Large scale conformal geometry

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X, X' metric spaces. Map $f : X \to X'$ is a coarse embedding if

$$\alpha(d(x,x')) \leq d(f(x),f(x')) \leq \omega(d(x,x')),$$

where $\alpha \to +\infty$, $\omega < +\infty$.

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New input: sharp invariants which take continuously many values.

Sample new result: there can be no **coarse conformal map** between Jorge Lauret's quasi-abelian groups S_A and S_B if

$$\frac{\Re e(\mathrm{trace}(A))}{\min \Re e(\mathrm{sp}(A))} > \frac{\Re(\mathrm{trace}(B))}{\min \Re e(\mathrm{sp}(B))} > 0.$$

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Mercator's map is *conformal* : angles between curves are preserved.

Distance distorsion blows up at poles.

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Corollary (Liouville)

Let $f : \mathbb{C} \to D$ *be a holomorphic function. Then* f *is constant.*

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Disappointing : conformal diffeomorphisms are rare.
In dimension $n \geq 3$, every conformal diffeomorphism between Euclidean domains is the restriction of a global conformal diffeomorphism of the sphere S^n, i.e. an element of the Möbius group $O(n+1,1)$.

Definition

A diffeomorphism between Riemannian manifolds is quasiconformal if its differential maps infinitesimal spheres to infinitesimal ellipsoids of bounded excentricity.

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Proof : quasi-Schwartz Lemma. A quasiconformal diffeomorphism f of the ball satisfies

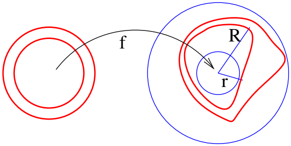
$$\frac{1}{L}d(x,x') - C \leq d(f(x),f(x')) \leq Ld(x,x') + C$$

with respect to hyperbolic distance. It is a *quasiisometry*, i.e. a large scale biLipschitz map.

Quasisymmetric : homeomorphism between metric spaces such that

$$\forall x,\ \forall x',\ \forall x'',\quad \frac{d(f(x),f(x'))}{d(f(x),f(x''))}\leq \eta(\frac{d(x,x')}{d(x,x'')}),$$

where $\eta\in Homeo(\mathbb{R}_+)$.



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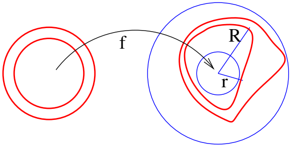

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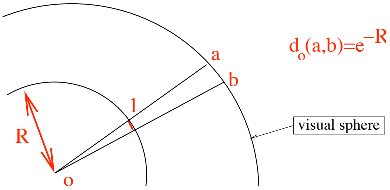
Remark. Quasisymmetric $\implies$ quasiconformal.

Theorem (H. Grötsch 1928)

Quasiconformal homeomorphisms of Euclidean spaces are quasisymmetric.

A *hyperbolic group* has an ideal boundary ∂G, a compact space equipped with a family of *visual distances*.



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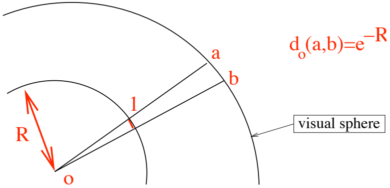

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Furthermore, visual distances are Ahlfors-regular.

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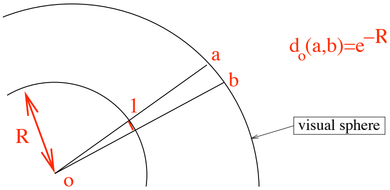

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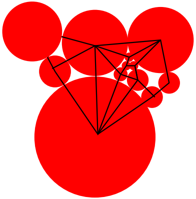
Whence the notion of *quasisymmetric gauge*: the equivalence class of Ahlfors-regular distances on ∂G which are quasisymmetric to a visual distance.

This "microscopic conformal structure" determines the large scale geometry of G.

Sphere packing : collection of balls with disjoint interiors.

Incidence graph : one vertex per ball, an edge when two balls touch.



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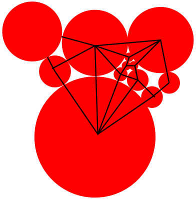

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Theorem (Koebe 1931)

A graph is packable in $\mathbb{R}^2$ if and only if it is simple and planar (i.e. embeddable into $\mathbb{R}^2$).

Interpretation : mesoscopic version of Riemann's conformal mapping theorem.

Notes.

- There are fast algorithms to compute Koebe's packing.
- Apply Koebe's theorem to the equal disk packing of a plane domain. The obtained discrete map converges to Riemann's conformal mapping as radius tends to 0.
- This is an efficient algorithmic way to compute an approximation of Riemann's conformal mapping.
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In higher dimensions, little is known.

Theorem (Benjamini-Schramm 2013)

The grid in $\mathbb{R}^n$ is packable in $\mathbb{R}^d$ iff $n \leq d$. There exist lattices of hyperbolic n-space whose Cayley graphs are packable in $\mathbb{R}^d$ only if $n \leq d$.

(N, ℓ, R, S)-**packings**: This means a countable collection of balls B_j of radii $\in [R, S]$ such that the collection of concentric balls ℓB_j has multiplicity $\leq N$.

Example: A usual disk or ball packing is a $(1, 1, 0, \infty)$-packing.

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Definition

A coarse conformal map is a map $f : X \to X'$ between metric spaces, enriched with a correpondance between balls $B \mapsto B'$ such that $f(B) \subset B'$, and there exist $R > 0$ such that $\forall S \geq R$ and $\ell' \geq 1$, there exist $N' \geq 1$ and $\ell \geq 1$ such that f maps $(1, \ell, R, S)$-packings to $(N', \ell', 0, \infty)$-packings.

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Coarse conformal maps can be precomposed with coarse embeddings and postcomposed with quasisymmetric homeomorphisms.

Examples.

- Every nilpotent group c.c. maps to some $\mathbb{R}^N$ (Assouad).
- Every hyperbolic group c.c. maps to some $O(N, 1)$ (Bonk-Schramm).
- $z \mapsto z|z|^{K-1}$ is coarsely conformal on $\mathbb{R}^N$.
- For every hyperbolic group G, the Poincaré model $G \to O(1, 1) \times \partial G$ is coarsely conformal.

Questions. What is the optimal N such that

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Partial answer. A necessary condition is that some dimension increases. "Non-squeezing theorem".

For nilpotent groups G, relevant dimension $d_1(G) = d_2(G)$ is exponent of volume growth.

For hyperbolic groups G, two candidates, conformal dimension of ideal boundary $d_2(G)$, or cohomological dimension (least p such that ℓ^p-cohomology does not vanish) $d_1(G)$.

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Open problem. Find more numerical invariants that must increase under coarse conformal maps.

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Lauret's quasi-abelian groups S_A with $\min \Re e(\mathrm{sp}(A)) > 0$ satisfy

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Isometry groups of Bourdon buildings satisfy $d = d_1(G) = d_2(G)$ that cover a dense subset of $[1, +\infty)$. Weird coarse (in fact, quasiisometric) embeddings exist among them. Theorem states that d must increase. Alternate proof of this due to Hume-Mackay-Tessera 2017, based on p-separation.

p-**energy**: if u is a map to a metric space,

$$E_{p,\ell,R,S}(u)=\sup_{(1,\ell,R,S)-packings\ \{B_j\}}\sum_j osc(u_{|B_j})^p.$$

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p-**modulus** of a curve family $\mathcal{F}$: it is the inf of p-energies of maps u to metric spaces such that $\mathrm{length}(u\circ\sigma)\geq 1$ for each curve $\sigma\in\mathcal{F}$.

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A finitely generated group G is p-parabolic iff the volume growth is polynomial of degree $d(G) \leq p$.

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Corollary. No c.c. map or coarse embedding $G \to G'$ if G' is nilpotent and G hyperbolic, or G nilpotent with $d(G) > d(G')$.

If G' is hyperbolic, compose the c.c. map $G \to G'$ with the *generalized Poincaré model* $G' \to \mathbb{R} \times \partial G'$. The resulting map is roughly conformal. Furthermore, $\mathbb{R} \times \partial G'$ is p-Ahlfors regular for every $p > \dim_{AR}(G')$.

Theorem (Benjamini-Schramm 2013 revisited)

Let Y be a compact, Q-Ahlfors-regular metric space. If there exists a roughly conformal map $f : X \to Y$, then

1. *either X is Q-parabolic,*
2. *or $\ell^Q \bar{H}^1(X) \neq 0$.*

If G is hyperbolic, it is p-parabolic for no p, so the obstruction comes from reduced ℓ^p cohomology, whence $d_1(G)$.

If G is nilpotent, it has vanishing reduced ℓ^p cohomology for all p, so the obstruction comes from p-parabolicity.

Cochains are functions $\kappa : X^{k+1} \to \mathbb{R}$. The ℓ^p *norm* of κ is the sup of sums $(\sum_j \sup |\kappa_{|B_j}|^p)^{1/p}$ over all $(1, \ell, R, S)$-packings (parameters ℓ, R, S are fixed).

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Proposition

For bounded geometry uniformly contractible Riemannian manifolds or simplicial complexes, for every $0 < R \leq S < \infty$, this coincides with previous definitions.

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If G is a hyperbolic group, $\ell^p H^1 = \ell^p \bar{H}^1$ vanishes for $p < d_1(G)$ and does not vanish for $p > d_2(G)$.

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5. If $\ell^Q \bar{H}^1(X) = 0$, these limits are Q-almost always the same, y.

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Proof.

1. Naturality : f transports finite energy maps. In particular, if Y is Q-Ahlfors-regular, $E_Q(f) < \infty$.
2. If $\ell^Q H^1(X) = 0$, every finite Q-energy map has a limit at infinity. Thus f has a limit $y \in Y$.
3. On $Y \setminus \{y\}$, there exists a finite energy function v_y that tends to $+\infty$ at y. Thus $v_y \circ f$ has finite energy and tends to $+\infty$, contradiction.
4. Finite energy maps have limits along almost every curve. So f has limits along Q-almost every curve.
5. If $\ell^Q \bar{H}^1(X) = 0$, these limits are Q-almost always the same, y.
6. Then $v_y \circ f$ has finite energy and tends to $+\infty$ along d-almost every curve. Thus Q-almost every curve = no curve, i.e. X is Q-parabolic.