The G-invariant spectrum and non-orbifold singularities [1]

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Objectives

The goal of this project is to study the inaudible properties of the G-invariant spectrum. We will:

- Define the G-invariant spectrum of the Laplacian on an orbit space M/G
- Generalize the Sunada-Pesce-Sutton technique to the G-invariant setting
- Construct pairs of isospectral non-isometric orbit spaces
- Study the geometry of these spaces to identify inaudible properties

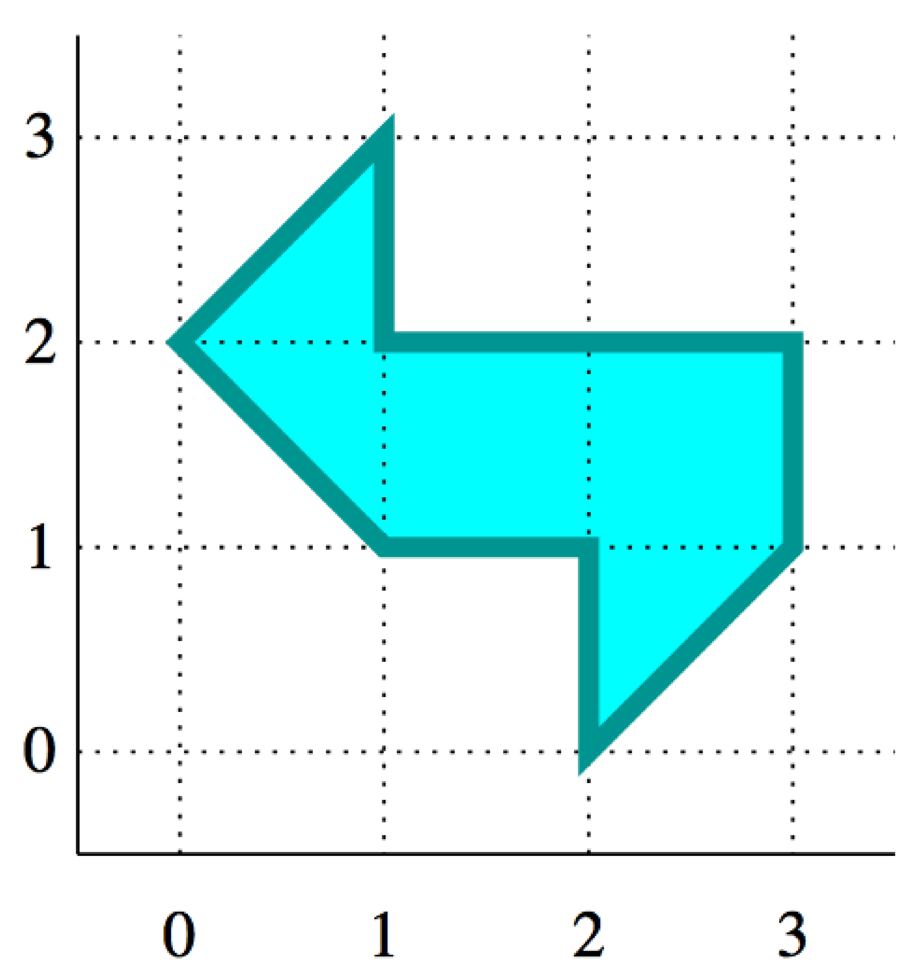
Introduction

When M is a compact Riemannian manifold one considers the eigenvalues of the Laplace-Beltrami operator Δ, i.e. those real numbers λ for which there exists a solution to the equation

$$\Delta(f) = \lambda f, \quad f \in C^\infty(M).$$

These eigenvalues form a discrete sequence of non-negative real numbers which we refer to as the spectrum of M. Given a compact subgroup of the isometry group $G \leq Isom(M)$ we consider the subsequence of eigenvalues that correspond to eigenfunctions which are constant on the G-orbits, again counting multiplicities. We will refer to this subsequence as the G-invariant spectrum of M. Given closed subgroups $H_i \leq G$ for $i \in \{1, 2\}$ we say that the quotient spaces M/H_1 and M/H_2 are isospectral if the H_i-invariant spectra are equal.

We are interested in the following inverse spectral questions: What information about the singular set of an orbit space M/G is encoded in its G-invariant spectrum? In particular, can one hear the existence of non-orbifold singularities, i.e. whether or not an orbit space is an orbifold? We note that the negative inverse spectral results from the manifold and orbifold settings hold in the more general setting of orbit spaces. It is therefore known that isotropy type [7] and the order of the maximal isotropy groups [6] are inaudible.



Sunada Technique

Negative inverse spectral results are realized by studying pairs of isospectral non-isometric spaces. The celebrated Sunada technique [8] provides a systematic method for producing such pairs. We generalize this technique to the G-invariant setting:

Definition: Two representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ of a Lie group G are said to be *equivalent* if there exists a vector space isomorphism $T : V_1 \to V_2$ such that $\rho_2(g) \circ T = T \circ \rho_1(g)$ for every $g \in G$.

Definition: Closed subgroups H_1, H_2 of a compact Lie group G are said to be *representation equivalent* if the quasi-regular representations $Ind_{H_1}^G(1_{H_1})$ and $Ind_{H_2}^G(1_{H_2})$ are equivalent.

Theorem: (G-invariant Sunada-Pesce-Sutton technique) Let M be a compact Riemannian manifold and $G \leq Isom(M)$ a compact Lie group. Suppose $H_1, H_2 \leq G$ are closed, representation equivalent subgroups. Then the orbit spaces M/H_1 and M/H_2 are isospectral in the sense that the H_i-invariant spectra of the Laplacian on M are equal.

Proof of Theorem B

We first apply principal isotropy reduction which yields the following smooth SRF isometries (note that these isometries do not preserve the spectra):

$$O_1 = S^{11}/U(3) = S^7/U(2)$$
$$O_2 = S^{11}/(Sp(1) \times SO(4)) = S^7/(Sp(1) \times O(2)).$$

It is shown in [4, Thm 1] that $S^7/U(2)$ is isometric to $S^3/\mathbb{Z}_2$, the 3-hemisphere of constant sectional curvature 4.

We show that the slice representation of the action is non-polar at points $v = (v_1, 0, 0) \in S^7$ from row D of Table 2, allowing us to conclude by [5, Theorem 1.1] that the image of this stratum is a non-orbifold point. The slice representation of the action at v is polar if and only if its restriction to the connected component of the identity is polar, cf. [4, Section 2.4]. We therefore consider the action of $Sp(1) \times SO(2)$ on S^7 which acts with isotropy $Id \times SO(2)$ at $v = (v_1, 0, 0) \in S^7$. The orbit through such a point is S^3 and its normal space is $\mathbb{C}^2$.

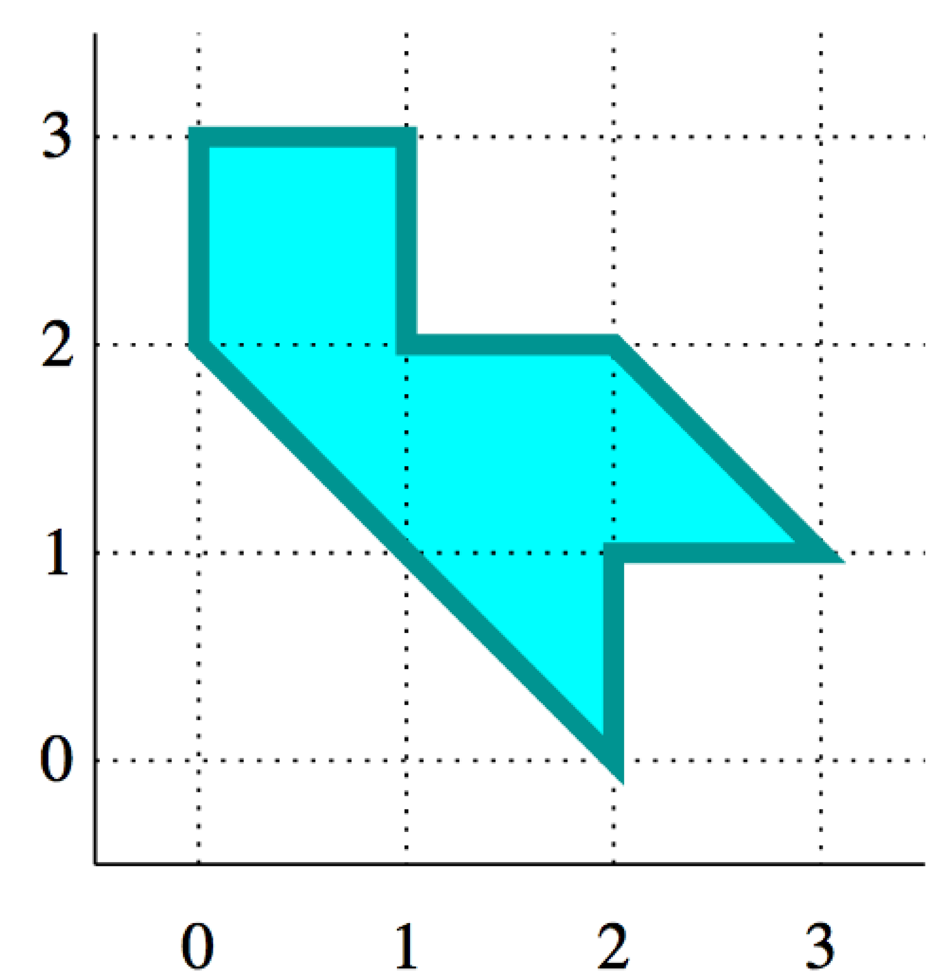
Main Results

Theorem A: Let the subgroups $H_1 = U(3)$ and $H_2 = Sp(1) \times SO(4)$ of $U(6) \leq Isom(S^{11})$ act on S^{11} via the embeddings given below. We have that the orbit spaces S^{11}/H_1 and S^{11}/H_2 are isospectral yet non-isometric.

Theorem B: The orbit space S^{11}/H_1 is smoothly SRF isometric to $S^3/\mathbb{Z}_2$, a hemisphere of constant sectional curvature, whereas S^{11}/H_2 admits a non-orbifold point and therefore has unbounded sectional curvature. We conclude that constant sectional curvature and the presence of non-orbifold singularities are inaudible properties of the G-invariant spectrum.

Proof of Theorem A

Fix embeddings where $A \in U(3)$ acts on $\mathbb{C}^6 = \mathbb{C}^3 \oplus \mathbb{C}^3$ as $(A, \bar{A})$ and $(B, C) \in Sp(1) \times SO(4)$ acts on $\mathbb{C}^6 = \mathbb{C}^2 \oplus \mathbb{C}^4$ as (B, C). Then [3, Theorem 1.5] shows that H_1 and H_2 are representation equivalent as subgroups of $SU(6) \leq Isom(S^{11})$. Isospectrality then follows immediately from the generalized Sunada technique.



The induced circle action is $z \cdot (z_1, z_2) = (zz_1, zz_2)$, which has trivial fixed point set and is therefore not polar [2, Prop 6.8]. Note that the slice representation at points from row B of the $S^7/U(2)$ table is a polar action. The orbit of such a point is again S^3 and the normal space is again a copy of $\mathbb{C}^2$. However now the slice representation is given by $z \cdot (z_1, z_2) = (z_1, zz_2)$ which has fixed point set $\mathbb{C} \times \{0\}$ and is a polar action.

Table 1: $O_1 = S^7/U(2)$

Row	Isotropy	qcodim	Points
A	Id	0	$v_1 \neq z \cdot \bar{v}_2$
B	$U(1)$	1	$v_1 = z \cdot \bar{v}_2$

$v = (v_1, v_2) \in S^7 \subset \mathbb{C}^2 \oplus \mathbb{C}^2$ and $z \in \mathbb{C}$

Table 2: $O_2 = S^7/(Sp(1) \times O(2))$

Row	Isotropy	qcodim	Points
A	$Id \times Id$	0	$v_1 \neq 0,\ v_2 \neq \lambda \cdot v_3$
B	$Id \times O(1)$	1	$v_1 \neq 0,\ v_2 = \lambda \cdot v_3$
C	$Sp(1) \times Id$	1	$v_1 = 0,\ v_2 \neq \lambda \cdot v_3$
D	$Id \times O(2)$	3	$v_1 \neq 0,\ v_2 = v_3 = 0$
E	$Sp(1) \times O(1)$	2	$v_1 = 0,\ v_2 = \lambda \cdot v_3$

$v = (v_1, v_2, v_3) \in S^7 \subset \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}$ and $\lambda \in \mathbb{R}$

Discussion

Although $S^{11}/U(3)$ and $S^3/\mathbb{Z}_2$ are smoothly SRF isometric, we can not conclude that these spaces are isospectral. Indeed, direct computation demonstrates that the Neumann spectrum on $S^3/\mathbb{Z}_2$ is distinct from the $U(3)$-invariant spectrum on S^{11}.

From the tables we can also conclude that isotropy type, maximal isotropy dimension, and the set of quotient codimensions of the strata are inaudible properties of the G-invariant spectrum.

The fact that constant sectional curvature is not determined by the G-invariant spectrum should be viewed in light of the positive spectral results in the manifold setting, where analysis of the asymptotic expansion of the heat trace has shown that constant sectional curvature is an audible property of the Laplace spectrum for manifolds of dimension less than six.

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Curvature of Reisnerr-Nordström Soliton determined as characteristic value

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Abstract

Present research paper focuses on the study of the gravitational field of Reisnerr-Nordström distorted metric. The technique of six dimensional formalism making an eigen equation gives rise to some decisive conclusions for the Gaussian curvature of Reisnerr-Nordström soliton. Further, we comparatively analyse the results for two and three dimensional hyper-surfaces.

1. Introduction

We have studied the concept of Ricci Soliton for the space-time of general relativity due to all important role of Ricci soliton in differential geometry and relativity. Hamilton defines a family $g_\lambda = g(\lambda; x)$ of Riemann metrics on a n-dimensional $(n \geq 3)$ smooth manifold M with parameter λ ranging in a time interval $J \subset \mathbb{R}$ including zero is called a Ricci flow if the Hamilton equations $\frac{\partial g_0}{\partial \lambda} = -2Ric_0$ of the Ricci flow (cf. [6], [7]) for $g_0 = g(0)$ and the Ricci tensor Ric_0 of the g_0 are satisfied. Corresponding to self similar solution of above equation, the notion of the Ricci soliton prevails, which is defined as a metric g_0 satisfying the equation $-2Ric_0 = \pounds_\xi g_0 + 2kg_0$ for vector field ξ on V_n and a constant k. The Ricci soliton is said to be steady (static) if $k = 0$, shrinking if $k < 0$ and expanding if $k > 0$. The metric g_0 is called a gradient Ricci soliton if $\xi = \nabla\phi$ i.e., gradient of some function ϕ.

For n-dimensional Riemannian manifold we can write general equation showing Ricci Soliton as

$$R_{ij} - \frac{1}{2}\pounds_\xi g_{ij} = kg_{ij} \tag{1}$$

Many applications of Ricci solitons are found in the literature, as one can see that Baleanu et. al. [4] obtain soliton equation for nonlinear Schrodinger equation (NNLSE). In fact they report the optical soliton solutions of NNLSE with parabolic law nonlinearity and time dependent coefficients which are the terms of velocity dipersion, linear and nonlinear terms and also non-local one. M.M. Akbar and E. Woolger [2] developed some examples for Ricci soliton. Corresponding to charged black hole metric (Reisnerr-Nordström black hole) Ali and Ahsan [3] developed a function equipped with metric tensor g_{ij} which solves the Einstein-free scalar field system satisfying equation (1). The Reisnerr-Nordström metric $ds^2 = g_{ij}dx^i dx^j \equiv dr^2 + h^2(r)d\Omega_k^2$ using the integrability conditions and taking $f'(r) = \frac{B}{[h(r)]^{n-1}}$ for $B =$ constant.

The soliton so developed for charged black hole is

$$ds^{*2} = dr^2 + (Ar^2)(d\theta^2 + sin^2\theta d\phi^2) - A^{\sqrt{2}}dt^2 \qquad (2),$$

while the original metric of a charged black hole (Reisnerr-Nordström metric) is given by $ds^{*2} = -Adt^2 + A^{-1}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2$ where $A = ((r^2 + e^2 - 2mr)/r^2)$.

In this paper we have worked on the geometry of charged metric and then elaborated the notions for its solitons in detail. By using the 6-dimensional formalism, the characteristic values of λ-tensor (i.e. $R_{AB} - \lambda g_{AB}$) has been given in this paper and an example of canonical form of the system is shown, also characterization of spacetime due to symmetric tensor R_{AB} is done . Further, the cases of 2 and 3-dimension for Reisnerr-Nordström soliton are discussed, in which Gaussian curvature is calculated and shown its dependence on characteristic value of λ-tensor. (For more see [8], [10])

2. Components of Christoffel symbol and Riemann Tensor

The non-zero components of the metric tensor, the Christoffel symbol and Riemann Curvature tensor for the metric (2) in spherical coordinates $x^\alpha \equiv (r, \theta, \phi, t)$ are given by (for formulas see [1])

$$g_{11} = 1, \; g_{44} = -((r^2 + e^2 - 2mr)/r^2)^{\sqrt{2}}, \qquad g_{22} = r^2 - 2mr + e^2, \; g_{33} = (r^2 - 2mr + e^2)\sin^2\theta \tag{3}$$

$$\begin{aligned}
&\Gamma^1_{22} = (m - r), \quad \Gamma^1_{33} = (m - r)\sin^2\theta \\
&\Gamma^1_{44} = \sqrt{2}\frac{(mr - e^2)}{r^3}A^{\sqrt{2}-1}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{r - m}{Ar^2} \\
&\Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{r - m}{Ar^2} \\
&\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta, \quad \Gamma^4_{14} = \Gamma^4_{41} = \frac{\sqrt{2}m}{r^2 - 2mr}
\end{aligned} \tag{4}$$

$$\begin{aligned}
&R_{1212} = (m^2 - e^2)/Ar^2, \; R_{2323} = (e^2 - m^2)\sin^2\theta, \\
&R_{1414} = (A^{\sqrt{2}(1-\sqrt{2})}r^6)[2(mr - e^2)^2 + \\
&\quad \sqrt{2}(-2mr^3 + (6m^2 + 3e^2)r^2) - 4mre^2 + e^4] \\
&R_{3131} = ((m^2 - e^2)\sin^2\theta)/Ar^2 \\
&R_{2424} = (-\sqrt{2}(mr - e^2)(m - r)A^{\sqrt{2}-1})/r^2 \\
&R_{3434} = (-\sqrt{2}(mr - e^2)(m - r)sin^2\theta A^{\sqrt{2}-1})r^2
\end{aligned} \tag{5}$$

3. Construction of Eigen Equation

We use the 6-dimensional formalism in the pseudo-Euclidean space $\mathbb{R}^6$ by making the identification [5]

$$\begin{array}{lcccccc} ij: & 23 & 31 & 12 & 14 & 24 & 34 \\ A: & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \tag{6}$$

We also make use of the identification as

$$g_{ik}g_{jl} - g_{il}g_{jk} = g_{ijkl} \to g_{AB} \tag{7}$$

where A, B= 1, 2, 3, 4, 5, 6 and g_{ij} are the components of the metric tensor at an arbitrary point (x^α) of the Reisnerr-Nordström soliton, whose metric is given by equation (2). The new metric tensor g_{AB} $(A, B$=1,2,3,4,5,6) is symmetric and non-singular. The non-zero components of the metric tensor g_{AB} for equation (2) in 6-dimensional formalism, by using formulation (7) are

$$\begin{aligned}
&g_{11}(x^\alpha) = (Ar^2)^2\sin^2\theta, \quad g_{22}(x^\alpha) = (Ar^2)\sin^2\theta, \\
&g_{33}(x^\alpha) = (Ar^2), \; g_{44}(x^\alpha) = -A^{\sqrt{2}}, \; g_{55}(x^\alpha) = \\
&-r^2A^{\sqrt{2}+1}, \; g_{66}(x^\alpha) = -(r^2 - 2mr)\sin^2\theta A^{\sqrt{2}}
\end{aligned} \tag{8}$$

Similarly, we can transform the components of the Riemann tensor as $R_{ijkl} \to R_{AB}$. Thus, for example R_{1212} can be written as R_{33} [using identification (6)]. So now all the non-zero components of the tensor R_{AB} under the identification (6) (associated components in equation (5)) are as

$$\begin{aligned}
&R_{11}(x^\alpha) = R_{2323}, \; R_{22}(x^\alpha) = R_{3131}, \; R_{33}(x^\alpha) = R_{1212}, \\
&R_{44}(x^\alpha) = R_{1414}, \; R_{55}(x^\alpha) = R_{2424}, \; R_{66}(x^\alpha) = R_{3434}
\end{aligned} \tag{9}$$

Next by using these components calculated above in 6-dimensional formalism we find a canonical form of the λ-tensor $R_{AB} - \lambda g_{AB}$, also then eigen values for the Reisnerr-Nordström soliton (2) will be calculated by solving the so constructed characteristic equation $| \; R_{AB} - \lambda g_{AB} \; |= 0$. Here, using Equations (8) and (9), the eigen values are as

$$\begin{aligned}
&\lambda_1(r) = (e^2 - m^2)/A^2r^4 = -\lambda_2(r) = -\lambda_3(r) \\
&\lambda_4(r) = \frac{-1}{A^2r^6}[2(mr - e^2)^2 + \\
&\sqrt{2}(-2mr^3 + (6m^2 + 3e^2)r^2) - 4mre^2 + e^4] \\
&\lambda_5(r) = (\sqrt{2}(mr - e^2)(m - r))/Ar^5 = \lambda_6(r)
\end{aligned} \tag{10}$$

Eigenvalues $\lambda_i, \; i = 1, \; 2, \; 3, \; 4, \; 5, \; 6$ obtained in equation (10) depend on parameters m and r. In other words, we can say that for these λ_i, the determinant of λ-tensor $R_{AB} - \lambda g_{AB}$ vanishes. Further, we can transform the system in canonical form for values of λ_i as

$g_{A'B'}$ = Diag$(1, \; 1, \; 1, \; -1 \; -1 \; -1)$ and

$$R_{A'B'} = \text{Diag}(\lambda_1(r), \; \lambda_2(r), \; \lambda_3(r), \; -\lambda_4(r), \; -\lambda_5(r), \; -\lambda_6(r)) \tag{11}$$

3.1. Result : The dimension of Jordan blocks in the canonical form of Ricci tensor in Equation(11), shows that for the Reisnerr-Nordström soliton the λ- tensor give rise to segre type $[(11)(11)]$ (cf., [9]) i.e. there are six lineally independent vectors two are corresponding to Jordan block of dimension two and next two are corresponding to next Jordan block of dimension two and respectively these are related to $\lambda_2 = \lambda_3$ and $\lambda_5 = \lambda_6$.

3.1. Example : Two Dimensional Hypersurface

If we take $\theta = 0$ or $\theta = \pi$ that is $d\theta = 0$, the Reisnerr-Nordström soliton, given by equation (2), reduces to the form

$${}^*ds^2 = dr^2 - ((r^2 - 2mr + e^2)/r^2)^{\sqrt{2}}dt^2 \tag{12}$$

equation (12) is a 2-dimensional surface now. The metric tensor ${}^*g$ in coordinates $x^\beta \equiv (r, \; t)$ is given by

$${}^*g_{ij}(x^\beta) = \begin{bmatrix} 1 & 0 \\ 0 & -((r^2 - 2mr + e^2)/r^2)^{\sqrt{2}} \end{bmatrix} \tag{13}$$

here $i, \; j = 1, \; 4$. Thus, the hypersurface for $\theta = 0$ or $\theta = \pi$ (i.e., ${}^*H_0$ or ${}^*H_\pi$) degenerates to two dimensional surface. The non-zero component of Riemann curvature tensor for equation (12) is unique and given by

$$\begin{aligned}
{}^*R_{1414}(x^\beta = &\frac{1}{(r^3 - 2mr^2 + re^2)^2}\left(\frac{r^2 - 2mr + e^2}{r^2}\right)^{\sqrt{2}}[2(mr - e^2)^2 \\
&+ \sqrt{2}(-2mr^3 + (6m^2 + 3e^2)r^2) - 4mre^2 + e^4]
\end{aligned}$$

so the Gaussian curvature ${}^*K$ for surface ${}^*H_0$ or ${}^*H_\pi$ is

$$\begin{aligned}
{}^*K(x^\beta) = &\frac{1}{(r^3 - 2mr^2 + re^2)^2}[2(mr - e^2)^2 + \\
&\sqrt{2}(-2mr^3 + (6m^2 + 3e^2)r^2) - 4mre^2 + e^4]
\end{aligned} \tag{14}$$

3.2. Result : Equations (10) and (14) show that curvature of the 2-dimensional surface of the Reisnerr-Nordström soliton is related to the eigen value $\lambda_4(r)$.

3.1. Note: Similar result we have obtained for the case $2m < r < \infty$, $0 < \theta < \pi$ and $\phi = 0$ in three dimension subspace.

3.3. Result : the curvature of the 3-dimensional space of Reisnerr-Nordström soliton can be expressed in terms of a λ-tensor which happens to be the solutions (eigen-values) of the characteristic equation $|R_{AB} - \lambda g_{AB}| = 0$.

4. Conclusion

In this paper we have worked out on gravitational field of Reisnerr-Nordström soliton by using characteristic of λ-tensor $R_{AB} - \lambda g_{AB}$, we have also discussed 2 and 3-dimensional cases. It is seen that Reisnerr-Nordström soliton, given by Ali and Ahsan [3] has different geometry as that of Reisnerr-Nordström metric. We see that the gravitational field for Reisnerr-Nordström soliton is of type $[(11)(11)]$ [equation (11)] in Segre symbols. For Reisnerr-Nordström soliton, not only the Gaussian curvature differ with that of Reisnerr-Nordström metric but also the dependence of curvature on eigen values of λ-tensor $R_{AB} - \lambda g_{AB}$ is not similar. Thus, the deformation in metric (along a λ-dependent diffeomorphism) of a spacetime is responsible for change in geometry or gravitational field.

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Boundary value problems for general first-order elliptic operators

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Setup

- M smooth manifold with smooth compact boundary $\Sigma = \partial M$;
- τ interior co-vectorfield along ∂M;
- μ smooth volume measure on M and ν induced smooth volume measure on Σ;
- (E, h^E), $(F, h^F) \to M$ Hermitian vector bundles over M;
- D first-order elliptic differential operator from E to F;
- D and D^* complete - i.e., $\mathrm{C}^\infty_c(E;F)$ and $\mathrm{C}^\infty_c(F;E)$ dense in $\mathrm{dom}(D_{\max})$ and $\mathrm{dom}(D^*_{\max})$ respectively.

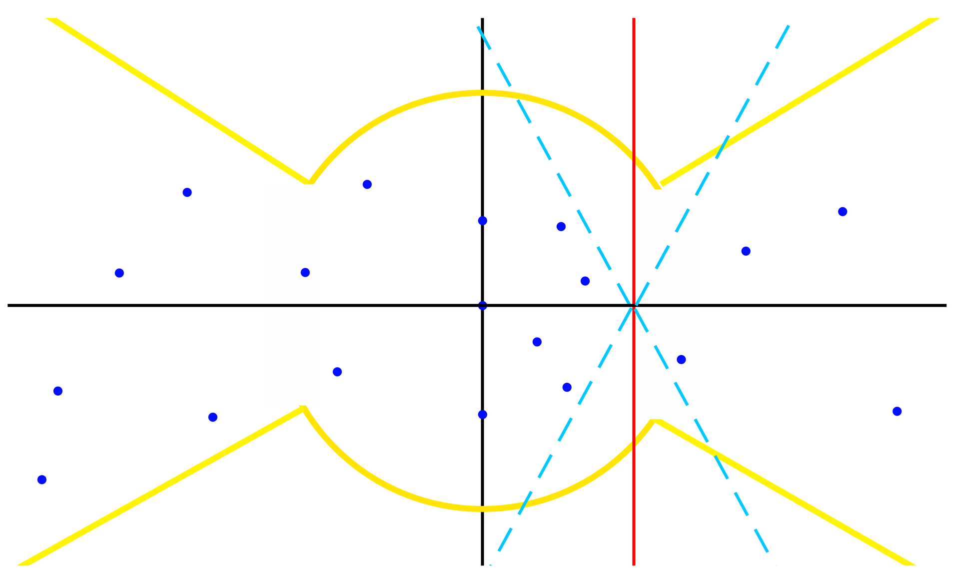
Adapted boundary operator

Principal symbol for D and D^*: $\sigma_D(x,\xi)$ and $\sigma_{D^*}(x,\xi)$, define $\sigma_0(x) := \sigma_D(x, \tau(x))$.

A and $\tilde{A}$ are *adapted boundary operators* (to D or D^* respectively) on $E_\Sigma := E|_\Sigma$ and $F_\Sigma := F|_\Sigma$ respectively if their principal symbols are given by:

$$\sigma_A(x,\xi) = \sigma_D(x,\tau(x))^{-1} \circ \sigma_D(x,\xi) \quad \text{and} \quad \sigma_{\tilde{A}}(x,\xi) = \sigma_{D^*}(x,\tau(x))^{-1} \circ \sigma_{D^*}(x,\xi).$$

- Exists and are elliptic differential operators of order 1.
- Unique up to an operator of order zero.
- Discrete spectrum, generally non-orthogonal eigenspaces.
- **No additional assumptions on A (i.e., self-adjointness) apart from ellipticity of D:**



Admissible cut $r \in \mathbb{R}$: the line $l_r := \{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta = r\}$ is not in the spectrum of A (yields $A_r := A - r$ invertible bi-sectorial).

An admissible cut always exists.

$\chi^\pm(A_r) : \mathrm{L}^2(E_\Sigma) \to \mathrm{L}^2(E_\Sigma)$ spectral projectors to the left and right of l_r - pseudos of order zero.

- Space: $\check{H}(A) := \chi^-(A_r)\mathrm{H}^{\frac{1}{2}}(E_\Sigma) \oplus \chi^+(A_r)\mathrm{H}^{-\frac{1}{2}}(E_\Sigma)$.
- Norm: $\|u\|^2_{\check{H}(A)} := \|\chi^-(A_r)u\|^2_{\mathrm{H}^{\frac{1}{2}}} + \|\chi^+(A_r)u\|^2_{\mathrm{H}^{-\frac{1}{2}}}$.
- *Norms corresponding to two different spectral cuts are comparable.*

Theorem 1: Maximal domains and $\check{H}(A)$, $\check{H}(\tilde{A})$ spaces

- $\mathrm{C}^\infty_c(E)$ is dense in $\mathrm{dom}(D_{\max})$ and $\mathrm{dom}((D^*)_{\max})$ with respect to corresponding graph norms.
- The trace maps $\mathrm{C}^\infty_c(E) \to \mathrm{C}^\infty_c(E_\Sigma)$ and $\mathrm{C}^\infty_c(F) \to \mathrm{C}^\infty_c(F_\Sigma)$ given by $u \mapsto u|_\Sigma$ extend uniquely to surjective bounded linear maps $\mathrm{dom}(D_{\max}) \to \check{H}(A)$ and $\mathrm{dom}((D^*)_{\max}) \to \check{H}(\tilde{A})$.
- The spaces

$$\mathrm{dom}(D_{\max}) \cap \mathrm{H}^1_{\mathrm{loc}}(E_\Sigma) = \left\{u \in \mathrm{dom}(D_{\max}) : u|_\Sigma \in \mathrm{H}^{\frac{1}{2}}(E_\Sigma)\right\}$$
$$\mathrm{dom}((D^*)_{\max}) \cap \mathrm{H}^1_{\mathrm{loc}}(F_\Sigma) = \left\{u \in \mathrm{dom}((D^*)_{\max}) : u|_\Sigma \in \mathrm{H}^{\frac{1}{2}}(F_\Sigma)\right\}.$$

- For all $u \in \mathrm{dom}(D_{\max})$ and $v \in \mathrm{dom}((D^*)_{\max})$,

$$\langle D_{\max}u, v\rangle_{\mathrm{L}^2(F)} - \langle u, (D^*)_{\max}v\rangle_{\mathrm{L}^2(E)} = -\left\langle \sigma_0 u|_\Sigma, v|_\Sigma\right\rangle_{\mathrm{L}^2(F_\Sigma)}.$$

Theorem 2: Higher regularity

$$\mathrm{dom}(D_{\max}) \cap \mathrm{H}^{k+1}_{\mathrm{loc}}(E) = \left\{u \in \mathrm{dom}(D_{\max}) : Du \in \mathrm{H}^k_{\mathrm{loc}}(F) \text{ and } \chi^+(A_r)(u|_\Sigma) \in \mathrm{H}^{k+\frac{1}{2}}(E_\Sigma)\right\}.$$

Proof ingredients of Theorems 1 and 2:

- Identification of $\mathrm{dom}(A_r) = \mathrm{dom}(A_r^*)$ by elliptic pseudo-differential operator theory.
- H^∞ functional calculus for the invertible sectorial operator $|A_r| := A_r \,\mathrm{sgn}(A_r)$.
- Semigroup theory and Kato square root problem methods: ellipticity via equivalent norm for which $|A_r|$ is maximal-accretive.
- Maximal regularity (via H^∞ functional calculus) for higher regularity.

Boundary conditions and the associated operator

A *closed* linear subspace $B \subset \check{H}(A)$ is called a *boundary condition* for D. Associated operator domains:

$$\mathrm{dom}(D_{B,\max}) = \left\{u \in \mathrm{dom}(D_{\max}) : u|_\Sigma \in B\right\}$$
$$\mathrm{dom}(D_B) = \left\{u \in \mathrm{dom}(D_{\max}) \cap \mathrm{H}^1_{\mathrm{loc}}(E_\Sigma) : u|_\Sigma \in B\right\},$$

and similarly for the formal adjoint D^* with A replaced by $\tilde{A}$.

- For boundary condition B, the operator D_B closed and between D_{cc} (on $\mathrm{C}^\infty_{cc}(E)$) and $D_{\max}$.
- D_c closed extension of D_{cc}, then $B := \left\{u|_\Sigma : u \in \mathrm{dom}(D_c)\right\}$ is a boundary condition and $D_c = D_{B,\max}$.
- Boundary condition $B \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$ if and only if $D_B = D_{B,\max}$.
- Adjoint boundary condition B^{ad} so that $D_B^{\mathrm{ad}} = D^*_{B^{\mathrm{ad}}}$:

$$B^{\mathrm{ad}} := \left\{v \in \check{H}(-\tilde{A}) : \langle\sigma_0 u, v\rangle_{\mathrm{L}^2(F_\Sigma)} = 0 \quad \forall u \in B\right\}$$

Elliptic boundary conditions

$B \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$ boundary condition is called *elliptic* if there exists an admissible cut $r \in \mathbb{R}$ and:

- $W_\pm$, $V_\pm$ are mutually complementary subspaces such that

$$V_\pm \oplus W_\pm = \chi^\pm(A_r)\mathrm{L}^2(E_\Sigma),$$

- $W_\pm$ are finite dimensional with $W_\pm, W_\pm^* \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$, and
- $g : V_- \to V_+$ bounded linear map with $g(V_-^{\frac{1}{2}}) \subset V_+^{\frac{1}{2}}$ and $g^*((V_+^*)^{\frac{1}{2}}) \subset (V_-^*)^{\frac{1}{2}}$ such that

$$B = W_+ \oplus \left\{v + gv : v \in V_-^{\frac{1}{2}}\right\}.$$

$B \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$ be a subspace, then the following are equivalent:

- B a boundary condition and $B^{\mathrm{ad}} \subset \mathrm{H}^{\frac{1}{2}}(F_\Sigma)$,
- the definition is satisfied for any admissible spectral cut $r \in \mathbb{R}$,
- B an elliptic boundary condition.

For elliptic boundary condition B, have B^{ad} elliptic boundary condition for D^* and

$$\sigma_0^*(B^{\mathrm{ad}}) = W_-^* \oplus \left\{u - g^*u : u \in (V_+^*)^{\frac{1}{2}}\right\}.$$

Pseudo-local and local boundary conditions

- For classical pseudo-differential projector P of order zero (not necessarily orthogonal), the space

$$B = P(\mathrm{H}^{\frac{1}{2}}(E_\Sigma))$$

is called a *pseudo-local boundary condition*.
- Boundary condition $B \subset \mathrm{H}^{\frac{1}{2}}(E_\Sigma)$ a *local boundary condition* if there exists a sub-bundle $E' \subset E_\Sigma$ such that

$$B = \mathrm{H}^{\frac{1}{2}}(E').$$

Theorem 3: Characterisation of pseudo-local boundary conditions

Given a pseudo-local boundary condition $B = P(\mathrm{H}^{\frac{1}{2}}(E_\Sigma))$, the following are equivalent:

- B an elliptic boundary condition,
- for admissible cut $r \in \mathbb{R}$, the operator

$$P - \chi^+(A_r) : \mathrm{L}^2(E_\Sigma) \to \mathrm{L}^2(E_\Sigma)$$

is Fredholm,
- for admissible cut $r \in \mathbb{R}$, the operator

$$P - \chi^+(A_r) : \mathrm{L}^2(E_\Sigma) \to \mathrm{L}^2(E_\Sigma)$$

is elliptic classical pseudo of order zero.

If B is a pseudo-local boundary condition and $D_B u$ is smooth, then u is smooth up to the boundary.

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Ricci Almost Solitons on semi-Riemannian Warped Products

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Abstract

It is shown that a gradient Ricci almost soliton on a warped product, $(B^n \times_h F^m, g, f, \lambda)$ whose potential function f depends on the fiber, is either a Ricci soliton or λ is not constant and the warped product, the base and the fiber are Einstein manifolds, which admit conformal vector fields. Assuming completeness, a classification is provided for the Ricci almost solitons on warped products, whose potential functions depend on the fiber. An important decomposition property of the potential function in terms of functions which depend either on the base or on the fiber is proven. In the case of a complete Ricci soliton, the potential function depends only on the base.

1. Basic Concepts and Notation

A (*gradient*) *Ricci almost soliton* (M, g, f, λ) is a semi-Riemannian manifold (M, g) with smooth functions $f, \lambda : M \to \mathbb{R}$ satisfying the following fundamental equation

$$Ric + \nabla\nabla f = \lambda g, \qquad (1)$$

The function $f : M \to \mathbb{R}$ is called *potential function*. This concept was introduced in [5], generalizing the notion of Ricci solitons.

Consider two semi-Riemannian manifolds (B^n, g_B) and (F^m, g_F). Given a smooth function $h : B \to (0, +\infty)$, we define the warped product $B \times_h F$ with warping function h, as the product manifold $B \times F$ endowed with the metric $g = g_B + h^2 g_F$, defined by

$$g = \pi^* g_B + (h \circ \pi)^2 \sigma^* g_F, \qquad (2)$$

where $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ are the canonical projections. So $B \times_h F = (B \times F, g)$ is a semi-Riemannian manifold of dimension $n + m$.

In what follows we will denote the connection, the Ricci curvature and other tensors defined using the metric g_B with a subscript B, as ∇_B, Ric_B. Similar notation will be considered for the metric g_F.

2. Characterization and Consequences

The theorem bellow says that when the potential function depends on the fiber then the fundamental equation (1) on a warped product reduces to a system of equations on the base and on the fiber, in the following way:

Theorem 1 *Let $B^n \times_h F^m$ be a non trivial warped product where the base (B^n, g_B) or the fiber (F^m, g_F) can be either a Riemannian or a semi-Riemannian manifold. Then $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci almost soliton, with f non constant on F if, and only if,*

$$f = \beta + h\varphi, \qquad (3)$$

where $\varphi : F \to \mathbb{R}$ is not constant and $\beta : B \to \mathbb{R}$ are differentiable functions such that

$$\begin{cases} \nabla_B\nabla_B h + ahg_B = 0, \\ Ric_B + \nabla_B\nabla_B\beta = [h^{-1}(\nabla_B h)\beta - bh^{-1} + (n-1)a]g_B, \\ \nabla_F\nabla_F\varphi + (c\varphi + b)g_F = 0, \\ Ric_F = (m-1)cg_F, \end{cases} \qquad (4)$$

for some constants $a,\ b,\ c \in \mathbb{R}$, the function λ is given by

$$\lambda = h^{-1}(\nabla_B h)\beta - bh^{-1} + (m+n-1)a - ah\varphi, \qquad (5)$$

and the constants a and c are related to h by the equation

$$|\nabla_B h|^2 + ah^2 = c. \qquad (6)$$

As an application of Theorem 1 we can prove that for a complete warped product Ricci solitons (that is, when λ is a constant) the potential function does not depend on the fiber.

Corollary 1 *Let $(B \times_h F, g, f, \lambda)$ be a Ricci soliton on a complete non trivial semi-Riemannian warped product. Then f does not depend on the fiber.*

Corollary 1 was considered also in [3] with a different approach. It shows that examples of Ricci solitons on complete semi-Riemannian warped products occur when the potential function depends only on the base.
Our next result characterizes Ricci almost solitons i.e., equation (1), on warped products, when the potential function depends only on the base.

Theorem 2 *Let $B^n \times_h F^m$ be a non trivial warped product where the base (B^n, g_B) or the fiber (F^m, g_F) can be either a Riemannian or a semi-Riemannian manifold. Then $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci almost soliton, with f constant on F if, and only if,*

$$\begin{cases} Ric_B + \nabla_B\nabla_B f - mh^{-1}\nabla_B\nabla_B h = \lambda g_B, \\ \lambda h^2 = h(\nabla_B h)f - (m-1)|\nabla_B h|^2 - h\Delta_B h + c(m-1), \\ Ric_F = c(m-1)g_F, \end{cases} \qquad (7)$$

for some constant $c \in \mathbb{R}$.

Remark 1 *The first and third equations in Theorem 1 say that the corresponding gradient fields are conformal vector fields.*

Remark 2 *The fourth equation of Theorem 1 and the third equation of Theorem 2 show that the fiber is an Eisntein manifold in both cases.*

3. Rigidity when f Depends on the Fiber

We say that a semi-Riemannian manifold (M, g) is a *Brinkmann space* if it admits a parallel light like vector field X, called a *Brinkmann field.*
We say that a vector field X is *improper* if there is an open set where X is light like. If there is no such an open set the field is said a *proper* vector field.

Theorem 3 *Let $B^n \times_h F^m$, $n \geq 2$, be a non trivial warped product where the base (B^n, g_B) is a semi-Riemannian manifold and the fiber (F^m, g_F) can be either a Riemannian or a semi-Riemannian manifold. Then $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci almost soliton, with f non constant on F and $\nabla_B h$ an improper vector field on B if, and only if, λ is constant and $f = \beta + h\varphi$, where $\varphi : F \to \mathbb{R}$ non constant and $\beta : B \to \mathbb{R}$ are smooth functions satisfying*

$$g(\nabla_B h, \nabla_B \beta) = \lambda h + b,$$
$$Ric_B + \nabla_B\nabla_B\beta = \lambda g_B,$$
$$\nabla_F\nabla_F\varphi + bg_F = 0$$

for a constant $b \in \mathbb{R}$, B is a Brinkmann space with $\nabla_B h$ as a Brinkmann field and F is Ricci flat. If in addition F is complete, then it is isometric to

1. *$\pm\mathbb{R} \times \bar{F}^{m-1}$, where $\bar{F}$ is Ricci flat, if $b = 0$;*
2. *$\mathbb{R}^m_\varepsilon$, if $b \neq 0$.*

The vector field $\nabla_B h$ is *non homothetic* if its local flow does not act by translations. The next result shows the rigidity of a Ricci almost soliton on a warped product when the potential function depends on the fiber and $\nabla_B h$ is a non homothetic vector field.

Theorem 4 *Let $B^n \times_h F^m$ be a non trivial warped product where the base (B^n, g_B) or the fiber (F^m, g_F) can be either a Riemannian or a semi-Riemannian manifold and suppose that $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci almost soliton with f non constant on F and $\nabla_B h$ a proper vector field. Then*

1. *If $\nabla_B h$ is homothetic, then λ is constant, i.e., it is a Ricci soliton;*
2. *If $\nabla_B h$ is non-homothetic, then λ is not constant, B, F and $B^n \times_h F^m$ are Einstein manifolds such that*

   $$Ric_{B\times_h F} = (n+m-1)ag,$$
   $$Ric_B = (n-1)ag_B,$$
   $$Ric_F = (m-1)cg_F,$$

   where the constants $a \neq 0$ and c are related to h by $|\nabla_B h|^2 + ah^2 = c$. Moreover, ∇f and $\nabla_B h$ are conformal gradient fields on $B^n \times_h F^m$ and on B^n, respectively, satisfying

   $$\nabla\nabla f + (af + a_0)g = 0,$$
   $$\nabla_B\nabla_B h + ahg_B = 0,$$
   $$\lambda = -af + a(m+n-1) - a_0,$$

   for some constant $a_0 \in \mathbb{R}$.

A direct corollary of both Theorem 3 and Theorem 4 is the following rigidity result. Other rigidity results can be found in [1], [2] or [5].

Corollary 2 *If $(B^n \times_h F^m, g)$ is a warped product Ricci almost soliton, with f non constant on F, then one of the following holds*

1. *λ is constant, i.e., it is a Ricci soliton;*
2. *λ is not constant, $(B^n \times_h F^m, g)$ is an Einstein manifold, $\nabla_B h$ is a proper and non-homothetic vector field and ∇f is conformal.*

4. Classification when f Depends on the Fiber

Einstein manifolds carrying conformal vector fields are classified and, using this classification, we will give a classification of complete Ricci almost solitons.
In order to state our classification result for Ricci almost solitons on complete semi-Riemannian warped products, we consider the following classes of n-dimensional complete semi-Riemannian Einstein manifolds:

Class I

1. $\mathbb{R} \times N^{n-1}$ where (N, g_N) is a complete semi-Riemannian Einstein manifold.
2. A Brinkman space of dimension $n \geq 3$, i.e. a semi-Riemannian manifold (M^n, g) admitting a parallel light like vector field.

Class II

1. $\mathbb{S}^n_\varepsilon(1/\sqrt{c})$, when $0 \leq \varepsilon \leq n-2$; the covering of $\mathbb{S}^n_{n-1}(1/\sqrt{c})$ when $\varepsilon = n-1$ and the upper part of $\mathbb{S}^n_n(1/\sqrt{c})$ when $\varepsilon = n$ with $c > 0$.
2. $\mathbb{H}^n_\varepsilon(1/\sqrt{|c|})$, when $2 \leq \varepsilon \leq n-1$; the covering of $\mathbb{H}^n_1(1/\sqrt{|c|})$ when $\varepsilon = 1$ and the upper part of $\mathbb{H}^n_0(1/\sqrt{|c|})$ when $\varepsilon = 0$, with $c < 0$.
3. $(\mathbb{R} \times N^{n-1}, \pm dt^2 + \cosh^2(\sqrt{|c|}\,t)g_N)$, where (N^{n-1}, g_N) is a semi-Riemannian Einstein manifold.
4. $(\mathbb{R} \times N^{n-1}, \pm dt^2 \pm e^{2\sqrt{|c|}\,t}g_N)$, where (N^{n-1}, g_N) is a Riemannian Einstein manifold,

The following result classifies the complete Ricci almost solitons on warped products, whose potential functions depend on the fiber.

Theorem 5 *Let $M^{n+m} = B^n \times_h F^m$ be a non trivial warped product where (B^n, g_B) or (F^m, g_F) can be either a Riemannian or a semi-Riemannian manifold. Then $(B^n \times_h F^m, g, f, \lambda)$ is a complete Ricci almost soliton with f non constant on F if, and only if, there exist constants $a \neq 0,\ a_0, c \in \mathbb{R}$ such that $f = a^{-1}(-\lambda + a(m+n-1) - a_0)$ and*

1. *if $n = 1$ then B^1 is isometric to $(\mathbb{R}, sgn\, a\, dt^2)$*

   $$h = \begin{cases} Ae^{\sqrt{|a|}t} & \text{if } c = 0, \\ \sqrt{|\frac{c}{a}|}[\cosh(\sqrt{|a|}t + B)] & \text{if } c \neq 0, \end{cases} \qquad (8)$$

   where $A \neq 0$ and $B \in \mathbb{R}$. Moreover, M is an Einstein manifold satisfying $Ric_M = (m+n-1)ag$ and if $m \geq 2$, F is an Einstein manifold satisfying $Ric_F = (m-1)cg_F$.
2. *If $n \geq 2$ and $m \geq 2$ then*
   - *M^{n+m} is an Eisntein manifold isometric either to a manifold of Class II.1 (resp. II.2) when $a > 0$ (resp. $a < 0$) and f has some critical point or it is isometric to a manifold of Class II.3 or II.4 if f has no critical points.*
   - *B is a complete Einstein manifold isometric either to a manifold of Class II.1 (resp. Class II.2) and index $\varepsilon_B = n$ (resp. $\varepsilon_B = 1$) if $a > 0$ (resp. $a < 0$) and h has critical points or to a manifold of Class II.3 or II.4 if h has no critical points.*
   - *F is a complete Einstein manifold isometric to either $\mathbb{R}^n_\varepsilon$, or to a manifolds of Class I when $c = 0$ and it is isometric to a manifold of Class II when $c \neq 0$.*
3. *Moreover, F^m, $m \geq 1$ is positive definite (resp. negative definite) if B^n, $n \geq 1$ is positive definite (resp. negative definite).*

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CERTAIN SUBMANIFOLDS OF COMPLEX SPACE FORMS

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The study of real hypersurfaces of Kählerian manifolds has been an important subject in geometry of submanifolds, especially when the ambient space is a complex space form. However, for arbitrary codimension, there are only a few recent results (see [2] for more details).

If a complex hypersurface M^n of a Kähler manifold $\overline{M}^{n+2}$ satisfies the condition $(*)$, then M^n is a totally geodesic submanifold.

Let M be a complete m–dimensional CR submanifold of maximal CR dimension of a complex space form $\overline{M}^{\frac{m+k}{2}}$. If the condition $(*)$ is satisfied, then one of the following three statements holds:

–M is a complete m–dimensional CR submanifold of CR dimension $\frac{m-1}{2}$ of a complex Euclidean space, and then M is isometric to $\mathbb{E}^m$, $\mathbb{S}^m$ or $\mathbb{S}^{2p+1}\times\mathbb{E}^{m-2p-1}$;

–M is a complete m–dimensional CR submanifold of CR dimension $\frac{m-1}{2}$ of a complex projective space and then M is isometric to $M^C_{p,q}$, for some p,q satisfying $2p+2q=m-1$;

– M is a complete m–dimensional CR submanifold of CR dimension $\frac{m-1}{2}$ of a complex hyperbolic space and then M is isometric to M^*_m or $M^H_{p,q}(r)$, for some p,q satisfying $2p+2q=m-1$.

Let M be a connected submanifold of real codimension two of a complex Euclidean space If M satisfies the condition $(*)$, then M is one of the following:

(1) n-dimensional sphere $\mathbf{S}^n$,
(2) n-dimensional Euclidean space $\mathbf{E}^n$,
(3) product manifold of an r-dimensional sphere and an $(n-r)$-dimensional Euclidean space $\mathbf{S}^r\times\mathbf{E}^{n-r}$, where r is an even number.
(4) CR submanifold of CR dimension $\frac{n-2}{2}$ with $\lambda=0$.

Let M^n be a submanifold of real codimension two of a complex Euclidean space with $\lambda=0$ which satisfies the condition $(*)$.

(I) If there exists a totally geodesic hypersurface M' of $\mathbf{C}^{\frac{n+2}{2}}$ such that $M\subset M'$, then M is one of the following:
(1) n-dimensional hyperplane $\mathbf{E}^n$,
(2) product manifold of an odd-dimensional sphere and a Euclidean space: $\mathbf{S}^{2p+1}\times\mathbf{E}^{n-2p-1}$.

(II) If there exists a totally umbilical hypersurface M' of $\mathbf{C}^{\frac{n+2}{2}}$, such that $M\subset M'$, then M is a product of two odd-dimensional spheres.

If for a real submanifold M of a complex manifold $(\overline{M},J)$, the holomorphic tangent space $H_x(M)=JT_x(M)\cap T_x(M)$ has constant dimension with respect to $x\in M$, the submanifold M is called a CR submanifold and the constant complex dimension is called the CR dimension of M. In [2] we collected the elementary facts about complex manifolds and their submanifolds and introduced the reader to the study of CR submanifolds of complex manifolds, especially complex projective space.

We assume that M satisfies the condition

$$h(FX,Y)+h(X,FY)=0,\quad \text{for all } X,Y\in T(M)\quad (*).$$

h is the second fundamental form of a submanifold and F is the structure tensor induced from the natural almost complex structure of a complex manifold.

In a complex projective space there exists neither totally geodesic nor totally umbilical real hypersurfaces. The surface $M^C_{p,q}$, called "generalized equator", is a quotient manifold $(\mathbf{S}^{2p+1}\times\mathbf{S}^{2q+1})/\mathbf{S}^1$. It is real hypersurface of a complex projective space, introduced by Lawson [5].

CR submanifolds M^m of maximal CR dimension of complex space forms $\overline{M}^{\frac{m+k}{2}}$,i.e. $\dim H_x(M)=m-1$:

$$\begin{aligned} J\imath X &= \imath FX+u(X)\xi\,,\\ J\xi &= -\imath U\,,\\ J\xi_a &= P\xi_a,\ a=1,\dots,k-1,\\ F^2X &= -X+u(X)U. \end{aligned}$$

Submanifolds of real codimension two of csf

$$\begin{aligned} J\imath X &= \imath FX+u^1(X)\xi_1+u^2(X)\xi_2,\\ J\xi_1 &= -\imath U_1+\lambda\xi_2,\\ J\xi_2 &= -\imath U_2-\lambda\xi_1,\\ F^2X &= -X+u^1(X)U_1+u^2(X)U_2. \end{aligned}$$

Let M^n be a submanifold of real codimension two of a complex projective space, which is not its totally geodesic complex hypersurface and let M satisfy the condition $(*)$. If there exists a real hypersurface $M^C_{p,q}$ such that $M\subset M^C_{p,q}$, then M is congruent to $\pi(S^{2p+1}\times S^{2r+1}\times S^{2s+1})$, where $p+q+s=\frac{n+1}{2}$.

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$$\begin{array}{ccccc} \pi^{-1}(M) & \xrightarrow{\imath_1'} & S^{2p+1}\times S^{2q+1} & \xrightarrow{\imath_2'} & S^{n+3}\\ \downarrow\pi & & \downarrow\pi & & \downarrow\pi\\ M & \xrightarrow[\imath_1]{} & M^C_{p,q} & \xrightarrow[\imath_2]{} & P^{\frac{n+2}{2}}(\mathbf{C}) \end{array}$$

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Generic simplicity of the eigenvalues of the drifting Laplacian on compact Riemannian manifolds

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1 Introduction

In 1976, K. Uhlenbeck showed that for a class of second order elliptic operators L_b defined on a compact Riemannian manifold M, the following generic property holds: all eigenvalues are simple, that is, multiplicity 1.
As an application, all eigenvalues of Δ_g are simple, for a generic metric g on M.
In this poster, we want to show that this same generic property holds for the operator drifting Laplacian

$$\Delta + < \nabla\eta, \nabla > \qquad (1)$$

for a generic drifting function $\eta \in B = \{\eta : M \to \mathbb{R}, \eta > 0\} \subset C^\infty(M)$, that is, there exists a residual set $\Gamma \subset B$ such that for $\eta \in \Gamma$ the operator

$$L_\eta = \Delta + < \nabla\eta, \nabla > \qquad (2)$$

has all eigenvalues simple, that is, multiplicity equal to 1 also.

2 Preliminaries

Definition 1 *A Fredholm operator $F : M \to N$ is a linear map between Banach spaces with closed image and finite dimensional kernel and cokernel.*

Definition 2 *The index of a Fredholm operator is the dimension of the kernel minus the cokernel*

Definition 3 *A Fredholm map is a differentiable map between Banach manifolds which has a Fredholm operator as derivative at every point.*

Definition 4 *By a residual set we mean a set of second category.*

Theorem 1 (Sard-Smale) *Let $F : M \to N$ be a Fredholm map between separable Banach manifolds. If F is C^r for $r > indexF$, then the regular values of F form a residual set in N.*

Definition 5 (Transversality) *A map $f : M \to N$ is transversal to a submanifold $Z \subset N$, if for all $x \in M$ with $f(x) \in Z$*

$$(df)_x(T_xM) + T_{f(x)}Z = T_{f(x)}N.$$

Theorem 2 (Transversality Theorem 1) *Let $\varphi : H \times B \to E$ be a C^k map, H, B and E Banach manifolds with H e E separable. If 0 is a regular value of φ and $\varphi_b = \varphi(, b)$ is a Fredholm map of index $< k$, then the set $\{b \in B : 0$ is a regular value of $\varphi_b\}$ is residual in B.*

3 Basic theory of elliptic operators

It is a result of basic theory of elliptic operators that if the coefficients of an elliptic operator L are C^k then:

1. The maps $(L + \lambda I) : H^p_k(M) \cap H^p_{1,0}(M) \to H^p_{k-2}(M)$, $k \geq 1$, are Fredholm of index zero;
2. The eigenfunctions of L are solutions $u \in H_{1,0}(M)$ of $(L + \lambda I)u = 0$, and by regularity theory they also will be in $H^p_k(M)$;
3. The eigenspaces are finite dimensional;
4. If L is self-adjoint then the eigenfunctions span $L^2(M)$.

4 The drifting Laplacian

Let (M, g) be a connected compact Riemannian manifold, provided with a weighted measure $dm = \epsilon^{-\eta}dM$, where dM is the original volume form of M, that is, dM is the volume form associated to metric g.
The function η belongs to the open set $B = \{\eta : M \to \mathbb{R}, \eta > 0\} \subset C^\infty(M)$.
We consider the following second order ellpitic operators:

$$\eta \to L_\eta = \Delta + < \nabla\eta, \nabla > \qquad (3)$$

where $\Delta = \Delta_g$, $\nabla = \nabla^g$ e $<.,.>=<.,.>_g$.
L_η is called $\eta - Laplacian$ or *drifting Laplacian*, and η is called *drifting function*.
We also highlight the following properties of the drifting Laplacian that are of extreme importance for the remainder of this work:

(i) $\eta - Laplacian$ is formally self-adjoint on Hilbert space $L^2(M, dm)$;
(ii) $\eta - Laplacian$ is elliptic.

5 Auxiliary Lemmas

We consider the unitary sphere:

$$S^p_k = \{u \in H^p_k(M) \cap H_{1,0}(M) : \int_M u^2 dm = 1\}$$

and the following map $\varphi : S^p_k \times \mathbb{R} \times B \to H^p_{k-2}(M)$ given by

$$\varphi(u, \lambda, \eta) = (L_\eta + \lambda I)u$$

from where we can consider the following map $\varphi_\eta = \varphi(.,.,\eta)$ where η is fixed.

Lemma 1 *φ_η is a Fredholm map of index zero.*

Lemma 2 *$(u, \lambda, \eta) \in \varphi^{-1}(0)$ if and only if u is an eigenfunction of L_η with eigenvalue λ. The u lies in a one dimensional eigenspace if and only if u is a regular point of φ_η.*

Lemma 3 *L_η has one-dimensional eigenspaces if and only if 0 is a regular value of φ_η.*

Lemma 4 *0 is a regular value of φ.*

6 Main Result

Theorem 3 *The set $\{\eta \in B : L_\eta$ has one-dimensional eigenspaces $\}$ is residual in B. In other words, the eigenvalues of L_η are generically simple.*

Proof: By Lemma 1, φ_η is Fredholm of index zero and by Lemma 4 0 is regular value of φ. Then, by Transversality Theorem 1

$$\{\eta \in B : 0 \text{ é valor regular de } \varphi_\eta\}$$

is residual in B. Since, by Lemma 3, 0 is regular value of φ_η if and only if L_η has one-dimensional eigenspaces, then the set

$$\{\eta \in B : L_\eta \text{ has one-dimensional eigenspaces } \}$$

is residual in B.

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Eigenstructure of Laplace operator on the equilateral triangle and its relation with hexagonal flat torus

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Abstract

This work is a study of Laplace operator and its eigenstructure over the equilateral triangle under Dirichlet boundary condition. The strategy is to solve the corresponding partial differential equation making a specific change of coordinate system in order to apply the method of separation of variables. It also is made a link between the equilateral triangle and the hexagonal flat torus. We follow closely the Mcartin work [8].

1 Introduction

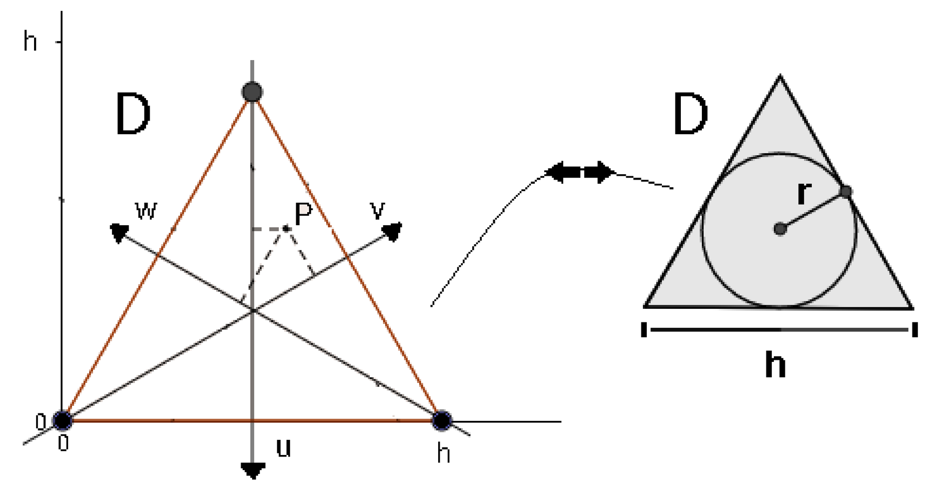
The study of Laplace operator on manifolds has several applications on physics, engineering and other fields. For example, the heat equation, variation rates, informations about the topology of surfaces and others geometric aspects.

Even in 2-dimensional manifolds such as the equilateral triangle, the analysis of the Laplacian spectrum is not simple and it is necessary to developed specific methods to study it.

We consider the following partial differential equation with Dirichlet boundary condition:

$$(a)\ \Delta T + k^2 T = 0, \quad (b)\ T \upharpoonleft_{\partial D} = 0 \tag{1}$$

where Δ denotes the usual Laplacian of euclidean spaces, the numbers k^2 are called eigenvalues and the real functions T defined over the equilateral triangle D are called eigenfunctions. D has side h and inner radius r as identified bellow:



2 Discussion and Results

We can relate the Cartesian coordinates $(x, y) \in D$ to triples (u, v, w) given by the relations $u = r - y$, $v = \frac{\sqrt{3}}{2}\left(x - \frac{h}{2}\right) + \frac{1}{2}(y - r)$ and $w = \frac{\sqrt{3}}{2}\left(\frac{h}{2} - x\right) + \frac{1}{2}(y - r)$ in order to obtain a new coordinate system (ξ, η) which has origin at the center of D. We define $\xi = u$ and $\eta = v - w$.

Once we have new coordinates, we can rewrite equation (1)(a) as:

$$\frac{\partial^2}{\partial \xi^2} T + 3 \frac{\partial^2}{\partial \eta^2} T + k^2 T = 0 \tag{2}$$

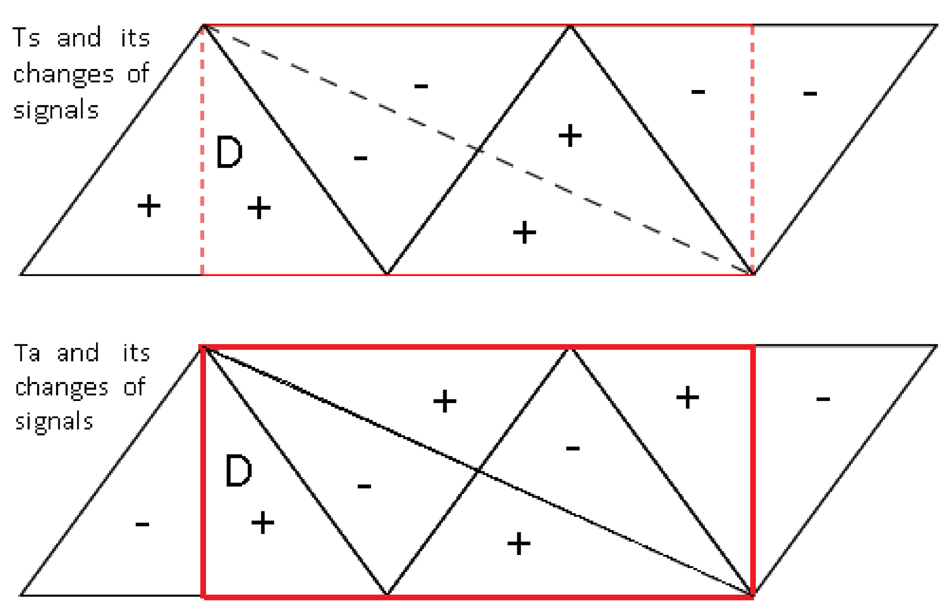
We will denote $T(\xi, \eta)$ as the eigenfunction T expressed on the system (ξ, η). Then we claim that $T(\xi, \eta)$ is eigenfunction if and only if $T_s(\xi, \eta) = \frac{T(\xi,\eta) + T(\xi,-\eta)}{2}$ and $T_a(\xi, \eta) = \frac{T(\xi,\eta) - T(\xi,-\eta)}{2}$ are eigenfunctions (note that $T = T_s + T_a$, $T_s(\xi, \eta) = T_s(\xi, -\eta)$ and $T_a(\xi, \eta) = -T_a(\xi, -\eta)$). They are symmetric and antisymmetric functions considered over the u axis, respectively. Therefore it is possible to study T_s and T_a individually. The next step is to apply the separable variables method on T i.e. $T(\xi, \eta) = f(\xi)g(\eta)$. It's the same that suppose $T_s(\xi, \eta) = f(\xi)g_s(\eta)$ and $T_a(\xi, \eta) = f(\xi)g_a(\eta)$, where g_s and g_a are symmetric and antisymmetric parts of g, respectively. Applying it to the equation (2), we obtain two ODE's. So taking account the Dirichlet boundary condition we get the following solutions T_s and T_a:

$$\begin{aligned} T_s^{m,n}(u, v, w) &= \sin\left(\tfrac{\pi l}{3r}(u + 2r)\right) \cos\left(\tfrac{\pi(m-n)}{9r}(v - w)\right) \\ &+ \sin\left(\tfrac{\pi m}{3r}(u + 2r)\right) \cos\left(\tfrac{\pi(n-l)}{9r}(v - w)\right) \\ &+ \sin\left(\tfrac{\pi n}{3r}(u + 2r)\right) \cos\left(\tfrac{\pi(l-m)}{9r}(v - w)\right) \\ T_a^{m,n}(u, v, w) &= \sin\left(\tfrac{\pi l}{3r}(u + 2r)\right) \sin\left(\tfrac{\pi(m-n)}{9r}(v - w)\right) \\ &+ \sin\left(\tfrac{\pi m}{3r}(u + 2r)\right) \sin\left(\tfrac{\pi(n-l)}{9r}(v - w)\right) \\ &+ \sin\left(\tfrac{\pi n}{3r}(u + 2r)\right) \sin\left(\tfrac{\pi(l-m)}{9r}(v - w)\right) \end{aligned}$$

$$k^2_{m,n} = \tfrac{2}{27}\left[\tfrac{\pi}{r}\right]^2 (l^2 + m^2 + n^2) = \tfrac{4}{27}\left[\tfrac{\pi}{r}\right]^2 (m^2 + mn + n^2)$$

for $m, n \in \mathbb{Z}$, satisfying $l + m + n = 0$; $|l| \neq |m| \neq |n| \neq |l|$ and in a such way that any eigenfunction is linear combination of those.

Another important result is that if we consider $\tilde{D}$ an equilateral triangle obtained by reflection over one of the sides of D, its eigenfunctions are almost the same. They differ each other only in a change of signal. Then, we can construct rectangles and parallelograms formed by equilateral triangles identifying the changed signals on T_s and T_a, as in the following pictures:

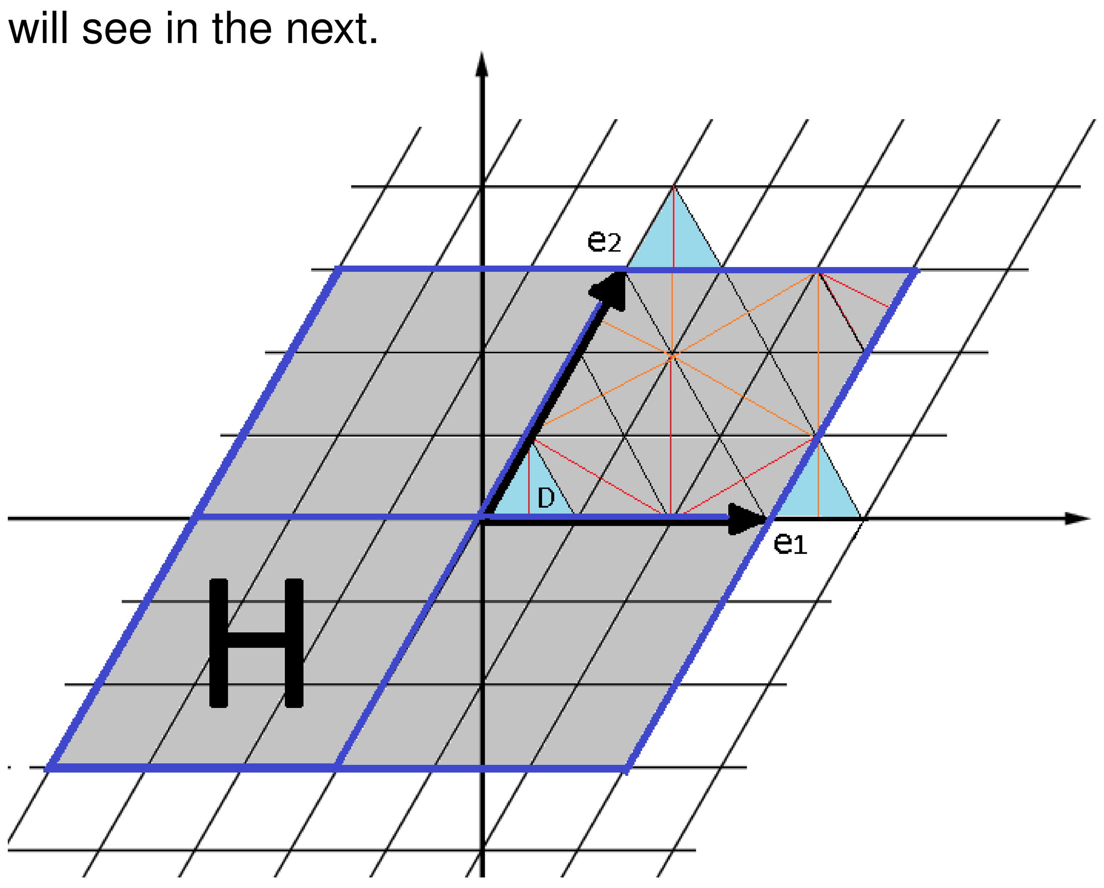


In the solid lines at the pictures we have the annulment of the eigenfunction while in the dashed lines we have the annulment of its normal derivative (this fact occurs specially on T_s eigenfunctions because its normal derivative has a similar behaviour to the T_a eigenfunctions).

This is a way to relate the solutions of the triangle to the solutions of a rectangle containing specific inner lines and boundary conditions.

By using the results showed above we can imply that we get a completeness of eigenfunctions at the triangle problem.

If we repeat this reflection process throughout the whole plan, it forms a lattice by parallelograms which are congruent to H (see next picture). So we can extend the eigenfunctions of the triangle to the whole plane as eigenfunctions of the hexagonal torus as we will see in the next.



Let $\Gamma := \mathbb{Z}e_1 + \mathbb{Z}e_2$ an additive group and R the equivalence relation over $\mathbb{R}^2$: xRy whenever exists $g \in \Gamma$ such that $x = y + g$. Therefore, the eigenfunctions are constant in each equivalence class and the relation R identifies all the parallelograms congruent to H as a unique parallelogram. In other words, each parallelogram congruent to H contains one representative element of each equivalence class.

The quotient $\mathbb{R}^2/\Gamma$ formed by this equivalence relation is known as hexagonal flat torus which has structure of a Riemann manifold and is locally isometric to $\mathbb{R}^2$. Hence it is possible to understand the notion of Laplacian, eigenfunctions and eigenvalues on the flat torus in a similar way comparing to euclidean domains.

The link between $\mathbb{R}^2/\Gamma$ and D is that every eigenfunction on D can be reflected throughout the parallelogram H with the same inner line conditions. Therefore they are related to eigenfunctions on $\mathbb{R}^2/\Gamma$. On the other hand, if we impose on $\mathbb{R}^2/\Gamma$ these inner lines to its corresponding conditions then we obtain eigenfunctions related to eigenfunctions on D. In this sense, we can see that the triangle spectrum is a subset of the spectrum of the torus.

3 Concluding remarks

The main goal of this work was to show a relation between the different settings exploring the symmetries between them. We exhibited a usual constructive process that obtain eigenvalues and eigenfunctions on equilateral triangle, establishing a link to a particular flat torus. We hope to apply similar procedure in non-euclidean manifolds.

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THE MEAN CURVATURE FLOW BY PARALLEL HYPERSURFACES

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Abstract

It is shown that a hypersurface of a space form is the initial data for a solution to the mean curvature flow by parallel hypersurfaces if, and only if, it is isoparametric. By solving an ordinary differential equation, explicit solutions are given for all isoparametric hypersurfaces of space forms. In particular, for such hypersurfaces of the sphere, the exact collapsing time into a focal submanifold is given in terms of its dimension, the principal curvatures and their multiplicities.

1. Basic Concepts and Notation

In what follows, $\mathbb{M}^{n+1}(\overline{\kappa})$ will be a space form of constant sectional curvature $\overline{\kappa}\in\{-1,0,1\}$, i. e., $\mathbb{R}^{n+1}$ if $\overline{\kappa}=0$, $\mathbb{S}^{n+1}\subset\mathbb{R}^{n+2}$ if $\overline{\kappa}=1$ and $\mathbb{H}^{n+1}\subset\mathbb{L}^{n+2}$ if $\overline{\kappa}=-1$, where $\mathbb{L}^{n+2}$ is the Lorentzian space. We consider $F:M^n\to\mathbb{M}^{n+1}(\overline{\kappa})$ a hypersurface immersed in the space form $\mathbb{M}^{n+1}(\overline{\kappa})$, with the induced metric $g(v,w)=\langle dF(v),dF(w)\rangle$, for all vector fields $v,\ w$ tangent to M. If $F(M)$ is oriented and N is a unit normal vector field, the *second fundamental form* of $F(M)$ is given by $h(v,w)=-\langle dN(v),dF(w)\rangle$. Let $e_1,...,e_n$ be orthonormal vector fields which are principal directions and let $\kappa_1,...,\kappa_n$, be the *principal curvatures* of $F(M)$ i.e., $g(e_\imath,e_\jmath)=\delta_{\imath\jmath}$ and $h(e_\imath,e_\jmath)=\kappa_\imath\delta_{\imath\jmath}$, for $1\le\imath,\jmath\le n$. We will denote the *mean curvature* by $H=\sum_{i=1}^n\kappa_\imath$. When the principal curvatures $\kappa_\imath$ of $F(M)$ do not depend on x, for all $\imath=1,...,n$, we say that $F(M)$ is an *isoparametric* hypersurface. From now on, we consider connected hypersurfaces.

Let $F:M^n\to\mathbb{M}^{n+1}(\overline{\kappa})$ be an oriented hypersurface with a unit normal vector field N. A one parameter family of hypersurfaces $\widehat{F}:M^n\times I\to\mathbb{M}^{n+1}(\overline{\kappa})$, $I\subset\mathbb{R}$, is a solution to the *mean curvature flow* (MCF) with initial condition F, if

$$\begin{cases}\dfrac{\partial}{\partial t}\widehat{F}(x,t)=\widehat{H}(x,t)\widehat{N}(x,t),\\ \widehat{F}(x,0)=F(x),\end{cases}\qquad(1)$$

where $\widehat{H}^t(.)=\widehat{H}(.,t)=\sum_{i=1}^n\widehat{k}_i^t$ is the mean curvature and $\widehat{N}^t(.)=\widehat{N}(.,t)$ is a unit normal vector field of $\widehat{F}^t(M)$. When F is a minimal hypersurface i.e. $H=0$, then the family $\widehat{F}(t,x)=F(x)$ gives a trivial solution to the MCF.

In this paper, we consider a special type of solution to the MCF by imposing that the hypersurfaces $\widehat{F}^t$ to be parallel. We first introduce the following notation

$$c(\xi)=\begin{cases}1, & \text{if } \overline{\kappa}=0,\\ \cos(\xi), & \text{if } \overline{\kappa}=1,\\ \cosh(\xi), & \text{if } \overline{\kappa}=-1,\end{cases}\quad\text{and } s(\xi)=\begin{cases}\xi, & \text{if } \overline{\kappa}=0\\ \sin(\xi), & \text{if } \overline{\kappa}=1,\\ \sinh(\xi), & \text{if } \overline{\kappa}=-1,\end{cases}\qquad(2)$$

Definition 1 Let $\widehat{F}:M^n\times I\to\mathbb{M}^{n+1}(\overline{\kappa})$ be a solution to the mean curvature flow in $\mathbb{M}^{n+1}(\overline{\kappa})$ with initial condition $F:M^n\to\mathbb{M}^{n+1}(\overline{\kappa})$. We say $\widehat{F}$ is a *solution to the mean curvature flow by parallel hypersurfaces* if there is a function $\xi:I\to\mathbb{R}$, such that $\xi(0)=0$ and

$$\widehat{F}^t(x)=c\big(\xi(t)\big)F(x)+s\big(\xi(t)\big)N(x),\qquad(3)$$

for all $t\in I$, where $c:\mathbb{R}\to\mathbb{R}$ and $s:\mathbb{R}\to\mathbb{R}$ are the functions defined in (2).

2. Main result

Theorem 1 *Let $F:M^n\to\mathbb{M}^{n+1}(\overline{\kappa})$ be a hypersurface in a space form $\mathbb{M}^{n+1}(\overline{\kappa})$. Then $F(M)$ is the initial data of a solution to the MCF by parallel hypersurfaces if, and only if, $F(M)$ is an isoparametric hypersurface.*

As a consequence of the proof of this theorem, given in Section 3, one obtains the MCF of the isoparametric hypersurfaces of space forms by solving an ordinary differential equation. Namely, we prove the following

Corollary 1 *Let $F:M^n\to\mathbb{M}^{n+1}(\overline{\kappa})$ be an isoparametric hypersurface, with unit normal vector field N and principal curvatures $\kappa_\imath$. Then the solution to the MCF with initial data F is given by (3) where s and c are the functions defined in (2) and $\xi(t)$ is the solution of*

$$\xi'(t)=\sum_{\imath=1}^n\frac{\overline{\kappa}s\left(\xi\left(t\right)\right)+\kappa_\imath c\left(\xi\left(t\right)\right)}{c\left(\xi\left(t\right)\right)-\kappa_\imath s\left(\xi\left(t\right)\right)},\qquad\xi(0)=0.$$

As an application, of Corollary 1, we obtain explicitly the MCF by parallel hypersurfaces of the isoparametric hypersurfaces of $\mathbb{R}^{n+1}$ and of $\mathbb{H}^{n+1}$ in Propositions 1-3. The MCFs for non minimal hypersurface of $\mathbb{S}^{n+1}$ with g distinct curvatures are given in Propositions 4-8.

We without the result for isoparametric hypersurfaces of the Euclidean space since it is well known.

3. MCF of Isoparametric Hypersurfaces of the Hyperbolic Space

Proposition 1 *Let $F:\mathbb{R}^n\to\mathbb{H}^{n+1}\subset\mathbb{L}^{n+2}$ be the immersion of a horosphere in the hyperbolic space, with unit normal vector field N and all principal curvatures $\kappa=\pm1$. Then, the solution to the MCF with initial data F is*

$$\widehat{F}^t(x)=\cosh(nt)F(x)+\kappa\sinh(nt)N(x),\qquad(4)$$

for all $t\in\mathbb{R}$. Moreover, $\widehat{F}^t(\mathbb{R}^n)$ is a horosphere for all $t\in\mathbb{R}$.

Proposition 2 *Let $F:M^n\to\mathbb{H}^{n+1}\subset\mathbb{L}^{n+2}$ be the immersion of a totally umbilic hypersurface in the hyperbolic space, with unit normal vector field N and all principal curvatures equal to κ where $\kappa\notin\{0,\pm1\}$. Then, the solution to the MCF with initial condition $F(M)$ is given by (3) where*

$$\cosh(\xi(t))=\frac{\kappa^2e^{-nt}-\sqrt{1-\kappa^2+\kappa^2e^{-2nt}}}{\kappa^2-1}$$

and

$$\sinh\xi(t)=\frac{\kappa e^{-nt}-\kappa\sqrt{1-\kappa^2+\kappa^2e^{-2nt}}}{\kappa^2-1}.$$

1. *If $0<|\kappa|<1$, then $\widehat{F}^t$ is defined for $t\in\mathbb{R}$ and it converges to a totally geodesic n-dimensional manifold when $t\to+\infty$.*
2. *If $|\kappa|>1$ then $\widehat{F}^t$ is defined for $t\in(-\infty,t^*)$, where $t^*=\frac{1}{2n}\ln\left(\frac{\kappa^2}{\kappa^2-1}\right)$ and it collapses to a point at t^*.*

Proposition 3 *Let $F:\mathbb{S}^{m_1}\times\mathbb{H}^{m_2}\to\mathbb{H}^{n+1}\subset\mathbb{L}^{n+2}$ be the immersion of a cylinder in the hyperbolic space, with m_1 principal curvatures equal to $\kappa_1>1$ and m_2 principal curvatures equal to κ_2, such that $\kappa_1\kappa_2=1$. Then the solution to the MCF with initial condition F, is given by (3) where*

$$\cosh(2\xi(t))=\frac{a\ell(t)-2\sqrt{q(t)}}{a^2-4},\quad\sinh(2\xi(t))=\frac{2\ell(t)-a\sqrt{q(t)}}{a^2-4}.$$

$$q(t)=\ell^2(t)-a^2+4;\quad\ell(t)=(a-b)e^{-2nt}+b,$$

$$a=\kappa_1+\kappa_2\quad\text{and}\quad b=-\frac{m_1-m_2}{n}(\kappa_1-\kappa_2).$$

$\widehat{F}^t$ is defined for all $t\in(-\infty,t^*)$ where $t^*=\frac{1}{2n}\ln\frac{m_1\kappa_1^2+m_2}{m_1(\kappa_1^2-1)}$. and it collapses into an m_2-dimensional focal submanifold at t^*.

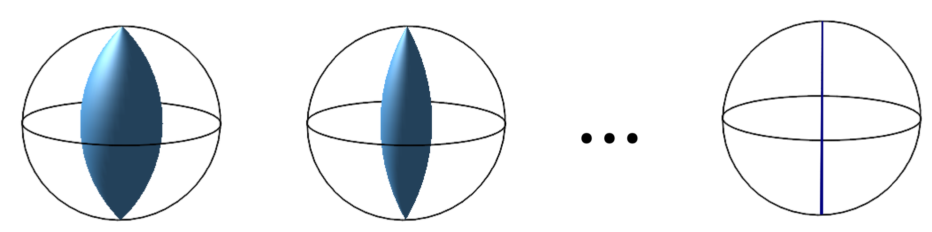

Figure 1: *MCF of Hyperbolic Cylinder*

4. MCF of Isoparametric Hypersurfaces of the Sphere

We will now consider the isoparametric hypersurfaces of the sphere. Munzner [4] showed that the number g of distinct principal curvatures, for an isoparametric hypersurface $M^n\subset\mathbb{S}^{n+1}$, is restricted to be $1,\ 2,\ 3,\ 4$ or 6.

Proposition 4 *Let $F:M^n\to\mathbb{S}^{n+1}\subset\mathbb{R}^{n+2}$ be the immersion of a totally umbilic hypersurface in $\mathbb{S}^{n+1}$, with unit normal vector field N and all principal curvatures are equal to $\kappa\neq0$. Then the solution to the MCF with F as initial data, is given by (3) where*

$$\cos(\xi(t))=\frac{\kappa^2e^{nt}+\sqrt{q(t)}}{\kappa^2+1},\qquad\sin(\xi(t))=\frac{\kappa e^{nt}-\kappa\sqrt{q(t)}}{\kappa^2+1},$$

and

$$q(t)=\kappa^2+1-\kappa^2e^{2nt}.$$

$\widehat{F}^t$ is defined for all $t\in(-\infty,t^*)$ where $t^*=\frac{1}{2n}\ln\left(\frac{\kappa^2+1}{\kappa^2}\right)$ and it collapses to a point at t^*.

Proposition 5 *Let $F:\mathbb{S}^l_{r_1}\times\mathbb{S}^{n-l}_{r_2}\to\mathbb{S}^{n+1}\subset R^{n+2}$ be an isoparametric hypersurface in $\mathbb{S}^{n+1}$, with two distinct principal curvatures κ_1 and κ_2 with multiplicities l and $n-l$ respectively. Then $\kappa_1\kappa_2=-1$ and assuming the immersion is not minimal, we may consider $\kappa_1>\sqrt{(n-l)/l}>1$. The solution to the MCF with initial data F, is $\widehat{F}^t$ given by (3) where*

$$\cos(2\xi(t))=\frac{a\,q(t)+2\sqrt{a^2+4-q^2(t)}}{a^2+4},$$

$$\sin(2\xi(t))=\frac{2q(t)-a\sqrt{a^2+4-q^2(t)}}{a^2+4}$$

and

$$a=\kappa_1+\kappa_2,\ b=-\frac{n-2l}{n}(\kappa_1-\kappa_2),\ q(t)=(a+b)e^{2nt}-b.$$

$\widehat{F}^t$ is defined for all $t\in[0,t^*)$, where $t^*=\frac{1}{2n}\ln\left(\frac{l(\kappa_1^2+1)}{l(\kappa_1^2+1)-n}\right)$ and it collapses into an $(n-l)$-dimensional focal submanifold of F at t^*.

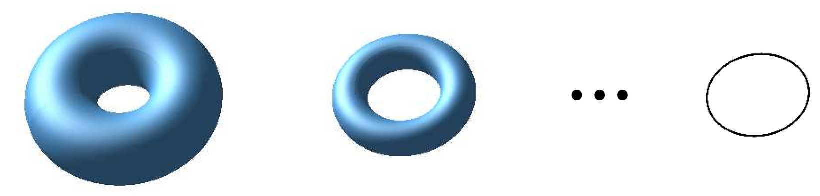

Figure 2: *MCF of Hopf Torus*

Proposition 6 *Let $F:M^n\to\mathbb{S}^{n+1}\subset R^{n+2}$ be a non minimal isoparametric hypersurface in $\mathbb{S}^{n+1}$, with unit normal vector field N and three distinct principal curvatures. $\kappa_1,\kappa_2,\kappa_3$. Then all the principal curvatures have the same multiplicity m, where $m=1,2,4$ or 8, i.e. $n=3m$. The solution to the MCF with initial data F, is $\widehat{F}^t$ given by (3) where*

$$\cos(3\xi(t))=\frac{a^2e^{9mt}+3\sqrt{q(t)}}{a^2+9},\ \sin(3\xi(t))=\frac{a(3e^{9mt}-\sqrt{q(t)})}{a^2+9},$$

$$a=\kappa_1+\kappa_2+\kappa_3=\frac{3\kappa_1(\kappa_1^2-3)}{3\kappa_1^2-1},\ q(t)=a^2+9-a^2e^{18mt}.$$

$\widehat{F}^t$ is defined for all $t\in[0,t^*)$, where $t^*=\dfrac{1}{18m}\ln\left(1+\dfrac{9}{a^2}\right)$ and it collapses into a $2m$-dimensional focal submanifold of $F(M)$ at t^*.

Proposition 7 *Let $F:M^n\to\mathbb{S}^{n+1}\subset R^{n+2}$ be a non minimal isoparametric hypersurface of $\mathbb{S}^{n+1}$, with unit normal vector field N and four distinct principal curvatures κ_j, with multiplicities m_j, $j=1,2,3,4$. Then we may consider*

$$\kappa_1>1,\qquad\kappa_2=\frac{\kappa_1-1}{\kappa_1+1},\qquad\kappa_3=\frac{-1}{\kappa_1},\qquad\kappa_4=\frac{-(\kappa_1+1)}{\kappa_1-1},$$

where the multiplicities m_j satisfy $m_1=m_3$ and $m_2=m_4$, $n=2(m_1+m_2)$. The solution to the MCF with initial data F, is $\widehat{F}^t$ given by (3) where

$$\cos(4\xi(t))=\frac{aq(t)+4\sqrt{a^2+16-q^2(t)}}{a^2+16},$$

$$\sin(4\xi(t))=\frac{4q(t)-a\sqrt{a^2+16-q^2(t)}}{a^2+16},$$

$$a=\sum_{j=1}^4\kappa_j=\frac{\kappa_1^4-6\kappa_1^2+1}{\kappa1(\kappa^2-1)},b=\frac{2(m_1-m_2)(\kappa_1^2+1)^2}{n\kappa_1(\kappa_1^2-1)}$$

$$\text{and}\qquad q(t)=(a+b)e^{4nt}-b.$$

Moreover, $\widehat{F}^t$ is defined for all $t\in[0,t^*)$, where $t^*=\dfrac{1}{4n}\ln\left(\dfrac{b+\sqrt{a^2+16}}{a+b}\right)$ and it collapses into (m_1+2m_2)-dimensional focal submanifold of $F(M)$.

Proposition 8 *Let $F:M^n\to\mathbb{S}^{n+1}\subset R^{n+2}$ be a non minimal isoparametric hypersurface in $\mathbb{S}^{n+1}$, with unit normal vector field N and six distinct principal curvatures κ_j, $j=1,...,6$. Then $n=6m$, where $m=1,2$, and we may consider $\kappa_1>\sqrt{3}$. The solution to the MCF with initial data F, is $\widehat{F}^t$ given by (3) where*

$$\cos(6\xi(t))=\frac{a^2e^{36mt}+6\sqrt{q(t)}}{a^2+36},\ \sin(6\xi(t))=\frac{a\left(6e^{36mt}-\sqrt{q(t)}\right)}{a^2+36},$$

where

$$a=\sum_{j=1}^6\kappa_j\text{ and }q(t)=a^2+36-a^2e^{72mt}.$$

which is defined for all $t\in[0,t^*)$, where $t^*=\dfrac{1}{72m}\ln\left(1+\dfrac{36}{a^2}\right)$. Moreover, the solution collapses into a $5m$-dimensional focal submanifold of $F(M)$ at t^*.

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SOLUTIONS FOR EQUATIONS INVOLVING THE INFINITY-LAPLACIAN

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0. ABSTRACT

In this work we study a parabolic equation involving the infinity-Laplacian from the point of view of Lie symmetries. We consider its radial form and, by using the method of separation of variables, we derive another one involving the Aronsson's nonlinear operator. All Lie point symmetries of these equations are found and by using the invariance group we are able to find exact solutions for the considered equations, some of them expressed in terms of the hypergeometric function.

1. INTRODUCTION/MOTIVATION

Let $D \subseteq \mathbb{R}^2$ be a convex region and $u \in C^1(D)\cap C^0(\overline{D})$. For any $n \in \mathbb{N}$, let

$$I_n(u) := \left(\int_D |\nabla u|^{2n}\right)^{\frac{1}{2n}}, \qquad (1)$$

where $\nabla u := (u_x, u_y)$. By supposing that u is a solution of the problem $\min I_n(u)$ then u satisfies the equation

$$|\nabla u|^{2(n-2)}\left[\frac{1}{2(n-1)}|\nabla u|^2(u_{xx}+u_{yy}) + u_x^2u_{xx} + 2u_xu_yu_{xy} + u_y^2u_{yy}\right] = 0. \qquad (2)$$

If $\nabla u \neq 0$ and n tends to infinity, equation (2) becomes

$$u_x^2u_{xx} + 2u_xu_yu_{xy} + u_y^2u_{yy} = 0, \qquad (3)$$

which was first derived by Aronsson [2]. Since then, such equation has been subject of intense research, see [3], [4], [11], [12], [15], [16] and references therein.

Some important results of [3]: $A(\Phi) = \Phi_x^2\Phi_{xx} + 2\Phi_x\Phi_y\Phi_{xy} + \Phi_y^2\Phi_{yy} = \frac{1}{2}grad\{(grad\,\Phi)^2\}\cdot grad\,\Phi$. *The condition* $A(\Phi) = 0$ *means that* $|grad\,\Phi|$ *is constant along every trajectory of the vector field grad* Φ *(called streamlines). If* u *is a solution of* $A(u) = 0$ *the curvature of a streamline is* $\frac{\pm|grad(|grad\,u|)|}{|grad\,u|}$.

(Lemma 1.) Let $u(x,y)$ *satisfy* $A(u) = 0$ *in a domain* D *and let grad* $u \neq 0$ *in* D. *If* C *is a streamline of* u *in* D, *then or the curvature of* C *is* $\neq 0$ *at all points of* C *or* C *is a straight line. Consequently, the streamlines of* u *are convex curves and straight lines.*

(Theorem 3.) Let $\Phi(x,y)$ *satisfy* $A(\Phi) = 0$, *and grad* $\Phi \neq 0$. *Consider the surface* $S : z = \Phi(x,y)$ *and the projections on this surface of the streamlines of* $\Phi(x,y)$. *These image curves are both asymptotic curves on* S *and helices with a common axis. namely the* z*-axis.*

Equation (3) is known as infinity Laplacian equation, Aronsson's Euler equation and Aronsson equation. Its left side is commonly written as $\triangle_\infty u = u_x^2u_{xx} + 2u_xu_yu_{xy} + u_y^2u_{yy}$ and the operator $\triangle_\infty$ is called infinity-Laplacian. Some authors have been considering a parabolic equation associated with the infinity-Laplacian. Namely, they have been studying n-dimensional versions of the equation

$$u_t = u_x^2u_{xx} + 2u_xu_yu_{xy} + u_y^2u_{yy} \qquad (4)$$

see [14] and [20]. In the paper [12] the Lie point symmetries of (3) were studied. In addition, some group invariant solutions to (3) were also obtained. Thus, inspired by the previous work on symmetry analysis of the Eq. (3), in this work we apply the same approach to (4) in order to

- find the Lie point symmetries of (4);
- construct the symmetry Lie algebra associated to the vector fields which generate the Lie point symmetries of (4);
- construct the adjoint representation of the Lie algebra associated to Eq. (4);
- construct exact solutions of the considered equation.

2. THEORY

Let $x \in M \subseteq \mathbb{R}^n$, M open, $u : M \to \mathbb{R}$. A Lie point symmetry generator of a PDE $F = F(x, u, \partial u, \cdots, \partial^m u) = 0$ of order m is a vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta(x,u)\frac{\partial}{\partial u}$$

on $M \times \mathbb{R}$ such that $X^{(m)}F = 0$ when $F = 0$ and

$$X^{(m)} := X + \eta_i^{(1)}(x,u,\partial u)\frac{\partial}{\partial u_i} + \cdots + \eta^{(m)}_{i_1\cdots i_m}(x,u,\partial u,\cdots,\partial^m u)\frac{\partial}{\partial u_{i_1\cdots i_m}}$$

$$\eta_i^{(1)} := D_i\eta - (D_i\xi^j)u_j,$$

$$\eta^{(j)}_{i_1\cdots i_j} := D_{i_j}\eta^{(j-1)}_{i_1\cdots i_{j-1}} - (D_{i_j}\xi^l)u_{i_1\cdots i_{j-1}l},\ 2 \le j \le m,$$

If $X \in \mathfrak{g}$ (symmetry Lie algebra associated to the vector fields which generate the Lie point symmetries of the equations we are interested) then it generates a one-parameter subgroup $\{\exp\epsilon X\}$, whose corresponding vector field on $\mathfrak{g}$ is $\mathrm{ad}X = \frac{d}{d\epsilon}\Big|_{\epsilon=0} Ad(\exp(\epsilon X))Y,\ Y \in \mathfrak{g}$, where

$$\mathrm{Ad}(\exp\epsilon X))Y = Y - \epsilon[X,Y] + \epsilon^2[X,[X,Y]] + \ldots.$$

3. RESULTS

In this sense, we shall proceed in the following way: Firstly we obtain the Lie point symmetries of (4), which are given by the following

Theorem 1: The Lie point symmetries of Eq. (4) are generated by the vector fields

$$X_1 = \frac{\partial}{\partial x},\quad X_2 = \frac{\partial}{\partial y},\quad X_3 = \frac{\partial}{\partial t},\quad X_4 = \frac{\partial}{\partial u},\quad X_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$

$$X_6 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u},\quad X_7 = 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

Proof.

$$X = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u}$$

$${\xi^3}_{xx} = 0,\ \eta_{xx} = 0,\ {\xi^3}_{xy} = 0,\ \eta_{xy} = 0,\ {\xi^3}_{xu} = 0,\ {\xi^3}_x = 0,\ \eta_x = 0,$$

$${\xi^3}_{yy} = 0,\ \eta_{yy} = 0,\ {\xi^3}_{yu} = 0,\ {\xi^3}_y = 0,\ \eta_y = 0,\ {\xi^2}_t = 0,\ {\xi^1}_t = 0,\ \eta_t = 0,$$

$${\xi^3}_{uu} = 0,\ {\xi^2}_{uu} = 0,\ {\xi^1}_{uu} = 0,\ {\xi^3}_u = 0,\ {\xi^2}_u = 0,\ {\xi^1}_u = 0,\ {\xi^1}_{yu}+{\xi^2}_{xu} = 0,$$

$${\xi^1}_y + {\xi^2}_x = 0,\ -2\eta_{xu} + {\xi^1}_{xx} = 0,\ -2\eta_{yu} + {\xi^2}_{yy} = 0,$$

$$-\eta_{uu} + 2{\xi^1}_{xu} = 0,\ -\eta_{uu} + 2{\xi^2}_{yu} = 0,\ -\eta_{uu} + {\xi^2}_{yu} + {\xi^1}_{xu} = 0,$$

$$-2\eta_{yu} + 2{\xi^1}_{xy} + {\xi^2}_{xx} = 0,\ {\xi^1}_{yy} - 2\eta_{xu} + 2{\xi^2}_{xy} = 0,\ -2\eta_u - {\xi^3}_t + 4{\xi^1}_x = 0,$$

$$-2\eta_u - {\xi^3}_t + 4{\xi^2}_y = 0,\ -4\eta_u - 2{\xi^3}_t + 4\left({\xi^2}_y + {\xi^1}_x\right) = 0$$

□

So we have the following one parameter groups g_i generated by the vector fields X_i:

$$g_1 : (x,y,t,u) \mapsto (x+\varepsilon,y,t,u),\quad g_2 : (x,y,t,u) \mapsto (x,y+\varepsilon,t,u),$$

$$g_3 : (x,y,t,u) \mapsto (x,y,t+\varepsilon,u),\quad g_4 : (x,y,t,u) \mapsto (x,y,t,u+\varepsilon),$$

$$g_5 : (x,y,t,u) \mapsto (x\cos\varepsilon + y\sin\varepsilon, -x\sin\varepsilon + y\cos\varepsilon, t, u),$$

$$g_6 : (x,y,t,u) \mapsto (e^\varepsilon x, e^\varepsilon y, t, e^{2\varepsilon}u),\quad g_7 : (x,y,t,u) \mapsto (x, y, e^{2\varepsilon}t, e^{-\varepsilon}u).$$

Below we determine the symmetry Lie algebra of Eq. (4):

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	0	0	0	$-X_2$	X_1	0
X_2	0	0	0	0	X_1	X_2	0
X_3	0	0	0	0	0	0	$2X_3$
X_4	0	0	0	0	0	$2X_4$	$-X_4$
X_5	X_2	$-X_1$	0	0	0	0	0
X_6	$-X_1$	$-X_2$	0	$-2X_4$	0	0	0
X_7	0	0	$-2X_3$	X_4	0	0	0

Invariant Solutions: if $u = f(x,y,t)$ is a solution of eq. (4) so are the functions

$$u^{(1)} = f(x+\varepsilon,y,t),\quad u^{(2)} = f(x,y+\varepsilon,t),$$

$$u^{(3)} = f(x,y,t+\varepsilon),\quad u^{(4)} = f(x,y,t) - \varepsilon,$$

$$u^{(5)} = f(x\cos\varepsilon + y\sin\varepsilon, -x\sin\varepsilon + y\cos\varepsilon, t),$$

$$u^{(6)} = e^{-2\varepsilon}f(e^\varepsilon x, e^\varepsilon y, t),\quad u^{(7)} = e^\varepsilon f(x, y, e^{2\varepsilon}t).$$

From the generator X_5 we conclude that the Eq. (4) is invariant under rotations, which allows us to find the radial form of the Eq. (4), that is

$$u_t = u_r^2u_{rr} \qquad (5)$$

This means that if $u = \phi(r,t)$ is a solution of (5) then

$$u(x,y,t) = \phi(\sqrt{x^2+y^2}, t) \qquad (6)$$

is a solution of (4). A natural question is to consider the symmetries of (5). Having this point in mind, our second result can now be announced:

Theorem 2: The Lie point symmetries of equation (5) is generated by the vector fields

$$R_1 = \frac{\partial}{\partial t},\ R_2 = \frac{\partial}{\partial r},\ R_3 = \frac{\partial}{\partial u},\ R_4 = r\frac{\partial}{\partial r} + 2u\frac{\partial}{\partial u},\ R_5 = 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \qquad (7)$$

Our next step is to apply the method of separation of variables to Eq. (4). Then, assume that $u(x,y,t) = T(t)v(x,y)$. By substituting this function into (4), a straightforward calculation shows that the functions v and T satisfy the equations

$$v_x^2v_{xx} + 2v_xv_yv_{xy} + v_y^2v_{yy} = kv(x,y); \qquad (8)$$
$$T'(t) = kT(t)^3,$$

where $k \neq 0$ is a constant. The solution of the last equation is

$$T(t) = \pm\frac{1}{\sqrt{a - 2kt}}, \qquad (9)$$

while our next result is:

Theorem 3: The Lie point symmetries of equation (8) is generated by the vector fields

$$V_1 = \frac{\partial}{\partial x},\ V_2 = \frac{\partial}{\partial y},\ V_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},\ V_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2v\frac{\partial}{\partial v}. \qquad (10)$$

4. ACTUAL STATUS OF THE WORK

From the analysis of the symmetry Lie algebra of equations (4), (5) and (9), and the adjoint representation of their Lie algebra, we find a list of simplified generators. For eq. (4) we have

$$\begin{aligned}
X &= a_4X_4 + a_5X_5 + \frac{1}{2}X_6 + X_7,\\
X &= a_5X_5 + a_6X_6 + X_7,\\
X &= a_3X_3 + a_5X_5 + X_6,\\
X &= a_3X_3 + a_4X_4 + X_5,\\
X &= a_1X_1 + a_2X_2 + a_3X_3 + X_4,\\
X &= a_1X_1 + a_2X_2 + X_3,\\
X &= a_1X_1 + X_2,\\
X &= X_1.
\end{aligned}$$

For eq.(5) we have

$$\begin{aligned}
R &= a_3R_3 + \frac{1}{2}R_4 + R_5, & R &= a_4R_4 + R_5,\\
R &= a_1R_1 + R_4, & R &= a_1R_1 + a_2R_2 + R_3,\\
R &= a_1R_1 + R_2, & R &= R_1.
\end{aligned}$$

For eq.(9) we have

$$\begin{aligned}
V &= a_2V_2 + a_3V_3 + V_4, & V &= V_3,\\
V &= a_1V_1 + a_2V_2 + V_3, & V &= a_1V_1 + V_2,\\
V &= V_1.
\end{aligned}$$

From some of these symmetry generators, we find the following exact solution of equation (4):

$$u(x,y,t) = \frac{c_1}{4}\arctan\left(\frac{\sqrt{x^2+y^2}}{\sqrt{c_1}\sqrt{t} - (x^2+y^2)}\right) + \frac{\sqrt{x^2+y^2}}{4\sqrt[4]{t}}\sqrt{c_1 - \frac{x^2+y^2}{\sqrt{t}}} + c_2,$$

which is a real valued solution on the region $\{(x,y,t); x^2+y^2 \le c_1\sqrt{t}, t > 0\}$, where it is assumed that $c_1 > 0$.

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Intrinsic and extrinsic geometry of hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$

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Abstract
The purpose of this poster is to present relations between intrinsic geometric properties of hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ and their extrinsic geometric structures. Geometric characterizations of conformally flat and radially flat hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ are given by means of their extrinsic geometry. Under suitable conditions on the shape operator, we classify conformally flat hypersurfaces in terms of rotation hypersurfaces. In addition, a close relation between radially flat hypersurfaces and semi-parallel hypersurfaces is established. These results lead to geometric descriptions of hypersurfaces with special intrinsic structures, such as Einstein metrics and Ricci solitons. We consider the geometry of Einstein hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ in order to obtain a complete classification for these hypersurfaces. We classify Ricci solitons $M^n \subset \mathbb{S}^n \times \mathbb{R}$ and $M^n \subset \mathbb{H}^n \times \mathbb{R}$ when the potential vector field is the projection on the tangent space of M^n of the unit vector field tangent to the second factor $\mathbb{R}$.

1 Introduction

A Riemannian manifold is conformally flat if each point has a neighborhood where the metric is conformal to a flat metric, i.e., a metric with zero sectional curvature. The investigation of conformally flat hypersurfaces in Riemannian manifolds, equipped with the induced metric, has been of interest for some time and the relationship between the intrinsic and extrinsic geometry has been considered by taking into account the geometry of the ambient space. When the ambient manifold is also conformally flat, Nishikawa and Maeda [7] have proved that n-dimensional conformally flat hypersurfaces must be ***quasi-umbilical***, i.e., one of the the principal curvatures has multiplicity at least $(n-1)$. In our case, we will see that rotation hypersurfaces are conformally flat. Conversely, conformally flat hypersurfaces, with additional conditions on the shape operator, are given by rotation hypersurfaces (Theorem 1).

On the other hand, radially flat Riemannian manifolds are the manifolds endowed with a smooth vector field X where the sectional curvatures vanish along planes that contain the vector field X. Radially flat Riemannian manifolds constitute an important class of metrics and were considered, for example, in the context of Ricci solitons [10, 9]. In this case, the vector field considered is the potential vector field of the soliton. It turns out that the radially flat condition can be seen, in some sense, as a weakening of the flatness condition and, consequently, more information about such metrics can be obtained. This situation will be seen in our context as a generalization of a result given in [2] for intrinsically flat rotation hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. Our main result regarding radially flat hypersurfaces is a close relation between the geometry of radially flat hypersurfaces and the geometry of semi-parallel hypersurfaces in such spaces (Theorem 2).

A Riemannian manifold is said to be Einstein if its Ricci tensor is a multiple of the metric. We classify the hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ with an Einstein structure. They are given by either a hypersurface with constant sectional curvature or a Riemannian product $M^{n-1} \times \mathbb{R}$, where $M^{n-1} \subset \mathbb{H}^n$ is a totally umbilical, not totally geodesic, hypersurface (Theorem 3).

A natural generalization of Einstein manifolds are the Ricci solitons. A Riemannian manifold (M, g) endowed with a smooth vector field V is a Ricci soliton if

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g = cg, \tag{1.1}$$

where c is a real constant and $\mathcal{L}_V g$ is the Lie derivative of g with respect to V. The vector field V is called potential vector field. The Ricci soliton is called shrinking when $c > 0$, steady when $c = 0$, and expanding when $c < 0$.

As a consequence of Theorem 2, we will see a relation between semi-parallel hypersurfaces and Ricci solitons. We also classify the Ricci solitons as hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ with potential vector field T. In this case, the hypersurface is either an Einstein manifold (in this case, the Ricci soliton will be called trivial) or an open part of a rotation hypersurface (Theorem 4).

2 Statement of the main results

In order to state our results, let us first establish some notation. Let $Q^n(\varepsilon)$ be the unit sphere $\mathbb{S}^n$, if $\varepsilon = 1$, or the hyperbolic space and $\mathbb{H}^n$ if $\varepsilon = -1$ and consider the manifold $Q^n(\varepsilon) \times \mathbb{R}$ given by:

$$\mathbb{S}^n \times \mathbb{R} = \left\{(x_1, \ldots, x_{n+2}) \in \mathbb{E}^{n+2} \,|\, x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\right\},$$
$$\mathbb{H}^n \times \mathbb{R} = \left\{(x_1, \ldots, x_{n+2}) \in \mathbb{L}^{n+2} \,|\, -x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = -1, x_1 > 0\right\},$$

with the metric induced by the ambient space, where $\mathbb{E}^{n+2}$ is the $(n+2)-$dimensional Euclidean space and $\mathbb{L}^{n+2}$ is the $(n+2)-$dimensional Lorentzian space with the canonical metric $ds^2 = -dx_1^2 + dx_2^2 + \ldots + dx_{n+2}^2$.

Let M^n be a hypersurface in $Q^n(\varepsilon) \times \mathbb{R}$ with unit normal N and let $\partial_{x_{n+2}}$ be the coordinate vector field of the second factor $\mathbb{R}$. The orthogonal projection of $\partial_{x_{n+2}}$ onto the tangent space of M^n will be denoted by T. Also, let θ be the angle function between N and $\partial_{x_{n+2}}$. Then we have the following decomposition

$$\partial_{x_{n+2}} = T + \cos\theta N.$$

Definition 1 ([2])**.** Consider a three-dimensional subspace P^3 of $\mathbb{E}^{n+2}$ resp. $\mathbb{L}^{n+2}$, containing the $x_{n+2}-$axis. Then $(Q(\varepsilon)^n \times \mathbb{R}) \cap P^3 = Q^1(\varepsilon) \times \mathbb{R}$. Let P^2 be a two-dimensional subspace of P^3, also through the $x_{n+2}-$axis. Denote by I the group of isometries of $\mathbb{E}^{n+2}$, resp. $\mathbb{L}^{n+2}$, which leave $Q(\varepsilon)^n \times \mathbb{R}$ globally invariant and which leave P^2 pointwise fixed. Finally, let α be a curve in $Q(\varepsilon)^1 \times \mathbb{R}$ which does not intersect P^2. The rotation hypersurface M^n in $Q(\varepsilon)^n \times \mathbb{R}$ with profile curve α and axis P^2 is defined as the $I-$orbit of α.

2.1 Conformally flat hypersurfaces

Theorem 1. *Let M^n, $n > 3$, be a hypersurface in $Q^n(\varepsilon) \times \mathbb{R}$. If M^n is a rotation hypersurface, then M^n is conformally flat. Conversely, if M^n is a conformally flat hypersurface, then either M^n is a totally umbilical hypersurface or its shape operator has two distinct eigenvalues of multiplicity $n-1$ and 1. In this case, M^n is locally congruent to a rotation hypersurface when one of following cases occurs:*

i) M^n is a totally umbilical hypersurface, which is not totally geodesic;

ii) the shape operator of M^n has two distinct eigenvalues λ and μ, of multiplicity 1 and $n-1$, respectively, and the vector field T is a principal direction.

Remark 1. The totally geodesic hypersurfaces in $Q^n(\varepsilon) \times \mathbb{R}$ are completely classified. They are given as an open part of $N^{n-1}(\varepsilon) \times \mathbb{R}$. with $N^{n-1}(\varepsilon)$ a totally geodesic hypersurface of $Q^n(\varepsilon)$, or an open part of $Q^n(\varepsilon) \times \{t_0\}$, for $t_0 \in \mathbb{R}$ (see these results in [12] and [1]). In this case, the totally geodesic hypersurface will be a rotation hypersurface only when $M^n = Q^{n-1}(\varepsilon) \times \mathbb{R}$.

2.2 Radially flat hypersurfaces

A hypersurface M^n in $Q^n(\varepsilon) \times \mathbb{R}$ will be called radially flat if the sectional curvatures along planes containing the vector field T vanish, i.e., $K_M(T, X) = 0$, for any vector field X. In addition, a hypersurface is said to be semi-parallel if the second fundamental form h and the curvature tensor R satisfy $h(R(X,Y)Z,W) + h(R(X,Y)W,Z) = 0$, for every X, Y, Z, W arbitrary vector fields tangent to M^n. Our result will provide an important intrinsic characterization for such hypersurfaces that were classified in [12] and [1]:

Theorem 2. *Let M^n, $n > 3$, be a hypersurface in $Q^n(\varepsilon) \times \mathbb{R}$. If M^n is radially flat and T is a principal direction, for a principal curvature $\lambda \neq 0$, then M^n is a semi-parallel, rotation hypersurface. Conversely, if M^n is a semi-parallel, not totally umbilical hypersurface, then M^n is radially flat.*

Remark 2. When M^n is radially flat and T is a principal direction, with principal curvature $\lambda = 0$, it follows by Gauss equation that $\cos\theta = 0$ and therefore $M^n = \overline{M}^{n-1} \times \mathbb{R}$, where $\overline{M}^{n-1}$ is a hypersurface of $Q^n(\varepsilon)$. It is no longer true, in general, that M^n in this case is semi-parallel. In fact, when M^n takes this form, it will be semi-parallel if, and only if, $\overline{M}^{n-1} \subset Q^n(\varepsilon)$ is semi-parallel (see [12, Theorem 5] and [1, Theorem 4.2]).

On the other hand, when M^n is a semi-parallel, totally umbilical hypersurface in $Q^n(\varepsilon) \times \mathbb{R}$, it does not follow directly that M^n is radially flat. In fact, M^n will be radially flat when:

a) M^n is an open part of the the totally geodesic $\mathbb{S}^{n-1} \times \mathbb{R}$. In fact, we must have the shape operator $S \equiv 0$ and $\cos\theta \equiv 0$.

b) M^n is a hypersurface in $\mathbb{H}^n \times \mathbb{R}$ with $\lambda^2 = \cos^2\theta$. Particularly, if $\lambda \equiv 0$, then M^n is is an open part of a totally geodesic $\mathbb{M}^{n-1} \times \mathbb{R}$, where $\mathbb{M}^{n-1} \subset \mathbb{H}^n$ is a totally geodesic hypersurface.

Let (M, g) a Ricci soltion with potential vector field V. If V is the gradient of a smooth function f, (M, g) is called gradient Ricci soliton and the function f is called potential function. Let us observe that the vector field T is actually a gradient vector field. In fact, if we express a point $p \in M^n$ as $p = (\varphi, h) \in Q^n(\varepsilon) \times \mathbb{R}$, then T is the gradient of the height function h. A gradient Ricci soliton is rigid if it is isometric to a quotient $N \times_\Gamma \mathbb{R}^k$ where N is an Einstein manifold, $f = \frac{c}{2}|x|^2$ on the Euclidean factor and Γ acts freely on N and by orthogonal transformations on $\mathbb{R}^k$ ([9, 10]). In [10, Theorem 1.2], Petersen and Wylie proved that a gradient Ricci soliton $\mathrm{Ric} + \mathrm{Hess}_f = cg$ is rigid if, and only if, it has constant scalar curvature and the sectional curvatures $K(X, \nabla f) = 0$, for any vector field. As a consequence of Theorem 2, we obtain when a hypersurface in $Q^n(\varepsilon) \times \mathbb{R}$ is a rigid gradient Ricci soliton:

Corollary 1. Let M^n, $n > 3$, be a Ricci soliton in $Q^n(\varepsilon) \times \mathbb{R}$ with potential vector field T and constant scalar curvature. If M^n is a rigid gradient Ricci soliton, and T is a principal direction for a principal curvature $\lambda \neq 0$, then M^n is a semi-parallel hypersurface. Conversely, If M^n is a semi-parallel, not totally umbilical hypersurface, then M^n is a rigid gradient Ricci soliton.

2.3 Einstein hypersurfaces and Ricci solitons

Theorem 3. *Let M^n, $n > 3$, be an Einstein hypersurface in $Q^n(\varepsilon) \times \mathbb{R}$. Then M^n is either*

a) a manifold with constant sectional curvature;

b) a product $M^{n-1} \times I$, where $M^{n-1} \subset \mathbb{H}^n$ is a totally umbilical, not totally geodesic, hypersurface.

In [6], hypersurfaces in $Q^n(\varepsilon) \times \mathbb{R}$, $n \geq 3$, with constant sectional curvature were completely classified. Therefore, the classification given by Theorem 3 is complete.

In what follows, a Ricci soliton will be called trivial if it is reduced to an Einstein manifold.

Theorem 4. *Let M^n, $n > 3$, be a Ricci soliton in $Q^n(\varepsilon) \times \mathbb{R}$, with potential vector field T. Then M^n is either*

a) a trivial Ricci soliton.

b) an open part of a rotation hypersurface.

Theorem 3 supplies a classification for the first case. In the second case, it follows by Theorem 1 that the hypersurface is conformally flat. Since T is the gradient of the height function h, we have a conformally flat gradient Ricci soliton and the classification of such solitons can be found in [3].

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A characterization of pseudo-parallel surfaces

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Abstract

In this work we give a characterization of pseudo-parallel surfaces in $\mathbb{S}^n_c \times \mathbb{R}$ and $\mathbb{H}^n_c \times \mathbb{R}$, extending an analogous result by Asperti-Lobos-Mercuri for the pseudo-parallel case in space forms. Moreover, when $n = 3$, we prove that any pseudo-parallel surface has flat normal bundle. We also give examples of pseudo-parallel surfaces which are neither semi-parallel nor pseudo-parallel surfaces in a slice. Finally, when $n \geq 4$ we give examples of pseudo-parallel surfaces with non vanishing normal curvature.

Preliminaries

We use $\mathbb{Q}^n_c$ with $c \neq 0$ to refer the sphere n-space $\mathbb{S}^n_c$ or the hyperbolic n-space $\mathbb{H}^n_c$. An isometric immersion $f : M^m \to \mathbb{Q}^n_c \times \mathbb{R}$ is said to be:

(i) *totally geodesic if* $\alpha = 0$;
(ii) *parallel if* $(\nabla_X \alpha) = 0$;
(iii) *semi-parallel if* $\tilde{R}(X, Y) \cdot \alpha = 0$;
(iv) *pseudo-parallel if* $\tilde{R}(X, Y) \cdot \alpha = \Phi X \wedge Y \cdot \alpha$,

for some smooth function Φ in M^m and any vector fields X, Y in M^m. Here, α denotes the second fundamental form of f and $\tilde{R} = R \oplus R^\perp$ denotes the curvature tensor of $\mathbb{Q}^n_c \times \mathbb{R}$. The concept of pseudo-parallel immersions was first introduced by Asperti-Lobos-Mercuri in [1] as a generalization of semi-parallel immersions. Also in [1], authors investigated pseudo-parallel surfaces in space forms. They obtained the following result:

Theorem (Asperti-Lobos-Mercuri [1])

Let $f : M^2 \to \mathbb{Q}^4_c$ be a surface with $R^\perp \neq 0$. Then f is pseudo-parallel if and only if f is superminimal, that is, f is a minimal immersion and is λ-isotropic.

Also, they classified such surfaces of codimension 3 and codimension 4 with constant pseudo-parallelism function.
We recall that an isometric immersion $f : M^n \to \tilde{M}^m$ is said to be *λ-isotropic* if $\|\alpha^f(X, X)\| = \lambda(p), \quad \forall X \in T_pM, \quad \forall p \in M^n$ with $\|X\| = 1$.
On the other hand, M. Sakaki studied surfaces in $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$, showing in [4] the following theorem:

Theorem (Sakaki [4])

Let $f : M^2 \to \mathbb{Q}^3_c \times \mathbb{R}$ a minimal surface with $c \neq 0$. If f is λ-isotropic at any point, then f is a totally geodesic immersion.

By the Fundamental Equations and pseudo-parallelism condition we get the relations:

$$R^\perp(e_1, e_2)\alpha_{11} = 2(\Phi - K)\alpha_{12}, \qquad (1)$$
$$R^\perp(e_1, e_2)\alpha_{12} = (K - \Phi)(\alpha_{11} - \alpha_{22}), \qquad (2)$$
$$R^\perp(e_1, e_2)\alpha_{22} = 2(K - \Phi)\alpha_{12}, \qquad (3)$$
$$K = c(1 - \|T\|^2) + \langle \alpha_{11}, \alpha_{22} \rangle - \|\alpha_{12}\|^2, \qquad (4)$$

where $\{e_1, e_2\}$ is an orthonormal frame of M^2, $\alpha_{ij} = \alpha(e_i, e_j)$, K is the Gaussian curvature of M^2 and is the tangent part of $\frac{\partial}{\partial t}$, the canonical unit vector field tangent to the second factor of $\mathbb{Q}^n_c \times \mathbb{R}$.

Proposition 1

Let $f : M^2 \to \mathbb{Q}^n_c \times \mathbb{R}$ be a surface with flat normal bundle. Then f is pseudo-parallel immersion.

Proof

Since f has flat normal bundle, by equations (1) to (3) we conclude that f is ϕ-pseudo-parallel by taking $\phi = K$, where K is the Gaussian curvature of M^2.

We have two propositions that is useful to construct examples of pseudo-parallel surfaces.

Proposition 2

Let $f : M^m \to \mathbb{Q}^n_c$ be an isometric immersion and let $j : \mathbb{Q}^n_c \to \mathbb{Q}^n_c \times \mathbb{R}$ be a totally geodesic immersion. If f is ϕ-pseudo-parallel, then $j \circ f$ is ϕ-pseudo-parallel.

Proposition 3

Let $f : M^m \to \mathbb{Q}^n_c \times \mathbb{R}$ be an isometric immersion and let $j : \mathbb{Q}^n_c \times \mathbb{R} \to \mathbb{Q}^{n+l}_c \times \mathbb{R}$ be a totally geodesic immersion. If f is ϕ-pseudo-parallel, then $j \circ f$ is ϕ-pseudo-parallel.

The Result

Theorem A

Let $f : M^2 \to \mathbb{Q}^n_c \times \mathbb{R}$ be a pseudo-parallel surface which does not have flat normal bundle on any open subset of M^2. Then $n \geq 4$, f is λ-isotropic and

$$K > \phi, \qquad (5)$$
$$\lambda^2 = 4K - 3\phi + c(\|T\|^2 - 1) > 0, \qquad (6)$$
$$\|H\|^2 = 3K - 2\phi + c(\|T\|^2 - 1) \geq 0, \qquad (7)$$

where K is the Gaussian curvature, λ is a smooth real-valued function on M^2, H is the mean curvature vector field of f and T is the tangent part $\frac{\partial}{\partial t}$, the canonic unit vector field tangent to the second factor of $\mathbb{Q}^n_c \times \mathbb{R}$.
Conversely, if f is λ-isotropic then f is pseudo-parallel.

Remark

Theorem A extends for $\mathbb{Q}^n_c \times \mathbb{R}$ a similar result of pseudo-paralell surfaces into space forms given by Asperti-Lobos-Mercuri in [1].

Some examples

For the parametrizations $f_i : \mathbb{R}^2 \to \mathbb{Q}^3_c \times \mathbb{R}$ below, we consider $0 < d < 1$, $k > 0$, $a \neq 0$ and $b \in \mathbb{R}$. The first example is a semi-parallel surface in $\mathbb{S}^3_c \times \mathbb{R}$ which is not parallel. The second and third are pseudo-parallel surfaces in $\mathbb{S}^3_c \times \mathbb{R}$ and $\mathbb{H}^3_c \times \mathbb{R}$, respectively, and both are not semi-parallel. In all the cases $0 < \|T\| < 1$, that is, f is not just an inclusion of a pseudo-parallel surface in $\mathbb{Q}^3_c$ into $\mathbb{Q}^3_c \times \mathbb{R}$.

$$f_1(u, v) = \frac{1}{\sqrt{c}}(\sqrt{1 - d^2}\cos\theta(u), \sqrt{1 - d^2}\sin\theta(u), d\cos v, d\sin v, kv),$$
$$f_2(u, v) = \frac{1}{\sqrt{c}}(d\cos u, d\sin u\cos v, d\sin u\sin v, \sqrt{1 - d^2}, au + b),$$
$$f_3(u, v) = \frac{1}{\sqrt{-c}}(d\cosh u, d\sinh u\cos v, d\sinh u\sin v, \sqrt{d^2 - 1}, au + b).$$

Let $f : \mathbb{R}^2 \to \mathbb{S}^5_c$ be the surface given by (see [2])

$$f(x, y) = \frac{2}{\sqrt{6c}}(\cos u\cos v, \cos u\sin v, \frac{\sqrt{2}}{2}cos(2u), \sin u\cos v, \sin u\sin v, \frac{\sqrt{2}}{2}\sin(2u)),$$

where $u = \sqrt{\frac{c}{2}}x$, $v = \frac{\sqrt{6c}}{2}y$. f is a pseudo-parallel immersion in $\mathbb{S}^5_c$ with $\phi = \frac{-c}{2}$.
Thus, if $i : \mathbb{S}^5_c \to \mathbb{S}^5_c \times \mathbb{R}$ is the totally geodesic inclusion given by $i(x) = (x, 0)$, by Proposition 2 we have that $i \circ f$ is a pseudo-parallel immersion in $\mathbb{S}^5_c \times \mathbb{R}$ with non vanishing normal curvature.

Question

Are there other examples, up to isometries, of pseudo parallel surfaces in $\mathbb{Q}^3_c \times \mathbb{R}$ ($c \neq 0$), which T is not a principal direction?
Is there an isometric immersion of a topological 2-sphere into $\mathbb{S}^4 \times \mathbb{R}$ that is not included in a slice?

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