Torus actions on positively curved manifolds

Burkhard Wilking

on joint work with Lee Kennard and Michael Wiemeler

Sao Paulo, July 25 2018

Positively curved manifolds

Examples with $K > 0$

Few known positively curved simply connected closed manifolds.

- The compact rank one symmetric spaces (CROSS'es), $\mathbb{S}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathrm{Ca}P^2$.
- All other known examples are in dimension 6, 7, 12, 13 and 24.

Examples with $K \geq 0$

There there is a relatively rich class of manifolds with $K \geq 0$.

- All compact homogeneous spaces G /K and biquotients G //K.
- The class is invariant under taking product and allows special gluing constructions (Cheeger, Grove-Ziller).

Few obstructions

There is not a single obstruction known that distinguishes the class of simply connected closed manifolds admitting positively curved metrics from the one admitting nonnegatively curved metrics.

Conjecture(Hopf 1930's)

$\mathbb{S}^2 \times \mathbb{S}^2$ does not admit $K > 0$.

Conjecture(Hopf 1930's)

An even dimensional positively curved manifold has positive Euler characteristic.

Symmetry program

Grove proposed in 1992 to study the problem under symmetry assumptions motivated in part by

Theorem (Hsiang, Kleiner '89)

If (M^4, g) is orientable with $K > 0$ and $\mathsf{S}^1 \subseteq Iso(M, g)$, then it is homeomorphic to $\mathbb{S}^4$ or $\mathbb{CP}^2$.

Theorem (Grove, Wilking '14)

If (M^4, g) is orientable with $K > 0$ and $\mathsf{S}^1 \subseteq Iso(M, g)$, then it is equivariantly diffeomorphic to $\mathbb{S}^4$ or $\mathbb{CP}^2$.

Fixed point components of 5-tori are rational CROSS'es

Theorem A

If (M^n, g) is orientable with $K > 0$ and $\mathrm{T}^5 \subseteq Iso(M, g)$, then every fixed point component F^f of T^5 has the rational homotopy type of $\mathbb{S}^f$, $\mathbb{CP}^{f/2}$ or $\mathbb{HP}^{f/4}$.

Corollary(Hopf conjecture for symrank 5)

A positively curved even dimensional manifold has positive Euler characteristic provided that the rank of its isometry group is ≥ 5.

- In previous proofs of the Hopf conjecture under symmetry assumptions (e.g. Amann, Kennard '14) the lower bound on the rank depended on the dimension of the ambient manifold.
- Recall that the fixed point set of T^5 is nonempty by Berger and has the same Eulercharacteristic as the ambient manifold.

Topology of the ambient manifold

Theorem B

If (M^{2n}, g) is orientable with $K > 0$, $b_{odd}(M^{2n}) = 0$ and $\text{rank}(Iso(M, g)) \geq 9$, then M^{2n} has the rational homotopy type of $\mathbb{S}^{2n}$, $\mathbb{CP}^n$ or $\mathbb{HP}^{n/2}$.

- "$b_{odd} = 0$" can be replaced by "M^{2n} is rationally elliptic".

Theorem C

Suppose T^{12} acts effectively and isometrically on (M^n, g) with $\pi_1(M^n) = 0$ and $K > 0$. Suppose there is a fixed point p such that no nontrivial finite istropy group occurs in a neighborhood of p. Then M^n has the rational homotopy type of $\mathbb{S}^n$, $\mathbb{CP}^{n/2}$ or $\mathbb{HP}^{n/4}$.

The following result follows from combining the connectedness Lemma(Wilking '03) and the 4-periodicity theorem (Kennard '13).

Theorem

Suppose an orientable manifold (M^n, g) with $K > 0$ contains two totally geodesic closed submanifolds $N_1^{n-k_1}$ and $N_2^{n-k_2}$ intersecting transversely with $k_1 + k_2 \leq n/2$. Then the rational cohomology ring of M^n is 4-periodic.

Representations of tori without finite isotropy groups

A representation $\rho\colon\ \mathrm{T}^d \to \mathrm{SO}(2)$ is determined uniquely by the Eulerclass $w_\rho \in H^2(B\,\mathrm{T}^d)$ of the $\mathbb{R}^2$ bundle $E\,\mathrm{T}^d \times_\rho \mathbb{R}^2$ over $B\,\mathrm{T}^d$.

Lemma

Let $\rho\colon\ \mathrm{T}^d \to \mathrm{SO}(k)$ be injective, $\rho_1, \ldots, \rho_h\colon\ \mathrm{T}^d \to \mathrm{SO}(2)$ a maximal collection of pairwise inequivalent subrepresentations, and let $w_1, \ldots, w_h \in H^2(B\,\mathrm{T}^d, \mathbb{Z})$ denote the induced elements. Then the following are equivalent

- There are no nontrivial finite isotropy groups for the induced action on $\mathbb{R}^k$.
- If $w_{i_1}, \ldots, w_{i_d}$ are linear independent, then they form a $\mathbb{Z}$-basis of $H^2(B\,\mathrm{T}^d, \mathbb{Z}) \cong \mathbb{Z}^d$.

Representations of tori without finite isotropy groups

To each such representation one can associate a finite subset $E \subseteq \mathbb{Z}^d$, with

- $E \cap -E = \varnothing$ and $\operatorname{span}_{\mathbb{Z}} E = \mathbb{Z}^d$
- Every d linear independent vectors in E form a $\mathbb{Z}$-basis of $\mathbb{Z}^d$.

Lemma

We can find a subgroup $\Gamma \subseteq \mathbb{Z}^d$ of corank one such that $E \setminus \Gamma$ is linear independent.

Corollary

$\#E \leq \frac{1}{2}d(d+1)$ with equality iff for some $\mathbb{Z}$-basis $b_1, \ldots, b_d$

$$E \cup -E = \{\pm b_i\} \cup \{b_i - b_j \mid i \neq j\}.$$

Representations of tori

Lemma

Let $T^{2d+1} \to SO(k)$ be injective. Then there is a d-dimensional subgroup $H \subseteq T^d$ such that the induced representation $T^{d+1} = T^{2d+1} \to SO(Fix(H))$ is faithful and of product type, that is there is a product decomposition $S\,0(2)^{d+1} = T^{d+1}$ such that the nontrivial irreducible subrepresentations are given by the projections to the (d+1) factors.