Submanifolds, Holonomy, and Homogeneous Geometry

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2. Euclidean submanifold geometry and holonomy.
3. Skew-torsion holonomy systems.
   - Applications to naturally reductive spaces.
4. The Onishchik index of a symmetric space.
5. The nullity of homogeneous spaces.

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These two topics will be often related, in a subtle way, via the so-called normal holonomy.

In this expository talk we will intend to give a panoramic view on the main results, in this context, obtained in the past thirty years. We will comment on recent developments and open problems in the area.

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On the one hand, this family of homogeneous spaces is so wide that from the geometry of such spaces one can infer general results in Riemannian geometry, e.g. de Rham decompositon theorem or properties of spaces with sectional curvatures of the same sign.

On the other hand, this family is so particular, that the geometry of symmetric spaces is very rich and non-generic.

Some of the most beautiful and important results in Riemannian geometry, are theorems that under mild non-generic assumptions imply symmetry.This is the case of the Berger holonomy theorem and the rank rigidity theorem of Ballmann/Burns-Spatzier.

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This last theorem can be generalized to the so-called *Skew-torsion holonomy theorem* (O.-Reggiani; Crelle, 2012) with strong applications to naturally reductive spaces.

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Euclidean submanifold geometry and holonomy

A similar rôle as symmetric spaces play in Riemannian geometry is played by all the orbits of s-representations i.e., the isotropy representation of semisimple symmetric spaces.

For example the isotropy representation of the symmetric space $\mathrm{SL}(n)/\mathrm{SO}(n)$ can be regarded as $\mathrm{SO}(n)$ acting by conjugation on the space of traceless symmetric matrices.

The isotropy representation of the Grassmannian $\mathrm{SO}(n+k)/\mathrm{SO}(n)\times\mathrm{SO}(k)$ can be regarded as the action of $\mathrm{SO}(n)\times\mathrm{SO}(k)$ on $\mathbb{R}^{n\times k}$ given by

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A symmetric space is characterized by the property that the parallel transport τ_c along any curve c, starting at p is achieved by the differential of an isometry g, i.e.,

$$dg_{|p} = \tau_c \qquad g \text{ is unique.} \qquad \text{(E. Cartan)}$$

On the other hand, an orbit $M = K.v \subset \mathbb{R}^N$ of an s-representation is characterized by a similar property, with respect to the normal connection. Namely, the normal parallel transport $\tau_c^\perp$, along any c in M, starting at p, can be achieved by the differential of some extrinsic isometry of M (O.-Sánchez; Crelle, 1991). That is, there exists an isometry g of $\mathbb{R}^N$ such that $g(M) = M$ and

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Informal Remark. The normal connection gives weaker information, in submanifold geometry, than the Levi-Civita connection in Riemannian Geometry.

Normal Holonomy Theorem (O.; PAMS, 1990) *The normal holonomy group of a Euclidean submanifold, acts on the normal space, up to the set of fixed vectors, as the isotropy representation of a semisimple symmetric space.*

The holonomy of a symmetric space, without Euclidean factor, coincides with the isotropy (represented on the tangent space). So, the normal holonomy theorem can be phrased as follows: *the normal holonomy representation of a Euclidean submanifold coincides with the holonomy representation of a symmetric space.* This means that the normal holonomy representation coincides with a non-exceptional Riemannian holonomy.

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The above theorem is not a *Berger-type theorem*, since it gives information only about the representation of the normal holonomy on the normal space, but not about the space (in this case the submanifold).

As we informally pointed out, the normal holonomy gives <u>weaker</u> information than the Riemannian holonomy. So, interesting applications of the normal holonomy, can only be given within a restrictive class of submanifolds, as, for instance, the following:

- 1) *Homogeneous submanifolds.* ...
- 2) *Submanifolds with constant principal curvatures.*
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Theorem (Thorbergsson; Ann. of Math., 1991). *Let M be a submanifold of the sphere with constant principal curvatures. Assume that the normal holonomy group of M acts irreducibly and non-transitively. Then M is an orbit of an s-respresentation.*

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The proof uses all the main techniques developed in this area. *We belive that this theorem would have strong applications in algebraic geometry.*

A. Di Scala and *F. Vittone* (Adv. Math. 2017) generalized the above theorem: *if we remove, in the above theorem, the completeness assumption, then one obtains the Mok's characteristic varieties.*

Open problem: what happens if the ambient space is the dual symmetric space of $\mathbb{C}P^n$, i.e. the complex hyperbolic space? It is conjectured, if the submanifold is complete, that the normal holonomy is always transitive.

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Conjecture. *If the normal holonomy group, of an irreducible and full homogeneous submanifold M^n of the sphere with $n \geq 2$, does not act transitively on the normal sphere, then M is an orbit of an s-representation.*

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Decompose the normal bundle of M as

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where $\nu_0 M$ is the maximal parallel and flat subbundle of νM.

Observe that the normal holonomy group acts on $\nu_0^{\perp} M$ as an s-representation.

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Corollary. Let $M^n = K.v$, $n \geq 2$, be a full irreducible Euclidean homogeneous submanifold. Then any parallel normal field is K-invariant.

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We recall the definition, by J. Simons, of a *holonomy system*: this is a triple, $[\mathbb{V}, R, G]$ where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $\mathrm{SO}(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$. Such a triple is said

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Simons proved the following Berger-type result which, as he pointed out, implies Berger holonomy theorem.

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In this case the triple $[\mathbb{V}, \theta, G]$ is called a skew-torsion holonomy system.

Such a $\mathcal{G}$-valued 1-form arises, usually, as the difference tensor $\nabla - \hat{\nabla}$ between the Levi-Civita connection and a metric connection $\hat{\nabla}$ with the same geodesics., i.e. connections with skew-torsion.

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Let $[\mathbb{V}, \Theta, G]$, $\Theta \neq 0$, *be an irreducible skew-torsion holonomy system with* $G \neq \mathrm{SO}(\mathbb{V})$. *Then* $[\mathbb{V}, \Theta, G]$ *is symmetric and non-transitive. Moreover,*

1. $(\mathbb{V}, [\ ,\ ])$ *is an orthogonal simple Lie algebra, of rank at least* 2, *with respect to the bracket* $[x, y] = \Theta_x y$;
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Theorem (O.-Reggiani; Crelle, 2012; Monath. Math., 2013). *Let M be a simply connected, irreducible naturally reductive Riemannian manifold. Assume that M is neither isometric to a sphere, nor to nor to a compact simple Lie group with a bi-invariant metric or its symmetric dual. Then the canonical connection is unique.* (In particular, any isometry is affine with respect to the canonical connection, which allows to determine the full isometry group).

The above theorem explains, in a geometric way, the following result (a question posed by *J. Wolf*, for the strongly isotropy irreducible spaces, and by *M. Wang* and *W. Ziller*).

Corollary (Wolf; Wang-Ziller). *Let $M^n = G/H$ be a compact, simply connected, irreducible homogeneous Riemannian manifold such that M is neither isometric to the sphere S^n, nor to a (simple) compact Lie group with a bi-invariant metric. Assume that M is isotropy irreducible with respect to the pair (G, H) (effective action). Then $G^o = \mathrm{Iso}(M)^o$.*

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Though a Lie triple system is an elementary algebraic object, explicit calculations with them can be tremendously complicated.

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Our starting point, for dealing with the index of a symmetric space, is the inequality in the following main result:

Theorem. *Let M be an irreducible Riemannian symmetric space. Then*

$$rk(M) \leq i(M).$$

Moreover, the equality holds if and only if, up to duality, $M = SL_{k+1}/SO_k$ or $M = SO^o_{k,n}/SO_kSO_n$

We prove the inequality $\mathrm{rank}(M) \leq i(M)$ by showing the following: *if Σ is a totally geodesic submanifold of a symmetric space M, then there exists a maximal flat F of M that intersects Σ transversally.*

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Slice Lemma. *Let Σ be a non-flat totally geodesic submanifold of an irreducible symmetric space $M = G/K$ of rank at least 2 (which contains $p = [e]$). Then the slice representation $\rho : (K^{\Sigma})^o \to SO(\nu_p \Sigma)$ is not trivial (or, equivalently, the restricted normal holonomy of Σ is not trivial).*

The above result is used to prove the following key result

Proposition. *Let Σ be a semisimple totally geodesic submanifold of a symmetric space $M = G/K$, with $p = [e] \in \Sigma$. Then Σ is reflective if and only if the full slice representation $\tilde{\rho} : \tilde{G}_p^{\Sigma} \to O(\nu_p \Sigma)$ is not injective.*

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Our main results with Jürgen Berndt were:

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List of symmetric spaces with unknown index (up to duality)

– $\mathrm{SO}^*_{2k+2}/\mathrm{U}_{k+1}$, $k \geq 5$ ($i_r = 2k$).

– $\mathrm{SU}^*_{2k+2}/\mathrm{Sp}_{k+1}$, $k \geq 3$ ($i_r = 4k$).

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The nullity of homogeneous spaces

In the last part of the talk we would like to comment on recents results with Antonio J. Di Scala and Francisco Vittone.

Given a Riemannian manifold M with Riemannian curvature tensor R and a point $p \in M$, the *nullity subspace* ν_p of M at p is defined as the subset of T_pM consisting of those vectors that annihilate R, i.e.,

$$\nu_p = \{v \in T_pM \, : \, R_{\cdot,\cdot} v \equiv 0\}.$$

The dimension of the nullity subspace is called the *index of nullity* at p In an open and dense subset of M, where this index is locally constant, the nullity define an integrable distribution with totally geodesic and flat leaves.

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The concept of nullity of the curvature tensor was first introduced and studied by Chern and Kuiper Ann of Math. 1952. Since then, many problems regarding manifolds with positive index of nullity (of the curvature tensor or curvarure-like tensors) have been approached by different authors, as *Dajzer, Florit, Ziller, ...,* obtaining strong geometric implications.

Nothing was known about the nullity of homogeneous Riemannian manifolds $M = G/H$.

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Let us explain our main result, obtained by geometric methods, about homogeneous Riemannian manifols $M = G/H$ with non-trivial nullity index.

This result implies that the *distribution of symmetry* $\mathfrak{s}$ of M is non-trivial. Moreover, its flat part $\mathfrak{s}_0$ must be non-trivial.

$$\mathfrak{s}_p = \{X_p : X \in \mathcal{K}(M) \text{is a transvection at } p\}$$

The distribution of symmetry was defined and studied by O.-Reggiani-Tamaru (Math. Z. 2014), Berndt-O.-Reggiani (JEMS 2017).

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Theorem (Di Scala-O.-Vittone). *Let $M = G/H$ be a homogeneous Riemannian manifold which does not split off a local flat factor and with a non-trivial nullity distribution. Then there exists a transvection $Y \neq 0$ at $p \in M$ such that:*

(**a**) *Y_p does not belong to the nullity space, but its Jacobi operator $R_{\cdot, Y_p} Y_p$ is zero.*

(**b**) $[Y, [Y, \mathcal{K}(M)]] = 0$. *Moreover,* $[Y, \mathcal{K}^G(M)] \neq 0$.

By making use of the Jacobson-Morozov theorem, we have:

Corollary (Di Scala-O.-Vittone). *Let $M = G/H$ be a homogeneous Riemannian manifold which does not split off a local flat factor and such that G is semisimple. Then the nullity distribution of M is trivial.*

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By making use of the main theorem we were able to construct the first known examples of homogeneous spaces with non-trivial nullity (in dimension 4, the minimal possible dimension). Namely, we construct a 1-parameter family of non-homothetic examples.

Questions:

Are there examples which are not topologically trivial?

Are there examples $M = G/H$, with G non-solvable?

Are there Khäler examples?

Are there (non-trivial) examples in any dimension $d \geq 5$?

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