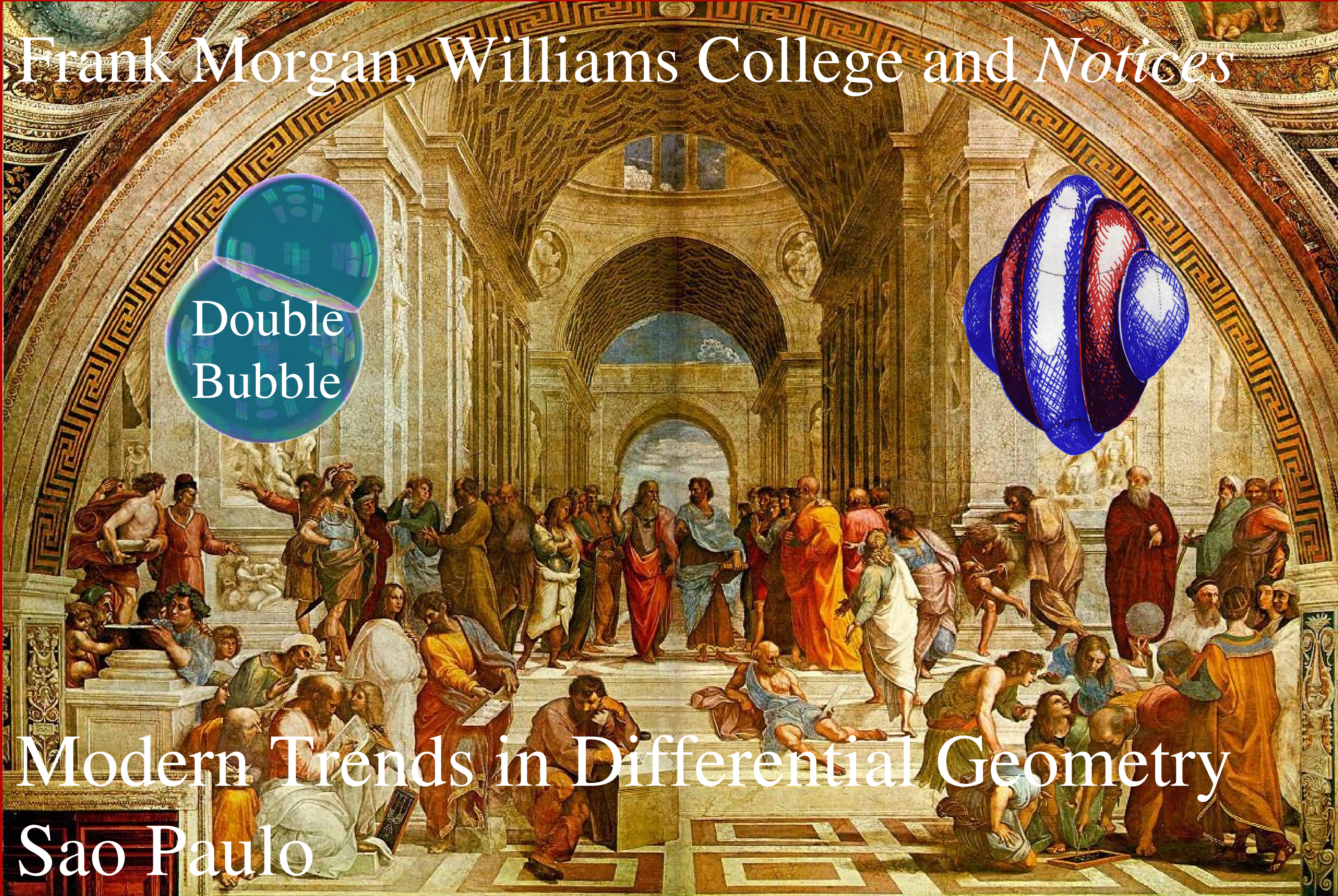
Isop. and Partitioning Problems

Frank Morgan, Williams College and *Notices*

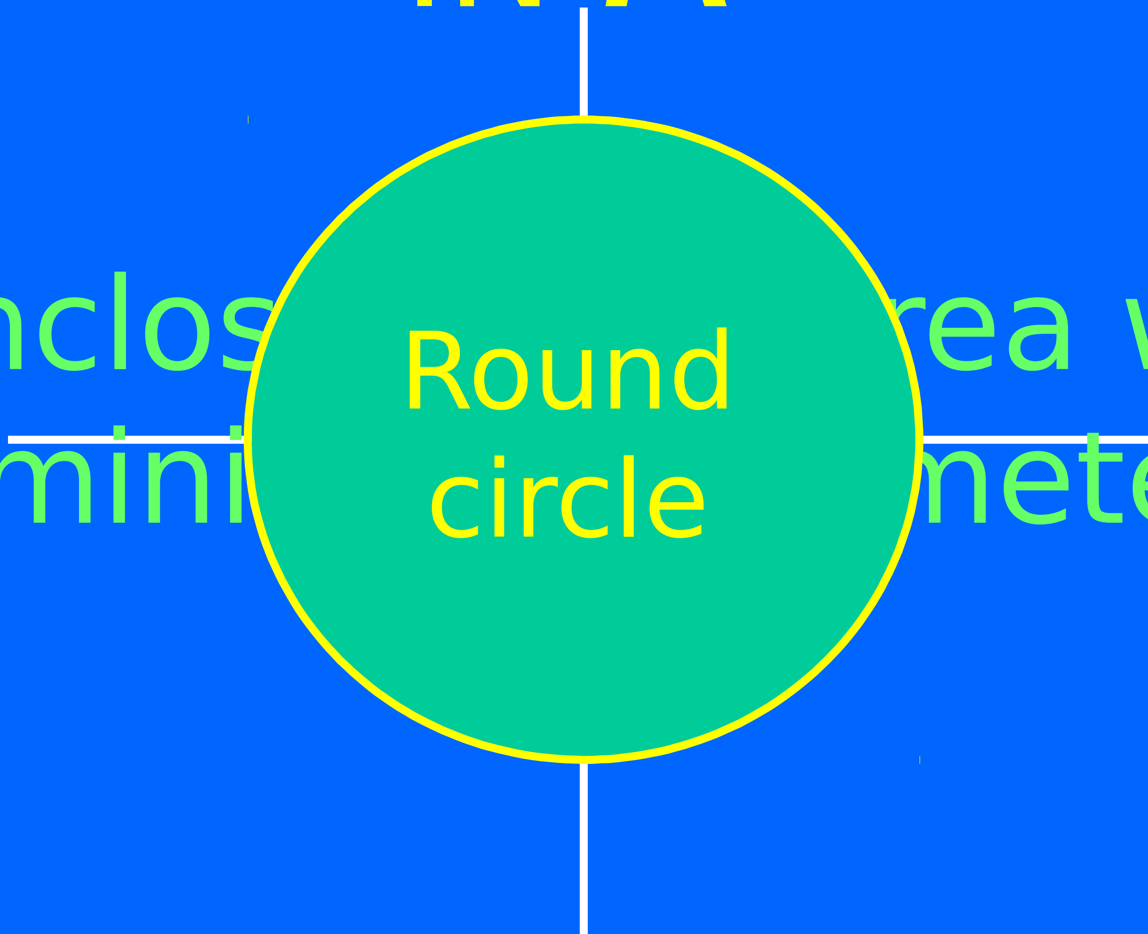


Modern Trends in Differential Geometry

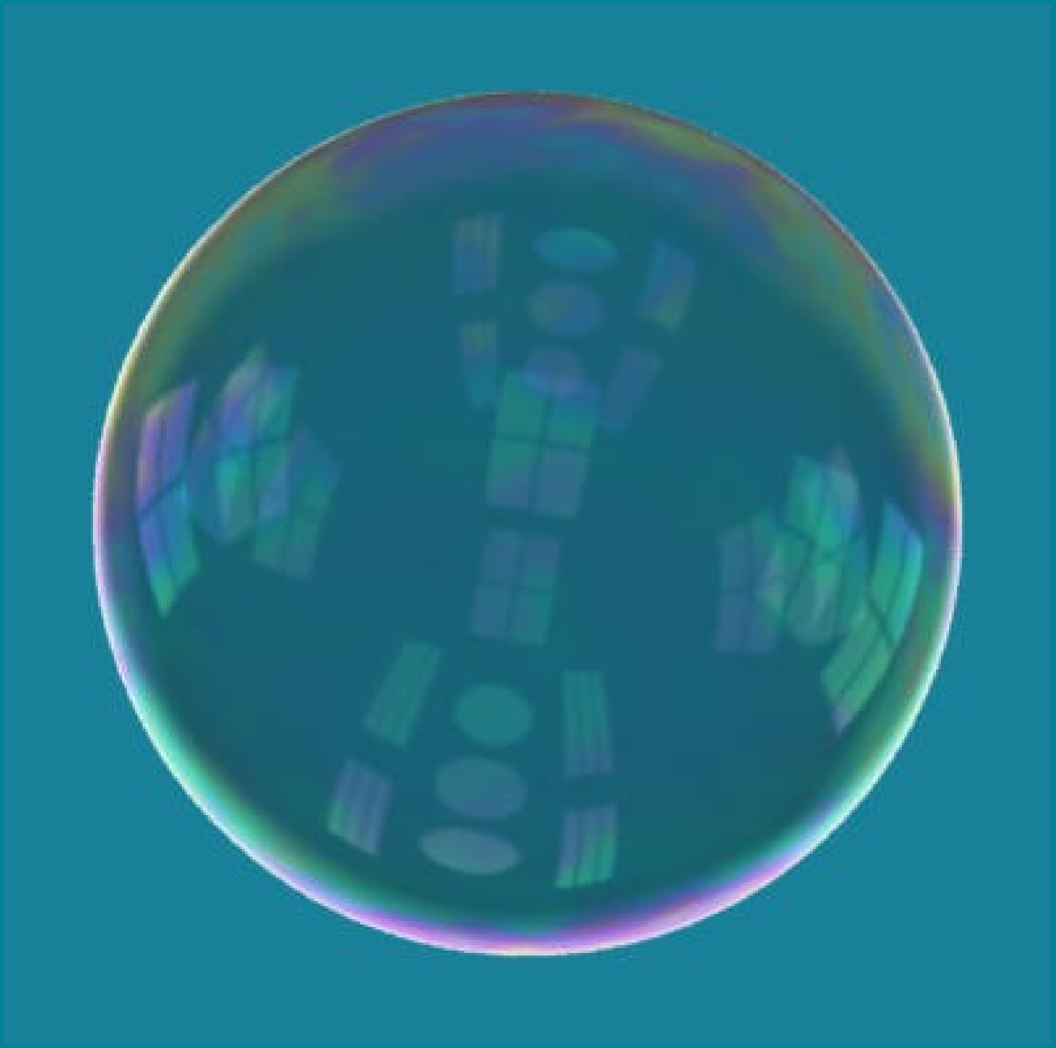
Sao Paulo

SINGLE BUBBLE IN R^2

Enclos … rea with
mini … meter.



SINGLE BUBBLE IN $\mathbf{R}^n$:
ROUND SPHERE by SYMMETRY



SIMILARLY
$\mathbf{S}^n$ and $\mathbf{H}^n$
Conj $\mathbf{CP}^2$

Diana Davis '07

GAUSS PLANE G^2

R^2 with Gaussian density

$$G^2$$

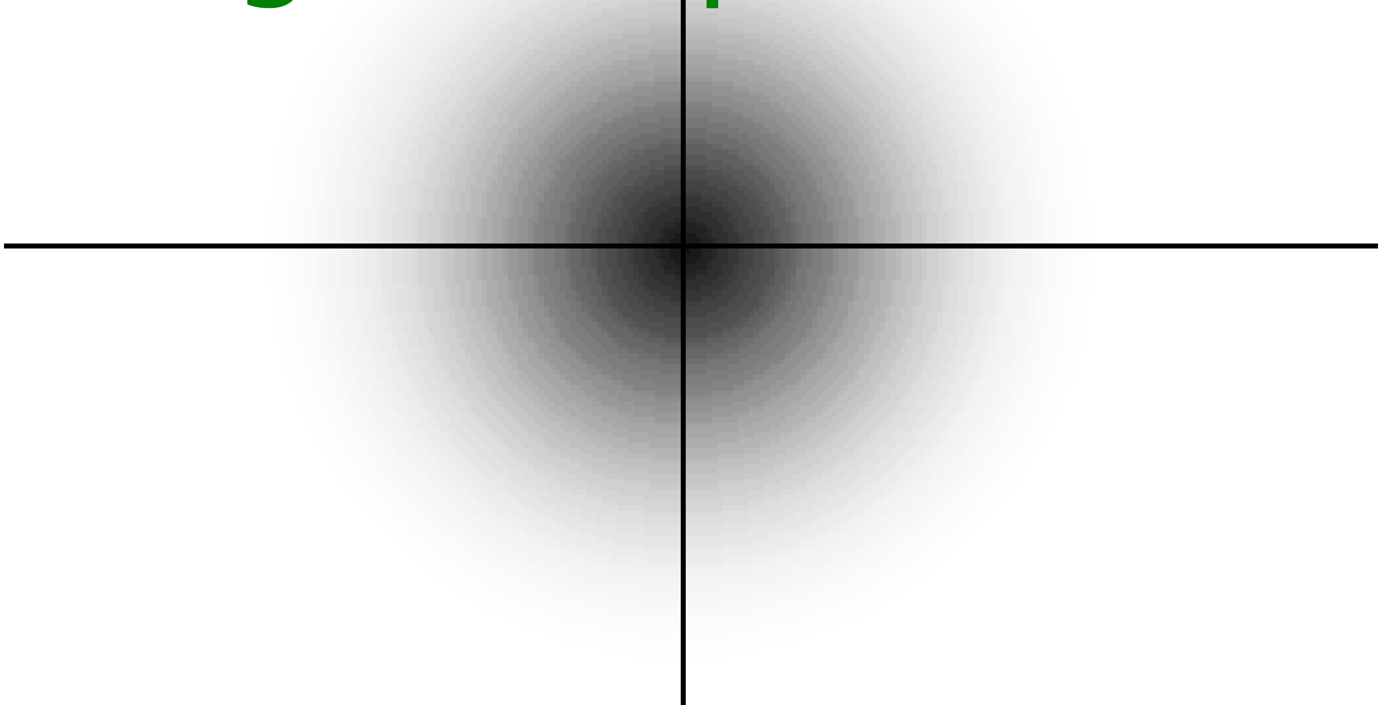
$$\Psi = (1/2\pi)\exp(-r^2/2)$$

Total weighted area one

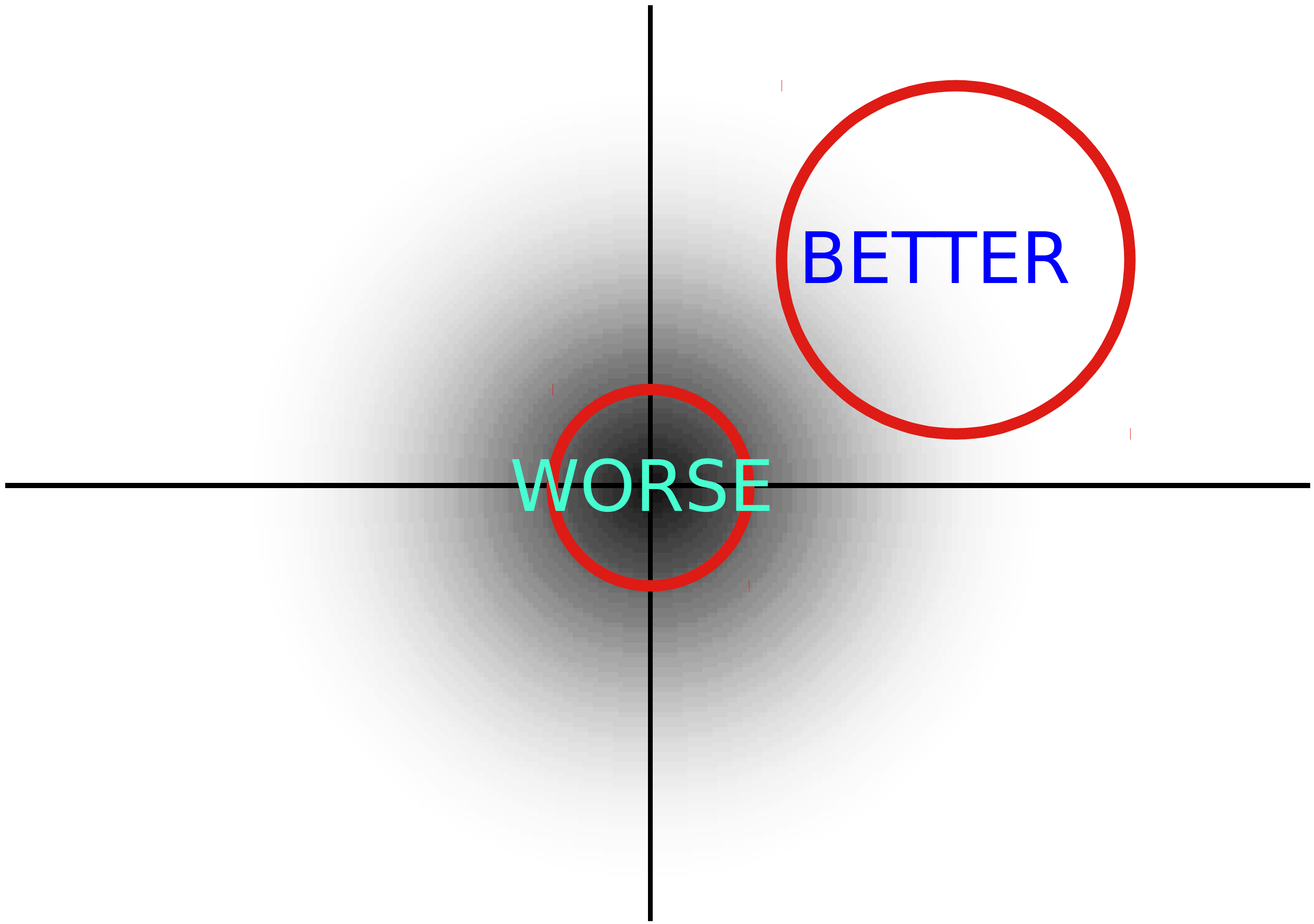
Brownian motion Stock option pricing Perelman's proof of Poincaré

Single Bubble in G^2

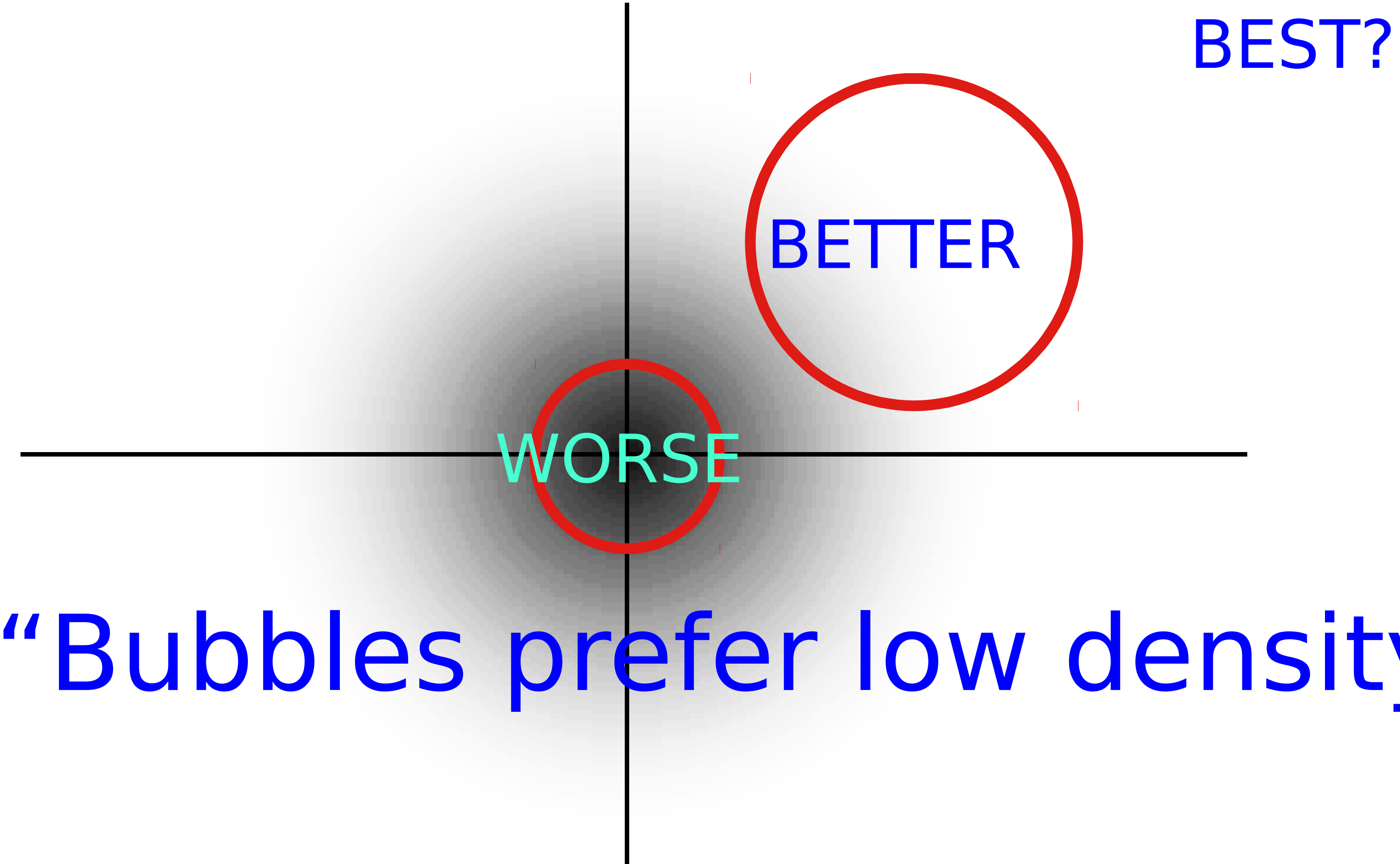
Enclose given (weighted) area with minimum (weighted) perimeter.



Bubble in G^2

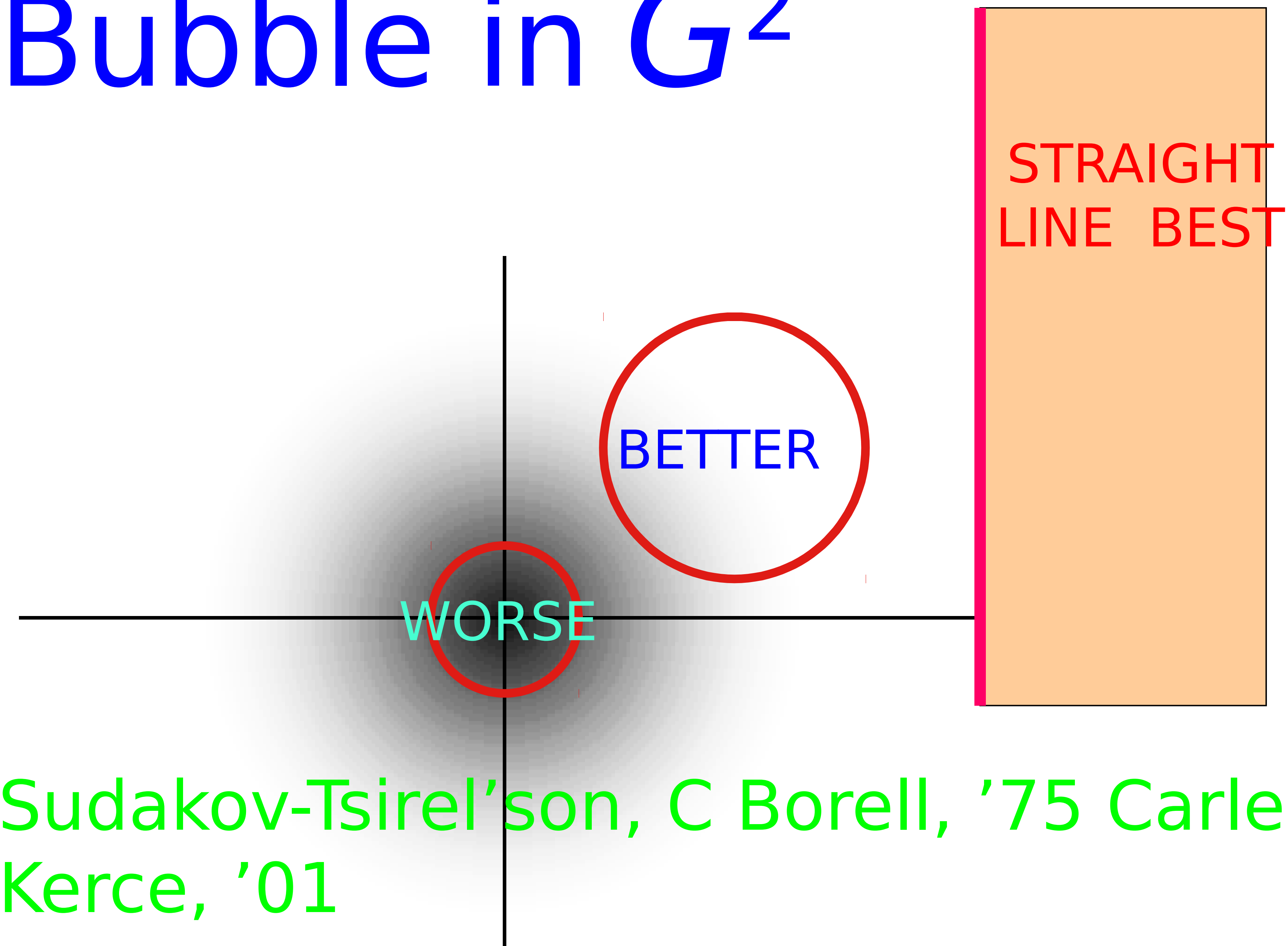


Bubble in G^2



“Bubbles prefer low density.”

Bubble in G^2



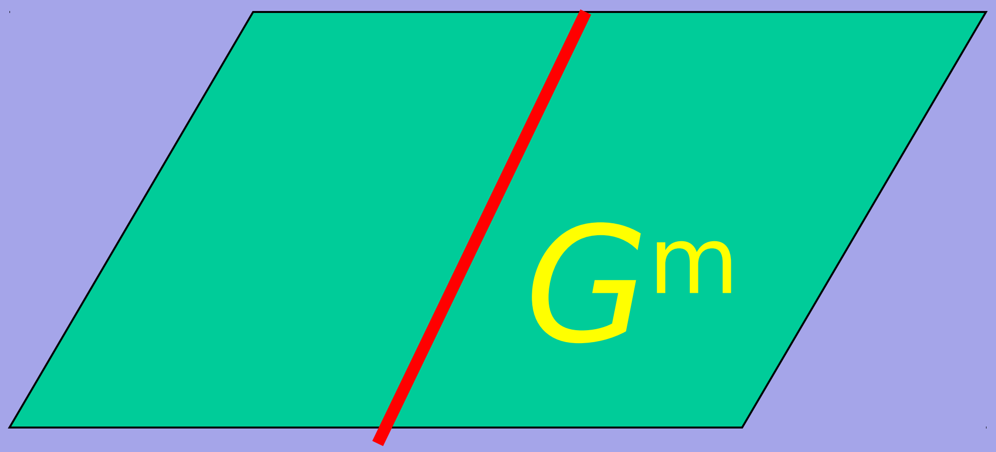
Sudakov-Tsirel'son, C Borell, '75 Carlen-Kerce, '01

In G^m, bubble is

HYPERPLANE

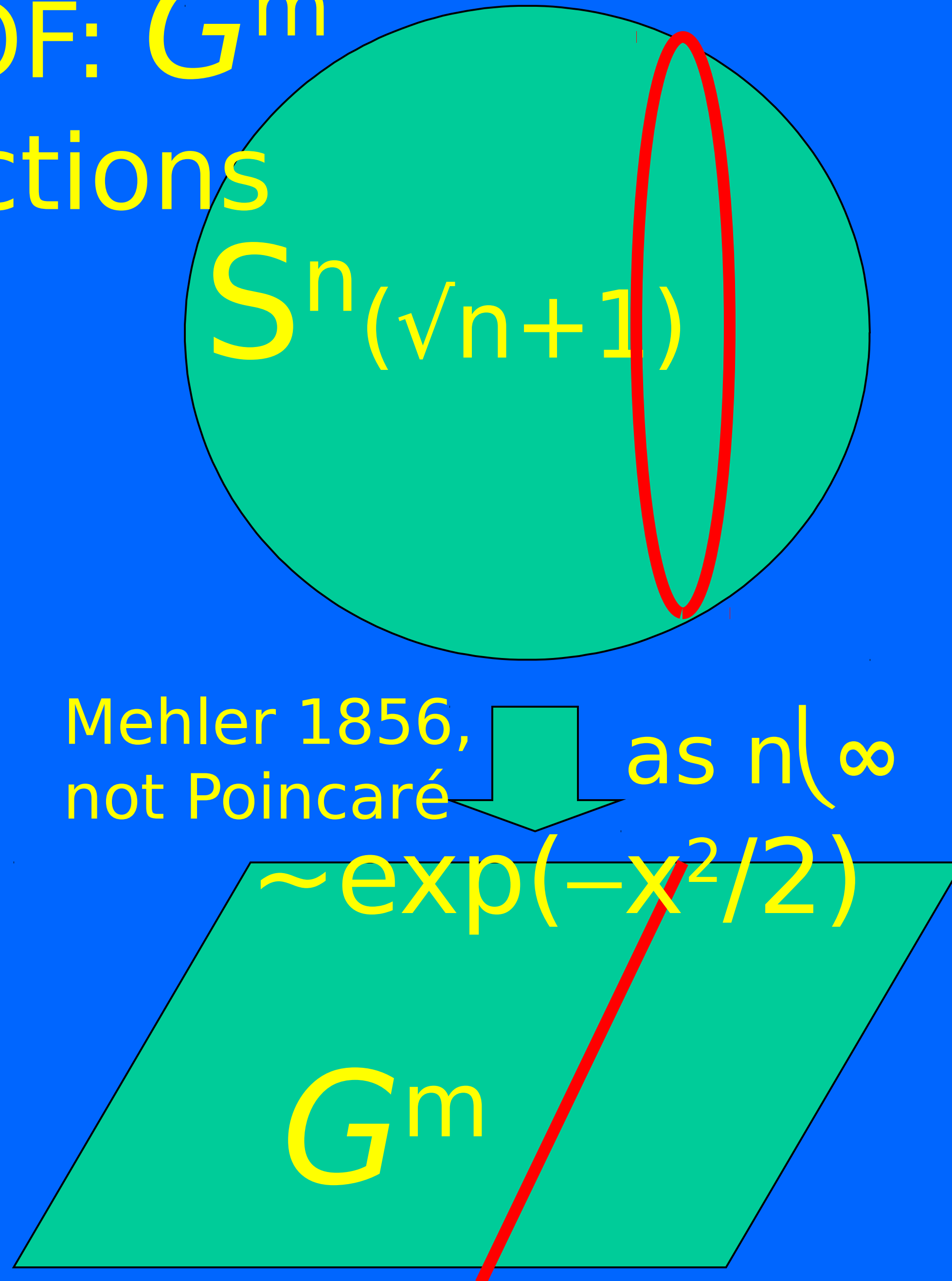
Sudakov-Tsirel'son,
Borell '75

Carlen-Kerce '01



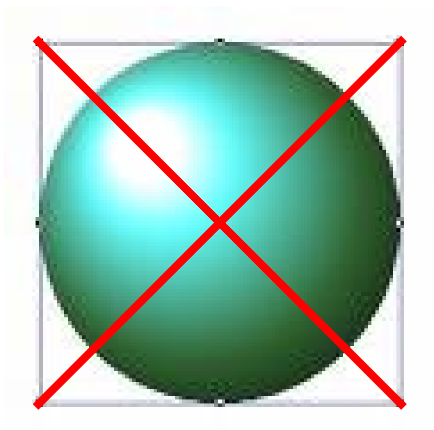
Used in Perelman's proof of Poincaré

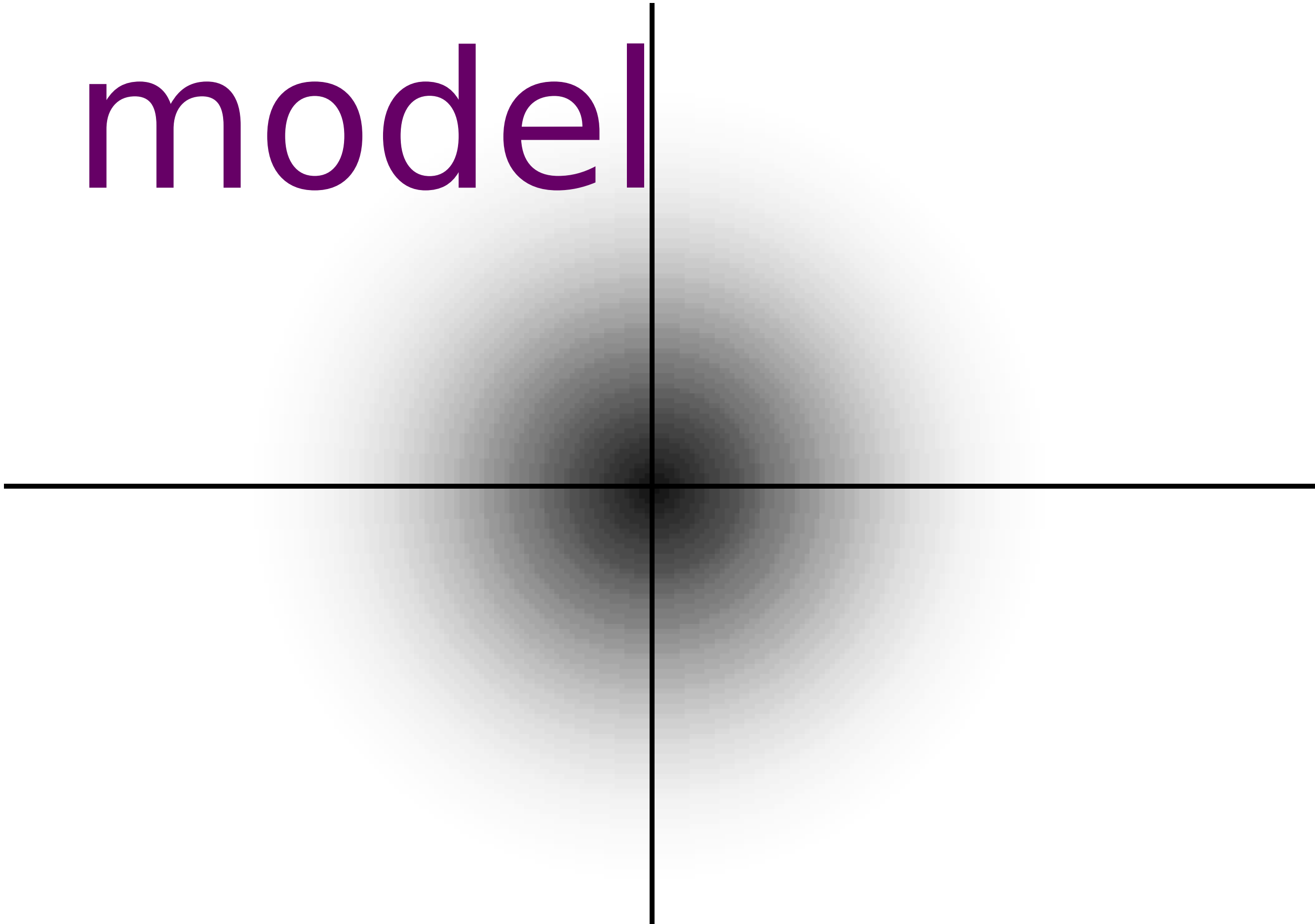
ORIGINAL PROOF: G^m is limit of projections of high-dim'l spheres.



Sudakov-Tsirel'son,
Borell '75

Gauss Space G^n replaces S^n as the model





« Manifolds with Density   Isoperimetric Inequality in Complement of Mean Convex Set Fails at Banff »

Manifolds with Density: Fuller References

16th March 2010, 09:09 am

SELECTED PUBLICATIONS IN THE HISTORY OF MANIFOLDS WITH DENSITY: ...

[1959] A. J. Stam, Some inequalities satisfied by the quantities of information of Fisher and Shannon, Indo. Control 2 (1959), 101–112, Eqn. 2.3. Gives a version of Gaussian log-Sobolev inequality, used by Perelman, often attributed to Gross [1975] or sometimes Federbush [1969].

[2018] Hiroki Nakajima, Takashi Shioya, Isoperimetric rigidity and distributions of 1-Lipschitz functions, https://arxiv.org/abs/1801.01302

[2018] Antonio Bueno, Translating solitons of the mean curvature flow in the space $\mathbb{H}2\times\mathbb{R}$, https://arxiv.org/abs/1803.02783

[2018] Elia Bruè, Daniele Semola, Regularity of Lagrangian flows over $RCD_*(K,N)$ spaces, https://arxiv.org/abs/1803.04387

[2018] Sebastiano Don, Davide Vittone, A compactness result for BV functions in metric spaces, https://arxiv.org/abs/1803.07545

[2018] Debora Impera, Michele Rimoldi, Alessandro Savo, Index and first Betti number of f-minimal hypersurfaces and self-shrinkers, https://arxiv.org/abs/1803.08268v1

[2018] Daisuke Kazukawa, A new condition for convergence of energy functionals and stability of lower Ricci curvature bound, https://arxiv.org/abs/1804.00407

[2018] Elia Bruè, Daniele Semola, Constancy of the dimension for RCD(K,N) spaces via regularity of Lagrangian flows, https://arxiv.org/abs/1804.07128

[2018] Shouhei Honda, Bakry-Émery conditions on almost smooth metric measure spaces, https://arxiv.org/abs/1804.07043

[2018] Ilaria Mondello (LAMA), J. Bertrand (IMT), C Ketterer, T. Richard (LAMA), Stratified spaces and synthetic Ricci curvature bounds, https://arxiv.org/abs/1804.08870

[2018] Antoni Kijowski, Characterization of mean value harmonic functions on norm induced metric measure spaces with weighted Lebesgue measure, https://arxiv.org/abs/1804.10005. "We conclude with a remarkable observation that strongly harmonic functions in R^*n* possess the mean value property with respect to infinitely many weight functions obtained from a given weight."

[2018] Angelo Alvino, Friedemann Brock, Francesco Chiacchio, Anna Mercaldo, Maria Rosaria Posteraro, The isoperimetric problem for a class of non-radial weights and applications, https://arxiv.org/abs/1805.02518v1

[2018] Jhovanny Muñoz Posso, A generalization of Sobolev trace inequality and Escobar-Riemann mapping type problem on smooth metric measure spaces, https://arxiv.org/abs/1805.03694 [part of PhD thesis under Fernando Codá´ Marques]

Welcome to my blog. I also have a blog at the Huffington Post and some posts on the math blog.

Frank Morgan

RECENT POSTS
Back To School at Age 35
Medgar Evers uses TOC to Stem Attrition
New Optimal Pentagonal Tilings
Berkshire Community College
Algebra Student Exhorts his Peers to Make the Most of their Opportunities

BLOGROLL
AMS Grad Student Blog
Math Blogs
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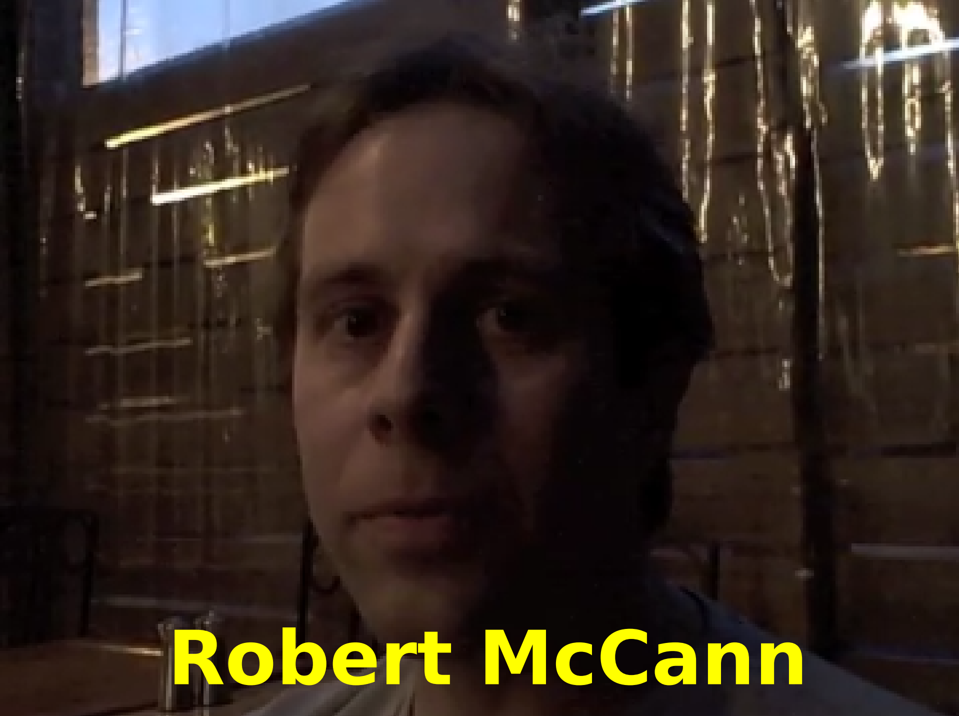
MARCH 2010
M T W T F S S
1 2 3 4 5 6 7
8 9 10 11 12 13 14


Robert McCann
U Toronto


Emanuel Milman
Technion

Robert McCann
U Toronto

Emanuel Milman
Technion

“Limits of manifolds”

“Projection
...
Gaussian”

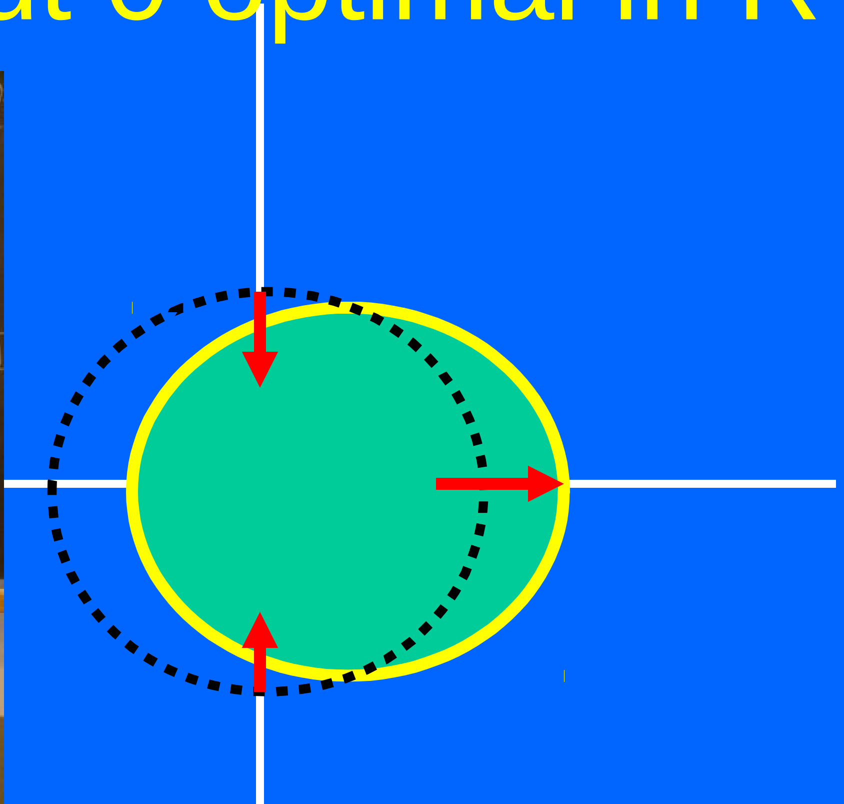
Log-convex Density Theorem

(Gregory Chambers 2018? Brakke conj 2004).

$\log \Psi(r)$ convex (e.g. $\exp +r^2$)

=> Spheres about 0 optimal in R^n

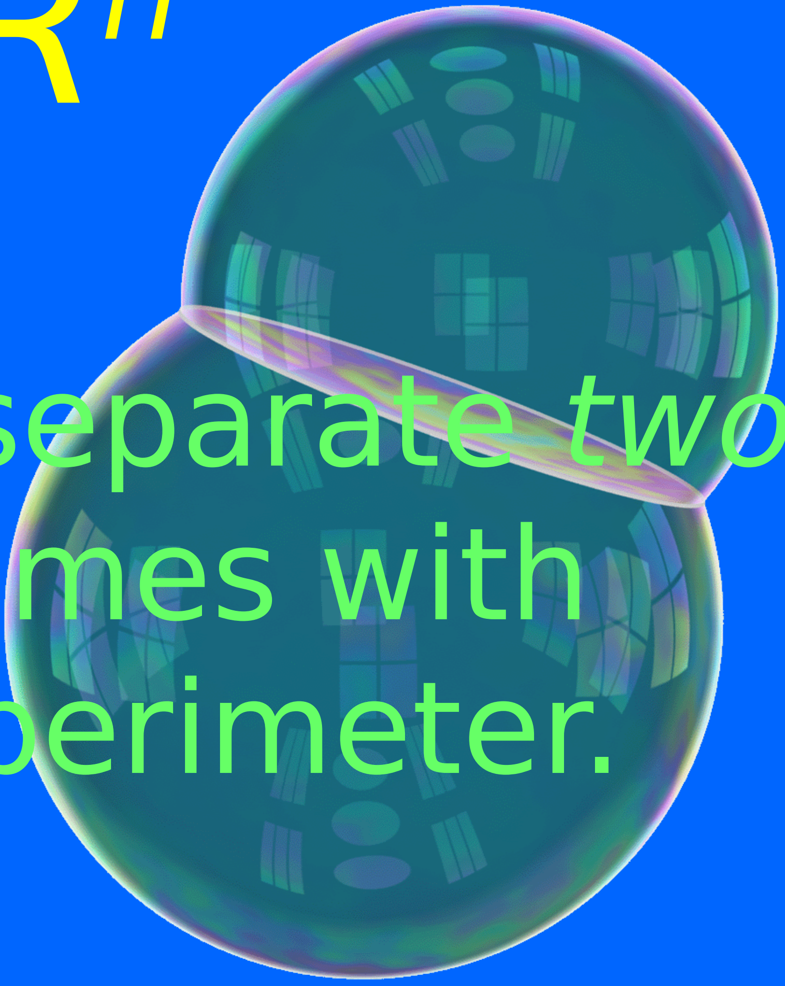
Pisa, June, 2014




Greg Chambers


Greg Chambers
“...symmetry...spiral
...very delicate
estimates...”

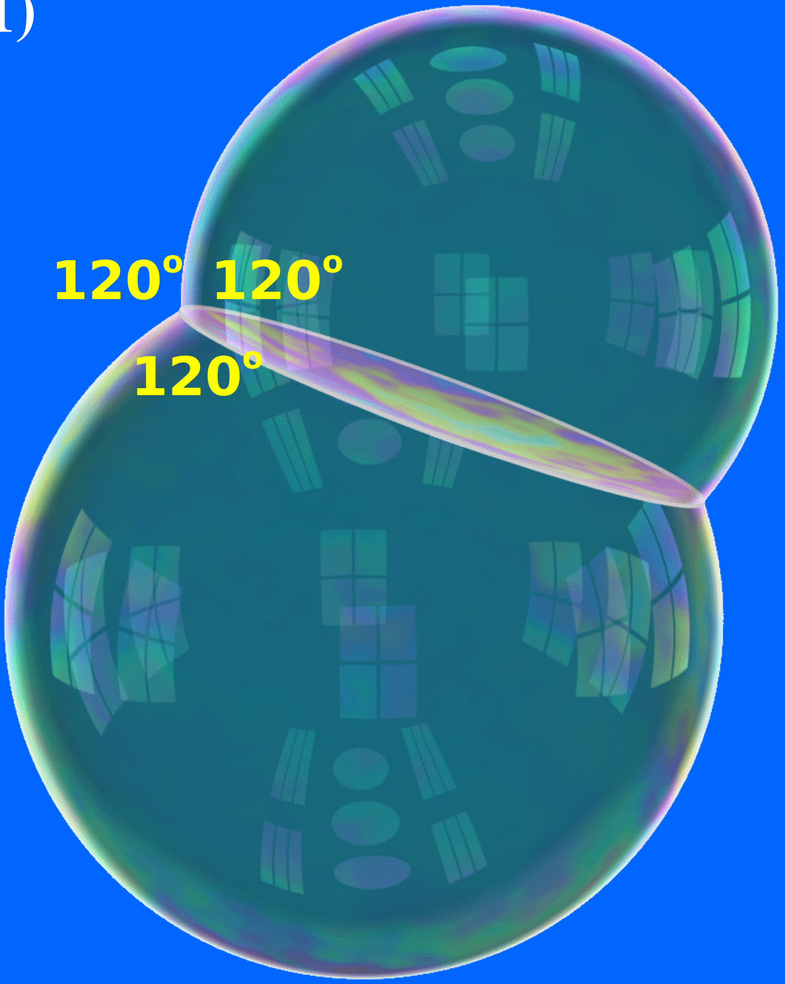
DOUBLE BUBBLE IN R^n



Enclose and separate *two* given volumes with minimum perimeter.

DOUBLE BUBBLE CONJECTURE

(Joel Foisy undergraduate thesis '91)

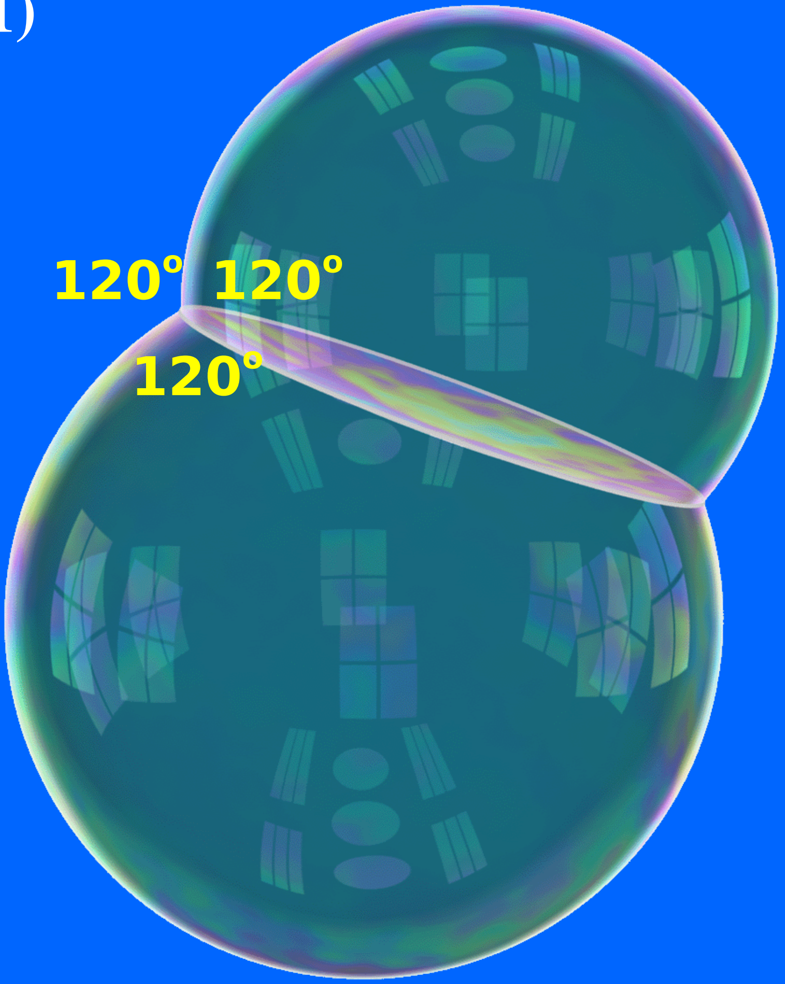


THE STANDARD DOUBLE BUBBLE IS THE MOST EFFICIENT SHAPE.

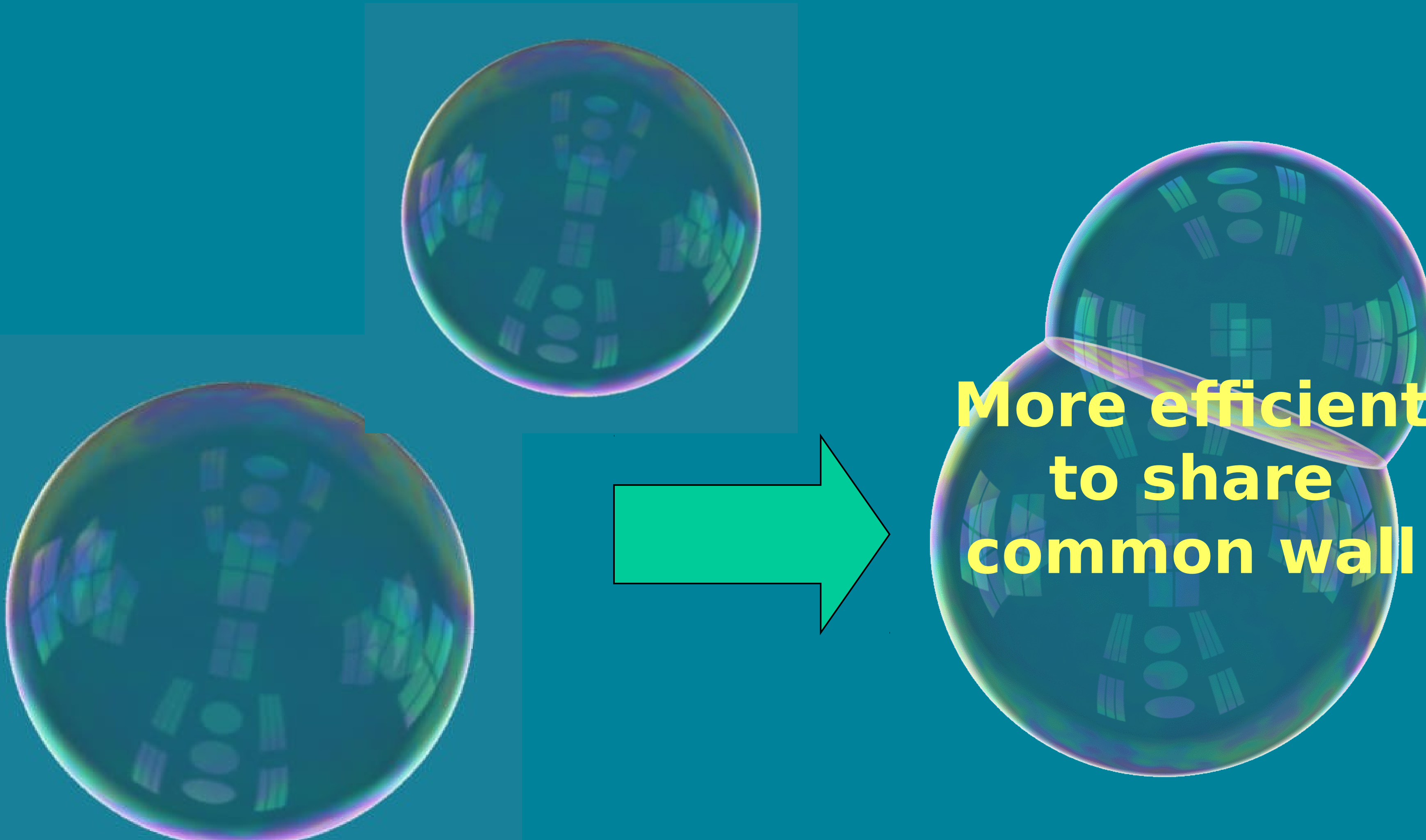
DOUBLE BUBBLE CONJECTURE

(Joel Foisy undergraduate thesis '91)

THE STANDARD DOUBLE BUBBLE IS THE MOST EFFICIENT SHAPE.



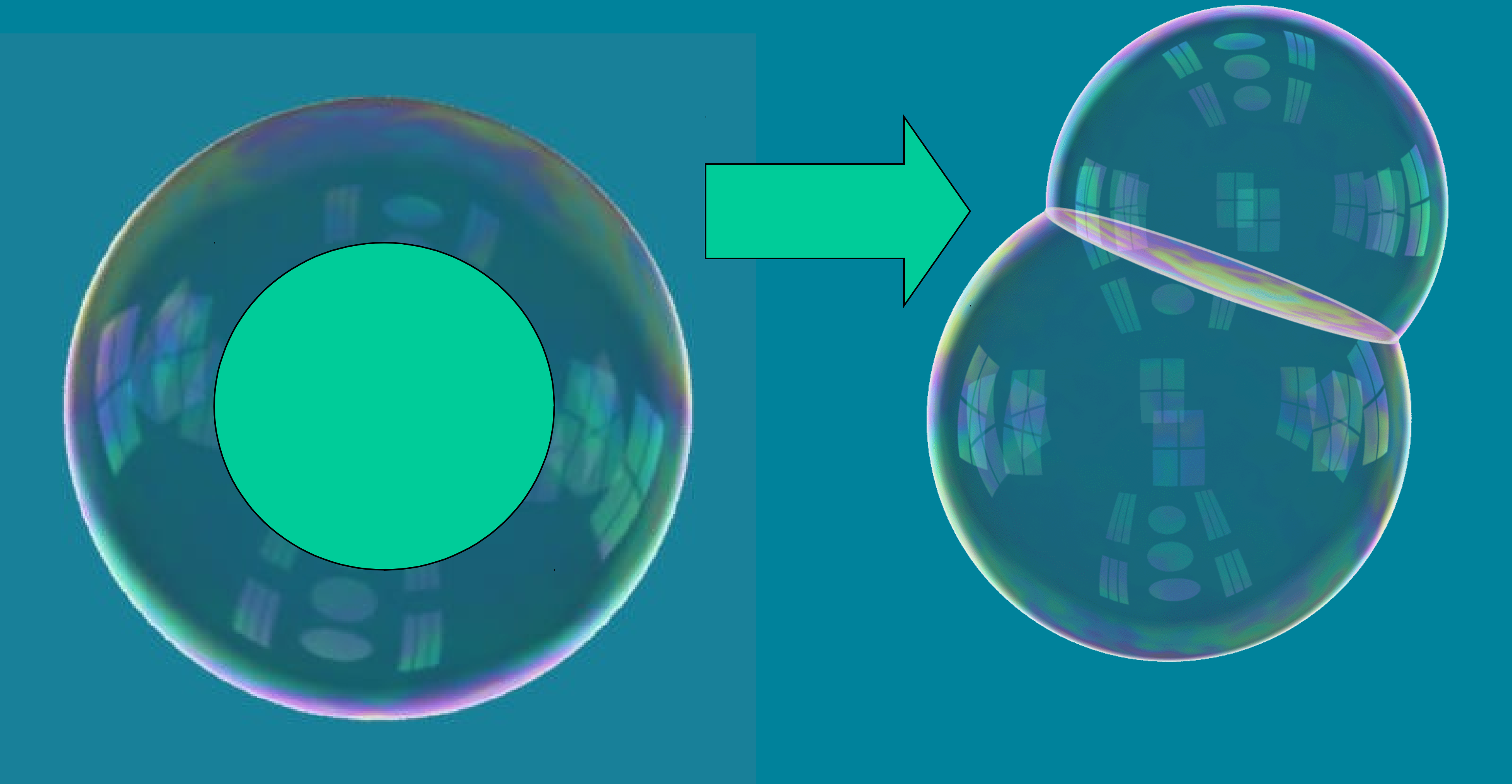
TWO SEPARATE BUBBLES ARE WASTEFUL:

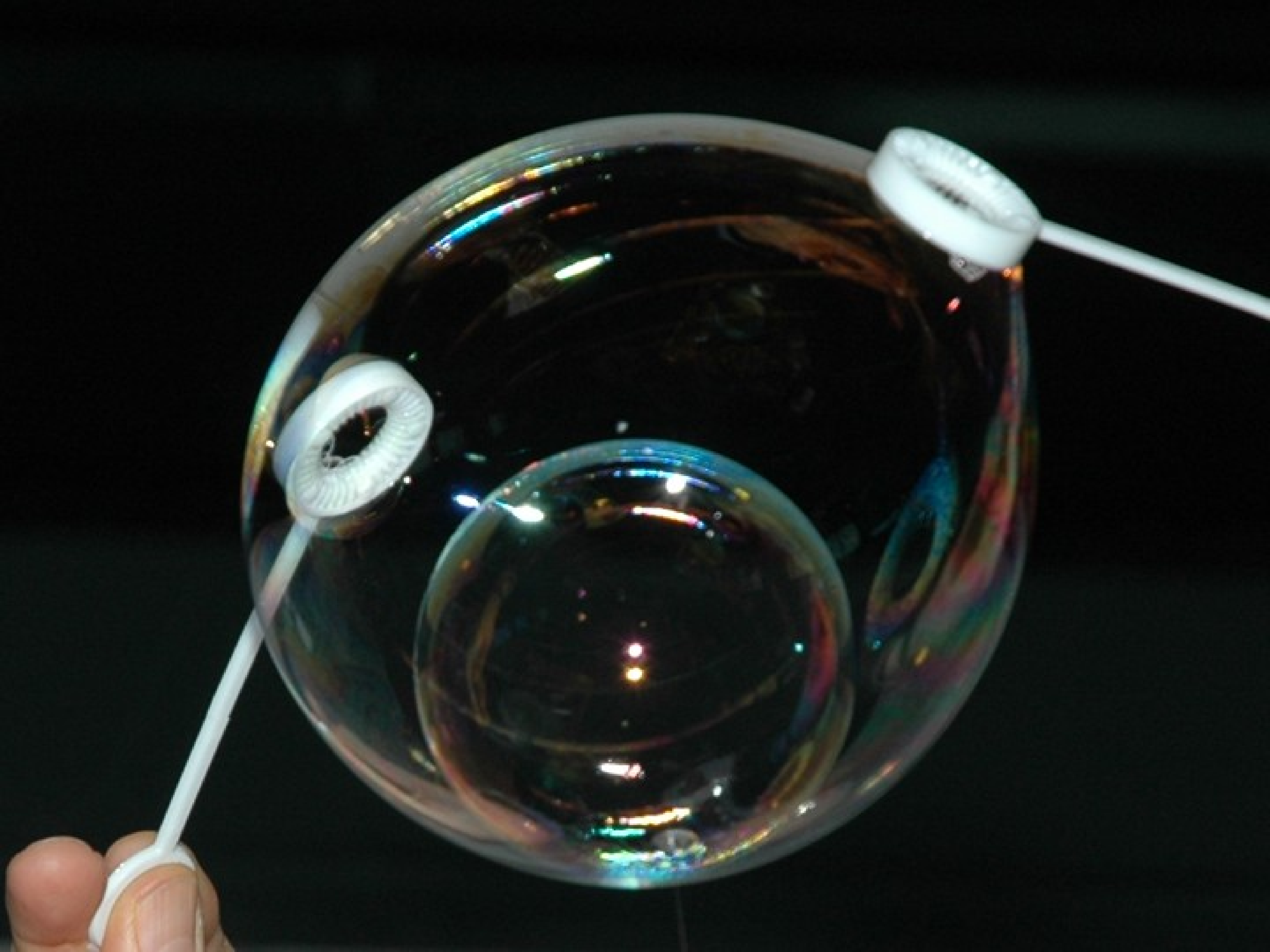


John M Sullivan

BUBBLE IN A BUBBLE EVEN WORSE:

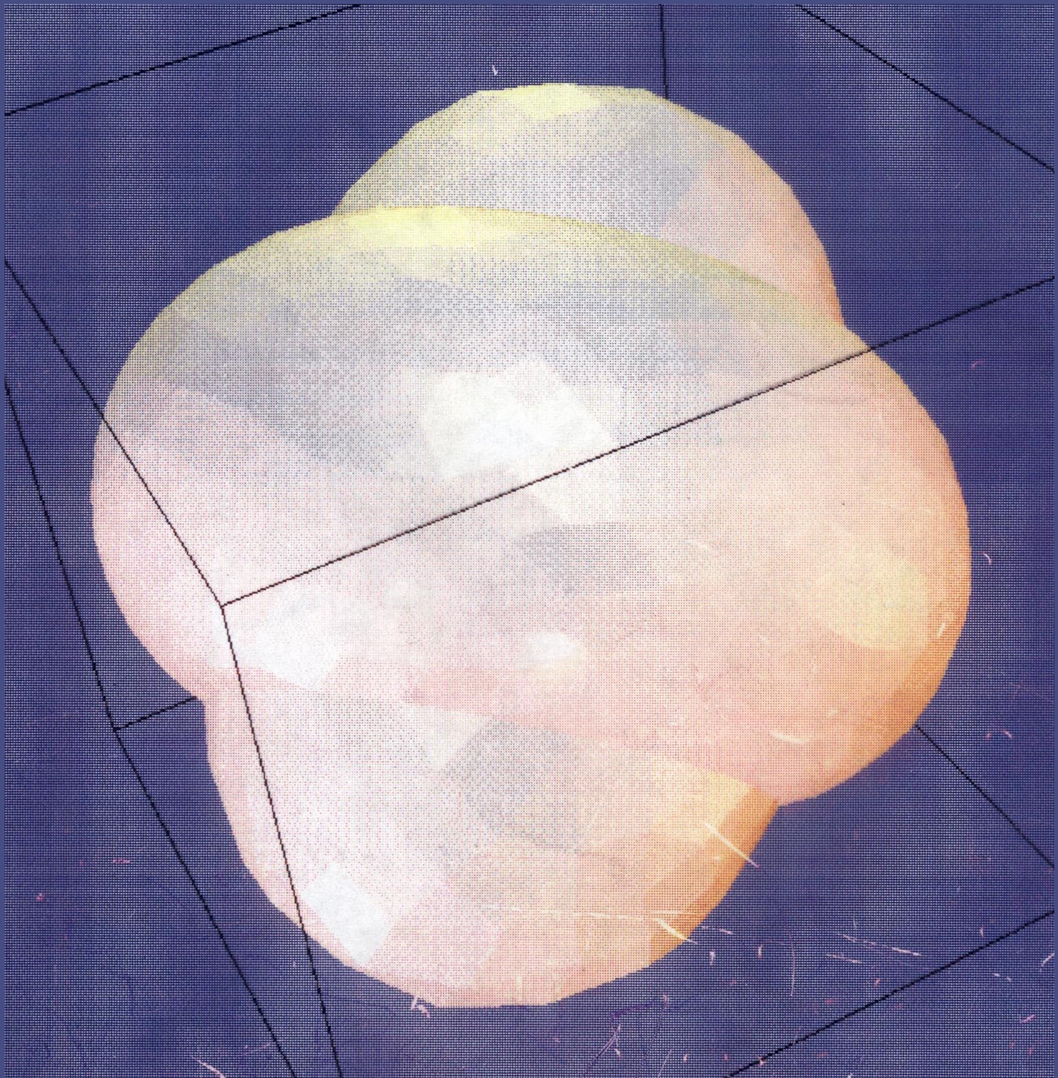
BUBBLE IN A BUBBLE EVEN WORSE:





NONSTANDARD DOUBLE BUBBLE

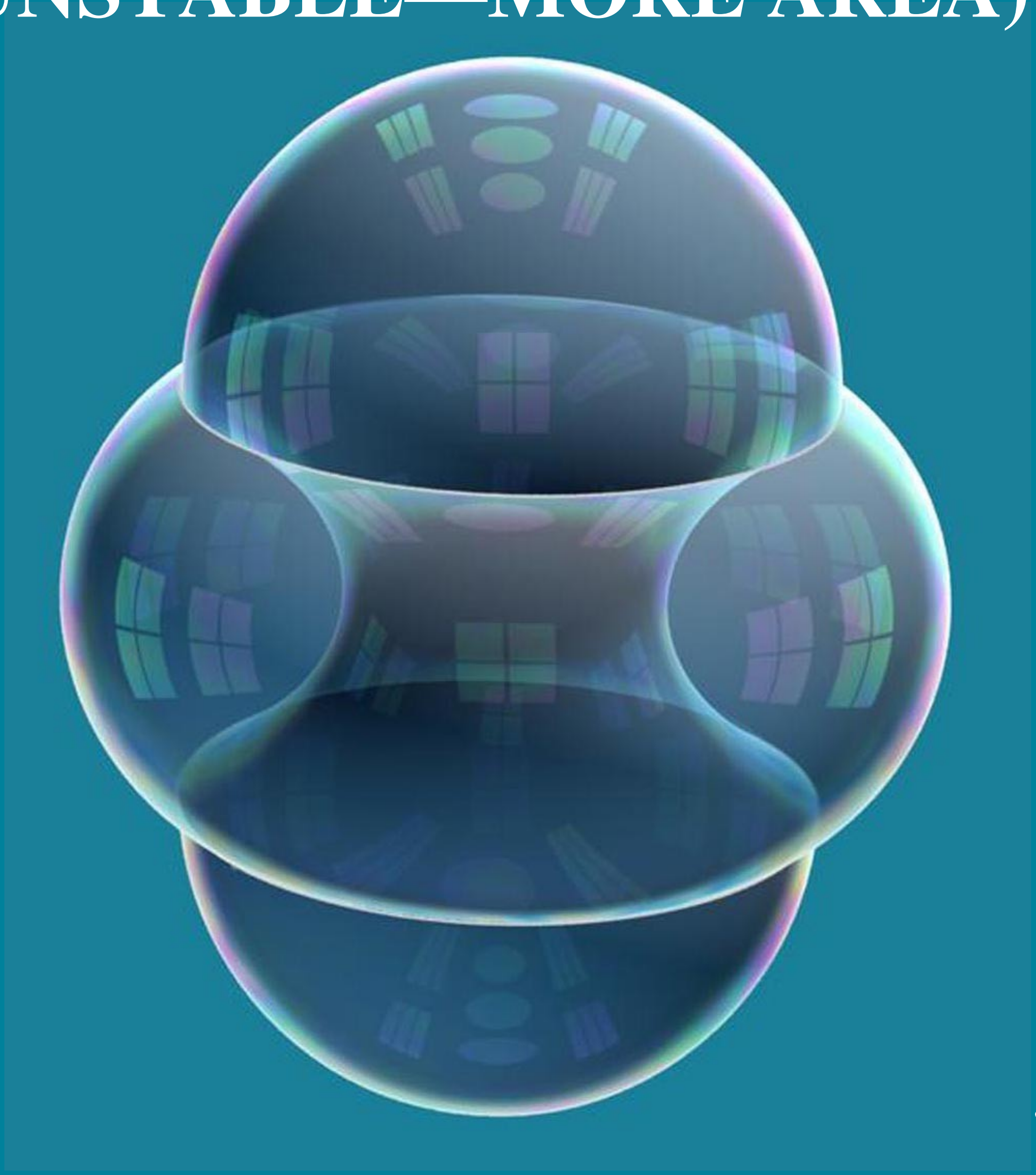
(UNSTABLE—MORE AREA)



Computer simulation by John M Sullivan

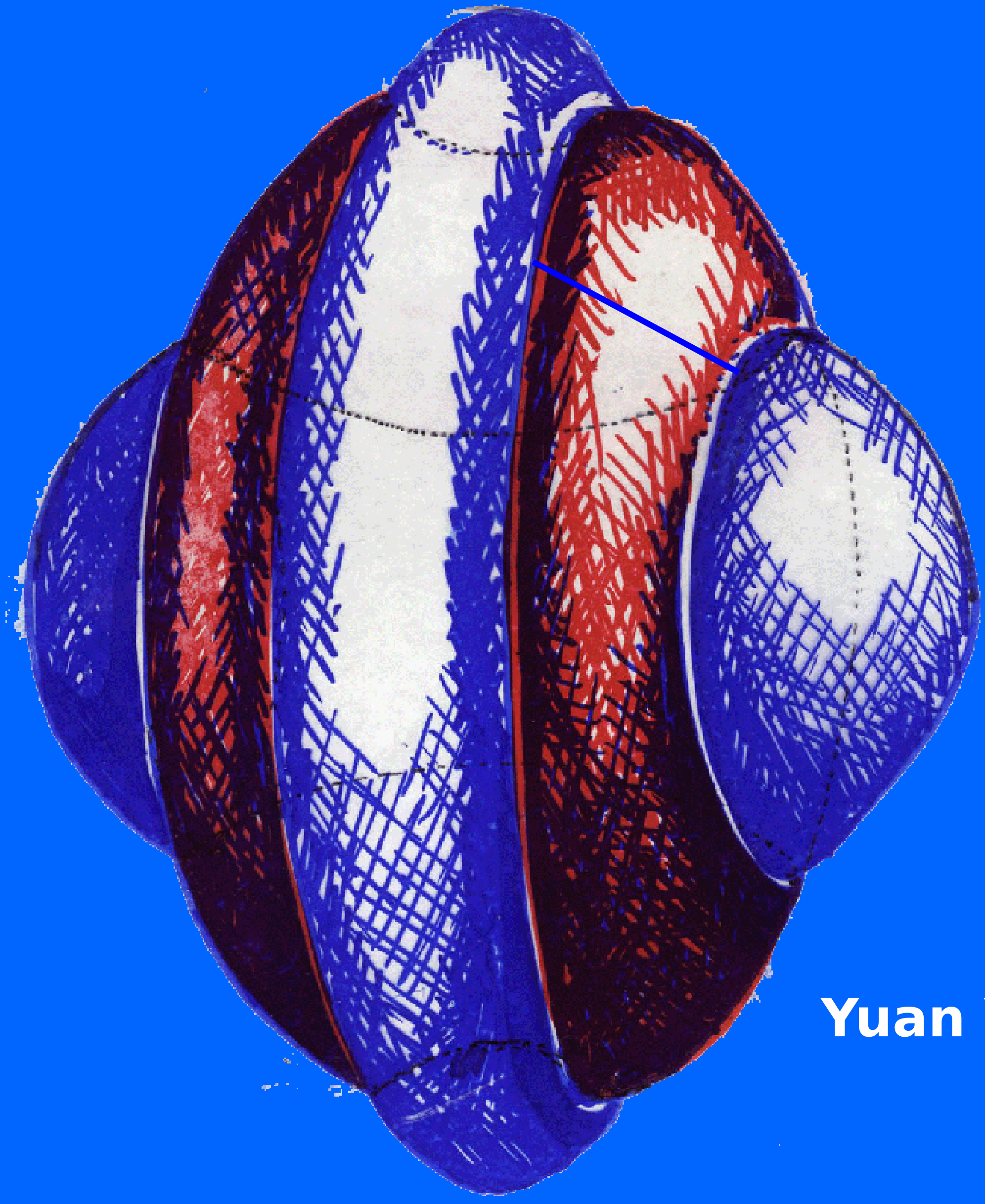
NONSTANDARD DOUBLE BUBBLE

(UNSTABLE—MORE AREA)



John M Sullivan

NONSTANDARD DOUBLE BUBBLE



Yuan Y Lai

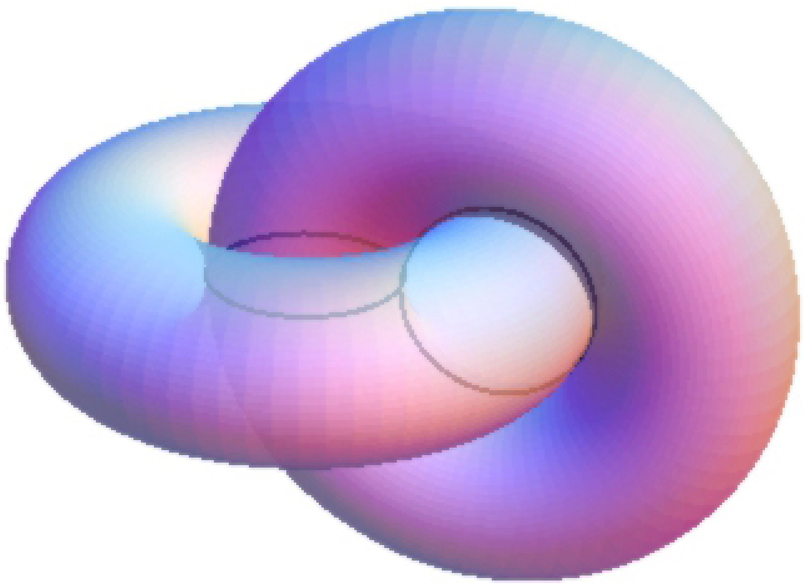
NONSTANDARD DOUBLE BUBBLE

Yuan Y Lai

NONSTANDARD DOUBLE BUBBLES

Jason Canterella and John M. Sullivan

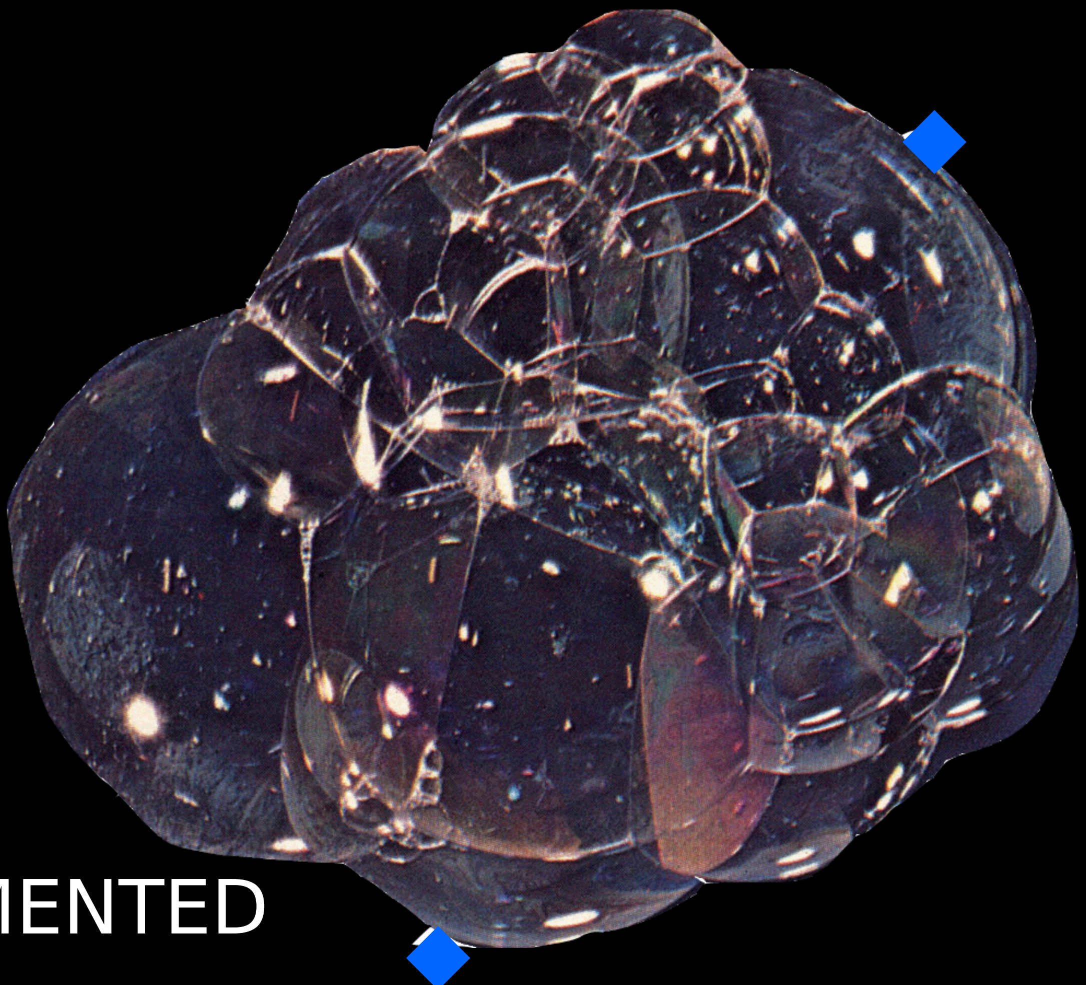
NONSTANDARD DOUBLE BUBBLES



mathoverflow.net/questions/70888/shortest-paths-on-linked-tori

Jason Canterella
John M. Sullivan

NONSTANDARD DOUBLE BUBBLE

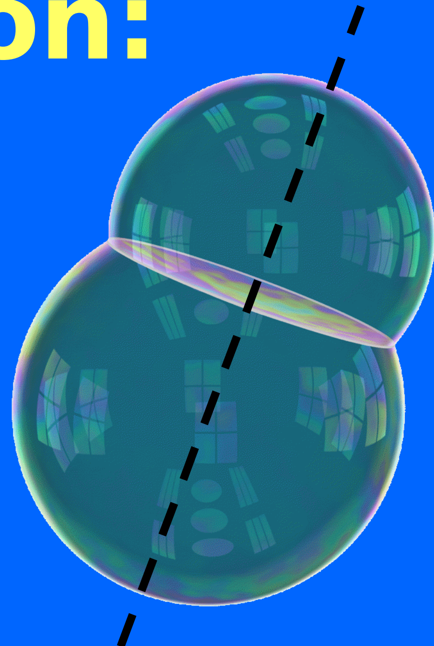


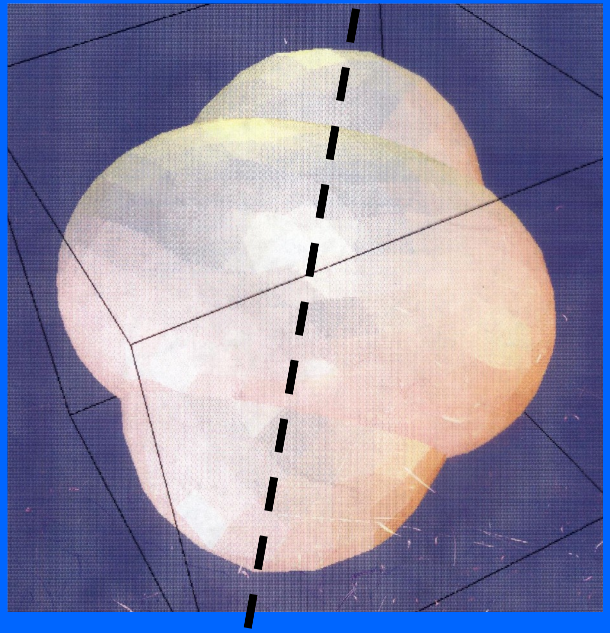
FRAGMENTED

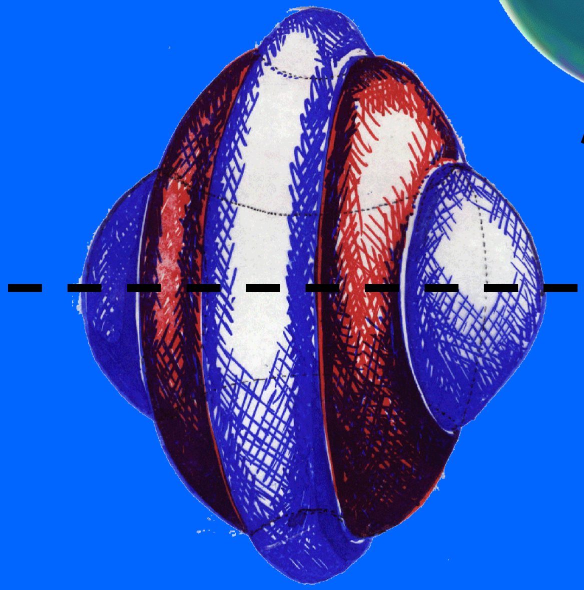
F. Goro

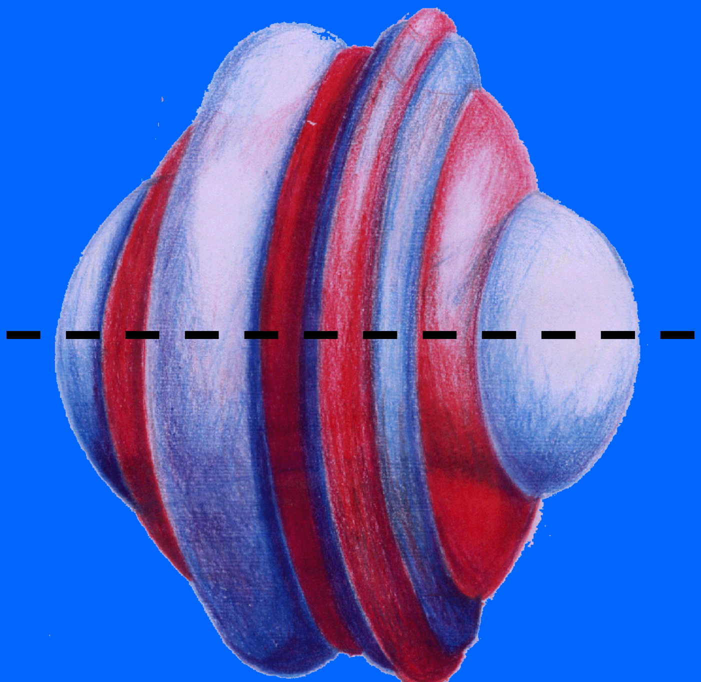
EMPTY SPACE INSIDE??

Prop. Surface of revolution:







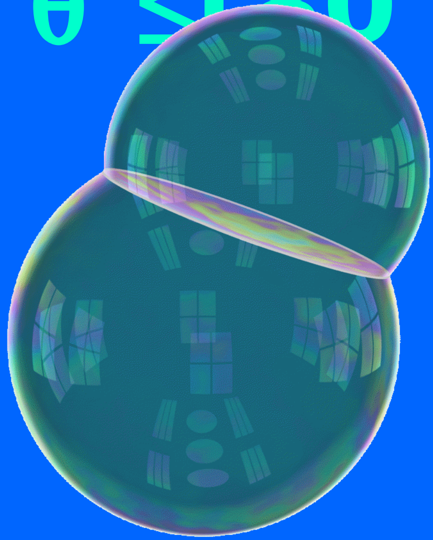


PROOF SURF OF REVOLUTION

LEMMA. Some vertical plane splits both volumes in half.

PROOF. **Consider vertical planes P_θ splitting 1st volume in half, $0 \leq \theta \leq 180°$**

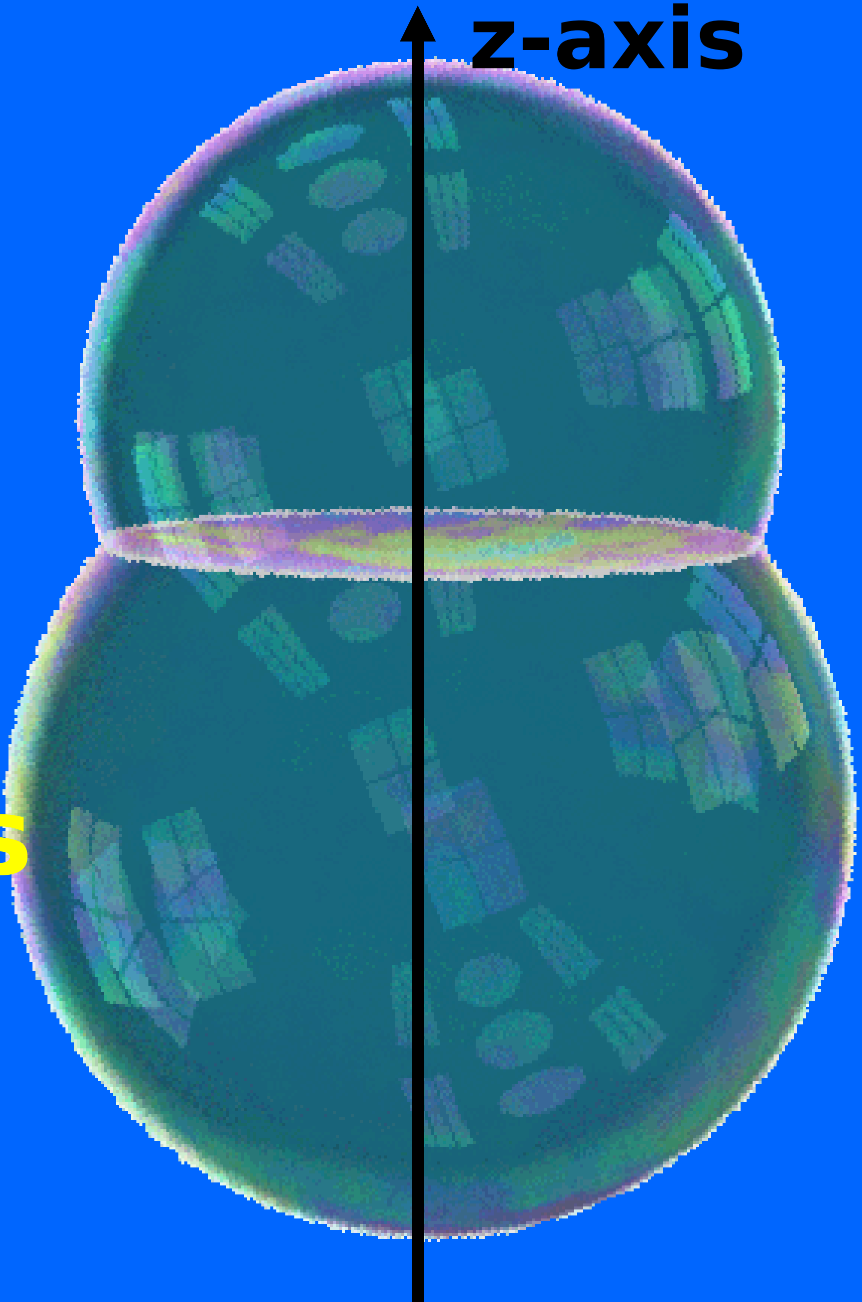
By the "Intermediate Value Theorem," one splits the 2nd



ONE PLANE SPLITS BOTH VOLS IN HALF

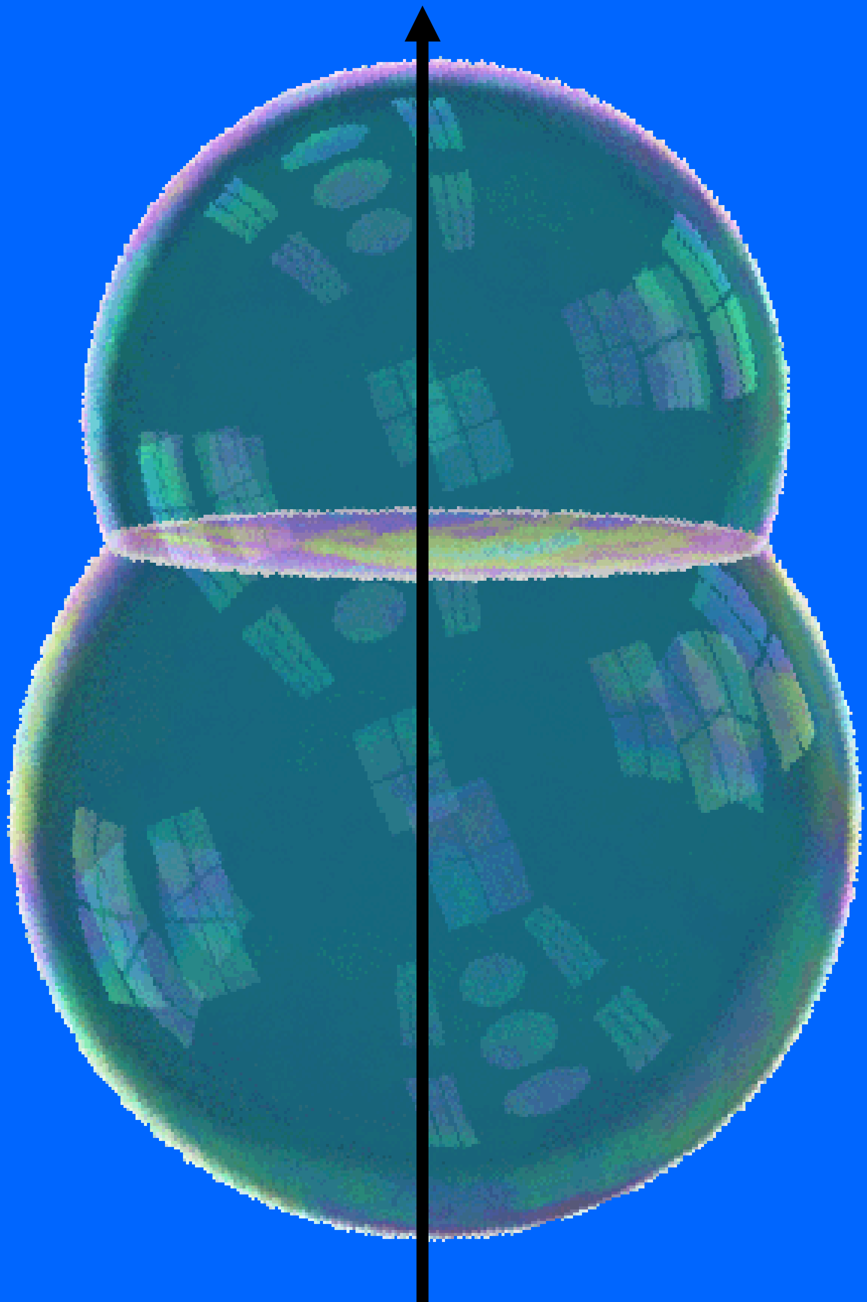
***Cor*. May assume x-z and y-z planes split volumes in half.**

Then z-axis turns out to be axis of revolution.



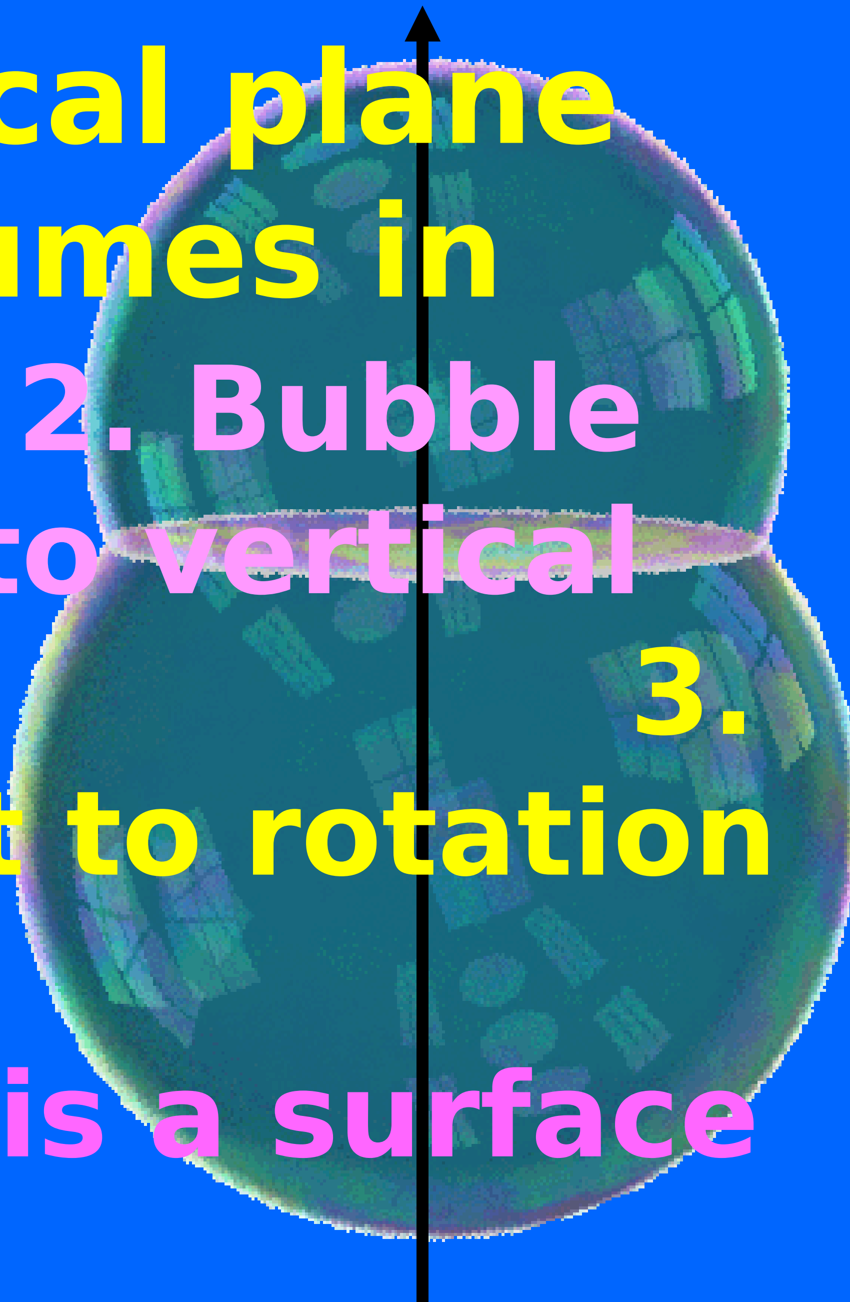
AXIS PLANES SPLIT BOTH VOLS IN HALF

***Cor 2*. May assume symmetry under $x \rightarrow x$, $y \rightarrow y$, 180° rotation.**



180° SYMMETRY =>

1. Every vertical plane splits the volumes in half. 2. Bubble perpendicular to vertical planes. 3. Bubble tangent to rotation vectorfield. 4. Bubble is a surface of revolution.



DBL BUB CONJ PROVED

by Hutchings, Morgan, Ritoré, and Ros

Annals Math. March 2002

SCIENCE

MATHEMATICS

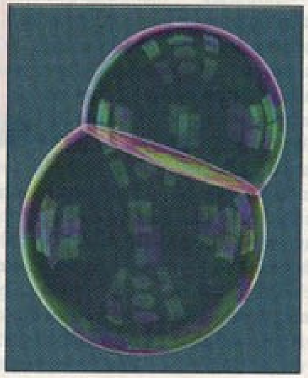
Rounding Out Solutions to Three Conjectures

Three long-standing puzzles involving spherical bodies—the configuration of double bubbles, stable orbits of three stars, and random packing of spheres in a box—have all been solved

Why Double Bubbles Form the Way They Do

Need to entertain a child? Try blowing soap bubbles. Need to keep a mathematician busy? Just ask why bubbles take the shapes they do. Individual soap bubbles, of course, are spherical, and for a very simple reason: Among all surfaces that enclose a given volume, the sphere has the least area (and in the grand scheme of things, nature inclines toward such minima). On the other hand, when two soap bubbles come together, they form a "double bubble," a simple complex of three partial spheres: two on the outside, with the third serving as a wall between the two compartments. Scientists have long considered it obvious that double bubbles behave this way for the same minimum-seeking reason—because no other shape encloses two given volumes with less total surface area. But mathematicians have countered with their usual vexing question: Where's the proof?

Now they have it. An international team of four mathematicians has announced a proof of the double bubble conjecture. By honing a new technique for analyzing the stability of competing shapes, Michael Hutchings of Stanford University, Frank Morgan of Williams College in Williamstown, Massachusetts, and Manuel Ritoré and Antonio Ros of the University of Granada have shown that only the standard shape is truly minimal—any other, supposedly area-minimizing shape can be ever so slightly twisted into a shape with even less area, a contradiction which rules out these other candidates.



Soap solution. Mathematicians prove that nature's way of forming double bubbles best.

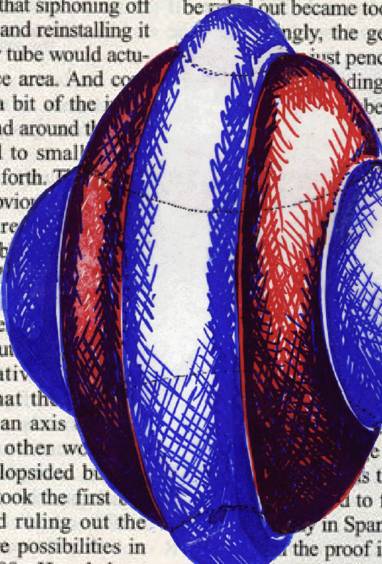
STANDARD DOUBLE BUBBLE BEST

What other shape could two bubbles possibly take? One candidate—or class of candidates—has one bubble wrapped around the other like an inner tube. But it could get even worse: Mathematically, there's no objection to splitting a volume into two separate pieces, so it's possible that siphoning off a bit of the central volume and reinstalling it as a "belt" around the inner tube would actually reduce the total surface area. And conceivably, then, siphoning a bit of the inner tube and placing it as a band around the belt would lead to smaller area yet, and so forth. Th… even any obvio… the true, are… double bub… "empty c… closed regi… long to eithe…

Just about… that's (relativ… prove is that th… must have an axis … metry—in other wo… can't have lopsided bu… Hutchings took the first … step toward ruling out the more bizarre possibilities in the early 1990s. He ruled out empty chambers and showed that the larger volume must be a single piece. Besides the standard double bubble, his results limited the possible solutions to ones consisting of a large inner tube around a small central region, perhaps with a set of one or more belts encircling the outside. Hutchings also found formulas that provide a bound on the number of belts, as a function of the ratio of the two volumes. In particular, if the two volumes are equal, or even nearly equal, there can be no belts, so the only alternative is a single inner tube around a central region.

Based on Hutchings's work, in 1995 Joel Hass of the University of California (UC), Davis, and Roger Schlafly, now at UC Santa Cruz, proved the double bubble conjecture for the equal-volume case. Their proof used computer calculations to show that any inner tube arrangement can be replaced by another with smaller area. "Ours was a comparison method," Hass explains. He and Schlafly found they could extend their results for volume ratios up to around 7:1, but beyond that the possible configurations to be ruled out became too complicated.

NOT:



with regions disconnected

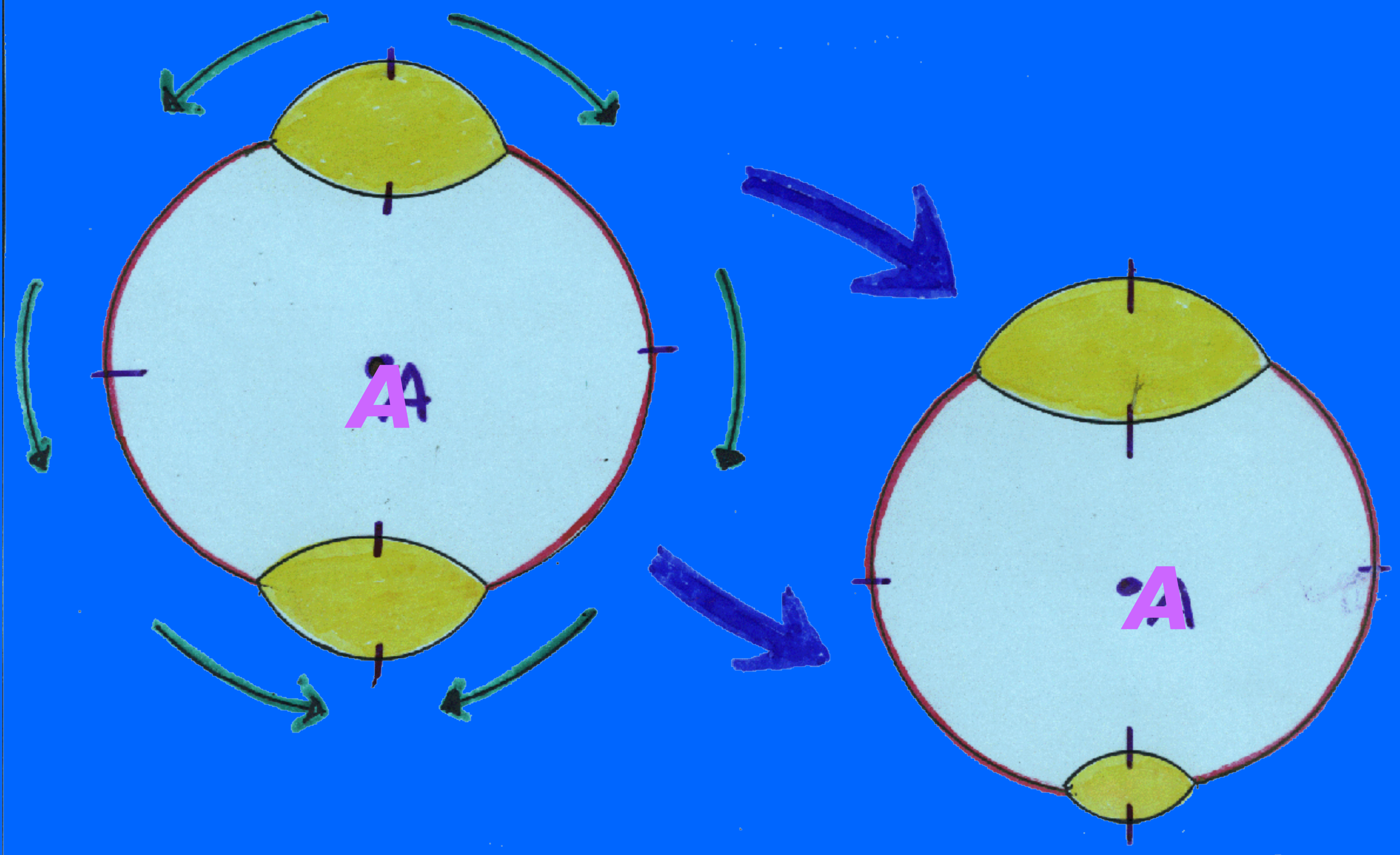
…ngly, the general proof requires …ust pencil and paper. The key …ding an "axis of instabili… …be arrangement. Twist… …round this axis—with …ringing out a wash… …in surface area, …tensible mini… …hat these re… …organ says. …icions, al… …ibility that …ation could …gument is new …es. The hardest …re to position the …at the twisting proce… …e the volumes of the …s the surface area. "For …d to frame the right ques… …y in Spanish."

…the proof is only now being announced, the main results were established last spring, when Morgan visited Granada during a sabbatical. Since then, a group of undergraduates in a summer research program at Williams College has extended the results to analogs of the double bubble conjecture in higher dimensions. (The two-dimensional double bubble conjecture was proved by an earlier group of undergraduates in 1990.) Ben Reichardt of Stanford, Yuan Lai of the Massachusetts Institute of Technology, and Cory Heilmann and Anita Spielman of Williams College have shown that an axis

CREDIT: J.M. SULLIVAN/UNIVERSITY OF ILLINOIS

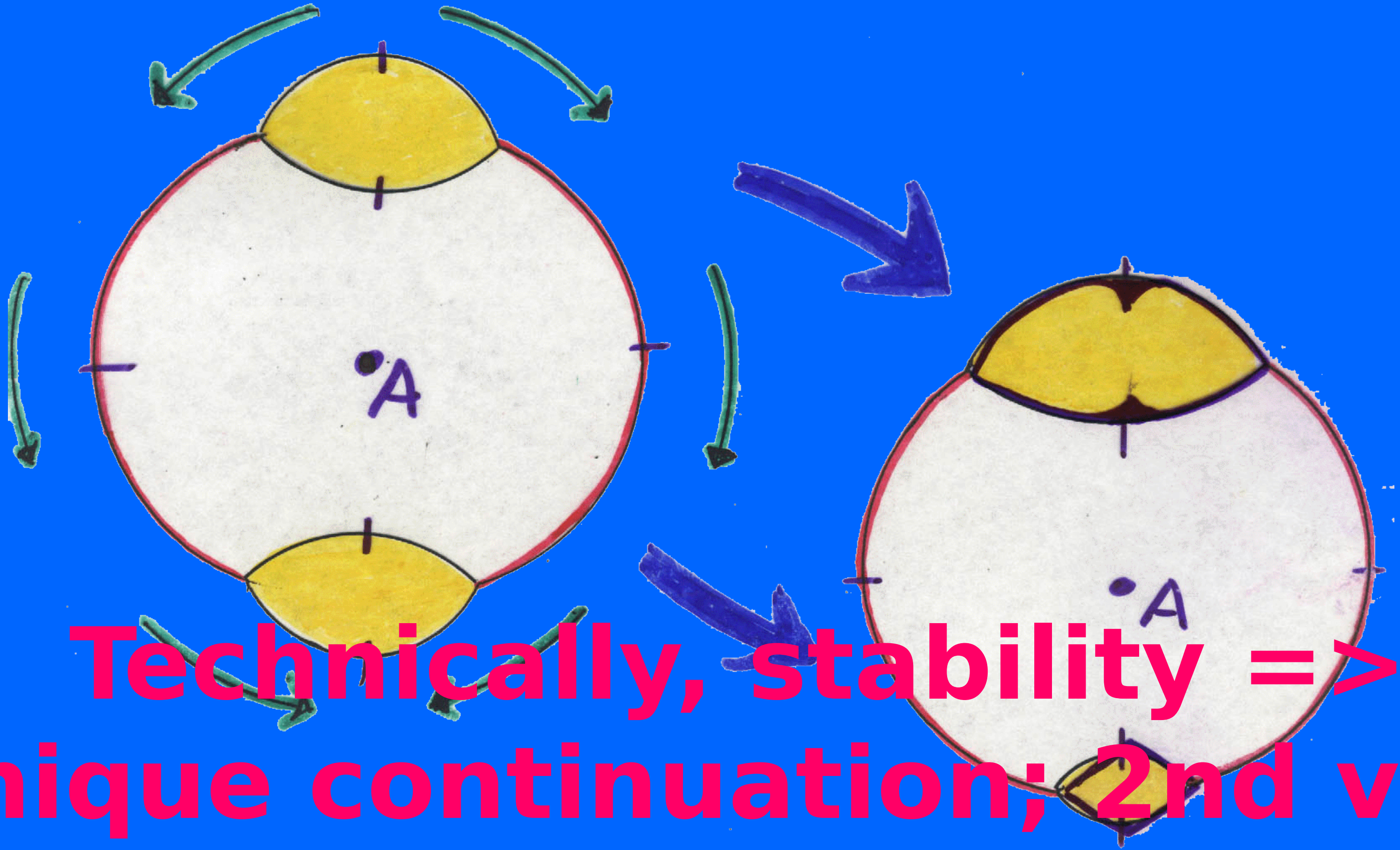
IDEA OF PROOF IF EACH REGION CONNECTED ("INSTABILITY")

ROTATE TWO "HALVES" OF BUBBLE IN DIFFERENT DIRECTIONS ABOUT CAREFULLY CHOSEN AXIS *A* TO PRESERVE VOLUMES



IDEA OF PROOF IF EACH REGION CONNECTED ("INSTABILITY")

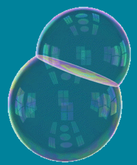
SMOOTH KINKS TO REDUCE PERIMETER:



Technically, stability => unique continuation; 2nd varn formula

Thm (Hutchings). Region of volume fraction *v* connected if Hutchings function positive:

Michael Hutchings
SMALL '92

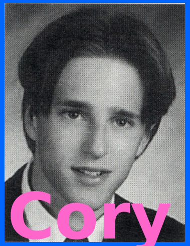
$F(v) =$
$2A(v/2) + A(1-v) + A(1) - 2A(v,1-$
$v)$   ). 

True for *v* > .2… in R^3.

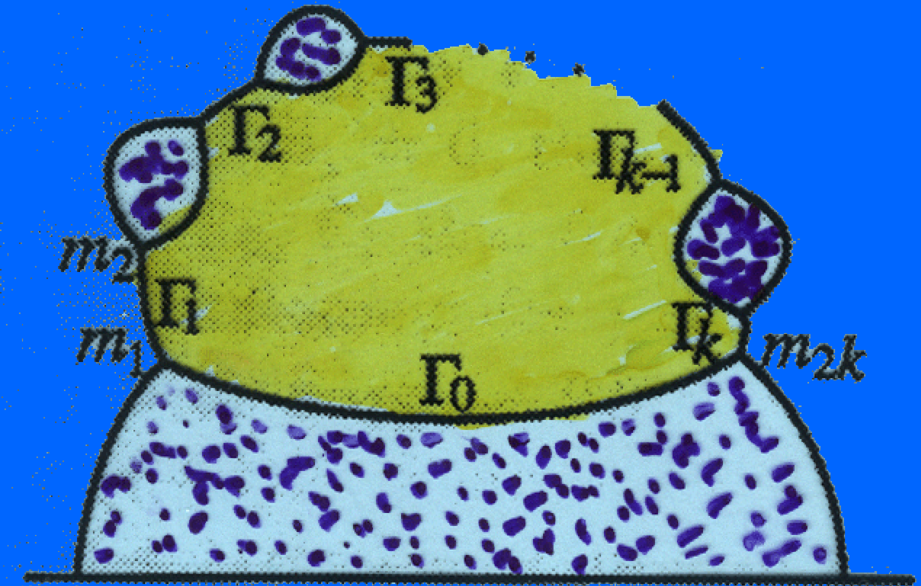
“SMALL” UND RES GEOM GROUP 1999

Ben Reichardt
Cory Heilmann
Yuan Lai
Anita Spielman

PROVED DBL BUB CONJ IN R^4



& R^n for certain volumes for which larger bubble is connected



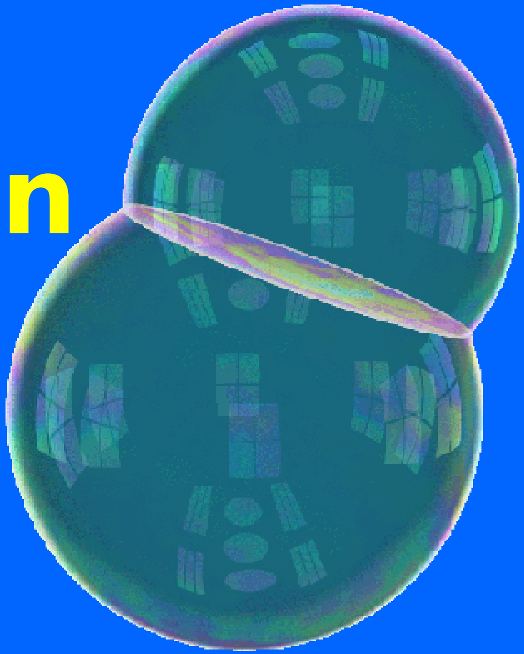
EXTNS TO S^3 & H^3 when both regions known connected

GEOM GPS 2001-2003

For S^3, all ≥ 10%.

For H^3, smaller at least 85% of larger.

(Ensuring each region connected.)

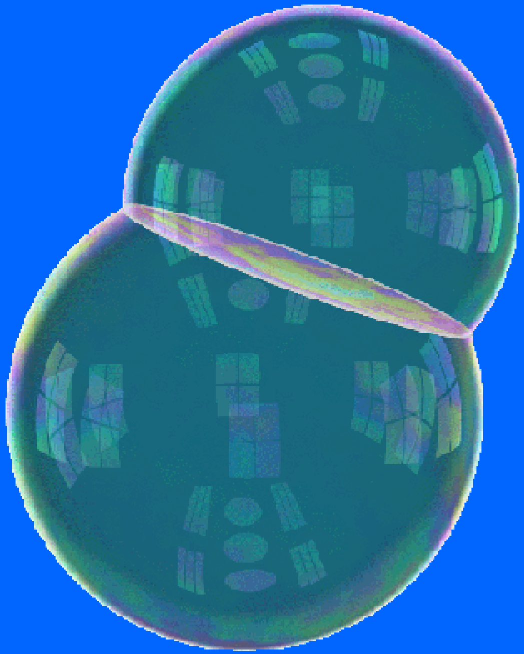


EXTNS TO S^3 & H^3

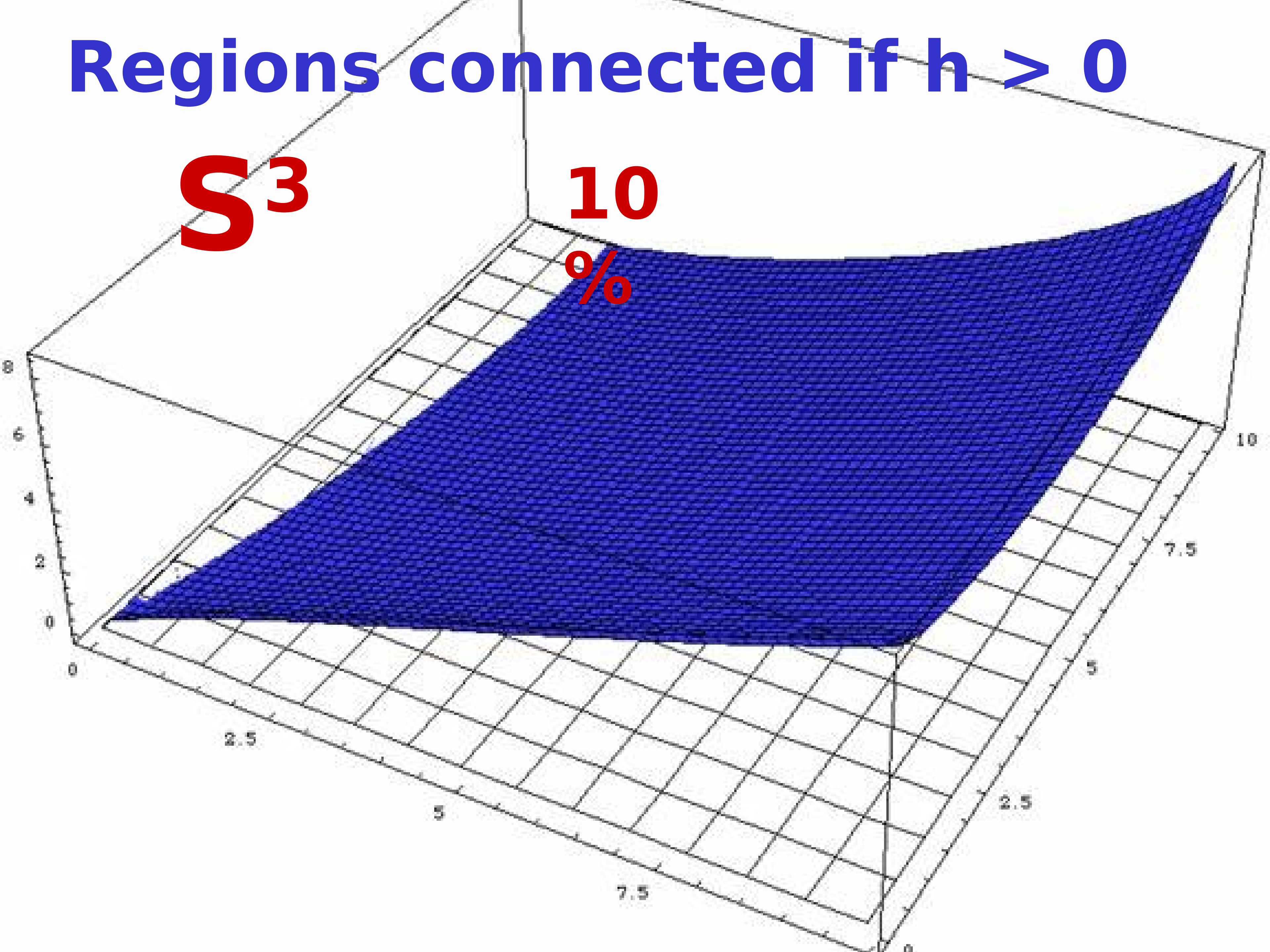
GEOM GPS 2001-2003

For S^3, all ≥ 10%.
For H^3, smaller at least 85% of larger.

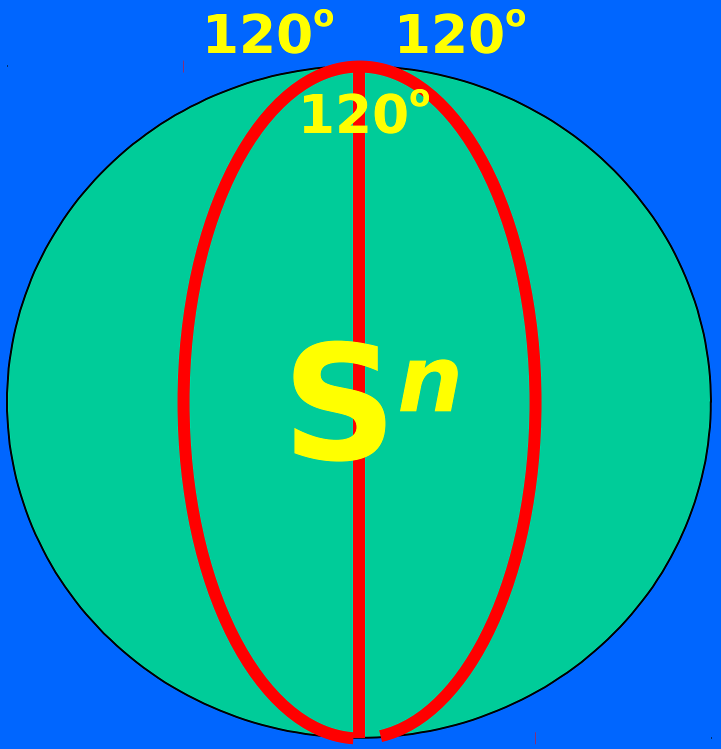
Verify Hutchings inequality numerically to show each region connected.



Regions connected if h > 0



S^n



Difficulty: how to verify crucial Hutchings inequality in all S^n?

GEOMETRY GROUPS 2004-5

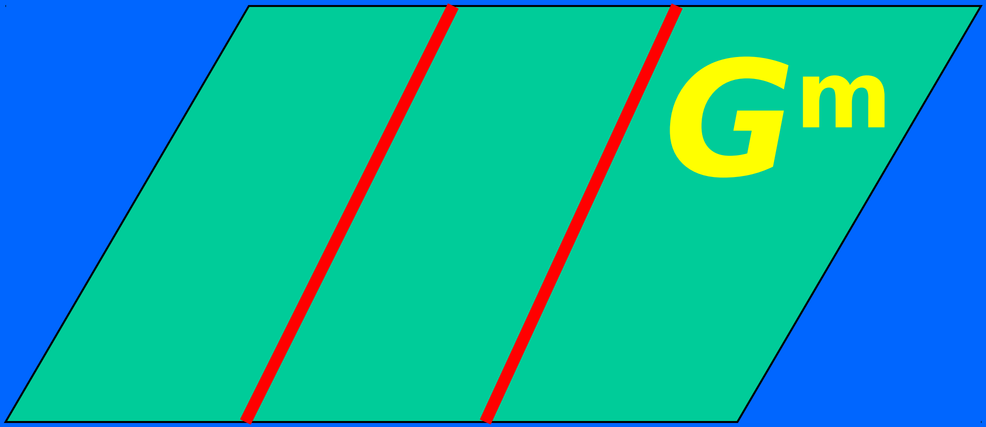
DOUBLE BUBBLES IN GAUSS SPACE

VERSAILLES P

GEOMETRY GROUPS 2004-5

DOUBLE BUBBLES IN GAUSS SPACE

Not 2 hyperplanes.

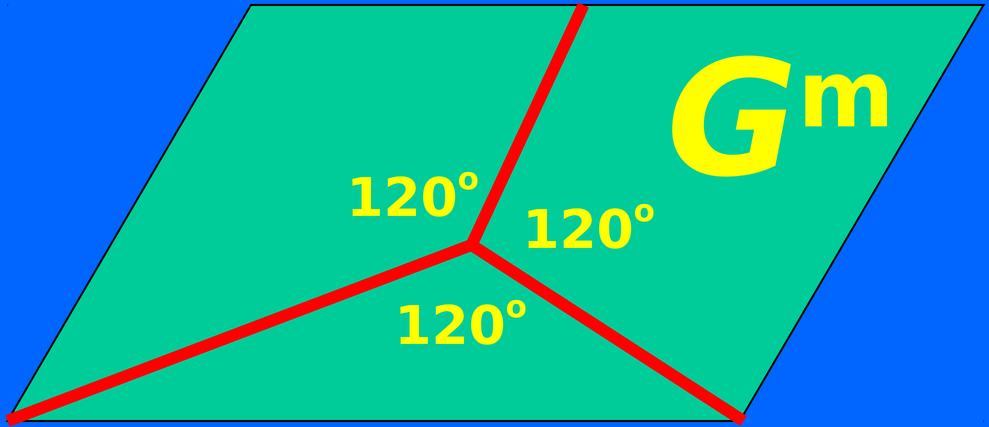


GEOMETRY GROUPS 2004-5

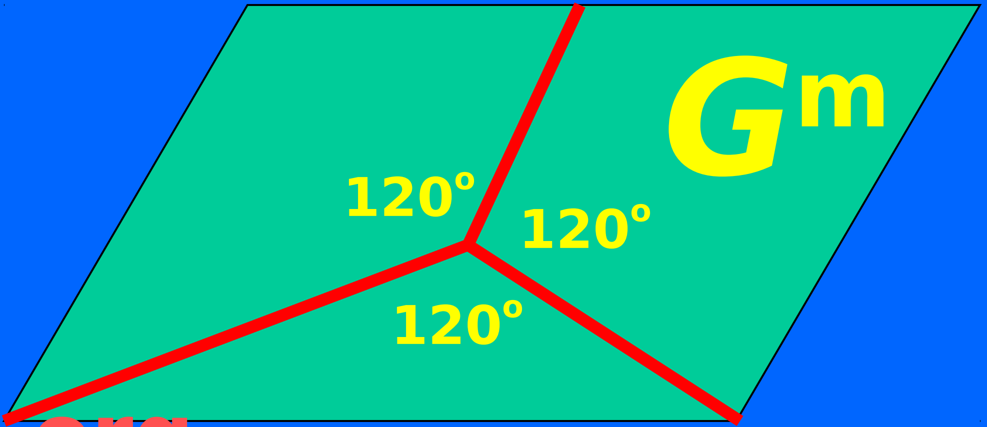
DOUBLE BUBBLES IN GAUSS SPACE

Conj.

3 half-hyperplanes meeting at 120°.



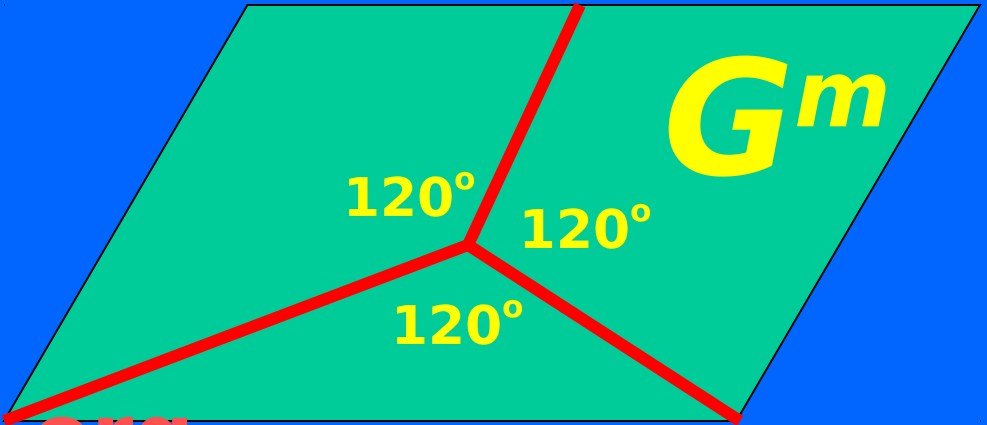
Good news: suffices to check Hutchings inequality in G^2.



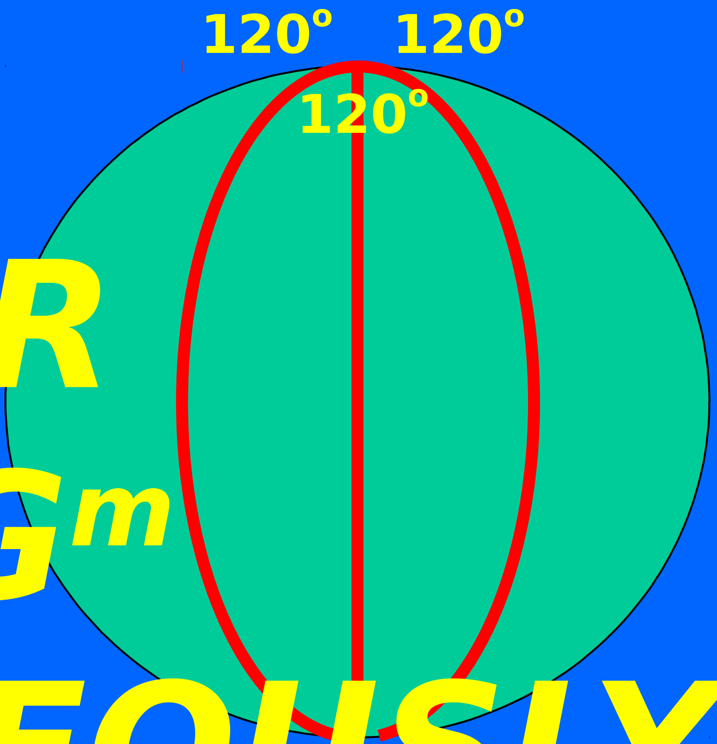
J. Corneli
SMALL '01
PlanetMath.org

Bad news: methods do not apply in G^m for lack of translational symmetry.

J. Corneli
SMALL '01
PlanetMath.org



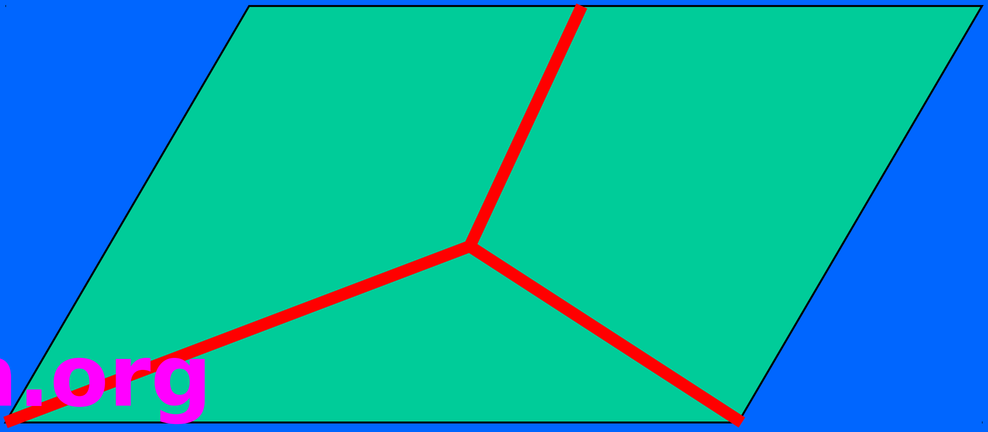
IDEA OF PROOF:



PROVE FOR

S^n AND G^m

SIMULTANEOUSLY



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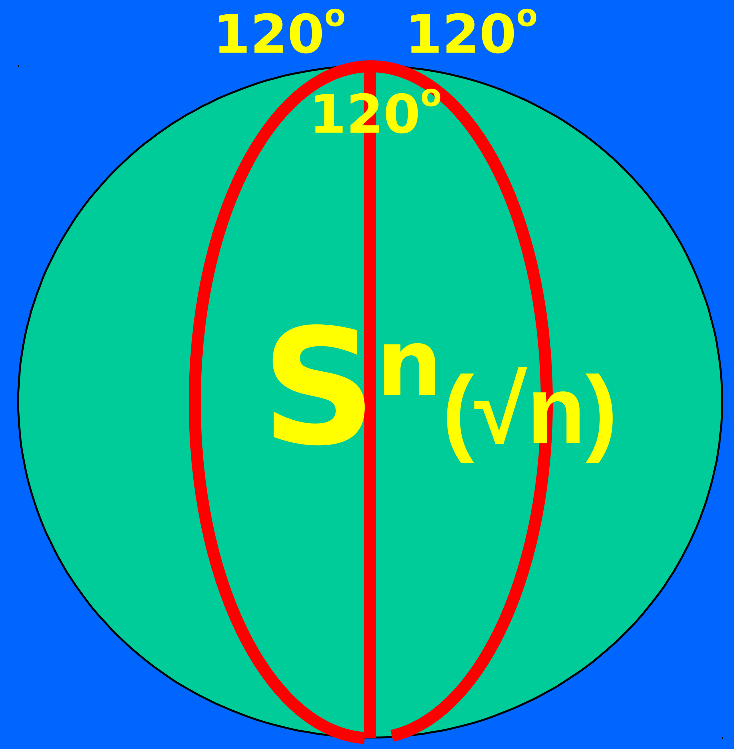
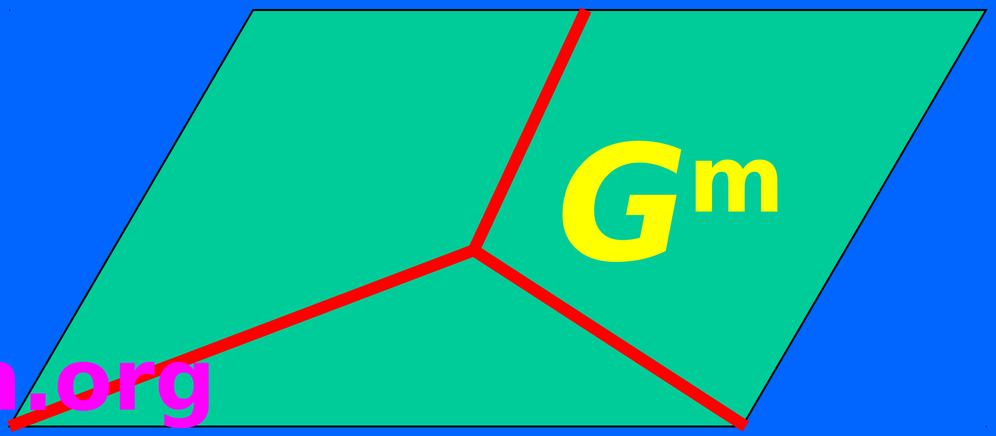
G^m is limit of projections of high-dim'l spheres.



Mehler 1856,
not Poincare

as $n \to \infty$



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IDEA OF PROOF:

1. **Verify Hutchings inequality in G^2.**
2. **Carry out proof in S^n.**
3. **Transfer result back to G^m.**

J. Corneli
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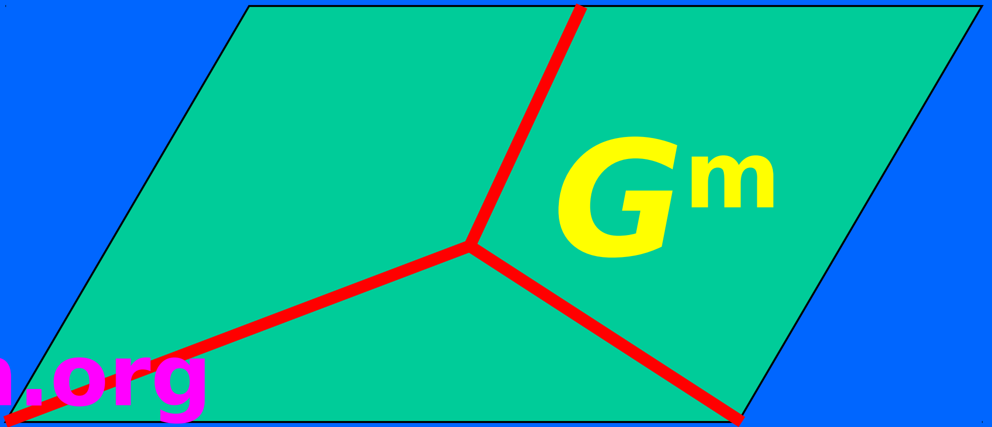
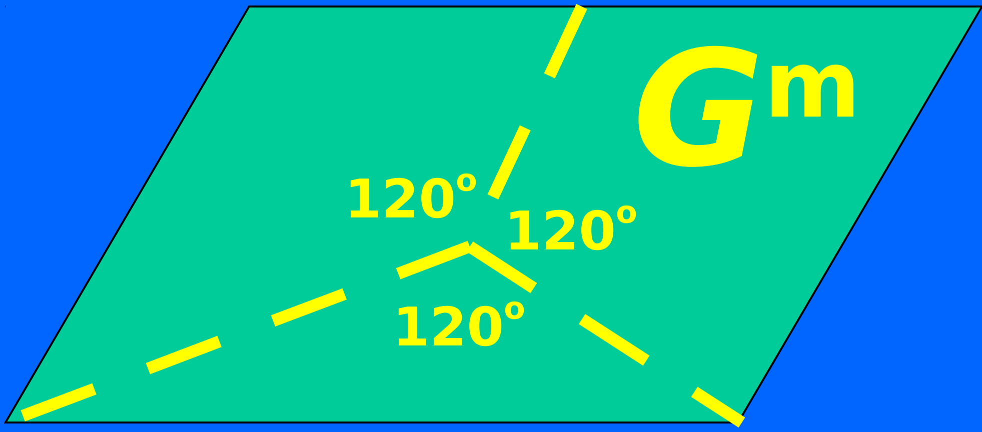


Proof.

Hutchings function *h* pos in G^2

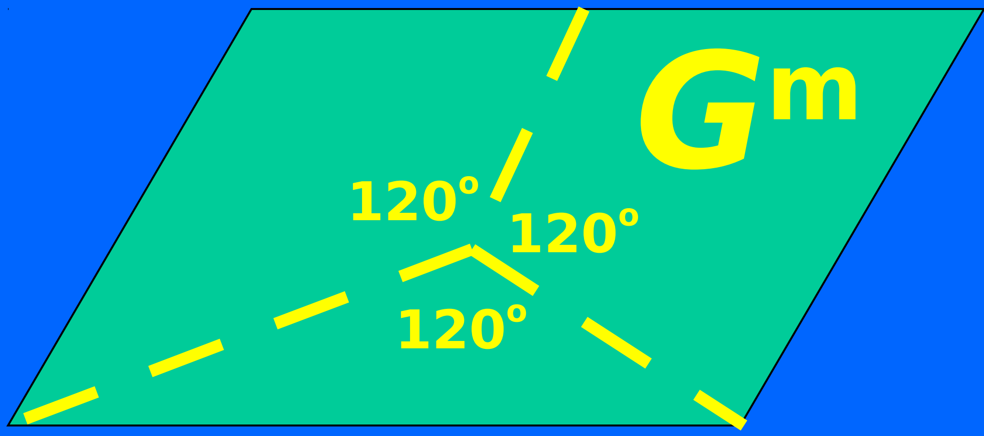
=> *h* pos in S^n for large and hence all n => Best dbl bub in S^n standard => Best dbl bub in G^m is hyperplanes at 120°

THREE VOLUMES IN GAUSSIAN G^m



Proved for nearly equal volumes (each .3 to .37).

THREE VOLUMES IN GAUSSIAN G^m



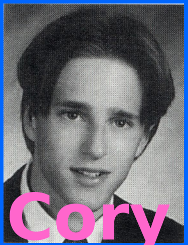
Proved for nearly equal volumes (each .3 to .37).

Proofs for all volumes require a different approach.

"SMALL" UND RES GEOM GROUP 1999

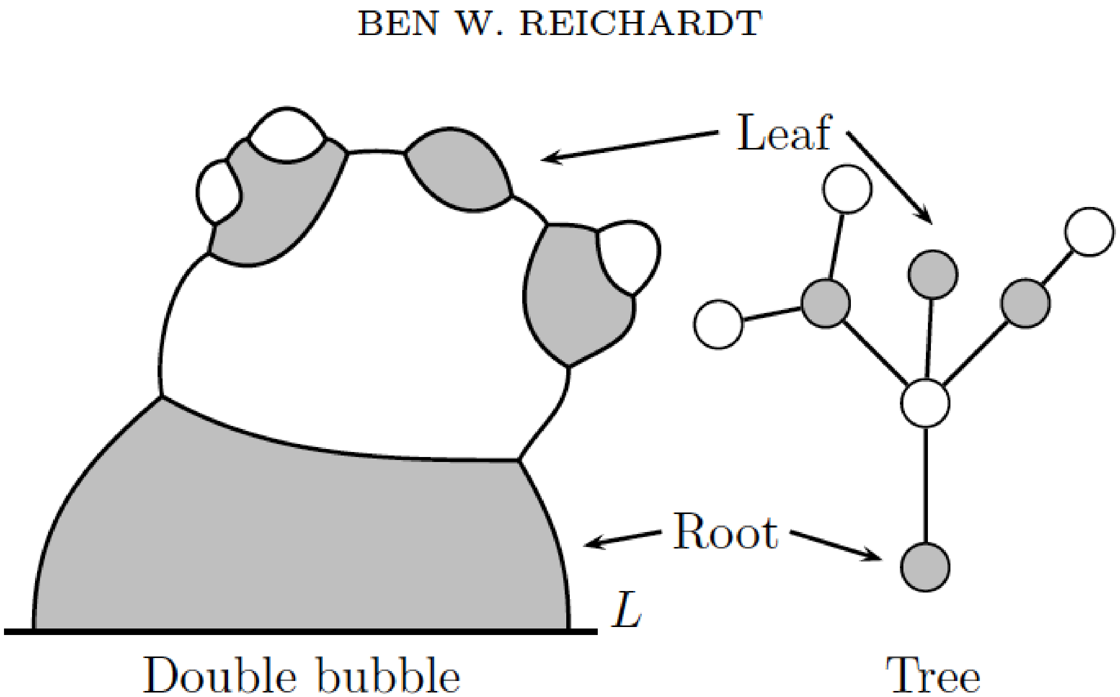
Ben Reichardt
Cory Heilmann
Yuan Lai
Anita Spielman

PROVED DBL BUB CONJ IN R^4



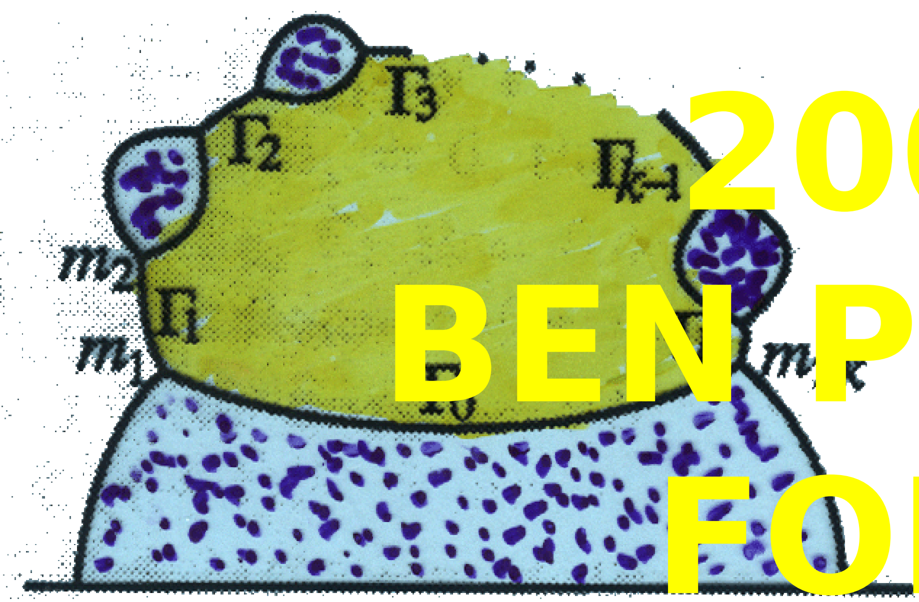
& R^n for certain volumes for which larger bubble is connected

2008: BEN PROVED FOR R^n



Neither region connected

Never extended to other spaces.



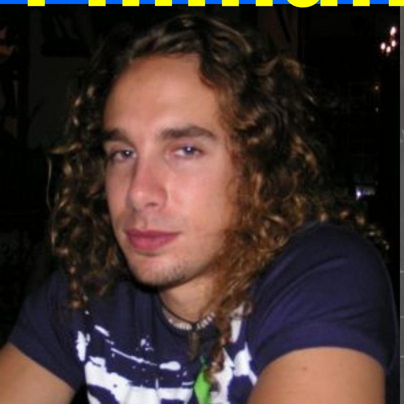
2008: BEN PROVED FOR R^n

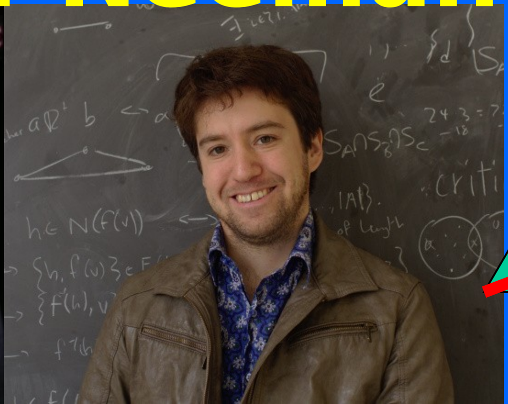
Gaussian Dbl Bub Thm.

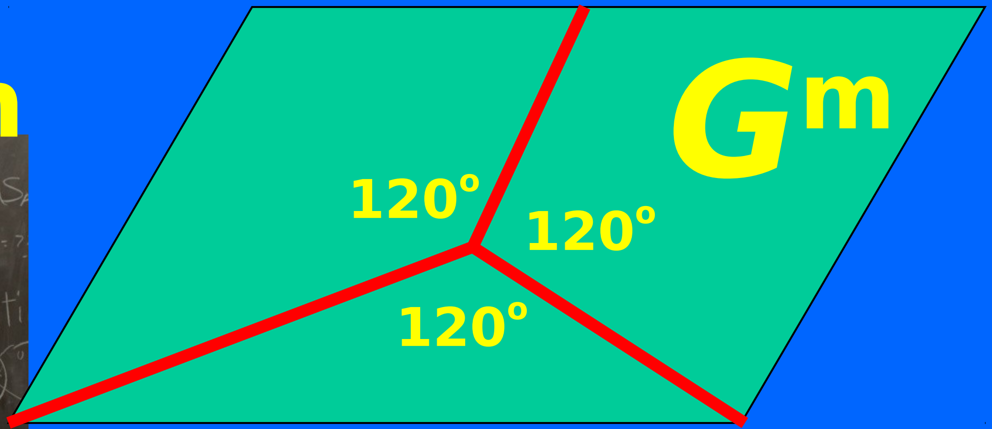
(Milman-Neeman, arXiv Jan’18). For any 3 volumes in G^m best partition is

3 HYPERPLANES AT 120°

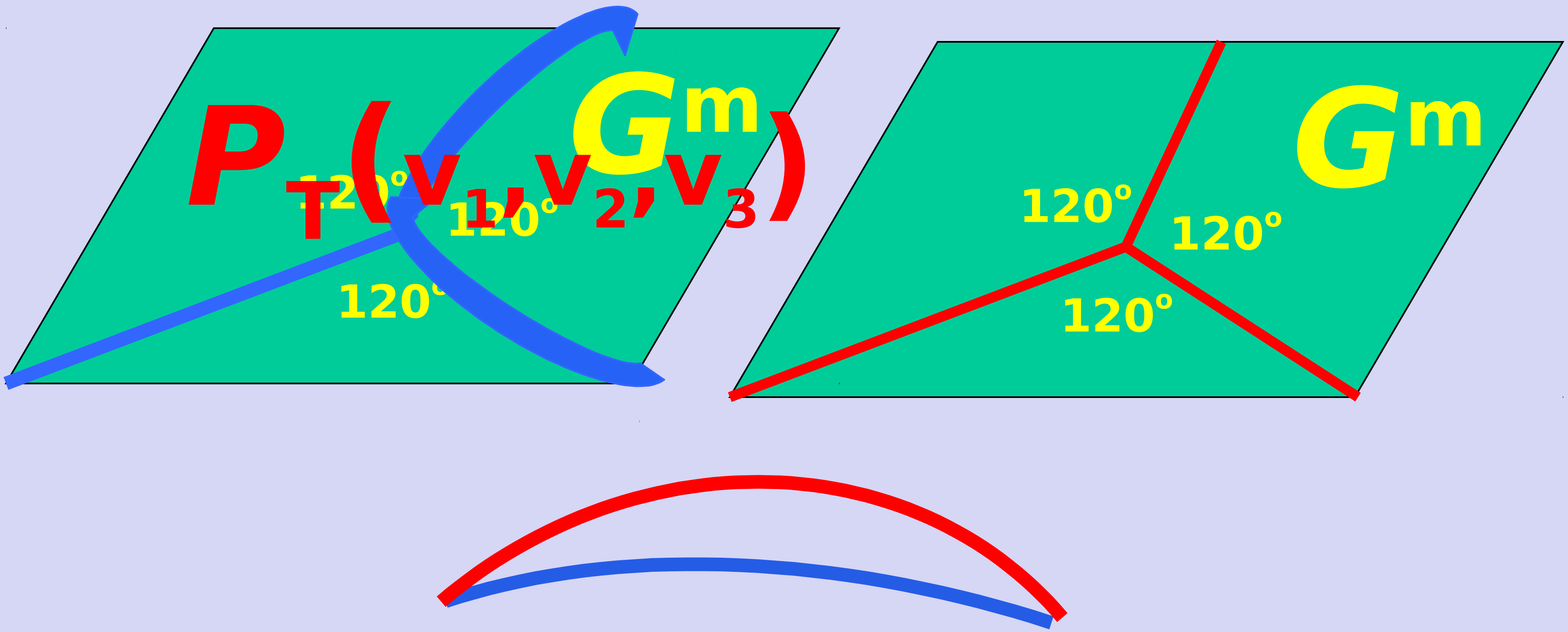
Milman-Neeman







$P_0(v_1, v_2, v_3) \leq$

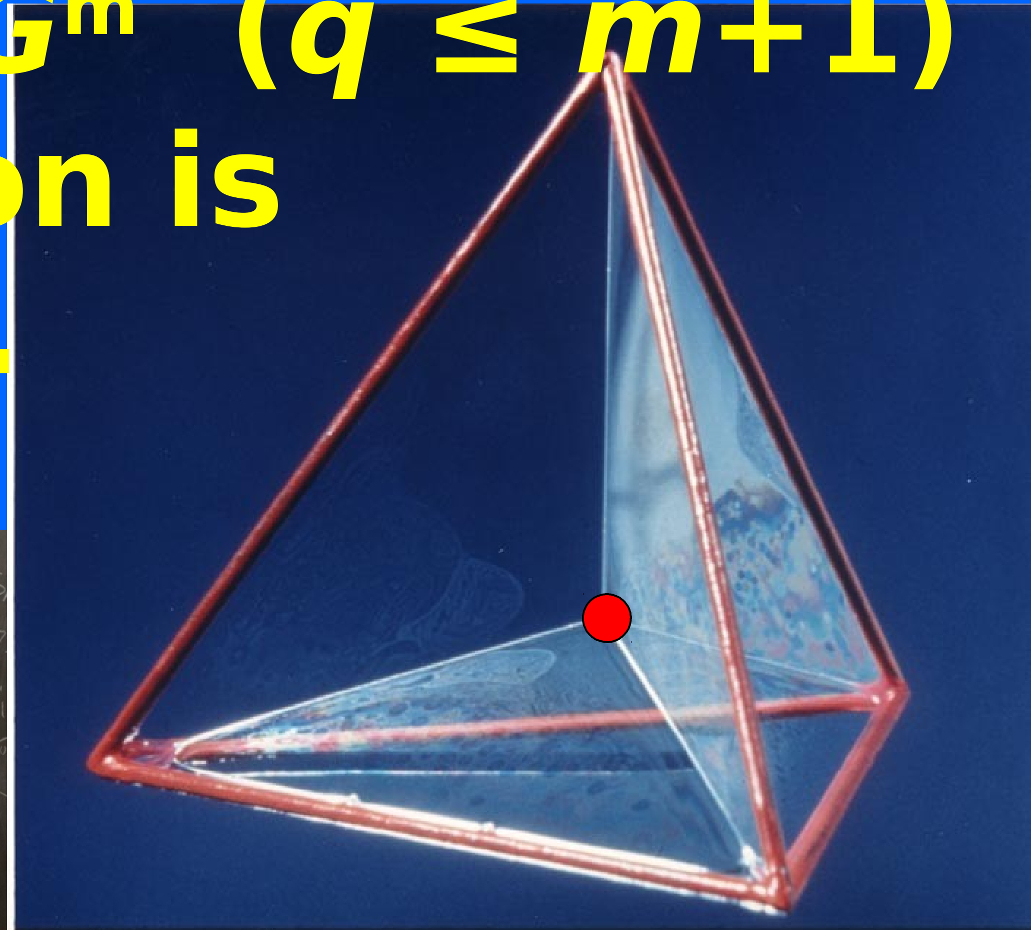


Prop. $P_0'' \leq P_T''$

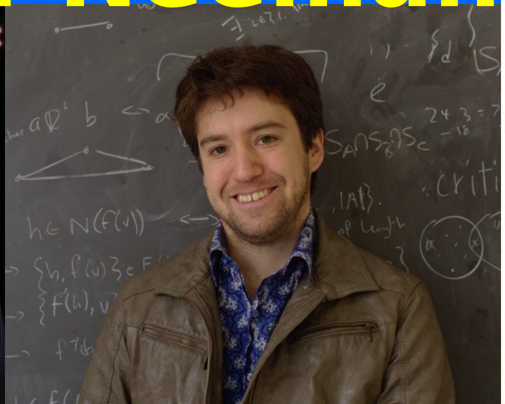
Contradiction.

Gaussian Multi-Bub Thm.

(Milman-Neeman, May '18). For any q volumes in G^m $(q \leq m+1)$ best partition is reg simplex.

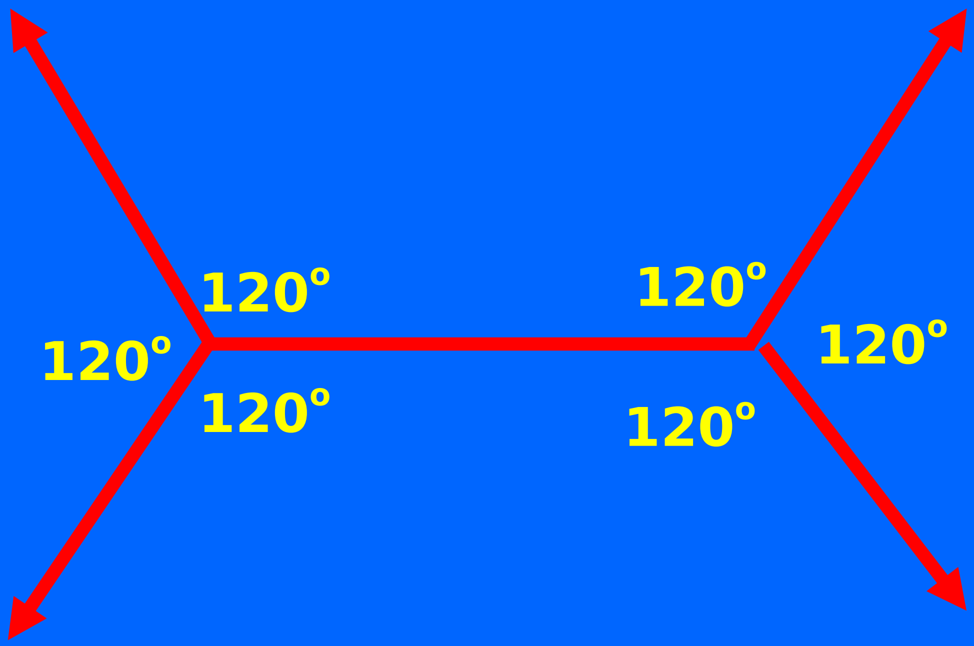


Milman-Neeman



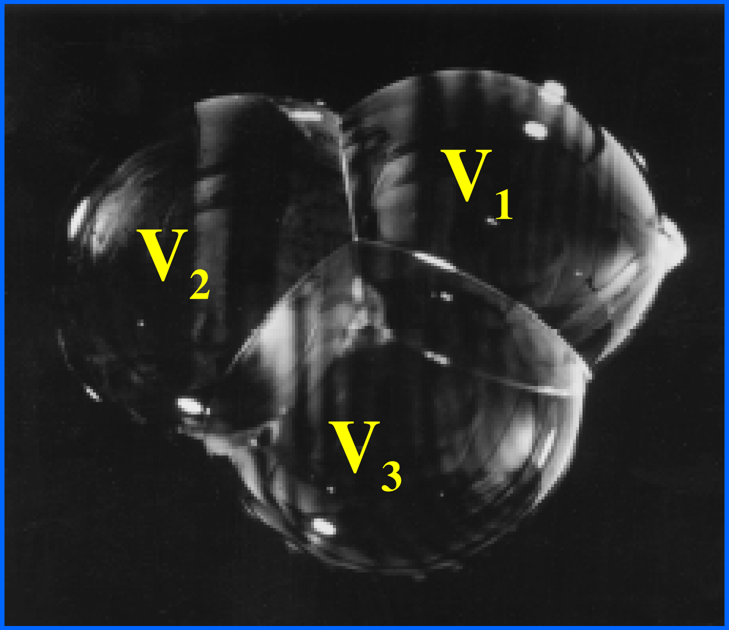
Conjecture in G^2

Best partition for 4 areas ($q = m+2$)

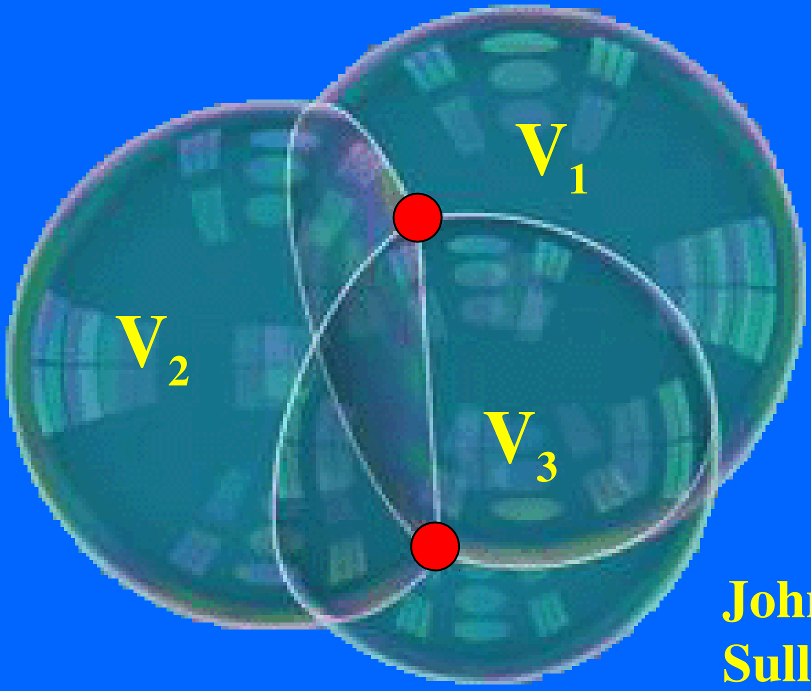


OPEN QUESTION IN R^n

IS THE STANDARD TRIPLE BUBBLE THE ABSOLUTE LEAST AREA SHAPE?

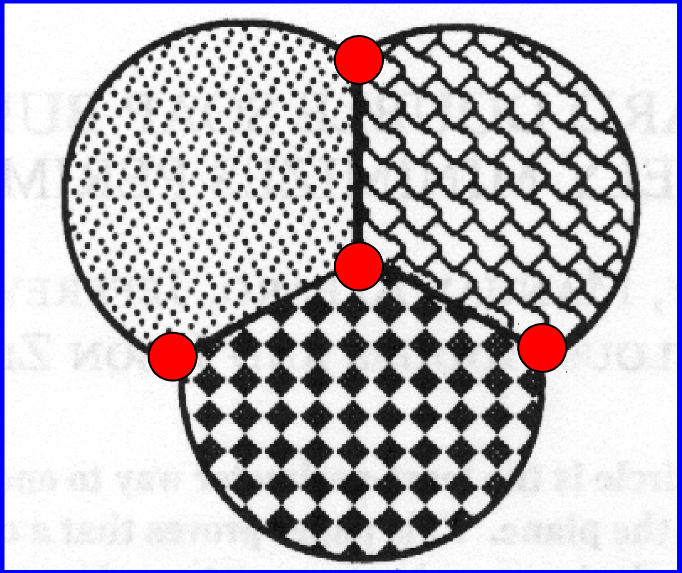


Jeremy Ackerman



John M Sullivan

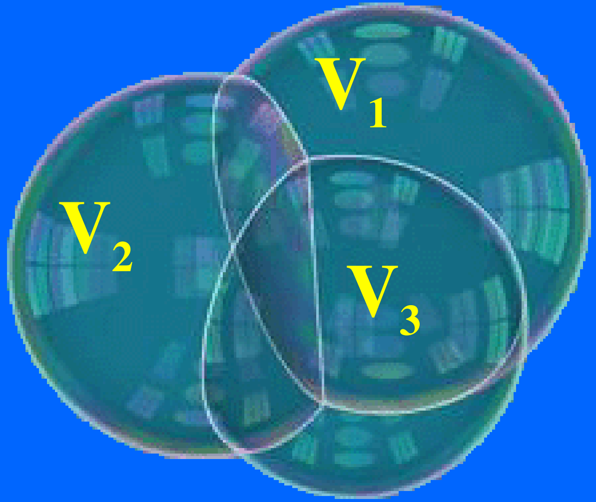
BEST PLANAR TRIPLE BUBBLE



Wacharin Wichiramala
PhD thesis, 2002
Hundreds of cases.

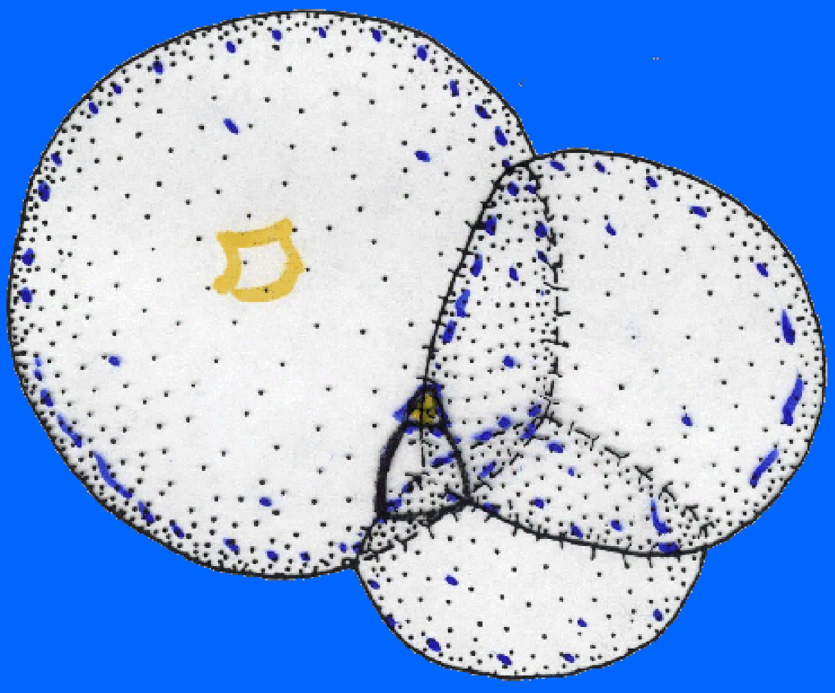
DO SOAP BUBBLE CLUSTERS FIND THE ABSOLUTE LEAST AREA SHAPE?

A. Yes, always.
B. No, not always.
C. No one knows for sure.

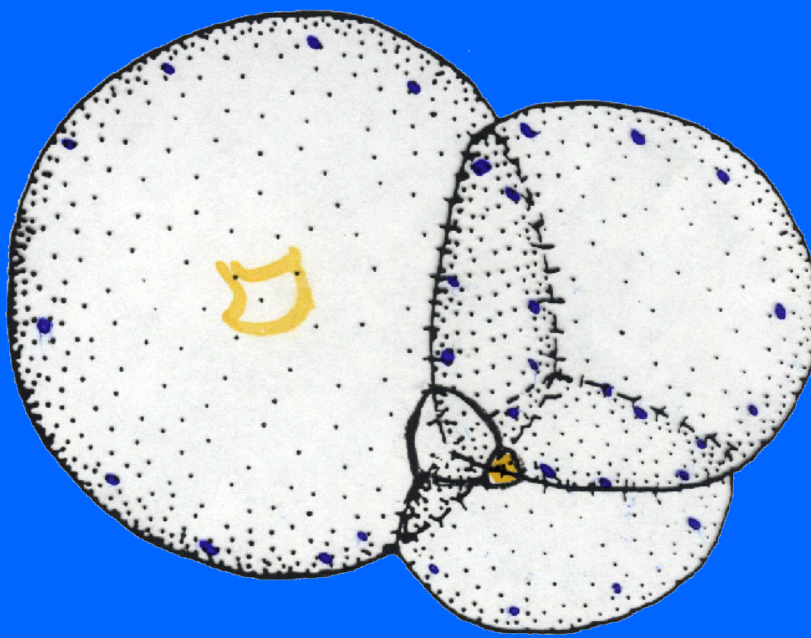


DO SOAP BUBBLE CLUSTERS FIND THE ABSOLUTE LEAST AREA SHAPE?

B. NO, NOT ALWAYS:

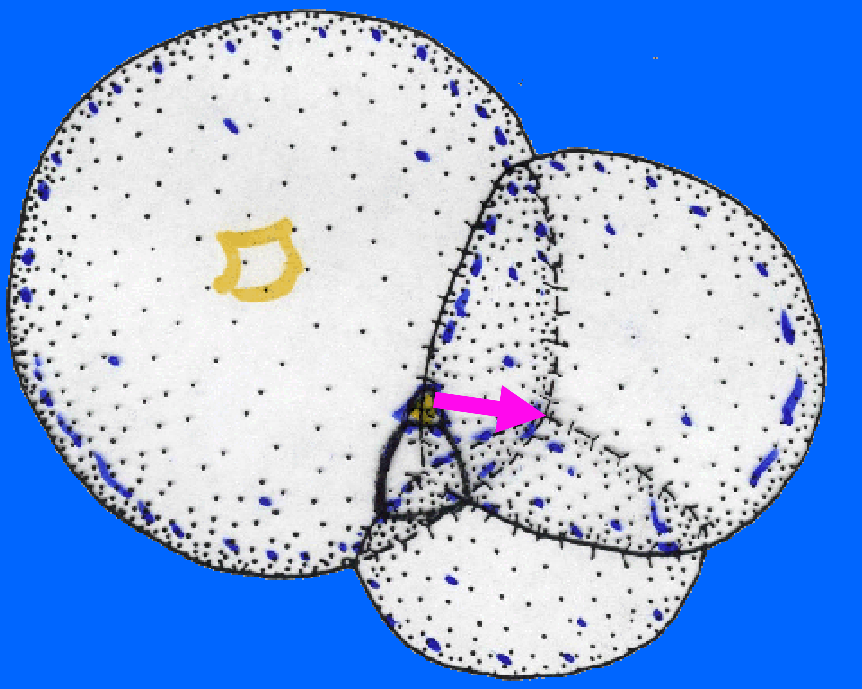


less area



more area

DO SOAP BUBBLE CLUSTERS FIND THE ABSOLUTE LEAST AREA SHAPE?



least area ?

No, better to put tiny fifth region in back, with three largest bubbles.

Best?

I think so, but it’s hard to know for sure.

Recall for R^n with radial log-*convex* density single bubble is sphere about origin.

$$e^{+r^2}$$

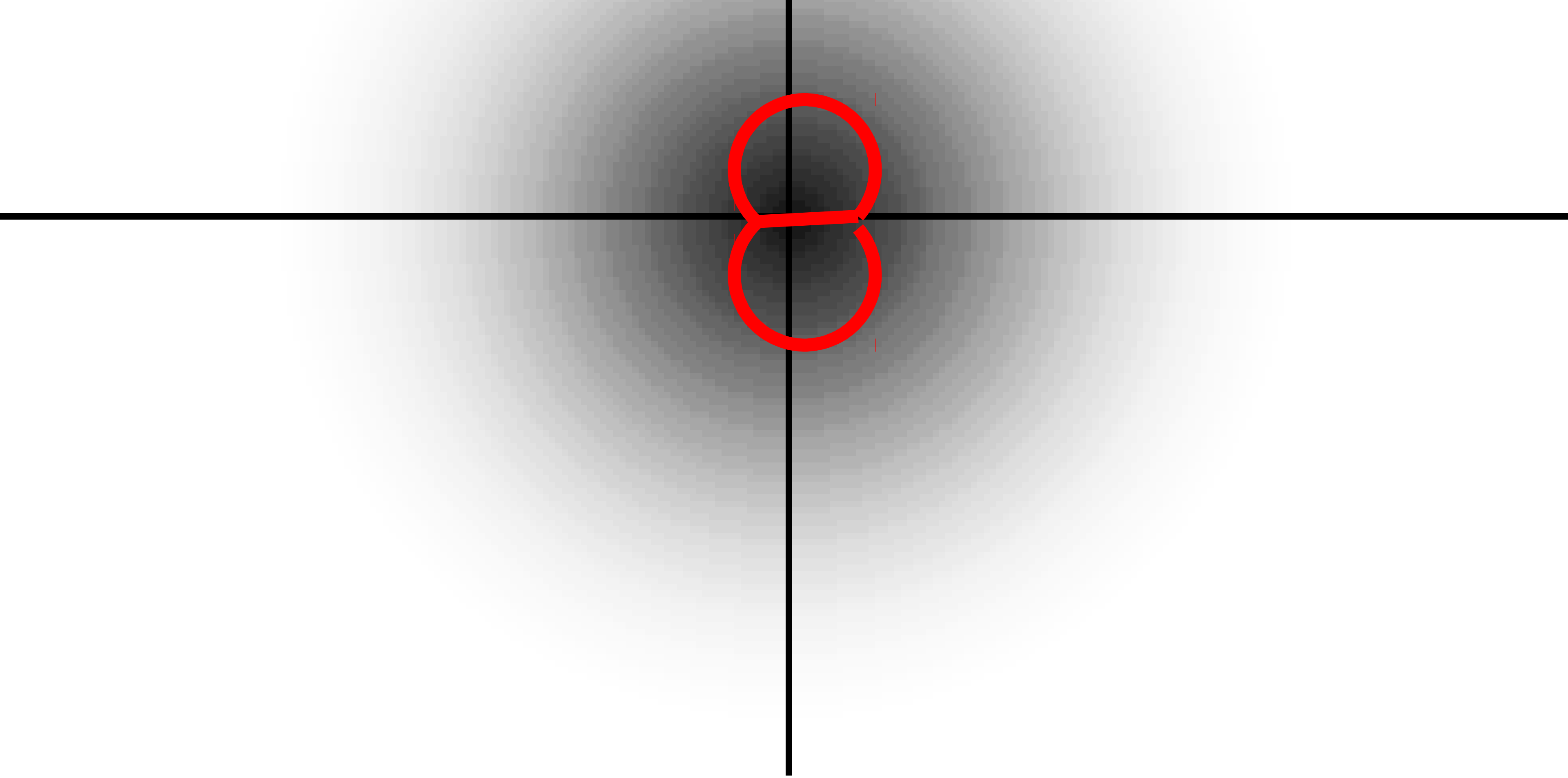
precisionballs.com

R^2 with density e^{r^2}

Double bubble problem: enclose and separate given *weighted* areas with minimum *weighted* e^{r^2} perimeter.

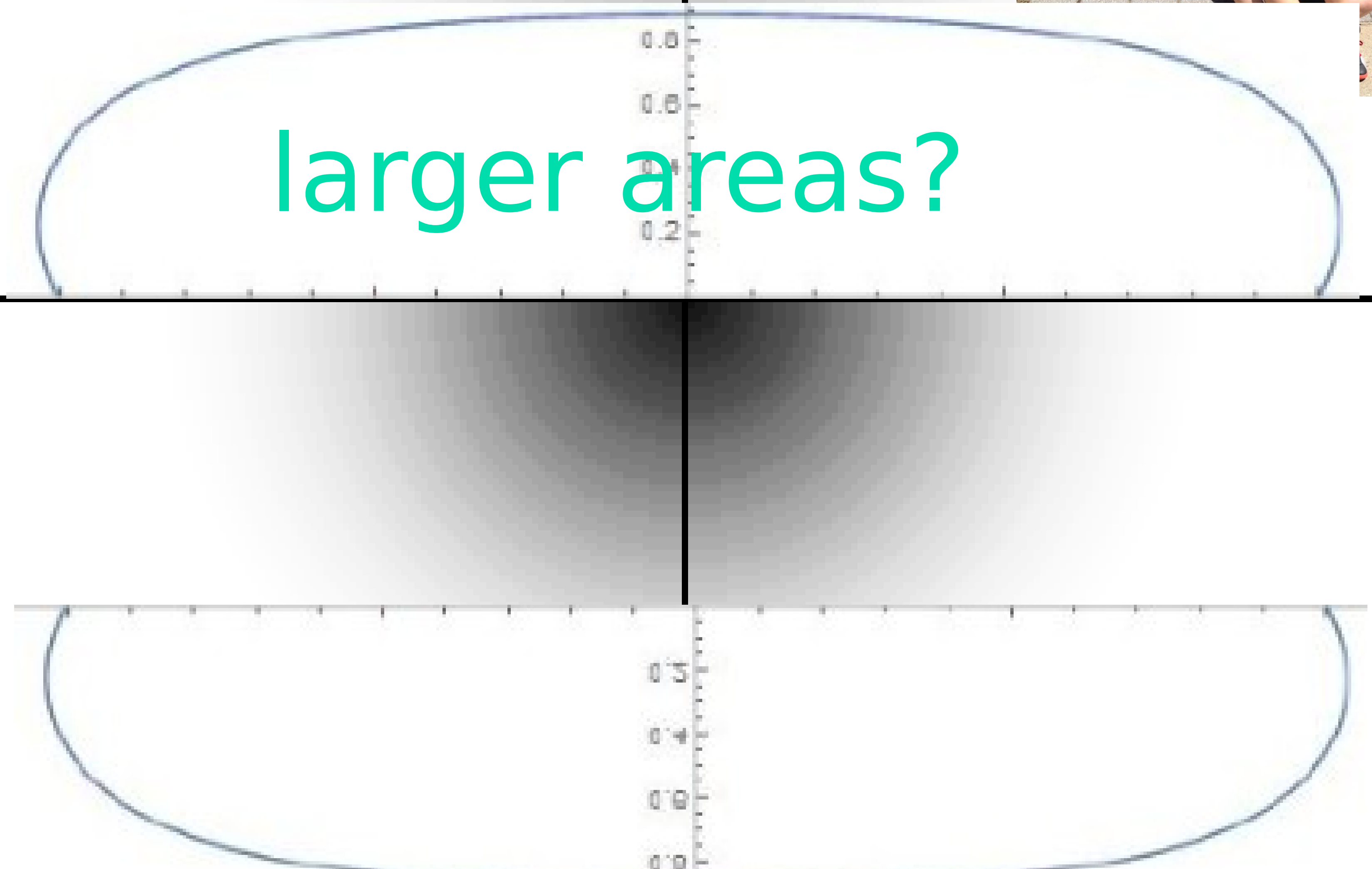
R^2 with density e^{r^2}

small areas



Diana Davis '07

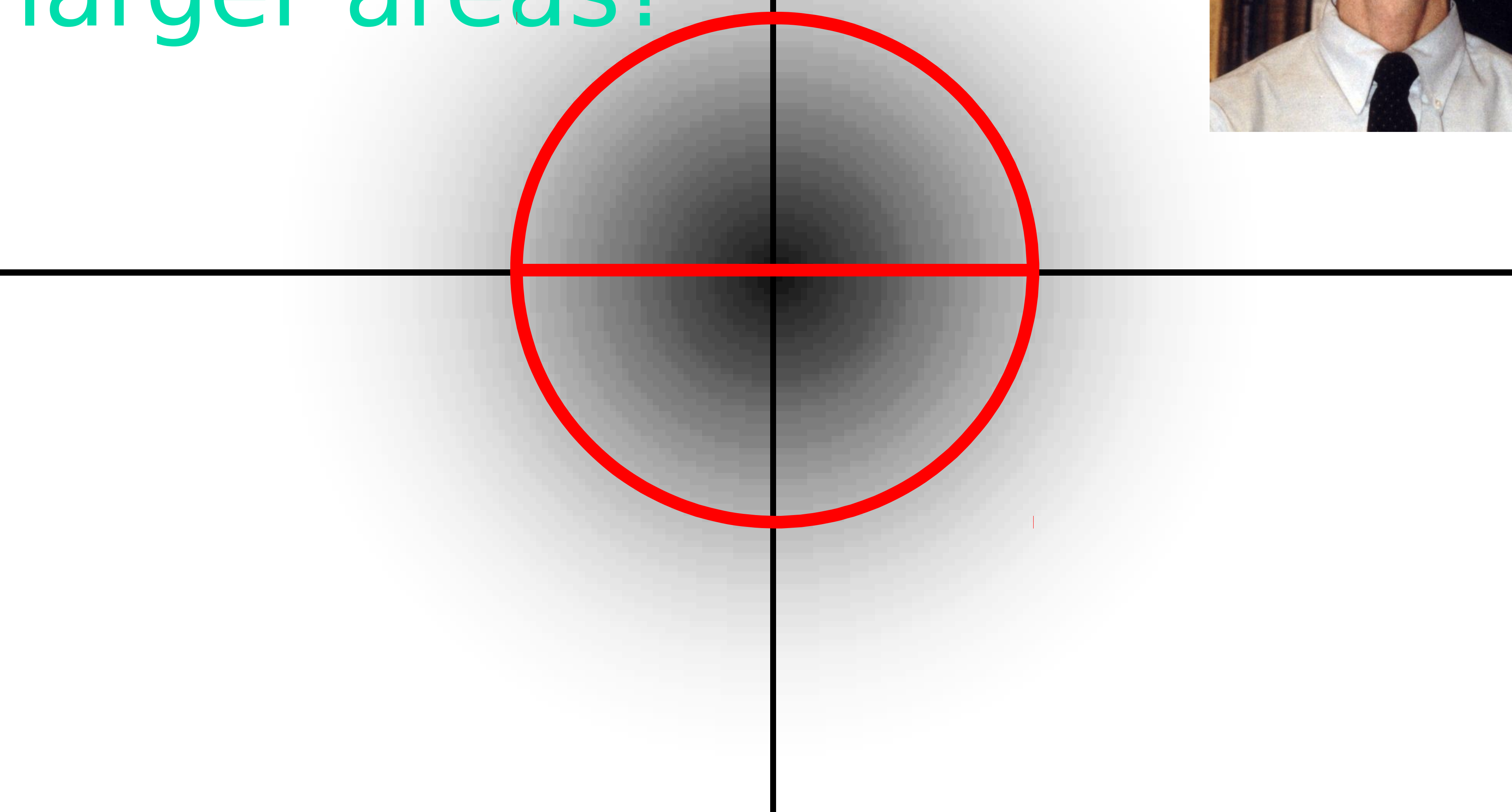
R^2 with density e^{r^2}

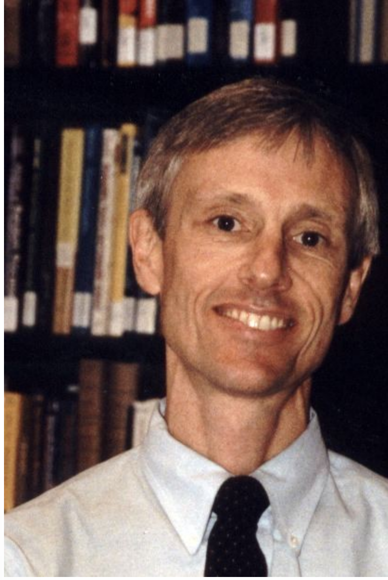


larger areas?

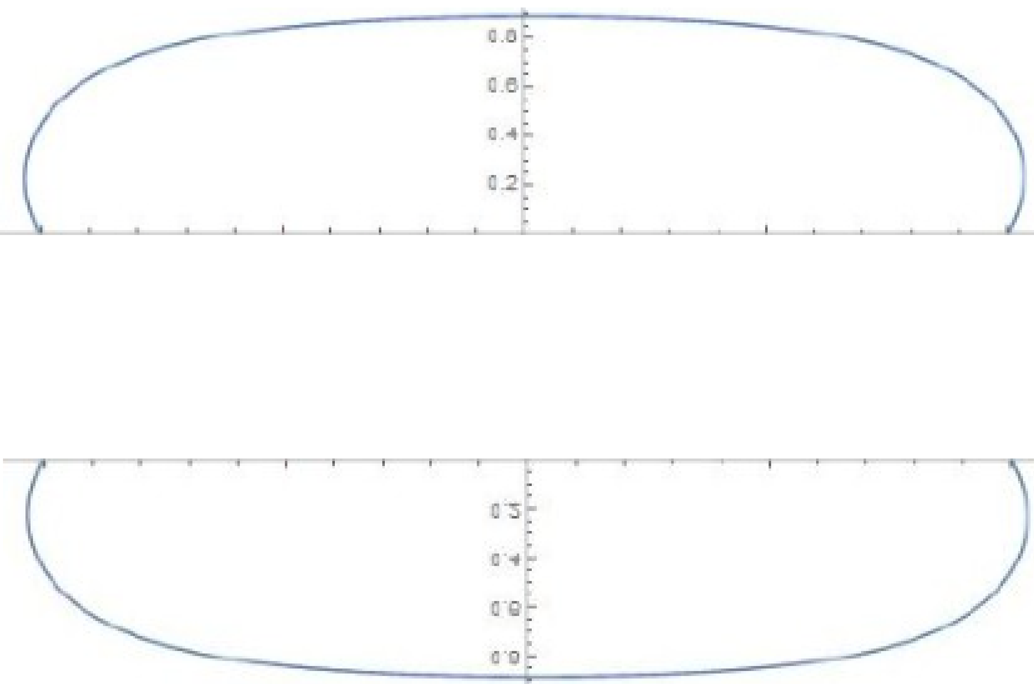
R^2 with density e^{r^2}

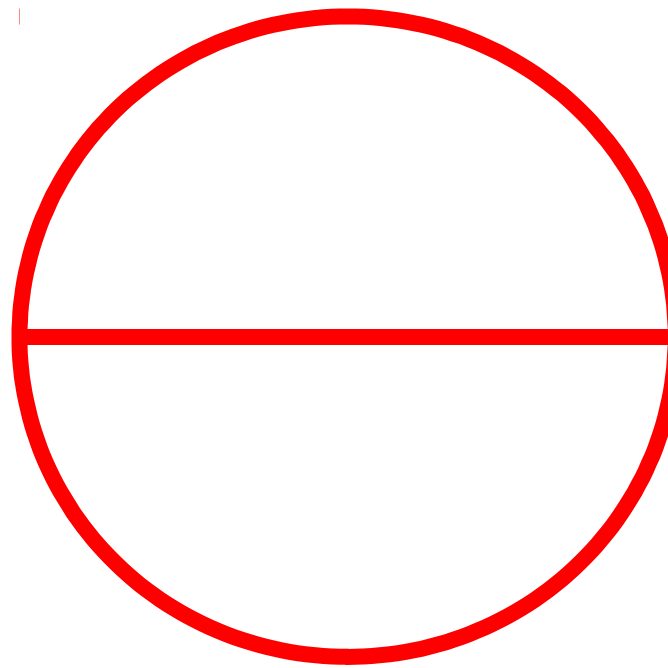
larger areas?

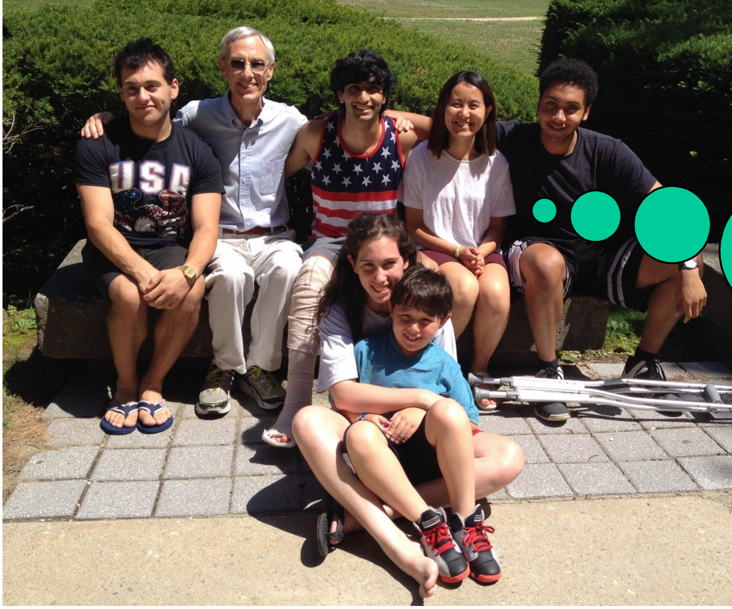




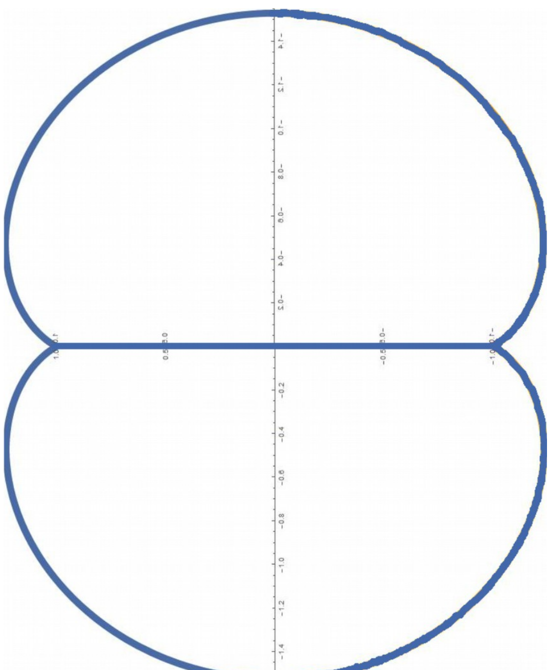
Who’s right?

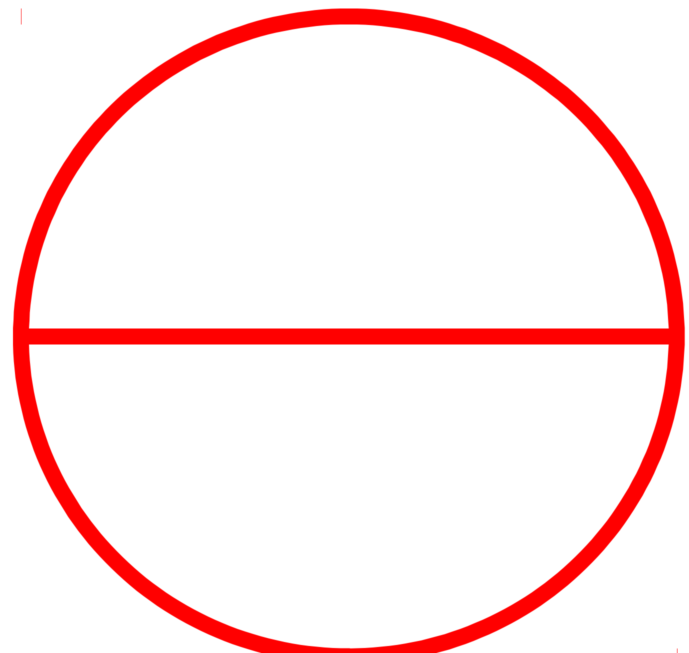


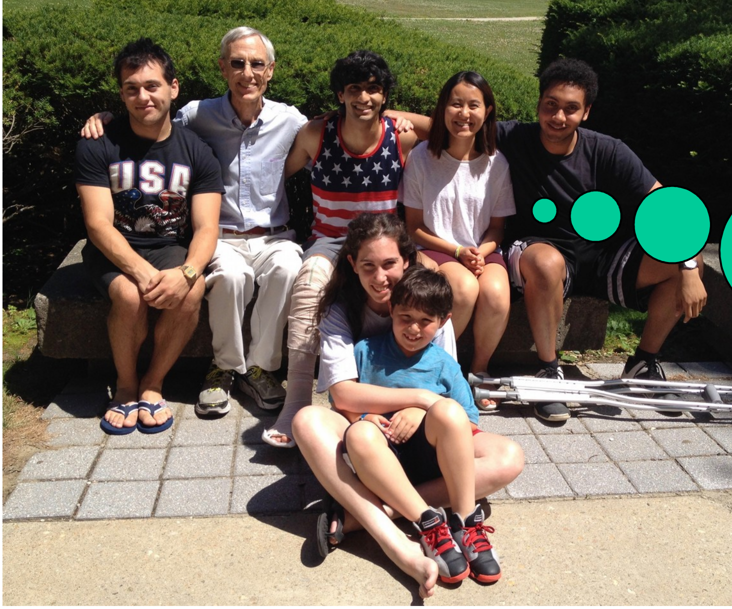




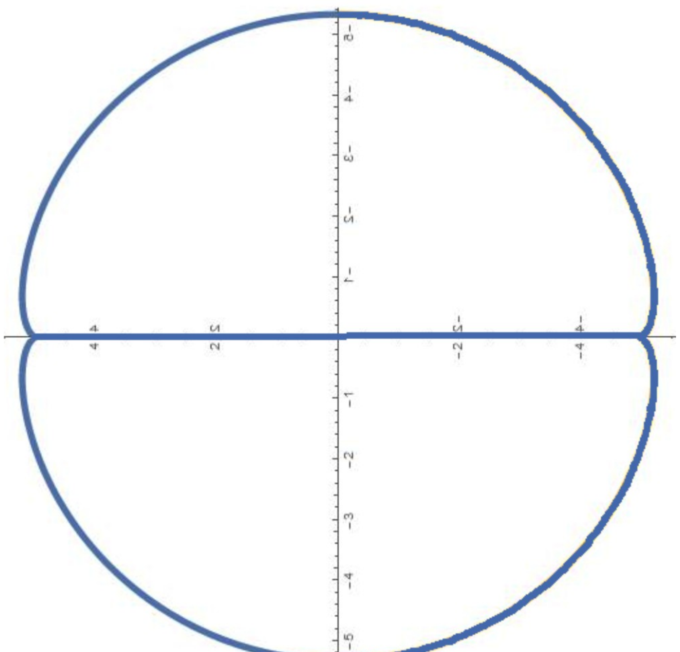
“You were right”

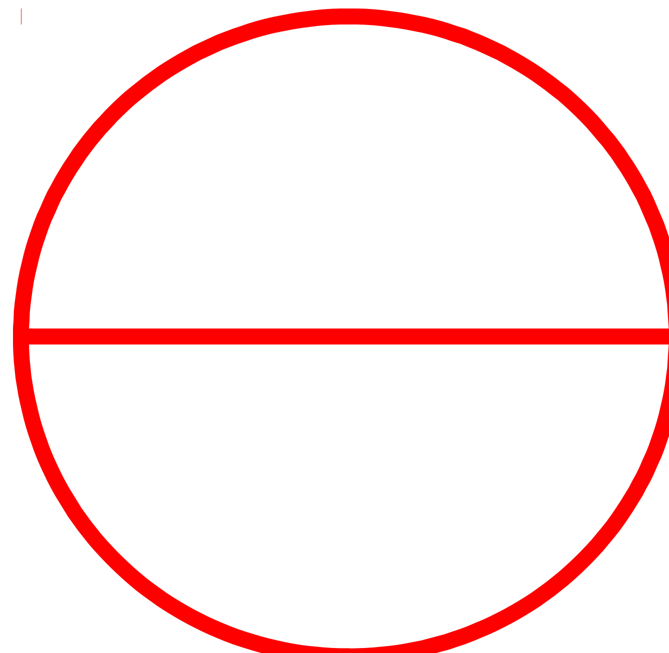


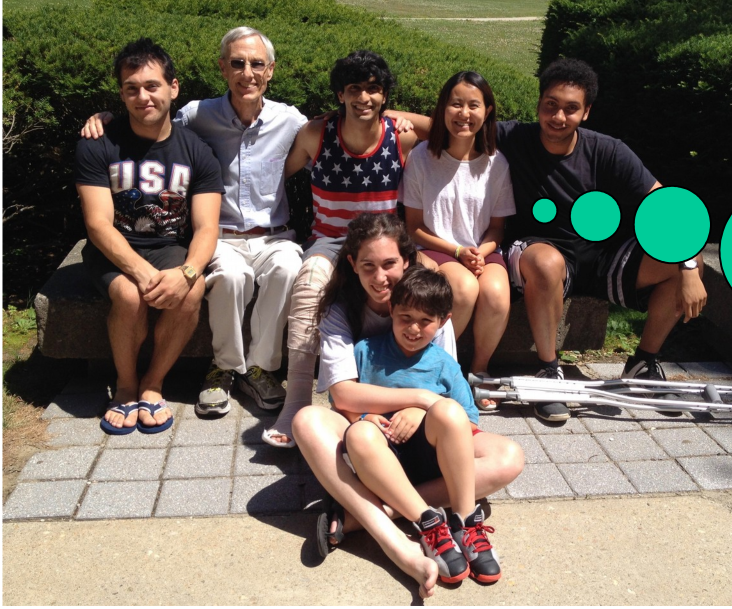




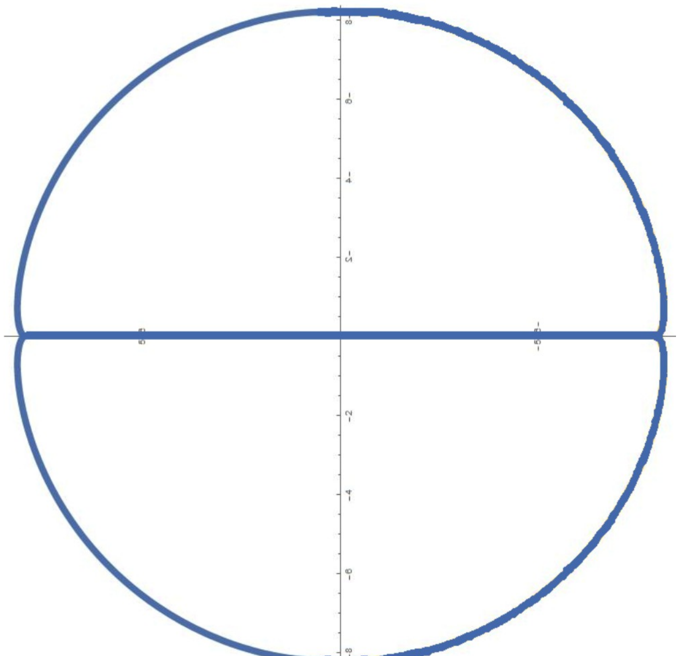

“You were right”

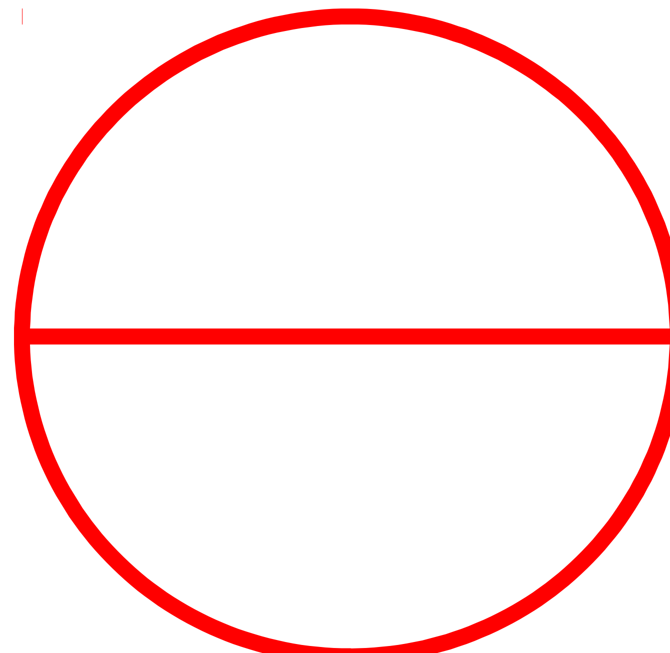


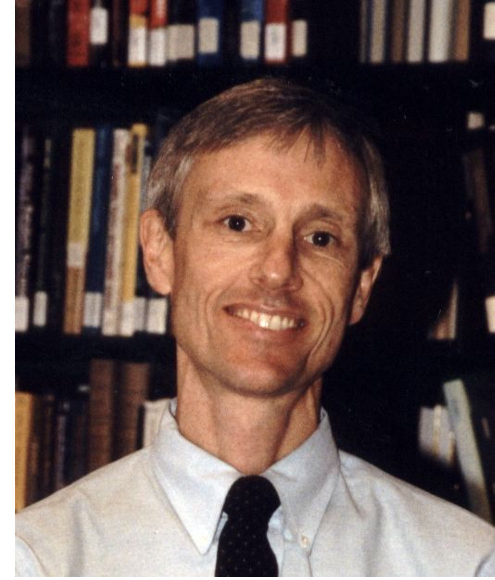


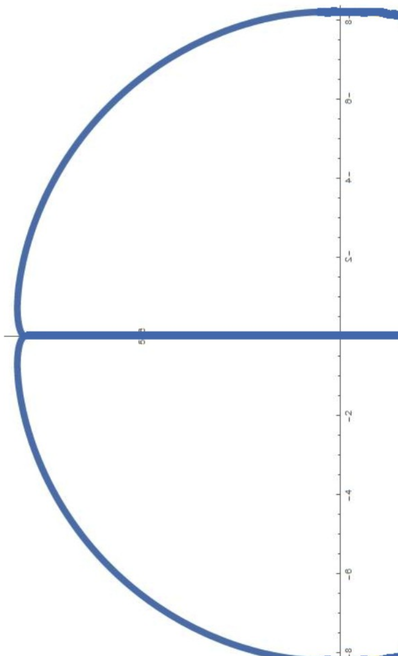


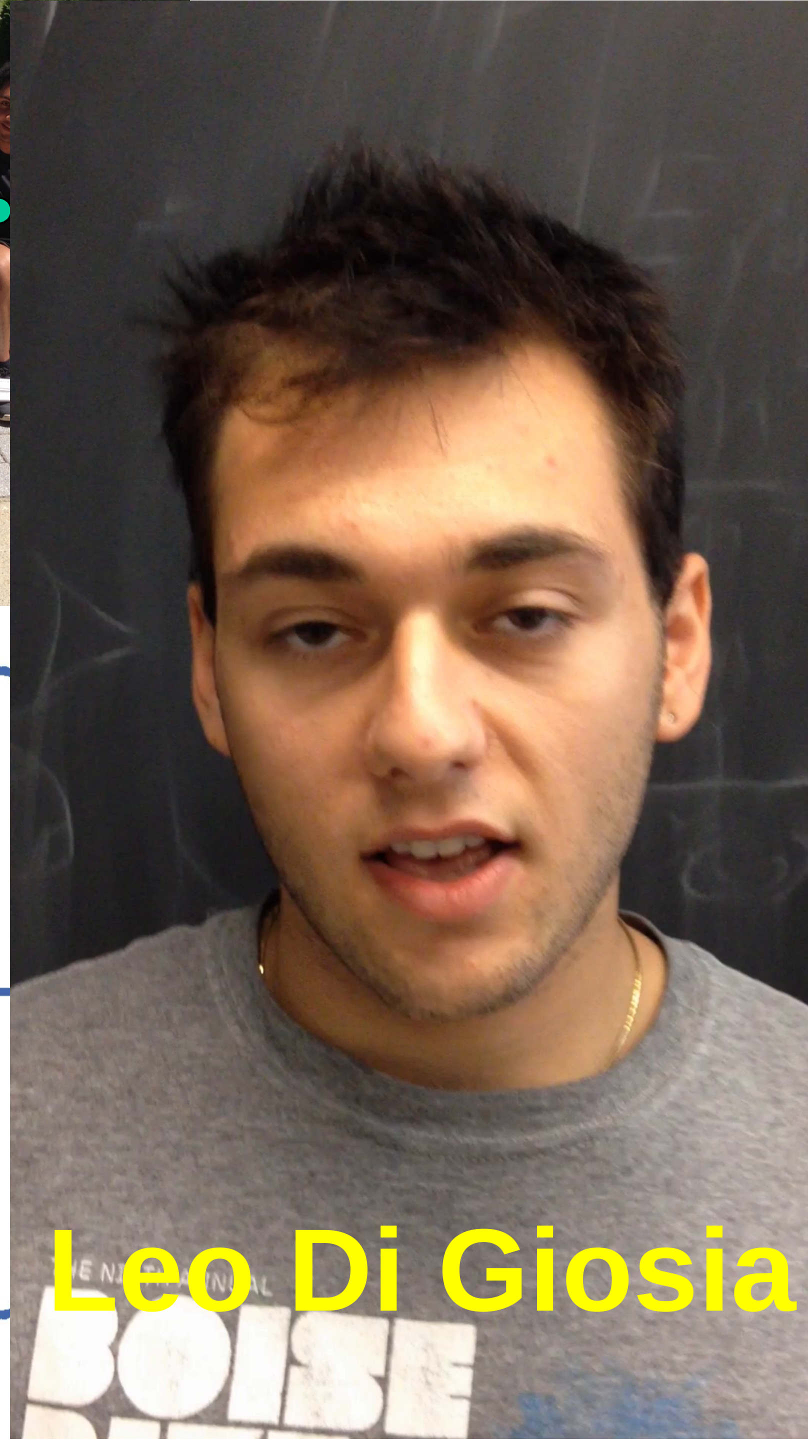

"You were right"

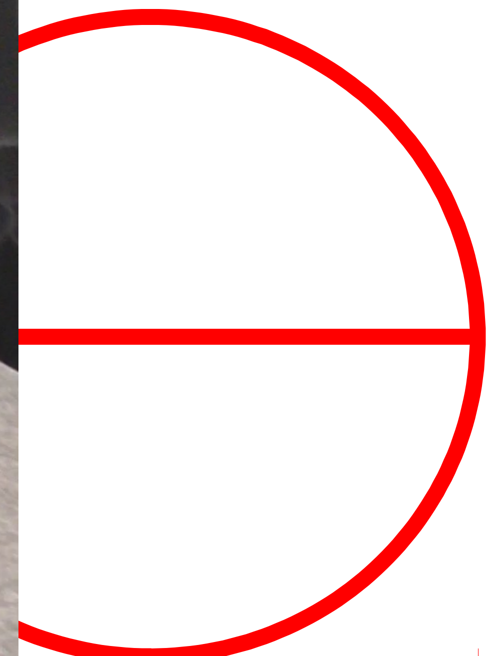




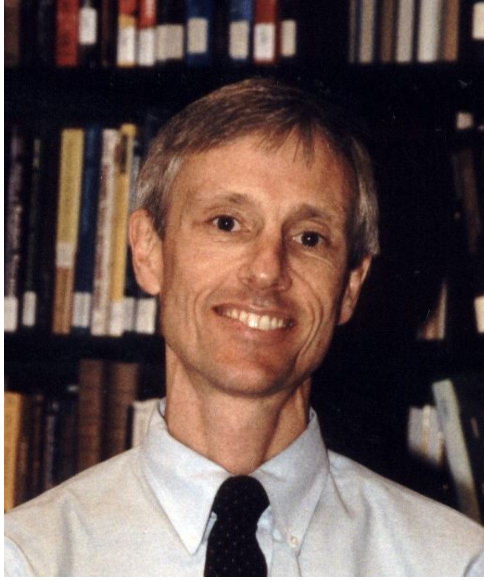


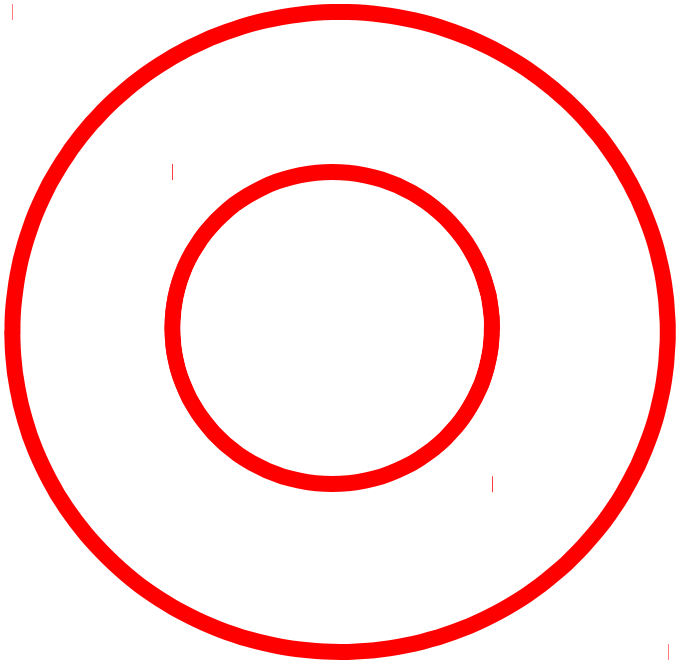


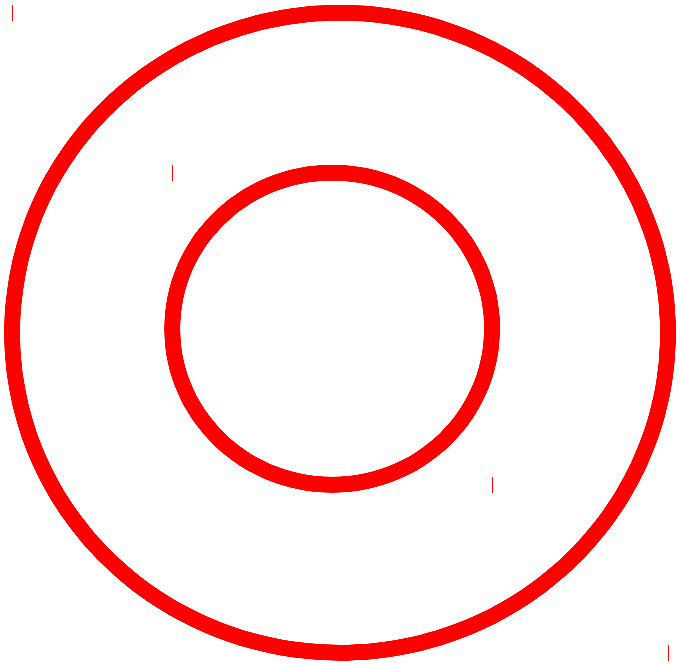

Leo Di Giosia



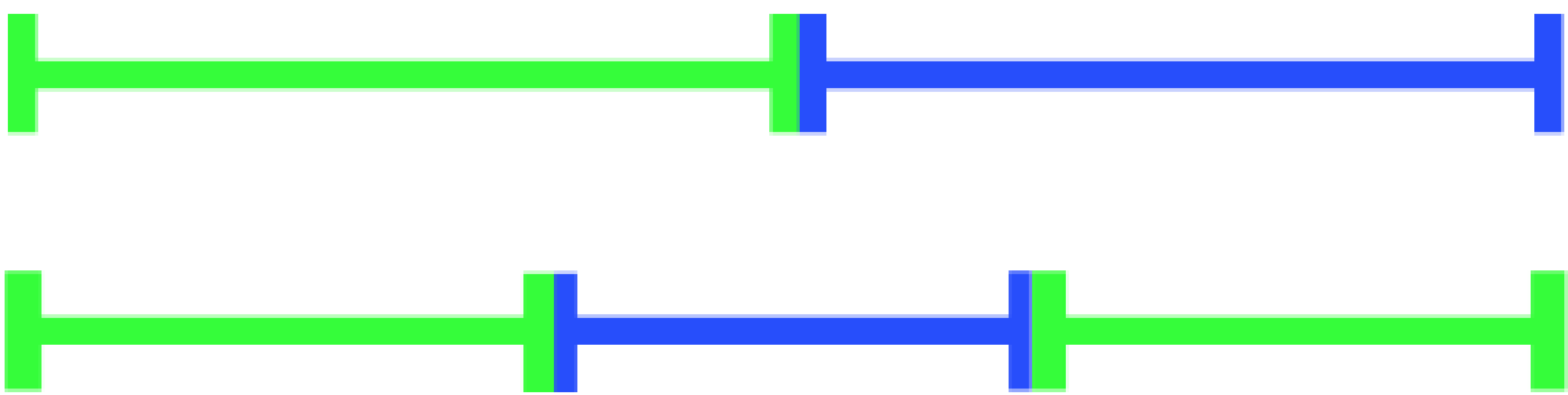
Some large unequal areas

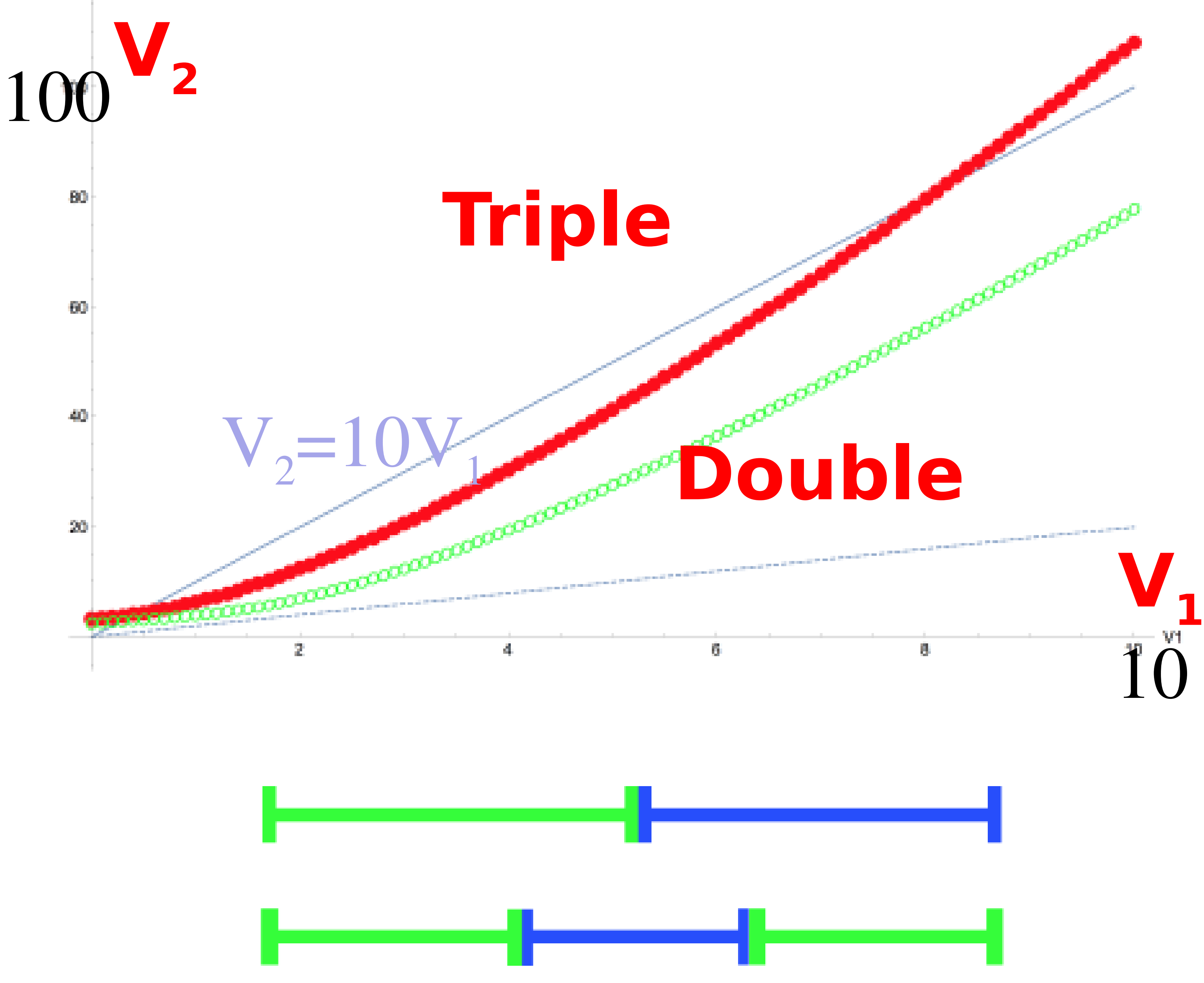


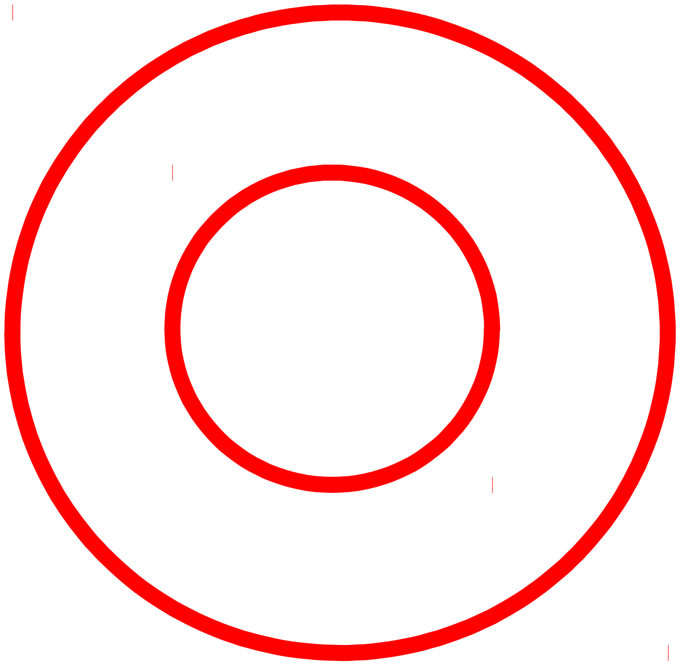




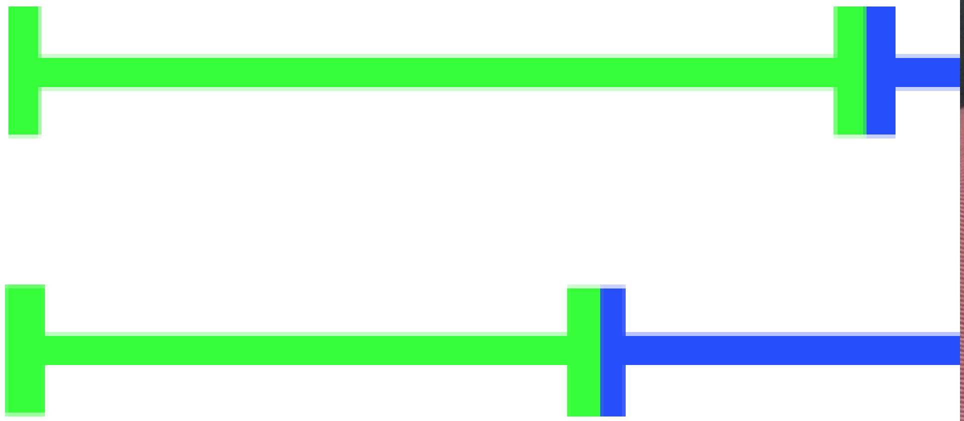
Proved in 1D for log-convex densities.




V2
100
Triple
V2=10V1
Double
V1
10



Prov
1D



Anal. Geom. Metr. Spaces 2018; 6:64–88 DE GRUYTER

Research Article Open Access

Eliot Bongiovanni, Leonardo Di Giosia, Alejandro Diaz, Jahangir Habib, Arjun Kakkar, Lea Kenigsberg, Dylanger Pittman, Nat Sothanaphan*, and Weitao Zhu

Double Bubbles on the Real Line with Log-Convex Density

WILLIAMS COLLEGE MATHEMATICS

math.williams.edu

small

UNDERGRADUATE RESEARCH

June 12 – August 12, 2017

The SMALL Undergraduate Research Project is a nine-week summer program (with a tenth week at home) in which undergraduates investigate open research problems in mathematics. One of the largest programs of its kind in the country, SMALL has been supported by grants from the NSF for Research Experiences for Undergraduates and by Williams College. Over 500 students have participated in the project since its inception in 1988. Students work in small groups directed by individual faculty members. Participants publish papers and present talks at research conferences based on work done in SMALL. Many have gone on to complete PhDs in Mathematics or related fields.

During off hours, students enjoy the many attractions of summer in the Berkshires: hiking, biking, plays, concerts, etc. Weekly lunches, teas, and casual sporting events bring SMALL students together with faculty and other students spending the summer doing research at Williams College. Students receive a stipend of about $4000. Several board plans are available at reasonable rates, with housing included for free.

Research Groups

The 2017 groups are Commutative Algebra (Susan Loepp), Geometry (Frank Morgan) Knot Theory (Colin Adams), Number Theory and Probability (Steven J. Miller), and Tropical Geometry (Ralph Morrison).

Students from around the world. Apply by early Feb.

For more information, see
http://math.williams.edu/small/
or email the program director (sjm1@williams.edu)

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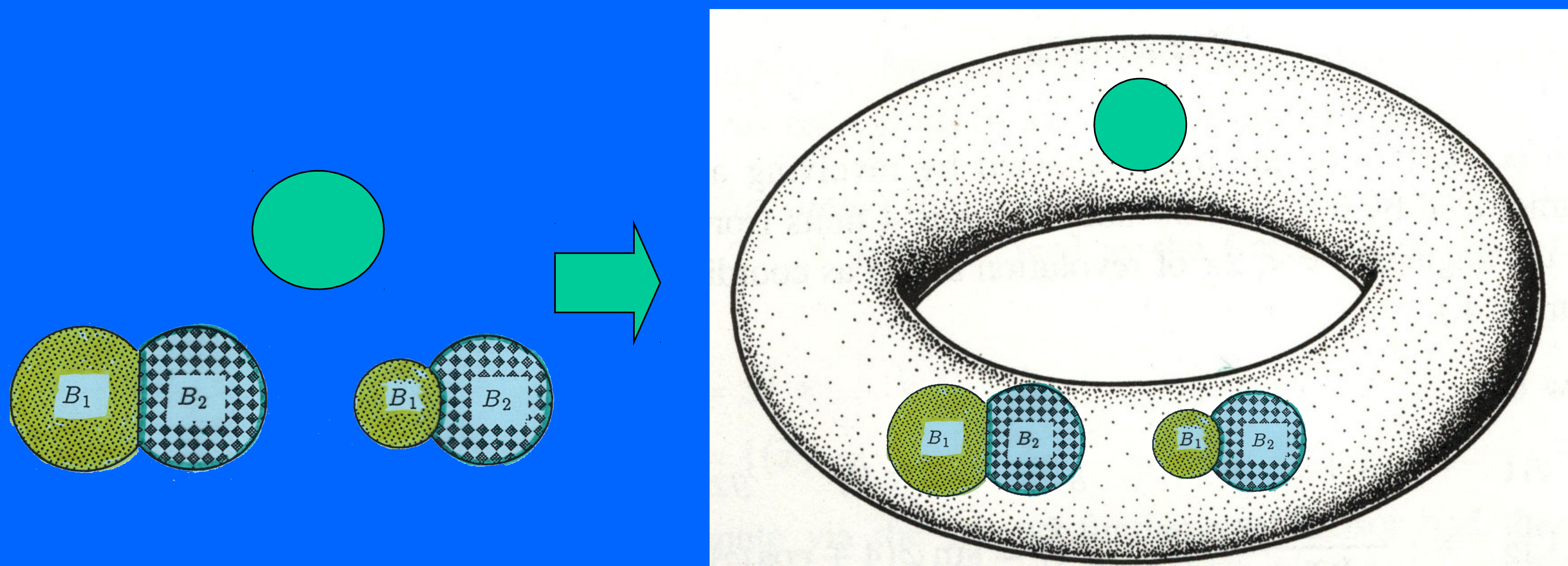
Research Experience for Undergraduates Summer Programs

The table below contains links to REU programs active during the summer season. Applicants should note that most application deadlines fall in February - March. Program directors: this list will be updated regularly. To update your entry, or to request a new entry, please send email to the AMS. Here is an article by Frank Morgan on what makes a good REU proposal.

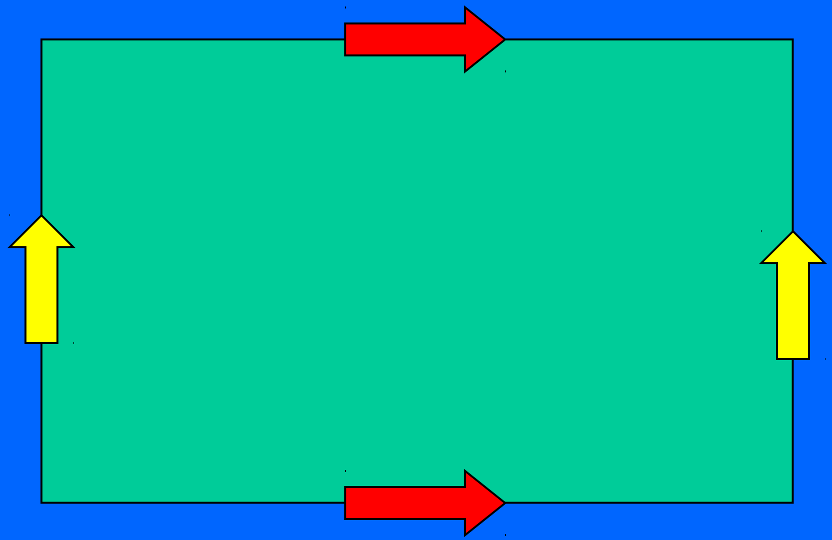
Most of the REU programs on this page handle their applications through the AMS service MathPrograms.org.

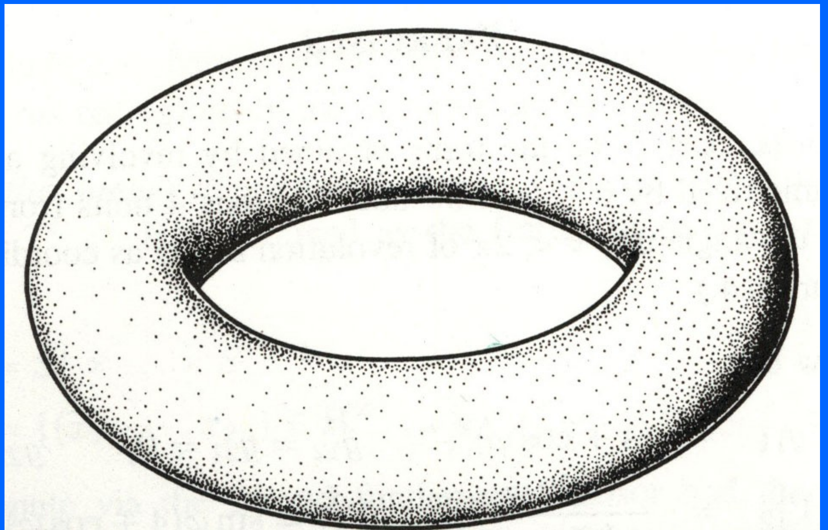
Lafayette College	Summer REU Program 2018

Bubbles in a Torus

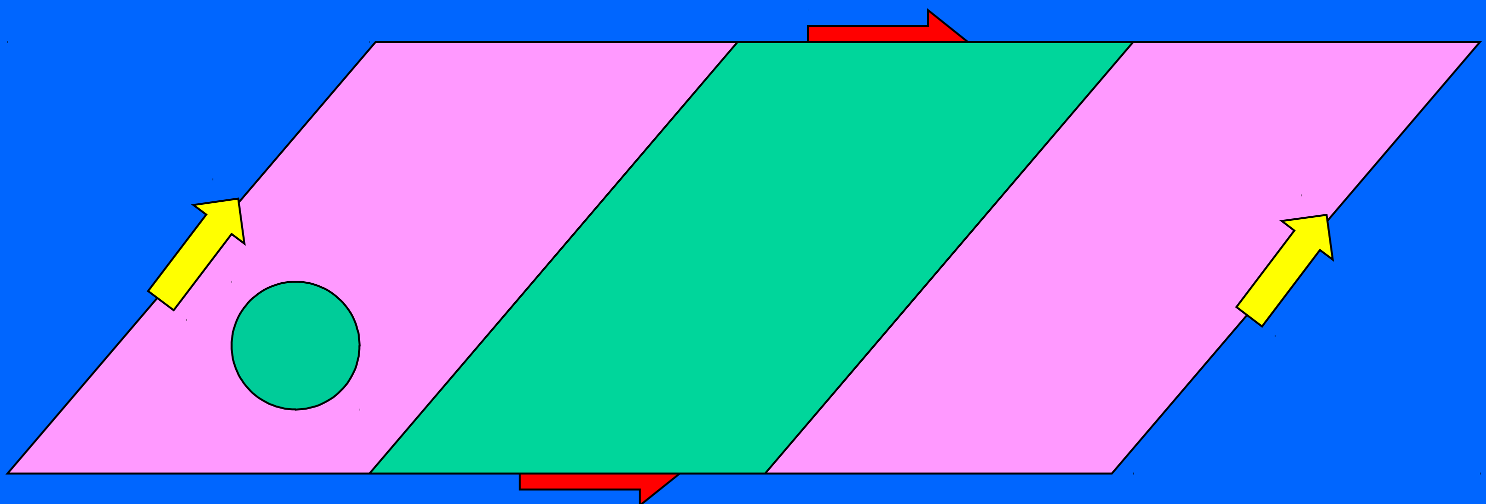


BUBBLES IN FLAT 2D TORUS



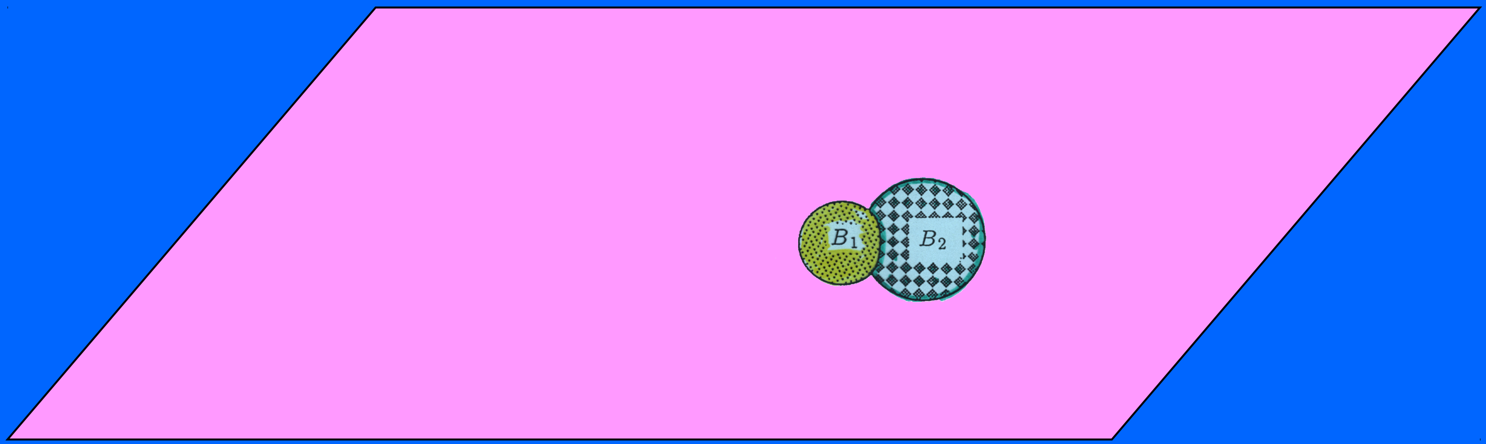


BEST SINGLE BUBBLE IN TORUS T^2:

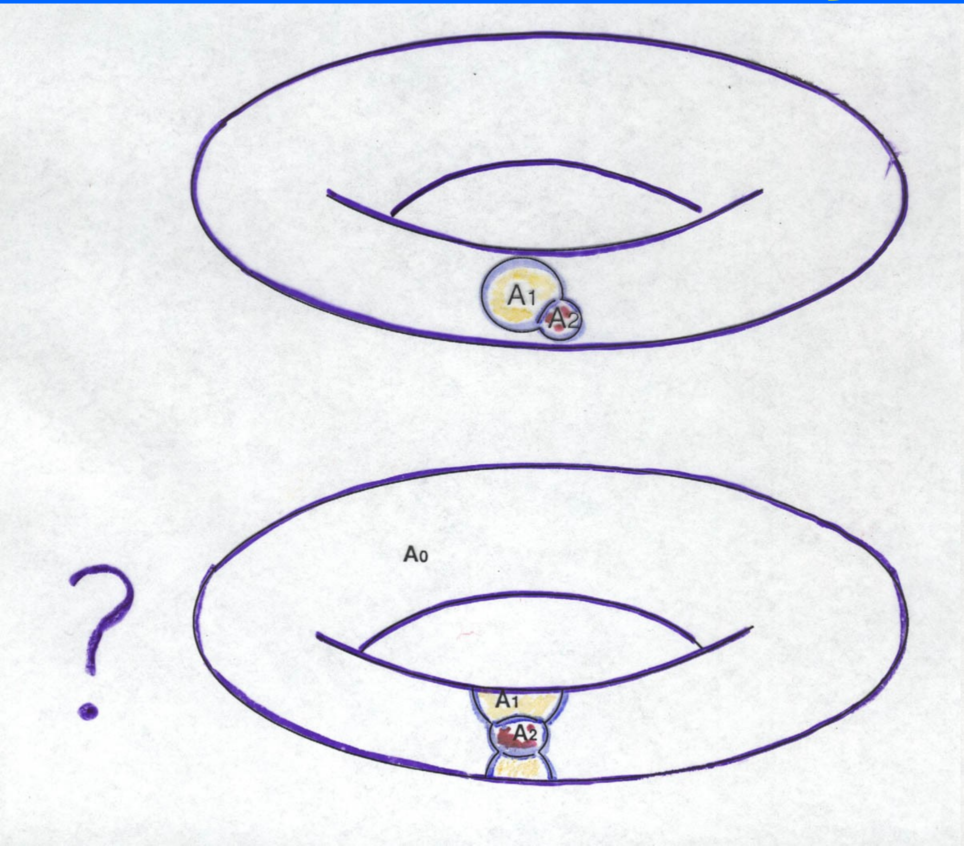


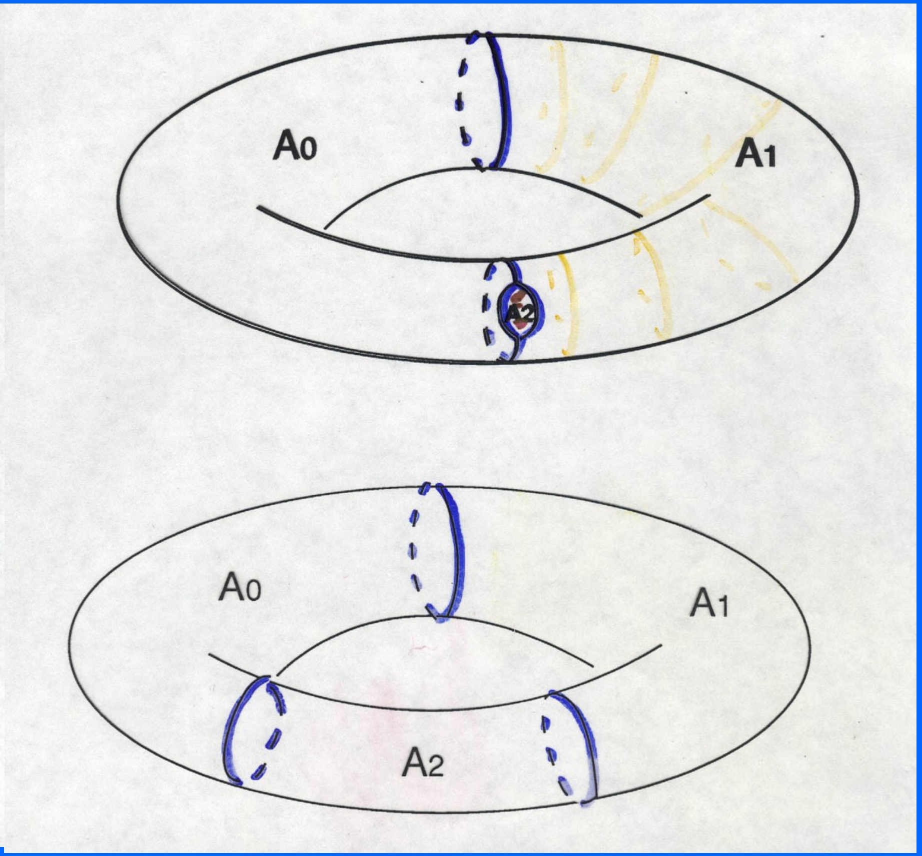
H. Howards '92

BEST DOUBLE BUBBLE IN TORUS T^2 ?



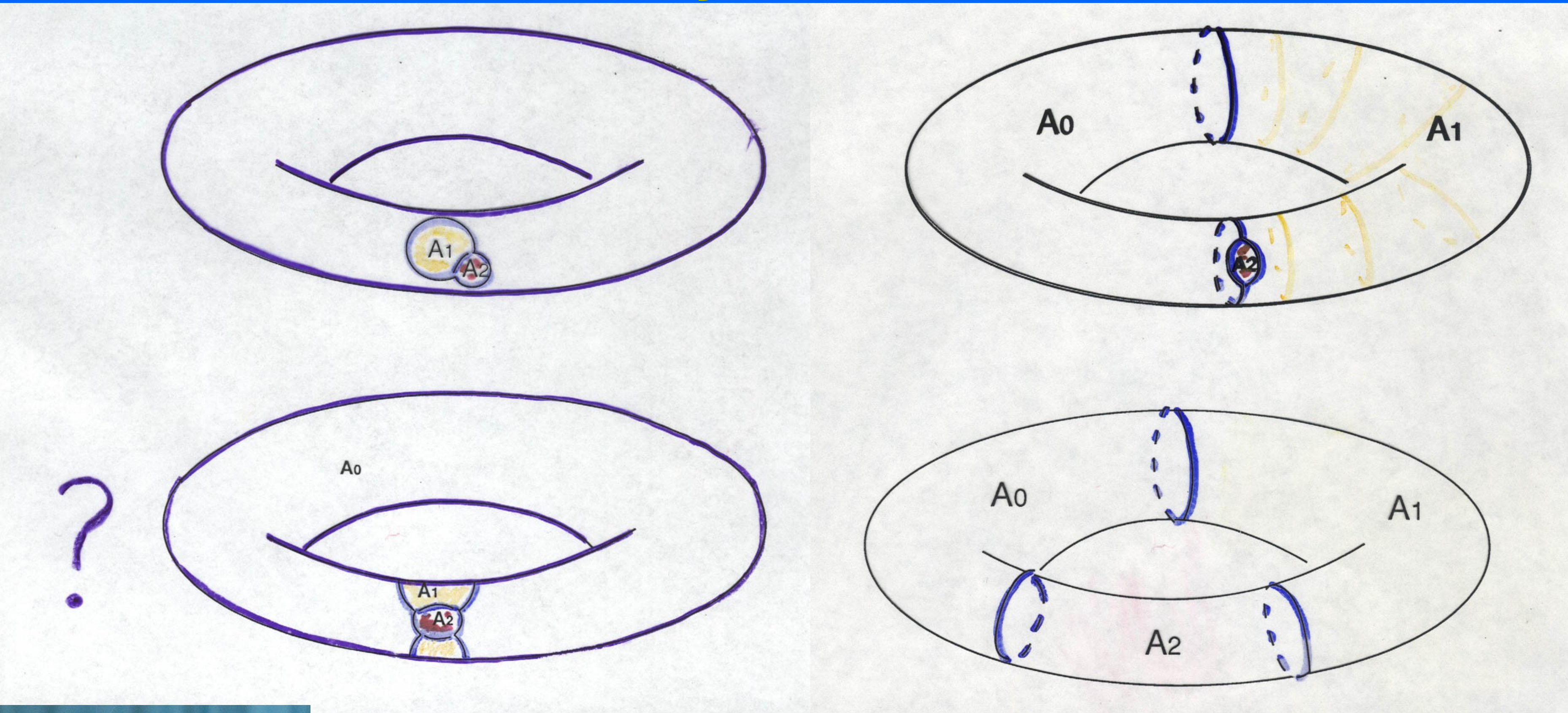
MASTERS' CONJ (1994)





CHAIN

MASTERS' CONJ (1994)



Ben Steinhurst

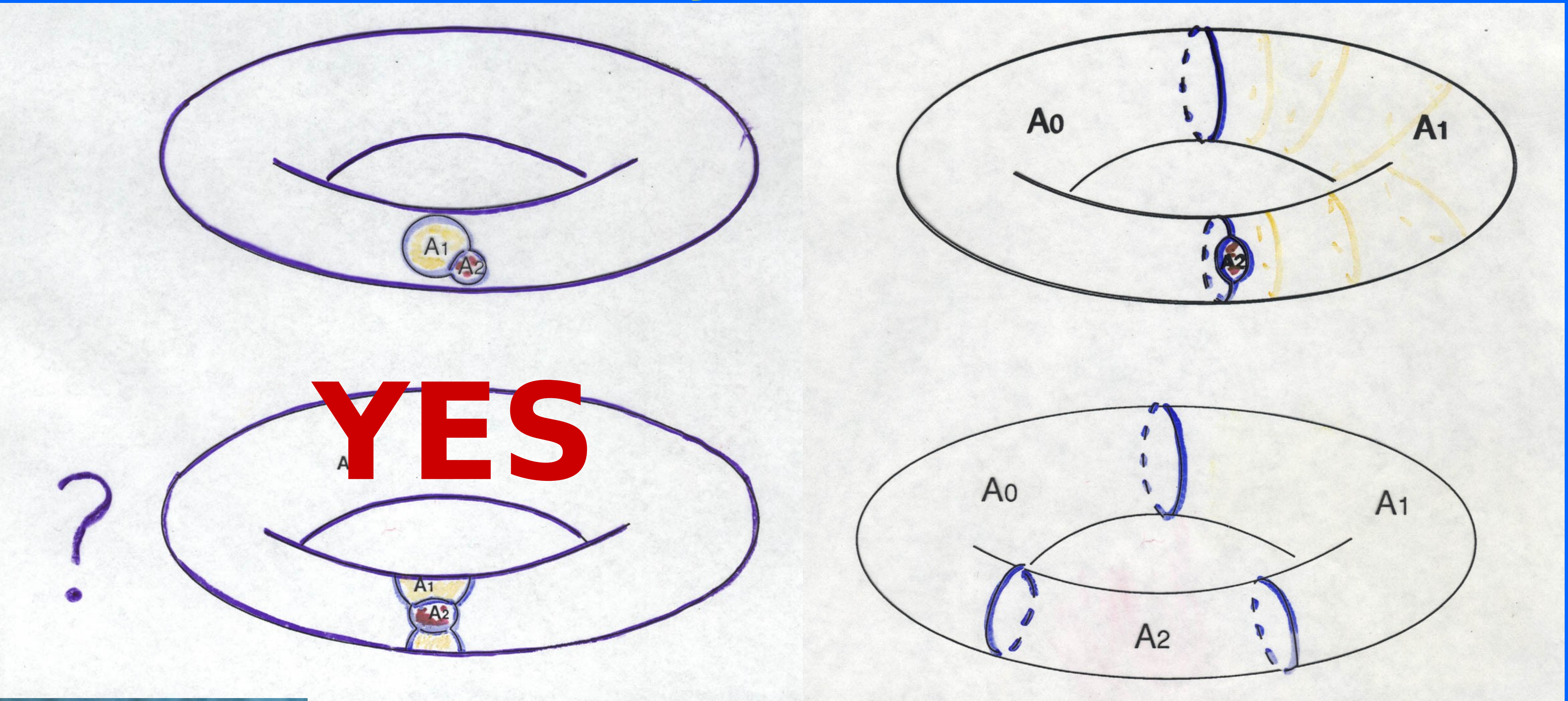
Joe Corneli

Nick Leger

Eric Schoenfeld

STUDIED BY GEOM GPS 2001-2002

MASTERS' CONJ (1994)

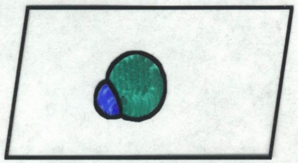


STUDIED BY GEOM GPS 2001-2002

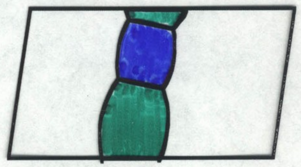
THEOREM:

Geom Gp ’01/’02

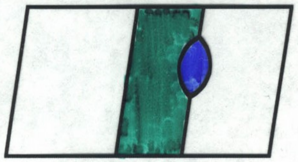
A perimeter-minimizing double bubble on the flat 2-torus is one of the following:



Standard Double Bubble

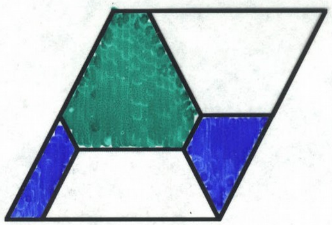


Standard Chain



Band Lens

Double Band



Standard Hexagon Tiling
(on a 60 degree rhombus)

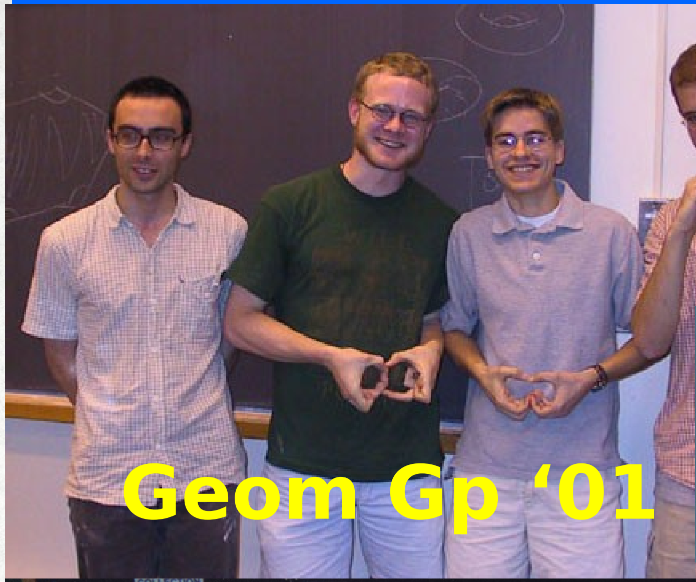


Figure 1: The six minimizer candidates on the flat two-torus. [#6. An octagon-square tiling.]

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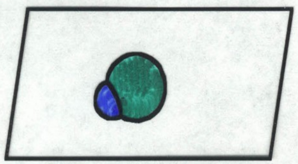
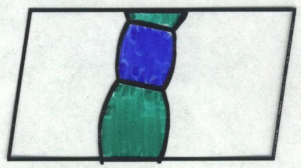
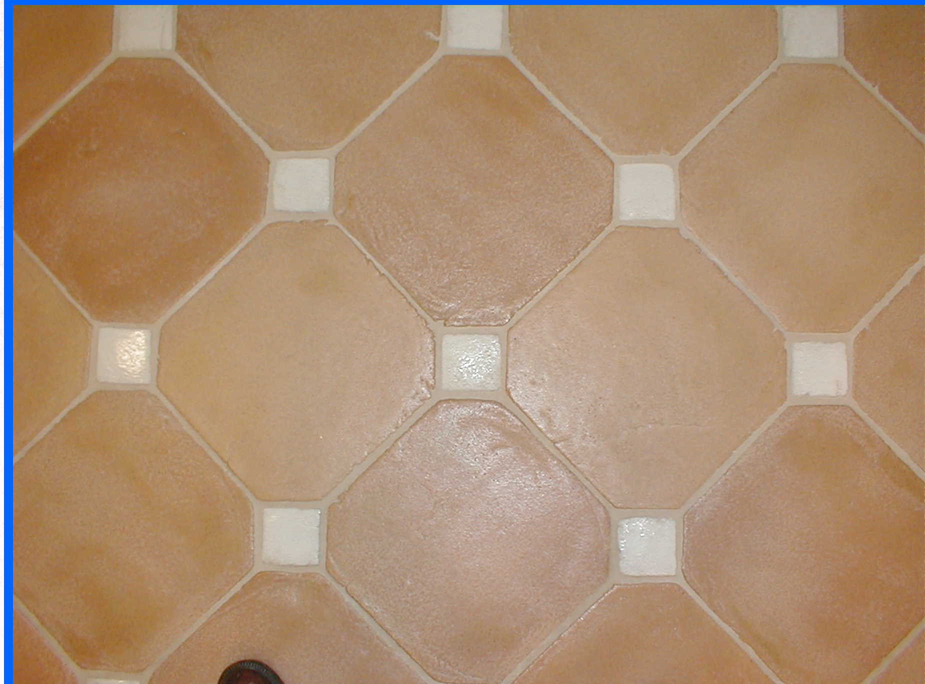


Standard Double Bubble



Standard Chain



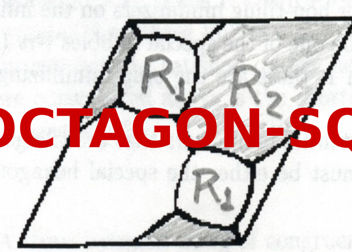


Figure 1: The six minimizer candidates on the flat two-torus. [#6. An octagon-square tiling.]

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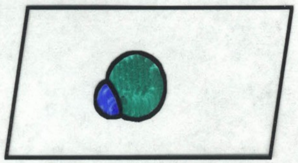
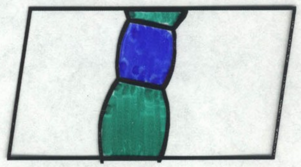
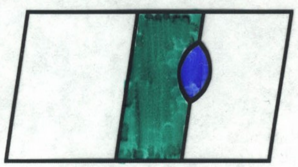


Standard Double Bubble



Standard Chain



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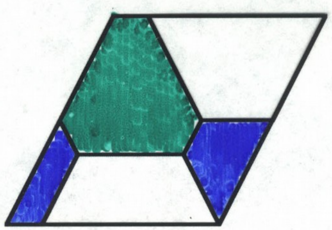
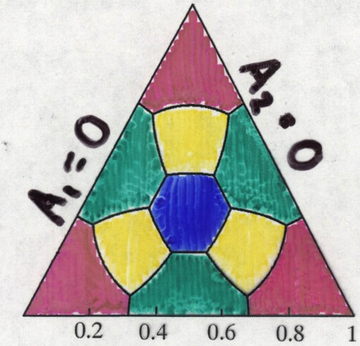
Double Band

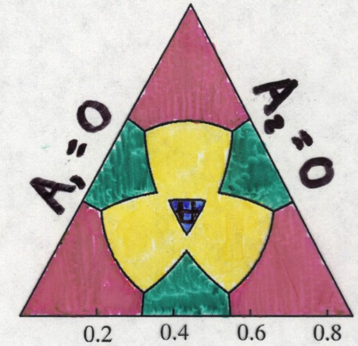


Standard Hexagon Tiling
(on a 60 degree rhombus)

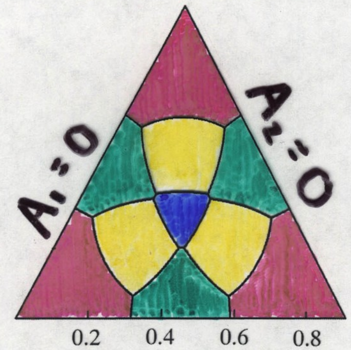
Winner Given (A_1, A_2)



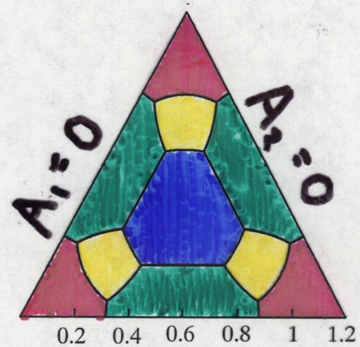
(a) Square torus ($\theta = \pi/2$)



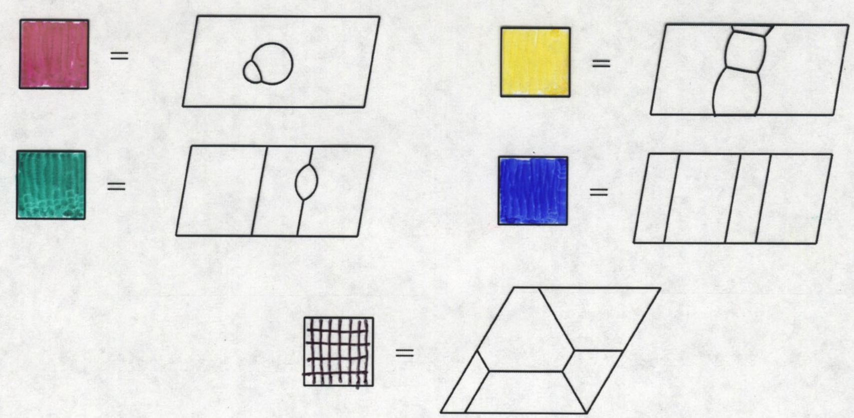
(b) Hexagonal torus ($\theta = \pi/3$)

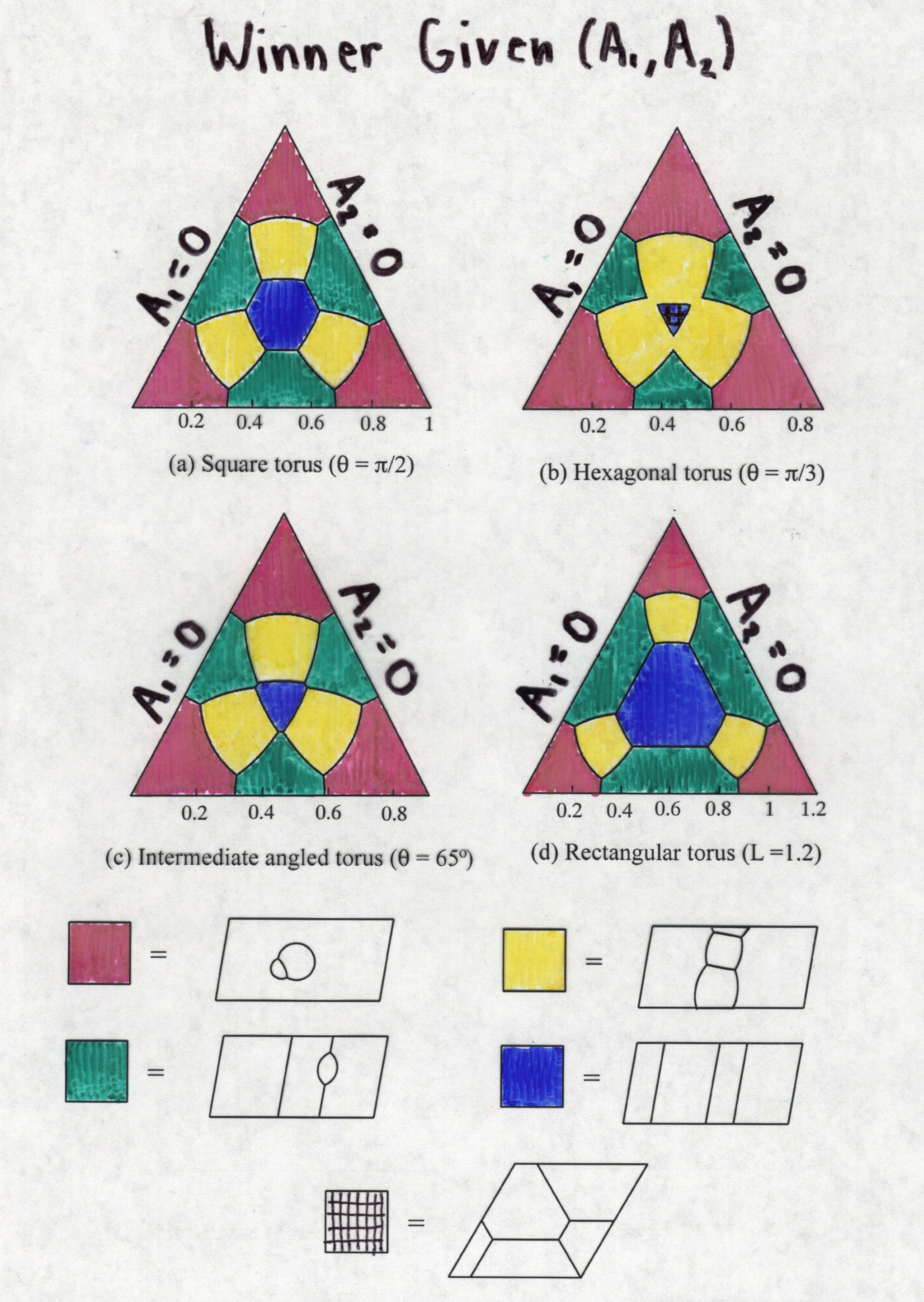


(c) Intermediate angled torus ($\theta = 65^o$)



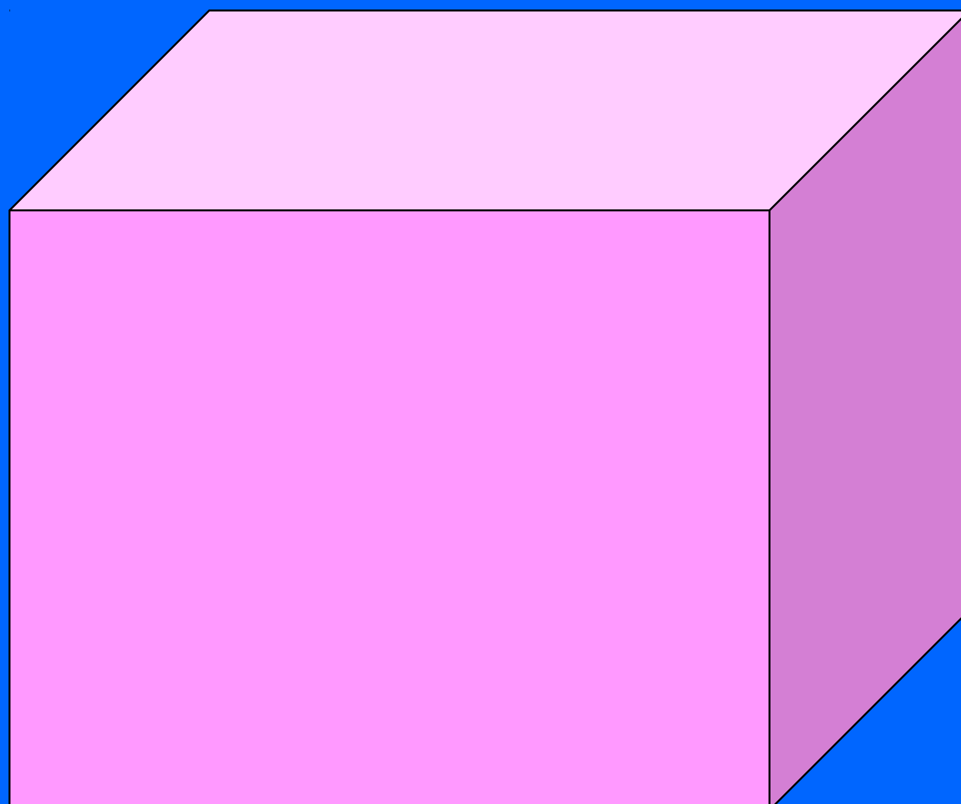
(d) Rectangular torus ($L = 1.2$)



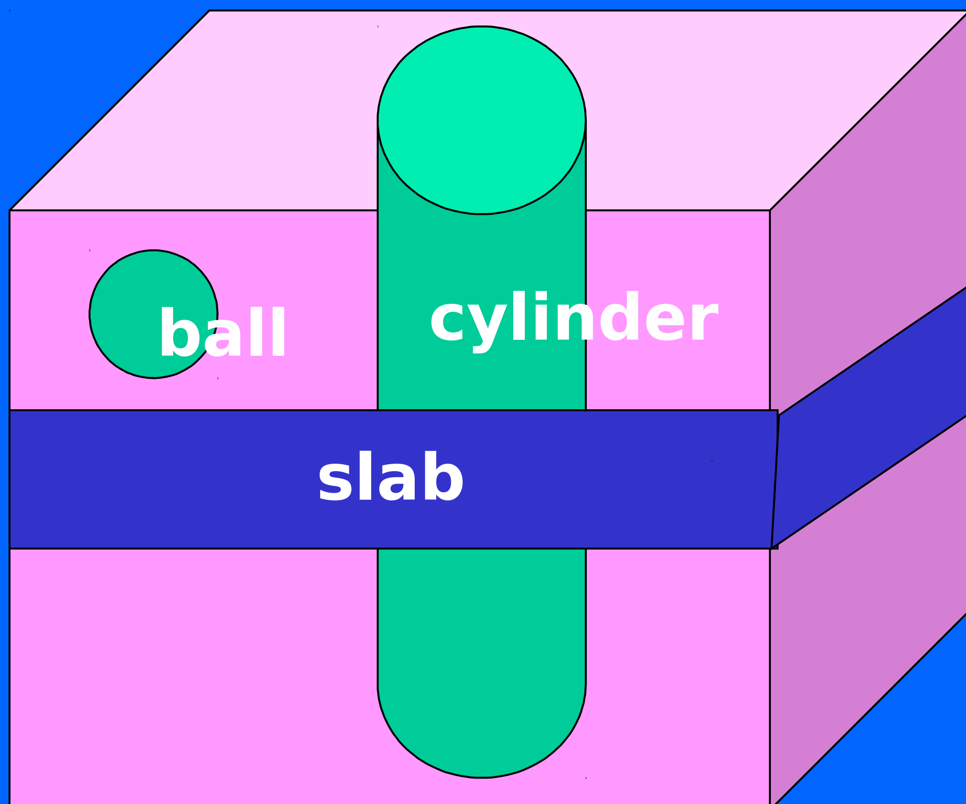


RE 37. This *Mathematica* plot of the intersection
eter plots for the standard double bubble and band
tes the process used to create the phase portraits ir
e plots suggest that perimeters do not fluctuate wild
n sets of unusual area pairs, increasing our confider
nase portraits are accurate.

BUBBLES ON 3D TORUS



BEST SINGLE BUBBLE ON 3D CUBIC TORUS

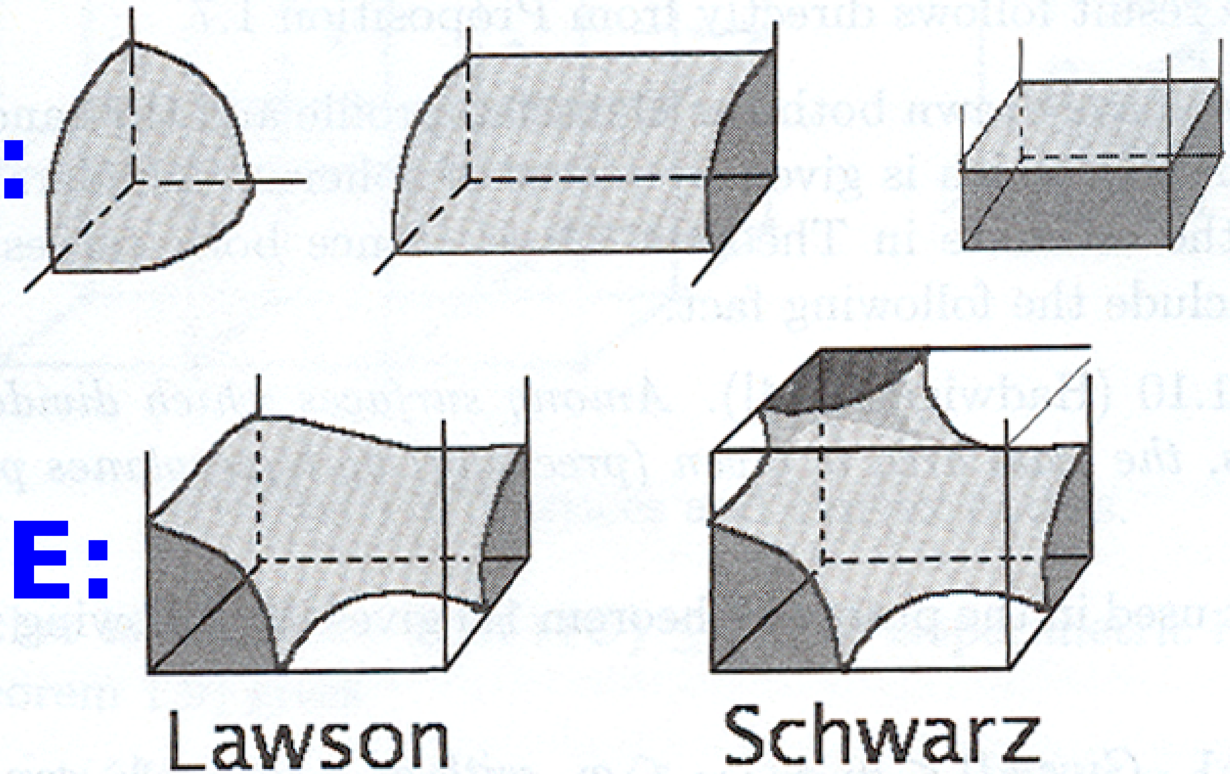


CONJECTURE (Ritoré and Ros)

1.5. Cubes and boxes. Consider the box $W = (0, a_1) \times \cdots \times (0, a_n) \subset \mathbb{R}^n$ with $0 < a_1 \leq \cdots \leq a_n$. The unit cube corresponds to the case $a_1 = \cdots = a_n = 1$. First we observe that the symmetrization argument of section §1.3 implies that the isoperimetric problem in W is equivalent (after reflection through the faces of the box) to the isoperimetric problem in the rectangular torus $T^n = \mathbb{R}^n/\Gamma$, where Γ is the lattice generated by the vectors $(2a_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 2a_n)$.

CONJ:

CLOSE:



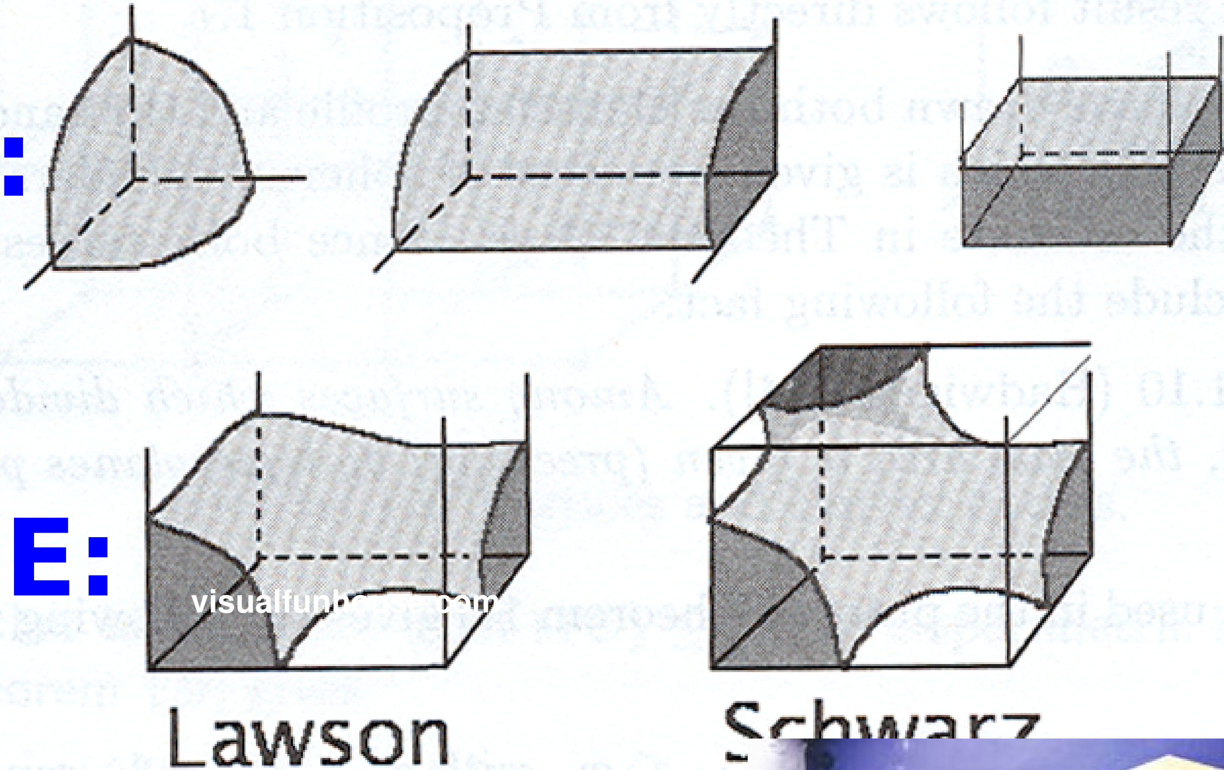
within .03%

FIGURE 9. Candidates for isoperimetric surfaces in the cube.

p. 183
Ros

CONJ:

CLOSE:

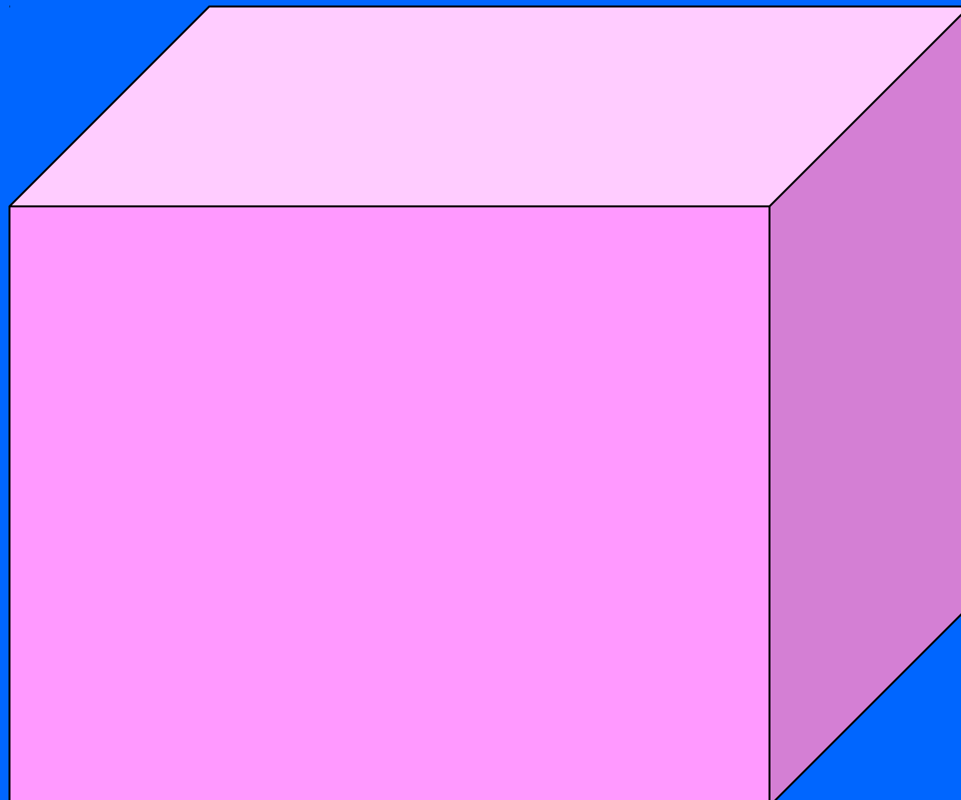


within .03%

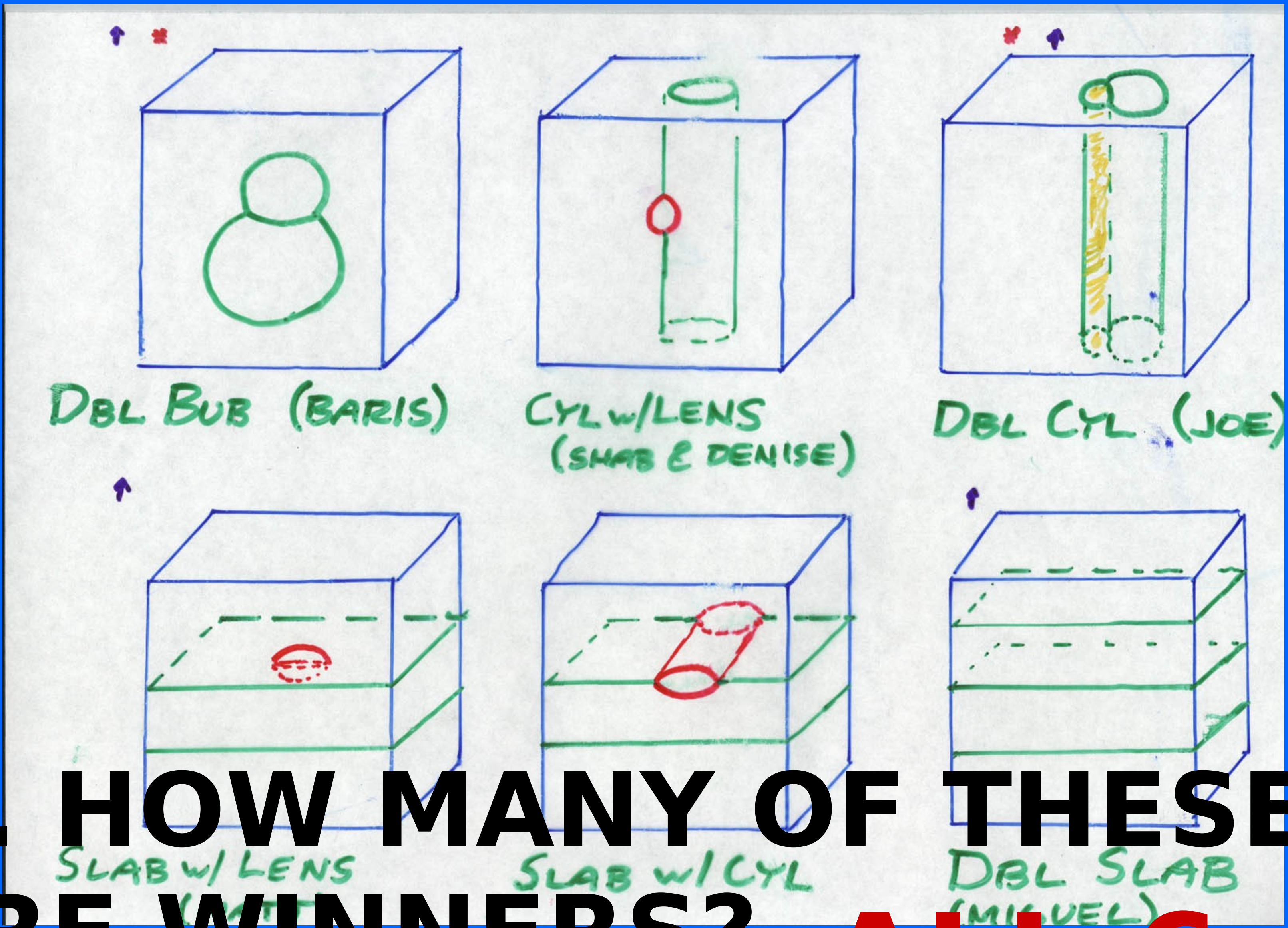
FIGURE 9. Candidates for isoperi

In skewed cube, beats conjecture by .02% (Romon).

BEST DOUBLE BUBBLE ON 3D TORUS



DBL BUBBLES ON 3D TORUS



Q. HOW MANY OF THESE ARE WINNERS?

ALL 6

ALL CANDs WIN EXCEPT SCARY GARY

DOUBLE BUBBLE CANDIDATES

	STANDARD DOUBLE BUBBLE	DELAUNEY SURFACE	DOUBLE CYLINDER	DOUBLE SLAB	CYLINDER W/ LENS	SLAB W/ CYLINDER	SLAB W/ LENS	SCARY GARY DBL BUBBLE
0.05 0.05	**1.241**	1.272	1.423	3.0	1.306	2.506	2.446	
0.10 0.10	1.971	**1.919**	2.013	3.0	2.015	2.706	2.741	
0.15 0.15	—	**2.461**	2.465	3.0	2.564	2.860	3.011	2.867
0.20 0.20	—	2.948	**2.846**	3.0	3.036	2.993	3.312	3.254
0.25 0.25	—	3.402	3.182	**3.0**	3.452	3.135	—	3.49
0.08 0.02	**1.197**	1.235	1.369	3.0	1.331	2.323	2.248	
0.16 0.04	1.901	**1.868**	1.936	3.0	1.871	2.448	2.384	
0.24 0.06	—	2.396	2.372	3.0	**2.326**	2.544	2.507	
0.32 0.08	—	2.871	2.738	3.0	2.717	2.630	**2.629**	
0.40 0.10	—	3.312	3.062	3.0	3.074	**2.709**	2.740	

“DOUBLE BUBBLES IN THE 3-TORUS”

BY…

JOE CORNELI

GENEVIEVE WALSH

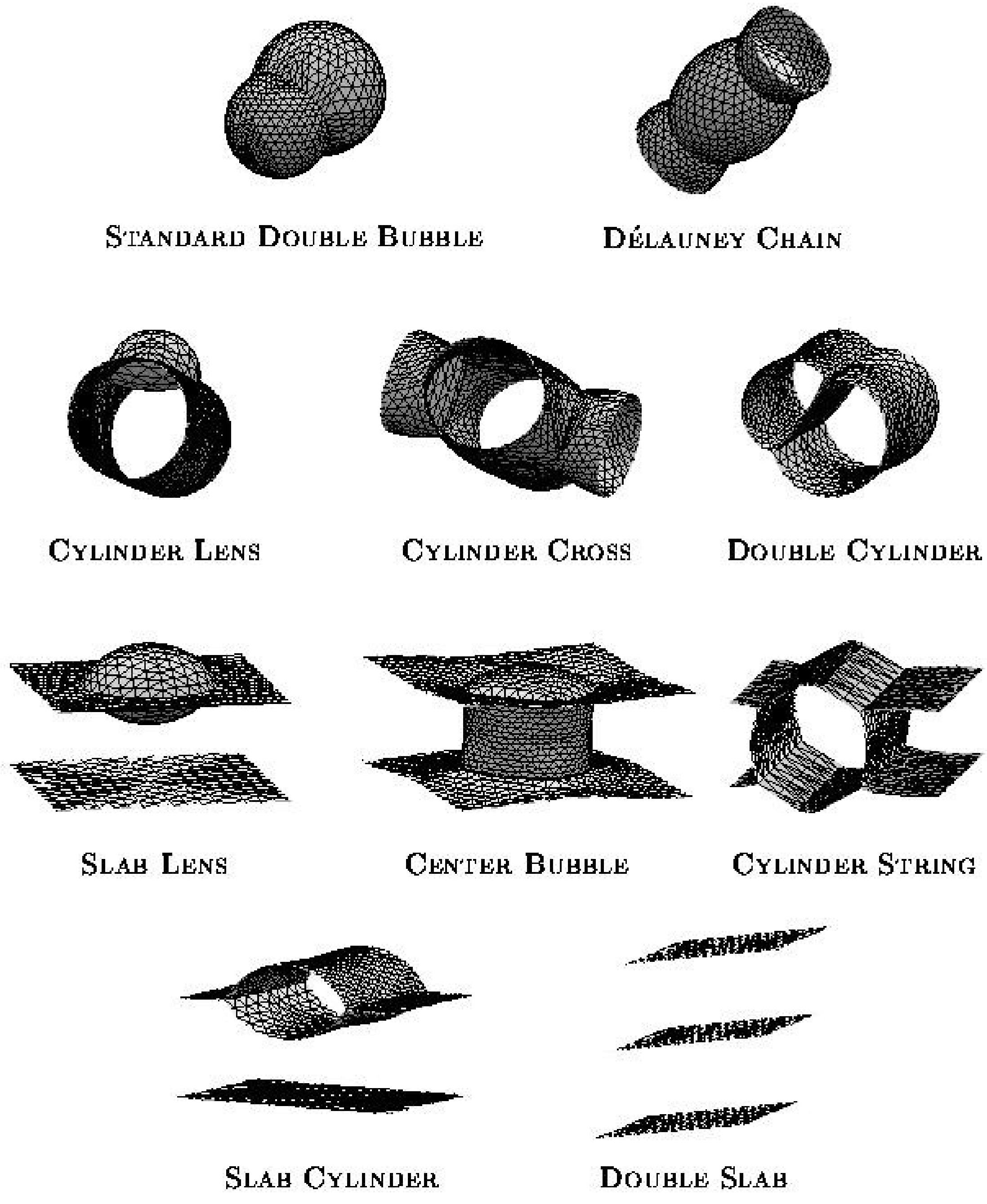
MIGUEL CARRIÓN

SHAB BEHESHTI

Exp Math 2003

DOUBLE BUBBLES IN THE 3-TORUS

TEN WINNING TYPES

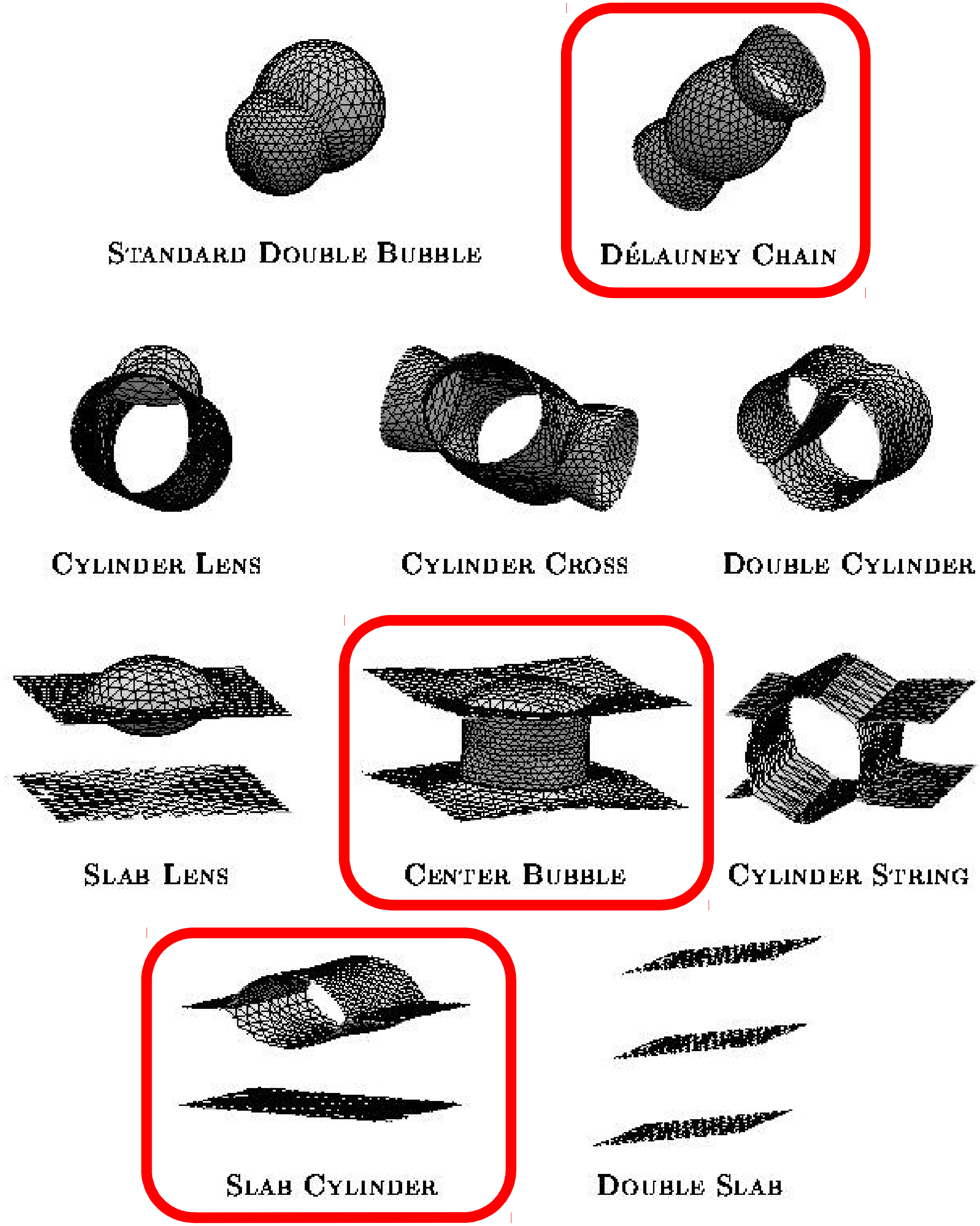


Millersville Student Research Team

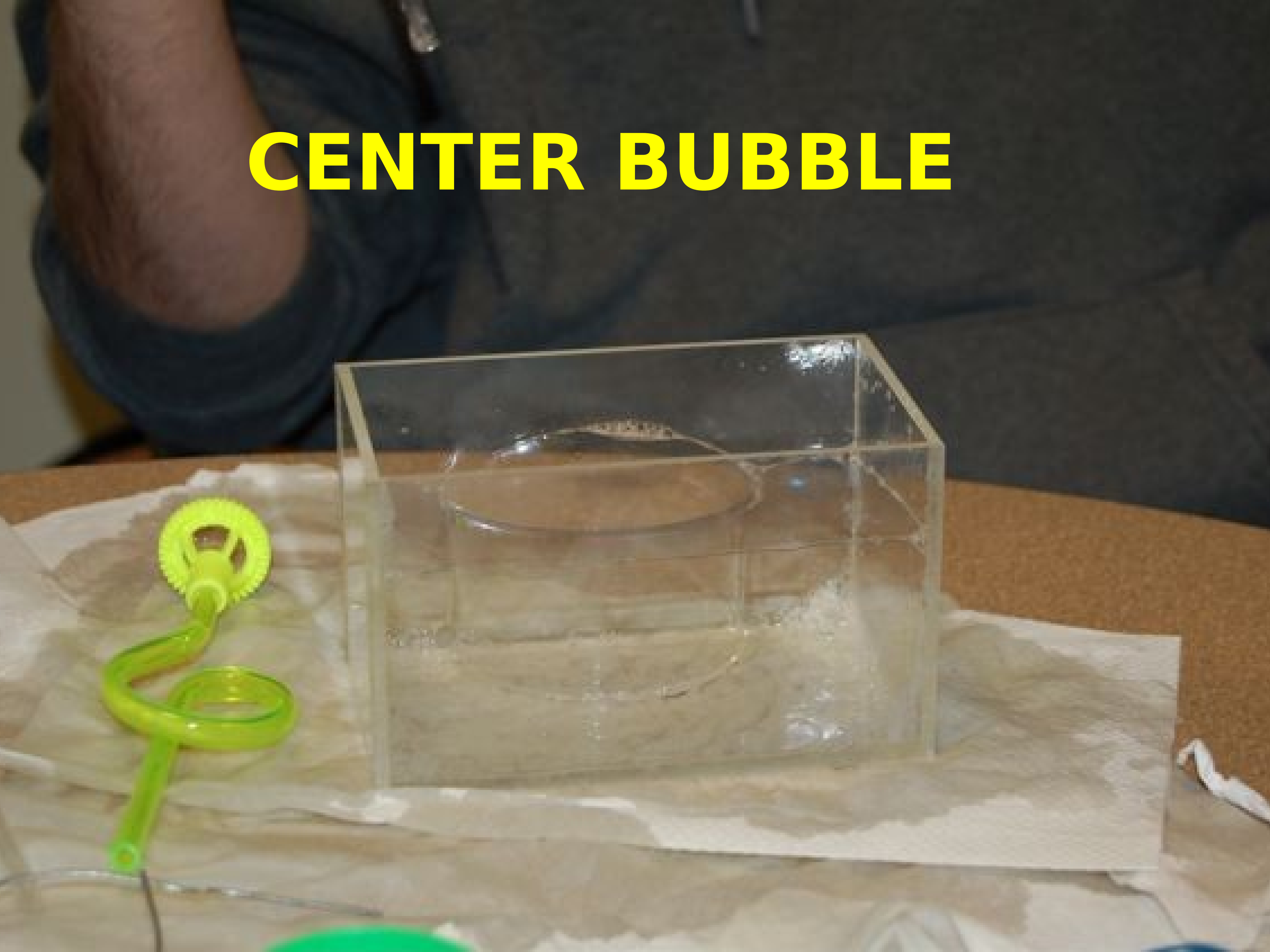
CHAIN

DOUBLE BUBBLES IN THE 3-TORUS

TEN WINNING TYPES




CYLINDER SLAB


CENTER BUBBLE

JOE CORNELI

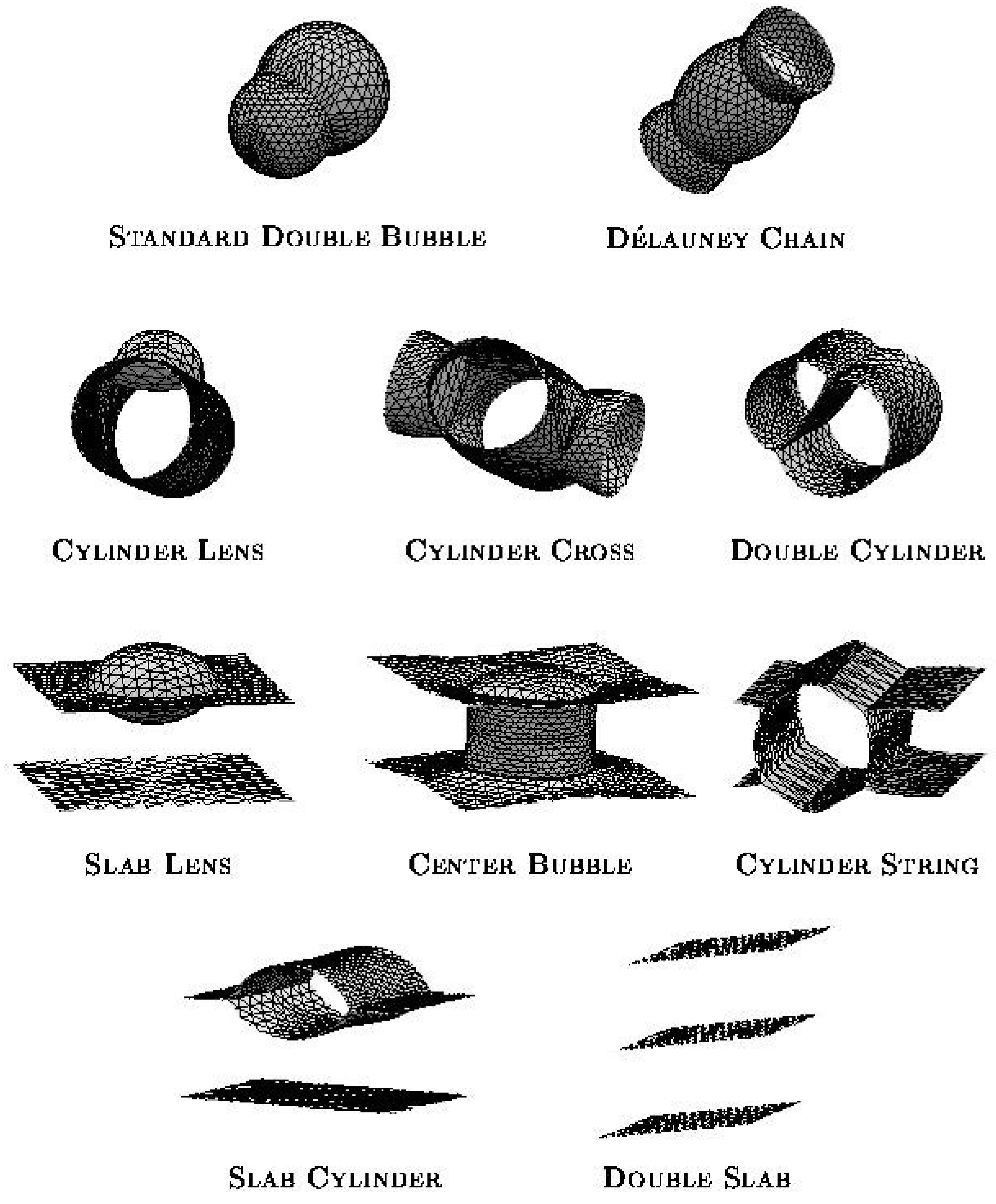
GENEVIEVE WALSH

“Those are awesome.”

MIGUEL CARRIÓN

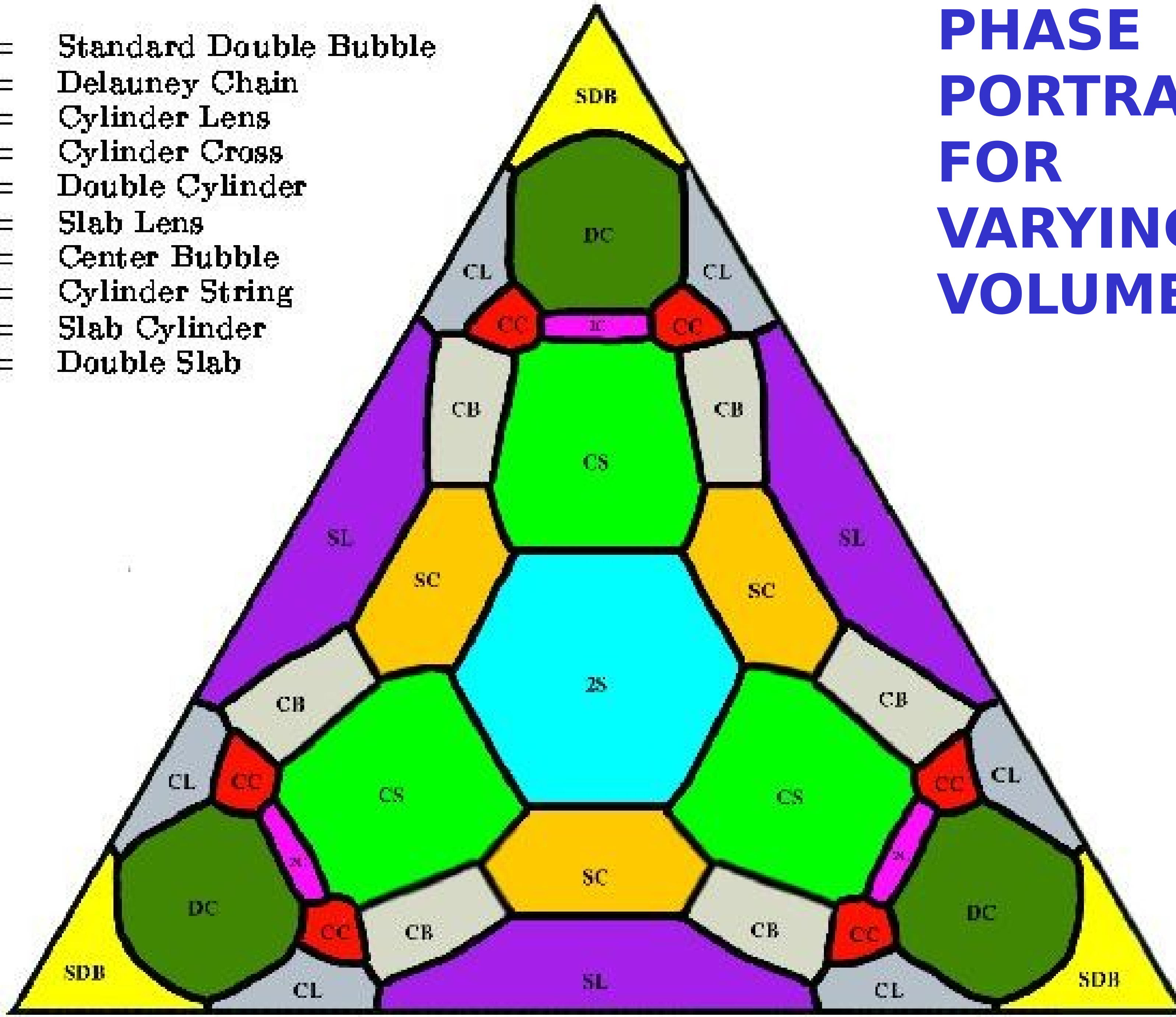
SHAB BEHESHTI

WHICH TYPE FOR WHICH VOLs?

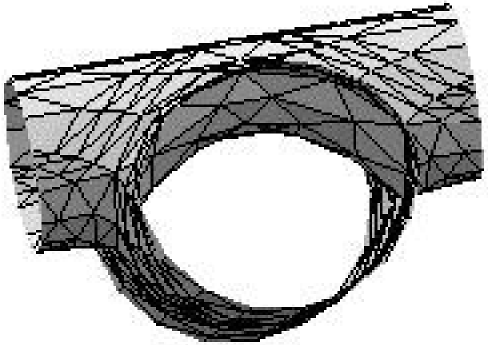


PHASE PORTRAI FOR VARYING VOLUMES

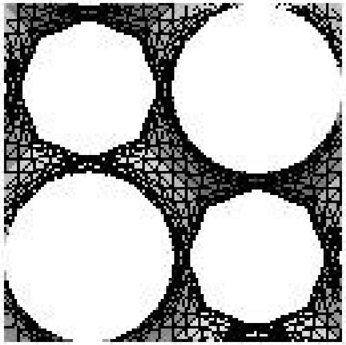
SDB = Standard Double Bubble
DC = Delauney Chain
CL = Cylinder Lens
CC = Cylinder Cross
2C = Double Cylinder
SL = Slab Lens
CB = Center Bubble
CS = Cylinder String
SC = Slab Cylinder
2S = Double Slab



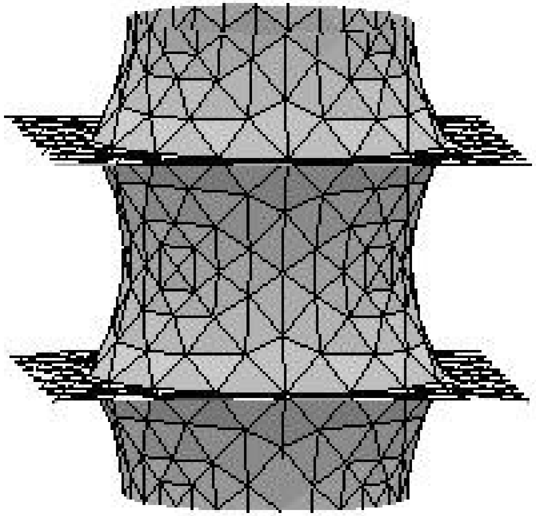
DOUBLE BUBBLES IN THE 3- TORUS



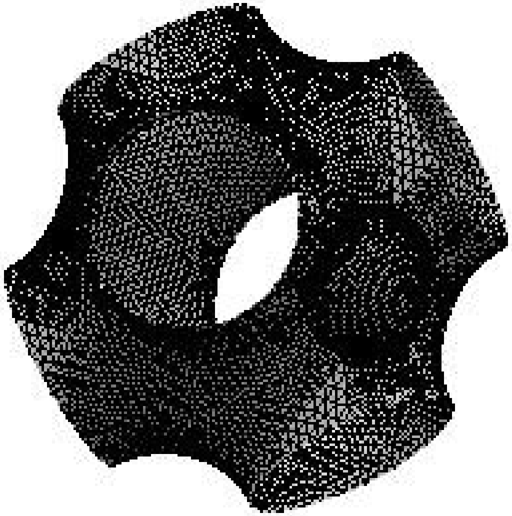
TRANSVERSE CYLINDERS



DOUBLE HYDRANT



CENTER CYLINDER

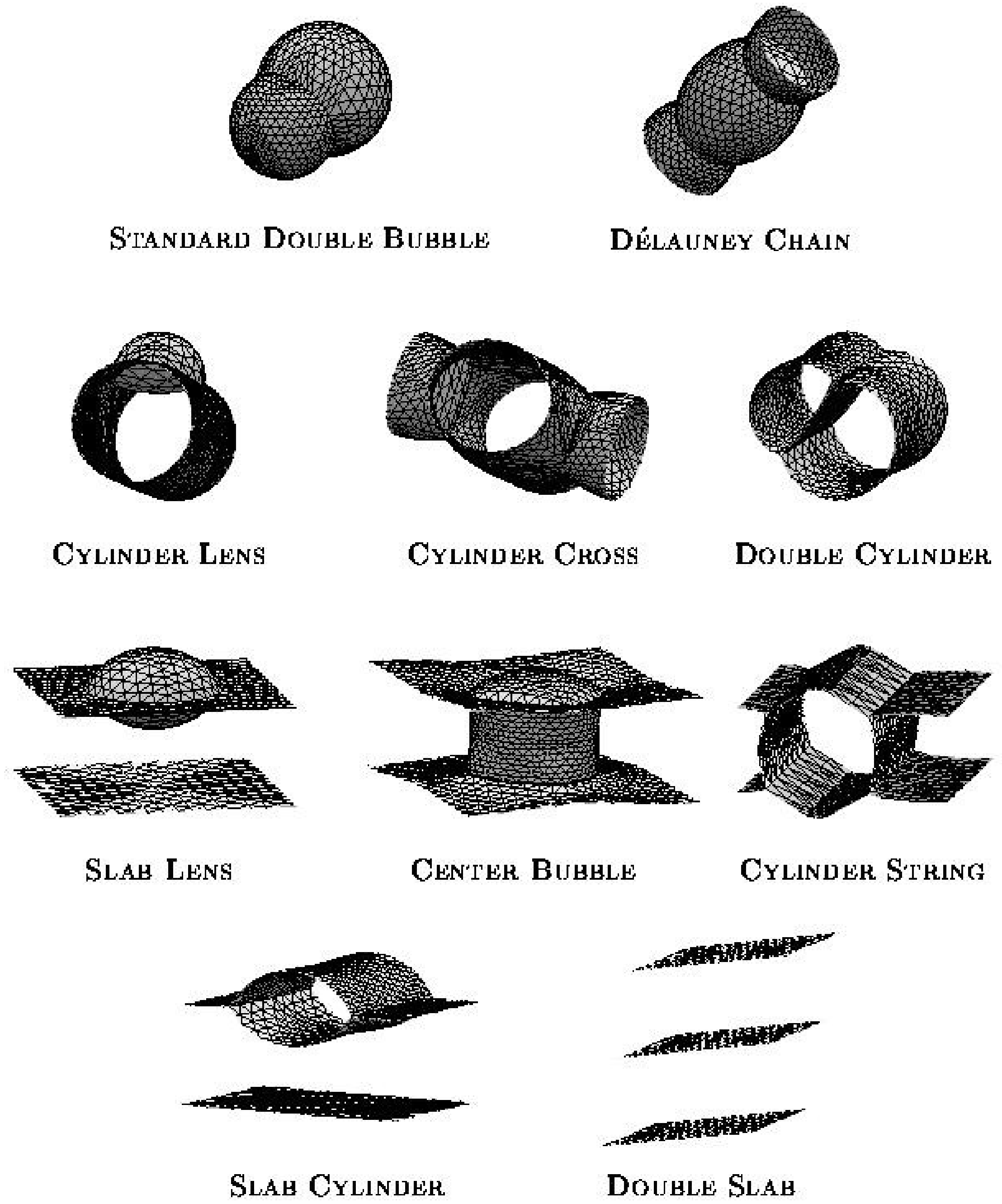


HYDRANT LENS

LOSING CANDIDATES

DOUBLE BUBBLES IN THE 3-TORUS

TEN WINNING TYPES (CONJ)



DOUBLE BUBBLES IN THE 3-TORUS

TEN WINNING TYPES (CONJ)

THANK YOU