Positive scalar curvature on manifolds with abelian fundamental groups

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- For finite fundamental groups no counterxample is known.

Special case: Odd order fundamental groups

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- $\pi_1(M) = (\mathbb{Z}/p)^r$ and M is p-atoral, (Botvinnik-Rosenberg 2001, H. 2016).

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Definition

M^d is p-atoral, if for all $k \geq 1$ and $c_1, \ldots, c_d \in H^1(M; \mathbb{Z}/p^k)$

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Note that M is p-atoral, if $\pi_1(M)$ is abelian and $\dim M > \mathrm{rk}(\pi_1(M))$.

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Theorem (H.)

Let $M^{d \geq 5}$ be a closed connected oriented non-spin manifold. Let $\pi_1(M)$ be abelian of odd order and let M be atoral (for all odd p). Then M carries a metric of positive scalar curvature.

Construction machine I: Positive bordism

For a topological space X let

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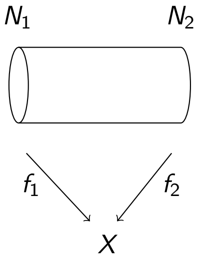

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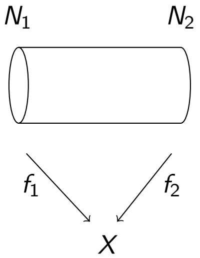

Let

$$\Omega_d^+(X) \subset \Omega_d(X)$$

only contain $[f : N \to X]$, N admitting a positive scalar curvature metric.

Bordism principle (Gromov-Lawson, Rosenberg-Stolz)

Let $M^{d \geq 5}$ be a closed oriented manifold and let $\phi : M \to B\pi_1(M)$ be the classifying map of the universal cover of M. Then the following assertions are equivalent.

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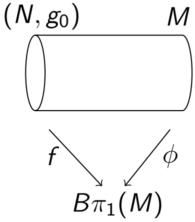

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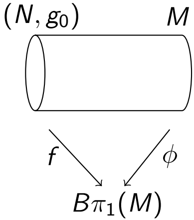

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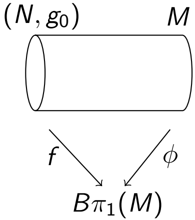

Here we assume:

- M is spin and we work with spin bordism or
- $\widetilde{M}$ is not spin and we work with oriented bordism.

Computation: Elementary abelian groups

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The reduced bordism $\widetilde{\Omega}_*(B(\mathbb{Z}/p)^r)$ is generated by "generalized products of lens spaces"

$$[L^{2m_1+1} \times \cdots \times L^{2m_k+1} \to B(\mathbb{Z}/p)^k \xrightarrow{B\phi} B(\mathbb{Z}/p)^r] \,.$$

Here $1 \leq k \leq r$ and

$$\phi : (\mathbb{Z}/p)^k \to (\mathbb{Z}/p)^r$$

is some group homomorphism. In particular all atoral classes in $\widetilde{\Omega}_*(B(\mathbb{Z}/p)^r)$ are positive.

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Unfortunaley $\Omega_*(B\pi_1(M))$ is very difficult to compute in general.

Manifolds with Baas-Sullivan singularities

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  $$\partial A(\omega) = \partial_0 A(\omega) \cup \dots \cup \partial_n A(\omega)\,,$$

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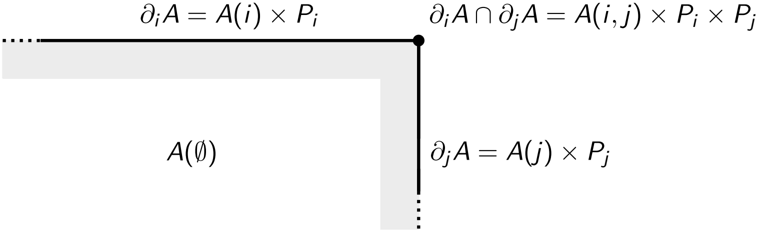
$\partial_0 A$ is the boundary of A. If A is compact and $\partial_0 A = \emptyset$, then A is closed.

Distinguished metrics on $\mathcal{P}$-manifolds



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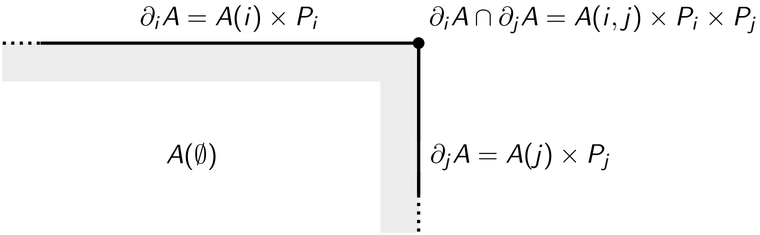

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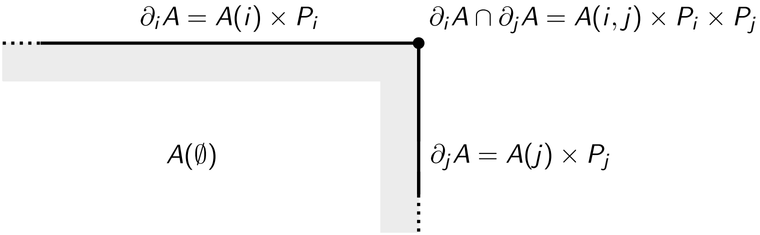

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Definition

A distinguished metric on a $\mathcal{P}$-manifold A is a family of metrics $g(\omega)$ on $A(\omega)$, $\omega \subset \{0, \ldots, n\}$, such that the following holds:

- For $i \notin \omega$ we have $g(\omega)|_{\partial_i A(\omega)} = g(\omega, i) \times h_i$.
- For $i \in \omega$, $i \neq 0$, the metric $g(\omega) \oplus h_i$ is of positive scalar curvature.

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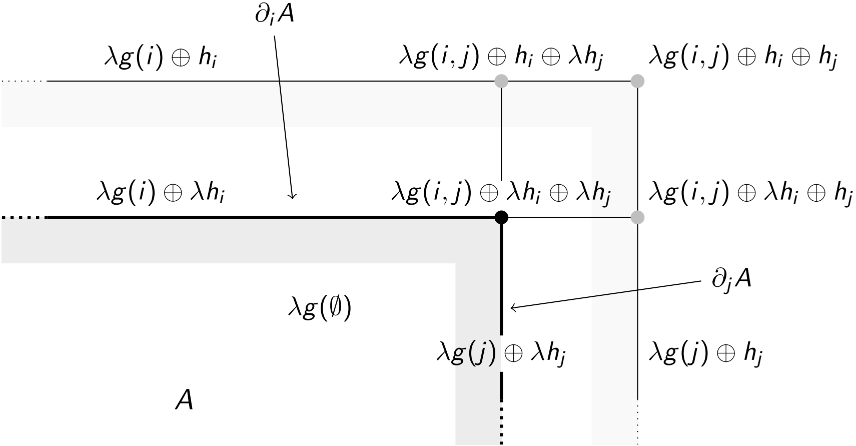

Choose $\lambda \gg 0$ and add a wide collar to A for interpolation.

Bordism with Baas-Sullivan singularities and homology

We can choose the singularity types in $\mathcal{P} = (P_1, P_2, \ldots)$ with

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A similar result holds for K-homology and Spin bordism.

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Theorem (S. Führing, 2013; H.)

Let $M^{d \geq 5}$ be oriented, smooth, non-spin and with $\pi_1(M)$ of odd order. Then the following are equivalent.

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A similar result holds for spin manifolds and positive K-homology. For $\mathcal{P}$-manifolds M we can prove a corresponding statement only in the non-spin case.

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For $\Gamma' = \Gamma \times \mathbb{Z}/p^\ell$ consider the Künneth exact sequence

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Generators for $d = 2m + 1$ are represented by lens spaces. Hence

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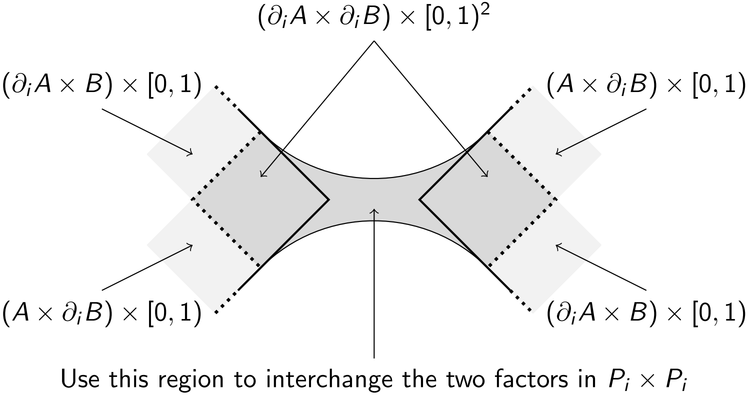
- If c_1 and c_2 are positive, then $\langle c_1, p^k, c_2 \rangle \subset H^+_{d_1+d_2+1}(B(\Gamma_1 \times \Gamma_2))$.

Interlude: Admissible products of $\mathcal{P}$-manifolds

- $\partial_i A = A(i) \times P_i$.
- $\partial_i B = B(i) \times P_i$.
- $\partial_i (A \times B) = (\partial_i A \times B) \cup (A \times \partial_i B)$.
- $(\partial_i A \times B) \cap (A \times \partial_i B) = \partial_i A \times \partial_i B = A(i) \times B(i) \times P_i \times P_i$.



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Theorem (H.)

Let Γ be an abelian p-group, p odd. Then all p-atoral classes in the image of $\mathrm{BPL}_*(B\Gamma) \to H_*(B\Gamma)$ are positive.