Groupoids

A Lie groupoid is a pair of submersions $s, t : \mathcal{G} \rightrightarrows X$, together with partial composition, identity and inversion maps satisfying the obvious axioms.



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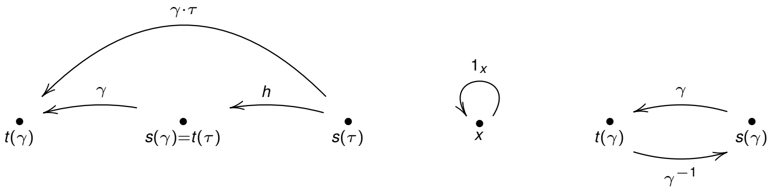

Examples:

- Pair groupoid: $X \times X \rightrightarrows X$;
- Lie groups: $G \rightrightarrows \{*\}$;
- Action groupoid: $G \times X \rightrightarrows X$.

Lie Groupoids

Given a Lie groupoid $\mathcal{G} \rightrightarrows X$:

- **source fibers** $s^{-1}(x)$ and **target fibers** $t^{-1}(x)$;
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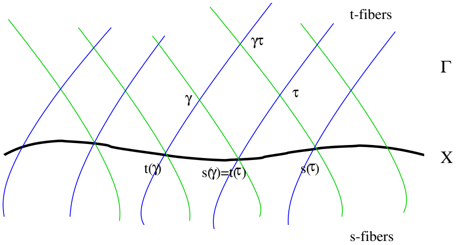

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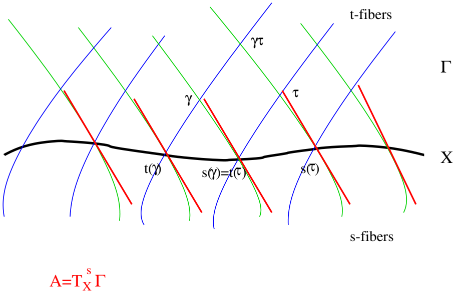

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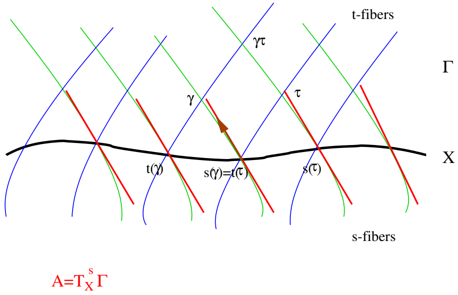

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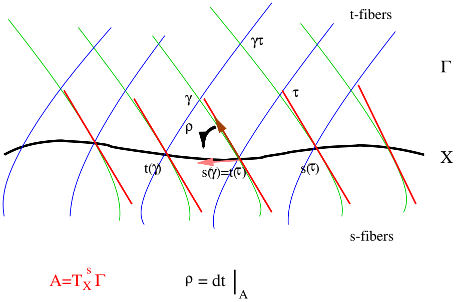

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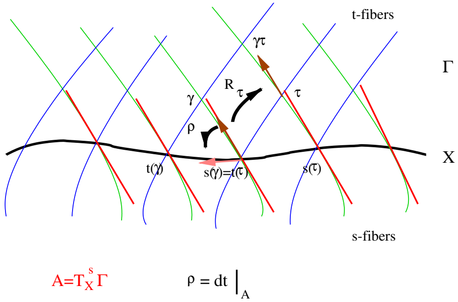

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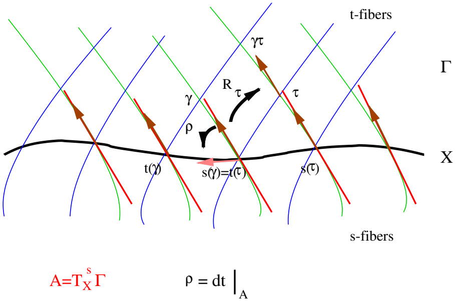

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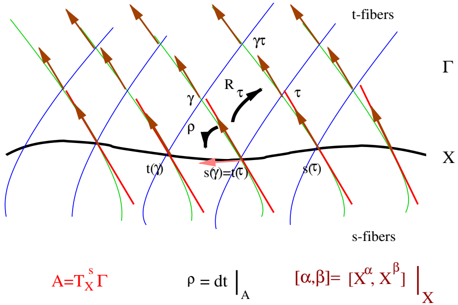
$$A := T^s_X\mathcal{G}, \quad \rho := \mathrm{d}t|_A, \quad [\ ,\ ]_A := \textbf{Lie bracket of } \mathfrak{X}_{\text{R-inv}}(\mathcal{G}) \equiv \Gamma(A).$$



Lie Groupoids

Examples:

- The pair groupoid $X \times X \rightrightarrows X$ and fundamental groupod $\Pi_1(X) \rightrightarrows X$ integrate the same Lie algebroid: $A = TX$;
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Basic Theorems:

- **Lie I:** Given a source connected Lie groupoid $\mathcal{G} \rightrightarrows X$ there is a unique source 1-connected Lie groupoid $\tilde{\mathcal{G}} \rightrightarrows X$ with the same Lie algebroid and a unique étale morphism of Lie groupoids $\tilde{\mathcal{G}} \to \mathcal{G}$;

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- **Lie III:** Not every Lie algebroid integrates to a Lie groupoid. Obstructions are completely understood [Crainic & RLF, 2003].

Lie Groupoids

For a Lie groupoid $\mathcal{G} \rightrightarrows X$ with algebroid $A \to X$, its Maurer-Cartan form is the s-foliated A-valued 1-form:

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Theorem (RLF & Struchiner, 2014)

If $A = X \times \mathbb{R}^n \to X$ is the trivial vector bundle, then the restrition $\omega_{MC}|_{s^{-1}(x)}$ to any source fiber $s^{-1}(x)$ is a coframe, and together with the target gives a Lie algebroid morphism:

$$\begin{array}{ccc} T(s^{-1}(x)) & \xrightarrow{\ \omega_{MC}|_{s^{-1}(x)}\ } & A \\ \downarrow & & \downarrow \\ s^{-1}(x) & \xrightarrow[\ t\ ]{} & X \end{array}$$

These solutions are universal: if a coframe (P, θ) induces an algebroid map $TP \to A$, there is a unique (local) isomorphism:

$$\begin{array}{ccccc} TP & & \overset{\phi_*}{\dashrightarrow} & & T^{\mathbf{s}}\mathcal{G} \\ \downarrow & \searrow^{\theta} & & \swarrow^{\omega_{MC}} & \downarrow \\ & & A & & \\ P & & \overset{\phi}{\dashrightarrow} & & \mathcal{G} \\ & \searrow_{h} & \downarrow & \swarrow_{\mathbf{t}} & \\ & & X & & \end{array}$$

Lie Algebroids and G-structures

Back to G-structures... notice that:

- The previous construction was about coframes ($\Leftrightarrow \{e\}$-structures);
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Key Remark: should take into consideration that the coframe in $P = \mathrm{F}_G(M)$ takes the special form (θ, η), where $\theta \in \Omega^1(P, \mathbb{R}^n)$ and $\eta \in \Omega^1(P, \mathfrak{g})$.

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Simplifying assumption: Henceforth, we assume that *G* is a compact, connected, Lie group.

Cartan Data and **Realizations** are formalized as follows:

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Let $G \subset \mathrm{GL}_n(\mathbb{R})$ and X a G-manifold. A Lie G-algebroid is a Lie algebroid $A \to X$:

- A is the trivial vector bundle with fiber $\mathbb{R}^n \oplus \mathfrak{g}$;
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$$\rho(u, \alpha) = F(u) + \psi(\alpha), \quad (u, \alpha) \in \mathbb{R}^n \oplus \mathfrak{g},$$
- the bracket on constant sections $(u, \alpha), (v, \beta) \in \Gamma(A)$ takes the form
$$[(u, \alpha), (v, \beta)] = (\alpha \cdot v - \beta \cdot u - c(u, v), [\alpha, \beta]_{\mathfrak{g}} - R(u, v)),$$
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Definition

A G-realization of a Lie G-algebroid $A \to X$ consists of a manifold P, equipped with a locally free, proper, G-action, together with an equivariant Lie algebroid map:

$$\begin{array}{ccc} TP & \xrightarrow{(\theta,\eta)} & A \\ \downarrow & & \downarrow \\ P & \xrightarrow{h} & X \end{array}$$

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In particular, each source fiber $s^{-1}(x)$ is a principal G-bundle over the orbifold

$$M = s^{-1}(x)/G.$$

Solving the classification problem

Theorem

If $\mathcal{G} \rightrightarrows X$ is a Lie G-groupoid integrating a Lie G-algebroid $A \to X$, then each source fiber $\mathbf{s}^{-1}(x)$ equipped with the restriction of the Maurer-Cartan form ω_{MC} yields a G-realization of A. Moreover, any G-realization of A is isomorphic to a G-invariant, open subset of one such G-realization (up to cover).

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For any value of (S_0, T_0, U_0), there is unique, up to isomorphism, (germ of) Bochner-Kähler orbifold (M, g, J, ω) whose invariants (S, T, U) take the value (S_0, T_0, U_0).

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Remarks:

- Finding complete solutions, depends on having G-integrations. There is an obstruction theory (G-monodromy) that solves this problem and does not require finding explicit G-integrations!
- Finding explicit solutions, depend on finding explicit G-integrations. One can recover in this way all known Bochner-Kähler metrics
- Similar results hold for other problems...

An explicit example : : Metrics of Hessian Curvature

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The associated classifying Lie G-algebroid is $A = \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, with Lie bracket and anchor:

$$[\alpha_1, \alpha_2] = -k\beta \quad [\alpha_1, \beta] = \alpha_2 \quad [\alpha_2, \beta] = -\alpha_1$$

$$\rho(\alpha_1) = k_1 \frac{\partial}{\partial k} + \frac{1}{2}(1-k^2)\frac{\partial}{\partial k_1}$$

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Computing the obstructions (infinitesimal G-monodromy):

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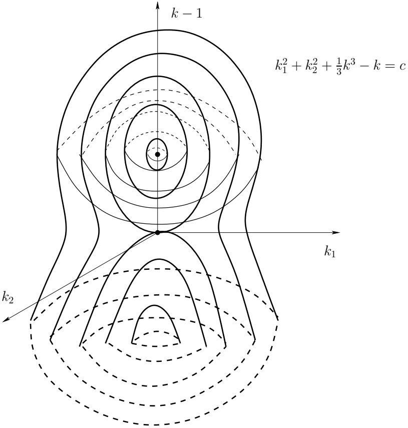
- At the two fixed points $(0, 0, 1)$ and $(0, 0, -1)$, there are solutions (constant curvature metrics);
- In the region filled by spheres there does not exist a G-integration for almost every leaf (but there exists G-integrations on some spheres);
- Over every other leaf in the other regions there exist G-integrations.



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THANK YOU!