Relative Volume Comparison along Ricci Flow

Gang Tian

Peking University

- Ricci flow and relative volume comparison
- Sketched proof for the relative volume comparison
- An application

Let M be a closed manifold of dimension n.

Consider the Ricci flow introduced by R. Hamilton in 1982:

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}(t) \tag{1}$$

where $\operatorname{Ric}(t)$ stands for the Ricci curvature tensor of $g(t)$.

Many powerful analytic tools for Ricci flow were developed in 90s, e.g.,

- Maximal principle for tensors (Hamilton)
- Differential Harnack (Hamilton)
- Derivative estimates of curvature (Shi)

In 2002, Perelman gave a non-collapsing estimate which is a breakthrough.

Perelman: For any $A > 0$ and dimension n, there exists $\kappa = \kappa(n, A) > 0$ with the following property. If $g(t)$, $0 \leq t \leq r_0^2$, is a solution to the Ricci flow on an n-dimensional manifold satisfying:

$$|Rm|(x,t) \;\leq\; r_0^{-2},\, \forall (x,t) \in B_{g(0)}(x_0, r_0) \times [0, r_0^2]$$

and

$$\mathrm{vol}\left(B_{g(0)}(x_0, r_0)\right) \;\geq\; A^{-1} r_0^n,$$

then, for $B_{g(r_0^2)}(x, r) \subset B_{g(r_0^2)}(x_0, A r_0)$ such that

$$|Rm|(y,t) \;\leq\; r^{-2},\, \forall (y,t) \in B_{g(r_0^2)}(x,r) \times [r_0^2 - r^2, r_0^2], \quad (2)$$

we have

$$\mathrm{vol}_{g(r_0^2)}\left(B_{g(r_0^2)}(x, r)\right) \;\geq\; \kappa r^n. \quad (3)$$

This theorem is proved by Perelman by using monotonicity of the reduced volume he introduced. It plays a crucial role in his proof towards the Poincaré Conjecture and was used to rule out possibility of developing finite-time singularity along surfaces of positive genera.

The assumption (2) on the parabolic ball can be weakened to be

$$R(y, r_0^2) \ \leq \ r_0^{-2}, \forall y \in B_{g(r_0^2)}(x, r) \tag{4}$$

on the time slice r_0^2.

This was first proved by Q. Zhang in his book and also by W. Bing more recently.

Such an improvement of Perelman's non-collapsing estimate is very useful and requires new ideas in its proof.

Z. L Zhang and I proved (2018):

For any n and $A \geq 1$, there exists $\kappa = \kappa(n, A) > 0$ such that the following holds. Let $g(t)$, $0 \leq t \leq r_0^2$, be a solution to the Ricci flow on an n-dimensional manifold M such that

$$|\operatorname{Ric}(t)| \leq r_0^{-2}, \text{ on } B_{g(0)}(x_0, r_0) \times [0, r_0^2]. \tag{5}$$

Then, for $B_{g(r_0^2)}(x, r) \subset B_{g(r_0^2)}(x_0, Ar_0)$ $(r \leq r_0)$ satisfying:

$$R(\cdot, r_0^2) \leq r^{-2} \text{ in } B_{g(r_0^2)}(x, r), \tag{6}$$

we have

$$\frac{\operatorname{vol}_{g(r_0^2)}(B_{g(r_0^2)}(x, r))}{r^n} \geq \kappa \frac{\operatorname{vol}_{g(0)}(B_{g(0)}(x_0, r_0))}{r_0^n}. \tag{7}$$

This theorem generalizes Perelman's non-local collapsing theorem.

It can be also regarded as a generalization of the Bishop-Gromov volume comparison: If (M, g) is a Riemannian manifold with $\mathrm{Ric}(g) \geq 0$, then for any $0 < r \leq r_0$,

$$\frac{\mathrm{vol}_g(B(x,r))}{r^n} \geq \frac{\mathrm{vol}_g(B(x,r_0))}{r_0^n}.$$

This volume comparison has played a crucial role in the study of Riemannian manifolds with Ricci curvature bounded from below, e.g., in the Cheeger-Colding theory.

The proof uses a localized version of the entropy, instead of the reduced volume as Perelman did.

The crucial difference between Perelman's theorem and ours is that the initial metric may collapse. This causes substantial difficulties, so that our proof is more involved than Perelman's non-collapsing.

Next we will show some ideas in the proof.

Let us recall Perelman's $\mathcal{W}$-functional

$$\mathcal{W}(g, f, \tau) = \int_M \left[\tau(R + |\nabla f|^2) + f - n\right](4\pi\tau)^{-n/2} e^{-f} dv.$$

Putting $u = (4\pi\tau)^{-n/4} e^{-f/2}$, we can rewrite it as

$$\tau \int_M (Ru^2 + 4|\nabla u|^2) dv - \int_M u^2 \log u^2 dv - \frac{n}{2} \log(4\pi\tau) - n. \tag{8}$$

Let Ω be any bounded domain of M. Define the local entropy

$$\mu_\Omega(g,\tau) = \inf\left\{\mathcal{W}(g,u,\tau)\,\Big|\, u\in C_0^\infty(\Omega),\ \int_\Omega u^2 = 1\right\}.$$

When $\Omega = M$, it is Perelman's original entropy which is monotonic along Ricci flow as Perelman showed. However, in general, local entropy is not monotonic. This causes new difficulties in using it.

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The local entropy relates closely to the local volume ratio.

Let $B(x,2r) \subset M$ be a metric ball with $\partial B(x,2r) \neq \emptyset$. If

$$\mathrm{Ric} \geq -r^{-2}, \quad \text{on} \quad B(x,2r), \tag{9}$$

then

$$\log \frac{\mathrm{vol}_g(B(x,r))}{r^n} \leq \inf_{0<\tau\leq r^2} \mu_{B(x,r)}(g,\tau) + C(n). \tag{10}$$

Conversely, if the scalar curvature

$$R \leq n\, r^{-2}, \quad \text{on} \quad B(x, r), \tag{11}$$

then,

$$\log \frac{\mathrm{vol}(B(x, r))}{r^n} \geq \inf_{0<\tau\leq r^2} \mu_{B(x,r)}(g, \tau) - C(n). \tag{12}$$

We may assume that $(M, g(t))$ exists on $[0,1]$ and $r_0 = 1$. Let $x_0 \in M$, $A \geq 1$ and u_A be a minimizer of $\mu_{B_{g(1)}(x_0,A)}(g(1),\tau)$ for some $0 < \tau \leq 1$. Let v be a solution to the conjugate heat equation of Ricci flow

$$-\frac{\partial}{\partial t}v = \triangle v - Rv, \quad v(1) = u_A^2. \tag{13}$$

Put $B_t = B_{g(t)}(x_0, 1)$. We have

$$\begin{aligned}\mu_{B_t}\big(g(t), \tau + 1 - t\big) &\leq \mu_{B_{g(1)}(x_0,A)}(g(1),\tau) \\ &+ C \cdot \left(\int_{B_{g(t)}(x_0,\frac{1}{4})} v(t) dv_{g(t)}\right)^{-1}.\end{aligned}$$

Let $B' = B_{g(1)}(x, r) \subset B_{g(1)}(x_0, A)$ at $t = 1$, then

$$\mu_{B_t}\big(g(t), \tau + 1 - t\big) \le \mu_{B'}(g(1), \tau) + C\left(\int_{B_{g(t)(x_0, \frac{1}{4})}} v(t)\, dv_{g(t)}\right)^{-1}.$$

If $\mathrm{Ric} \geq -1$ in $B_{g(t)}(x_0, 2)$ and $R \leq r^{-2}$ in B', then, at time t,

$$\log \mathrm{vol}(B_t) \leq \log \frac{\mathrm{vol}(B')}{r^n} + C(n) + C\left(\int_{B_{g(t)(x_0, \frac{r}{2})}} v(t) dv_{g(t)}\right)^{-1}. \tag{14}$$

How to estimate lower bound of $\int_{B_{g(t)(x_0, \frac{r}{2})}} v(t) dv_{g(t)}$?

We will need a new estimate on the following heat kernel.

Let $H(x,t;y,s)$ $(0 \le t < s \le 1)$ be the heat kernel to the conjugate heat equation which satisfies

$$-\frac{\partial H}{\partial t} = \Delta_{g(t),x} H - R(x,t) H \tag{15}$$

with $\lim_{t\to s} H(x,t;y,s) = \delta_{g(s),y}(x)$. Also we have

$$\frac{\partial H}{\partial s} = \Delta_{g(s),y} H, \quad \lim_{s\to t} H(x,t;y,s) = \delta_{g(t),x}(y). \tag{16}$$

Here $\delta_{g(t),x}$ denotes the Dirac function concentrated at (x,t) w.r.t. the measure induced by $g(t)$.

The solution v satisfies:

$$v(x,t) = \int_M H(x,t;y,1) \cdot v(y)\, dv_{g(1)}(y) \tag{17}$$

for any $t < 1$. So it suffices to prove lower bound of the conjugate heat kernel.

Tian-Z.L.Zhang (2018): There exists $0 < t_0 = t_0(n) \leq \frac{1}{200}$ such that for any $A \geq 1$ and $0 < t \leq t_0$,

$$H(x, 1-t; y, 1) \geq \frac{C(n,A)^{-1}}{\mathrm{vol}_{g(1)}(B_{g(1)}(x_0, \sqrt{t}))} \tag{18}$$

for any $x \in B_{g(1)}(x_0, e^{-2})$ and $y \in B_{g(1)}(x_0, A\sqrt{t})$.

Now we give an application of our relative volume comparison.

Let X be a compact Kähler manifold of complex dimension m with a Kähler metric g. In complex coordinate $z_1, \cdots, z_m)$, g is represented by a positive Hermitian matrix-valued function $(g_{i\bar{j}})$, moreover, its associated Kähler form ω is closed, where

$$\omega = \sqrt{-1}\, g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j.$$

Then the Ricci flow reads (up to a scaling)

$$\frac{\partial \omega}{\partial t} = -\operatorname{Ric}, \tag{19}$$

where $\operatorname{Ric} = \sqrt{-1}\, R_{i\bar{j}} dz_i \wedge d\bar{z}_j$ is the Ricci form of ω.

Tian-Zhou Zhang (2006): For any initial metric ω_0, the Kähler-Ricci flow exists up to

$$T = \sup\{t > 0 \mid [\omega_0] - 2\pi t c_1 > 0\}. \tag{20}$$

Hence, if $K_X = \det(T^*X)$ is nef, then the Kähler-Ricci flow has a global solution for all time.

Suppose that X is a smooth minimal model, i.e., $K_X \geq 0$. Then

• When the Kodaira dimension $\kappa = 0$, i.e., X is Calabi-Yau, Cao proved (1986) that the Kähler-Ricci flow converges to the unique Ricci flat metric in $[\omega_0]$.

• When $\kappa = n$, i.e., X is of general type, we consider the ”normalized Kähler-Ricci flow” on X:

$$\frac{\partial \omega}{\partial t} = - \operatorname{Ric} - \omega. \tag{21}$$

We observe that if $\omega(t)$ is a soultion, then

$$[\omega(t)] = e^{-t}[\omega_0] + (1 - e^{-t}) 2\pi K_X.$$

When $\kappa = n$, we have:

- Tsuji and Tian-Zhang: $\omega(t)$ converges to the unique Kähler-Einstein current ω_{KE} in $2\pi K_X$; the convergence is smooth on the ample locus of the canonical class K_X.

- J. Song: the Kähler-Einstein current ω_{KE} defines a metric on the canonical model X_{can}.

- When dimension $n \leq 3$, Tian-Z.L.Zhang proved that $(X, \omega(t))$ converges globally to (X_{can}, ω_{KE}) in the Cheeger-Gromov sense. (Guo-Song-Weinkove gave another proof when $n = 2$.)

- B. Wang: $(X, \omega(t))$ has uniform diameter bound and has a limit space. The limit should be (X_{can}, ω_{KE}) due to the AMMP, proposed by Song-Tian.

When $0 < \kappa < n$, we have

- Song-Tian proved the convergence of the Kähler-Ricci flow to $\pi^*\omega_{GKE}$ in the current sense; they also proved the C^0-convergence on the potential level. If X is an elliptic surface, we have the $C^{1,\alpha}_{loc}$-convergence of potentials on $X_{reg} = \pi^{-1}(X_{can} \backslash S)$ for any $\alpha < 1$.
- Fong-Zhang proved the $C^{1,\alpha}$-convergence of potentials when X is a global submersion over X_{can} and the Gromov-Hausdorff convergence in this special case.
- Tosatti-Weinkove-Yang improved Fong-Zhang estimate and showed that the metric $\omega(t)$ converges to $\pi^*\omega_{GKE}$ in the C^0_{loc}-topology on X_{reg}.
- When the generic fibres are tori, Fong-Zhang, Tosatti-Zhang proved the smooth convergence of $\omega(t)$ to $\pi^*\omega_{GKE}$ on X_{reg}.

Let $X_{GKE} = \overline{(X_{can} \backslash S, \omega_{GKE})}$, the metric completion of the regular set.

It was conjectured by Song-Tian that $(X, \omega(t))$ converges in the Gromov-Hausdorff topology to the generalized Kähler-Einstein space X_{GKE}.

It is known from Fong-Zhang and Tosatti-Weinkove-Yang that $(X, \omega(t))$ collapses the fibres locally uniformly on X_{reg}. The difficulty is how to control the size of the singular fibres under the Kähler-Ricci flow.

The relative volume comparison along Ricci flow opens up a way of overcoming this difficulty in controlling the singular fibres.

In 2018, Tian-Z.L. Zhang proved:

Let X be a compact Kähler manifold with $K_X \geq 0$ and Kodaira dimension 1. Suppose a Kähler-Ricci flow $\omega(t)$ on X satisfies

$$|\,\mathrm{Ric}\,| \leq \Lambda, \quad \text{on } \pi^{-1}(U) \times [0, \infty), \tag{22}$$

where Λ is a uniform constant and $U \subset X_{can} \backslash S$. Then $(X, \omega(t))$ converges in the Gromov-Hausdorff topology to the generalized Kähler-Einstein metric space X_{can}.

To prove the above, we first recall a result of Y.S. Zhang:

When $\kappa = 1$, the generalized Kähler-Einstein space X_{GKE} is compact. So, combining with the C^0-convergence of metric on the regular set X_{reg}, under the Kähler-Ricci flow, the singular fibres locates ”in a finite region”, with a uniformly bounded distance to U.

By using the relative volume comparison, we further show:

Choose any base point $x_0 \in U$, the collapsing rate of the singular fibres is controlled by the collapsing rate of regular fibres at x_0. Then our theorem follows.

Let X be a Kähler manifold with $K_X \geq 0$ and Kodaira dimension 1. If the generic fibres of $\pi : X \to X_{can}$ are tori, then any Kähler-Ricci flow on X converges in the Gromov-Hausdorff topology to the generalized Kähler-Einstein metric space X_{GKE}.

In particular, any Kähler-Ricci flow on a smooth minimal elliptic surface of Kodaira dimension 1 converges in the Gromov-Hausdorff topology to the generalized Kähler-Einstein metric space X_{GKE}.

Final Remarks:

• The last statement was conjectured by Song-Tian in 2006.

• The case of higher Kodaira dimension is not totally clear. One difficulty is about the diameter bound of the generalized Kähler-Einstein current ω_{GKE}. Another technical difficulty is about the Ricci curvature estimate under Kähler-Ricci flow.