On Generalizations of the Parking Permit Problem and Network Leasing Problems

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Abstract

We propose a variant of the parking permit problem, called multi parking permit problem, in which an arbitrary demand is given at each instant and one may buy multiple permits to serve it. We show how to reduce this problem to the parking permit problem, while losing a constant cost factor. We obtain a 4-approximation algorithm and, for the online setting, a deterministic $\mathrm{O}(K)$-competitive algorithm and a randomized $\mathrm{O}(\lg K)$-competitive algorithm, where K is the number of permit types. For a leasing variant of the Steiner network problem, these results imply $\mathrm{O}(\lg n)$-approximation and online $\mathrm{O}(\lg K \lg |V|)$-competitive algorithms, where n is the number of requests and $|V|$ is the size of the input metric. Also, our technique turns into polynomial-time the pseudo-polynomial algorithms by Hu, Ludwig, Richa and Schmid for the 2D parking permit problem. For a leasing variant of the buy-at-bulk network design problem, these results imply: (i) an algorithm which improves the best previous approximation, and (ii) the first competitive online algorithm.

Keywords: leasing optimization, Steiner network, buy-at-bulk network design, approximation algorithms, competitive online algorithms.

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1 Introduction

In the **parking permit problem** (PP), proposed by Meyerson [10], we have K types of permits with lengths $\delta_1, \ldots, \delta_K$ and costs $\gamma_1, \ldots, \gamma_K$, and we are given a sequence $r_0, \ldots, r_{T-1} \in \{0, 1\}$. The goal is to find a minimum-cost set $S \subseteq [K] \times \{0, \ldots, T-1\}$ of permits such that, for each t with $r_t = 1$, we have some permit $(k, \hat{t}) \in S$ satisfying $t \in [\hat{t}, \hat{t} + \delta_k)$. PP has a polynomial exact dynamic programming algorithm. In the online version, T is unknown and $r_0, \ldots, r_{T-1}$ are revealed one at a time, and the problem has deterministic $\mathrm{O}(K)$-competitive algorithm and $\Omega(K)$ lower bound, as well as randomized $\mathrm{O}(\lg K)$-competitive algorithm and $\Omega(\lg K)$ lower bound [10]. We assume

$$1 = \delta_1 < \delta_2 < \ldots < \delta_K \text{ and } \gamma_k/\delta_k < \gamma_\ell/\delta_\ell \text{ for } k > \ell. \tag{1}$$

PP is the seminal problem of the **leasing optimization model**, in which each resource may be leased for different periods of time, and it is more cost-effective to lease resources for longer periods. This contrasts with traditional optimization models, in which acquired resources last for unlimited duration. Leasing optimization may be applied to both offline and online problems. Some literature has been devoted to leasing variants of optimization problems [2,11,1]. Also, variants of PP and related problems were studied [9,8,6]. A traditional problem whose leasing variant was studied by Meyerson [10] is the Steiner forest problem. In the **Steiner leasing problem** (SLe), each edge can be leased for finite periods of time. Meyerson presented a relationship between SLe and PP: if the input metric is a tree, SLe reduces to solve PP for each edge. Using the technique of approximating a metric by a tree metric [4,5], a solution for a generic input can be obtained, losing some guarantee of quality. This idea can be formalized as follows.

Theorem 1.1 ([4,5]) *Given a minimization problem on a finite metric (V, d) whose objective function is a non-negative linear combination of distances in d, if there is an α-competitive algorithm for the special case of tree metrics, then there is a randomized $\mathrm{O}(\alpha \lg |V|)$-competitive algorithm for the general case.*

In this paper, we extend this approach to solve leasing variants of two network design problems. In the **Steiner network problem** (SN) [7,12], we are given pairs of vertices and a demand $r(u, v)$ for each pair (u, v), and we wish to buy a minimum-cost multiset of edges that contains $r(u, v)$ edge-disjoints (u, v)-paths, for each pair (u, v). In the **buy-at-bulk network design problem** (BaBND) [3], we also have pairs of vertices with demands, but we can install cables on each edge with different capacities per length, and we wish to install a minimum-cost multiset of cables that meet the total demand.

We propose the **multi parking permit problem** (MPP) and we show how to reduce it to PP, while losing a constant cost factor. Thus, we obtain an 8-approximation algorithm, as well as deterministic $\mathrm{O}(K)$-competitive and randomized $\mathrm{O}(\lg K)$-competitive online algorithms for MPP. We also have a 4-approximation algorithm based on dynamic programming with binary search. The online results are asymptotically optimal, since the lower bounds for PP also apply to MPP. The leasing variant of SN, the **Steiner network leasing problem** (SNLE), reduces to MPP if the input metric is a tree; thus, by Theorem 1.1, we obtain a $\mathrm{O}(\lg n)$-approximation algorithm and a $\mathrm{O}(\lg K \lg |V|)$-competitive online algorithm, where n is the number of request pairs and $|V|$ is the input metric size. We do not know previous results for the online case; for the offline scenario, there is a $\mathrm{O}(K)$-approximation algorithm by Anthony and Gupta for a generalization of the single-source case of SNLE [2].

Hu *et al.* [6] proposed the **2D parking permit problem** (2DPP) and gave constant-approximation and deterministic online $\mathrm{O}(K)$-competitive algorithms, but those are pseudo-polynomial. The technique we developed can be used to turn them into polynomial-time algorithms. The non-orthogonal leasing variant of BABND reduces to 2DPP if the input metric is a tree, so our results imply $\mathrm{O}(\lg n)$-approximation and online $\mathrm{O}(K \lg |V|)$-competitive randomized algorithms for non-orthogonal leasing BABND. We do not know previous results for the online case, and our result improves the $\mathrm{O}(K \lg n)$-approximation by Anthony and Gupta [2] for the non-orthogonal multi-source case. Due to space constraints, we omit the presentation of these results.

2 The Multi Parking Permit Problem

MPP is a variant of PP in which a demand greater than one can be given for each instant, i.e., we receive a sequence $r_0, \ldots, r_{T-1} \in \mathbb{Z}_+$. Moreover, multiple copies of the same permit can be bought. We wish to find a minimum-cost multiset of permits $S \subseteq [K] \times \{0, \ldots, T-1\}$ such that $|\{(k, \hat{t}) \in S : t \in [\hat{t}, \hat{t} + \delta_k)\}| \geq r_t$ for each t. We assume that permits satisfy Equation (1).

It is useful to adopt the following hypothesis, similar to [10,2], which only implies a constant factor increase to the cost of any solution for MPP.

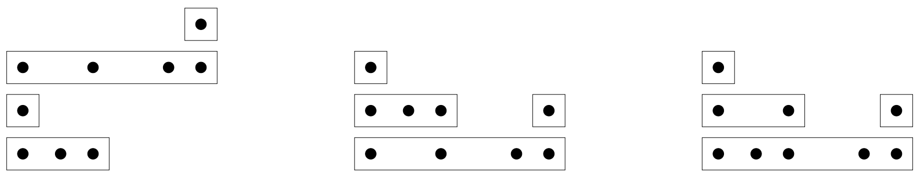
Hypothesis 2.1 (Interval Model (IM)) *For $k = 2, \ldots, K$, δ_k divides δ_{k-1}, and permits of type k can only begin at instants $c\,\delta_k$, for $c \in \mathbb{Z}_+$.*

Lemma 2.2 *If there is an α-competitive algorithm for instances that satisfy IM, then there is a 4α-competitive algorithm for generic instances.*

We prove that, under IM, there is an optimum solution with the following property. As in the Hanoi tower problem, permits that overlap in time are stacked in non-increasing order of length, with lower permits serving the maximum demand possible. More precisely, we have the following.

Lemma 2.3 (Hanoi tower property (HTP)) *Let $I = (T, K, \gamma, \delta, r)$ be an instance of* MPP *that satisfies* IM*. Let $R := \max_{t=0,\ldots,T-1} r_t$ and, for $j = 1, \ldots, R$, let $I^j := (T, K, \gamma, \delta, r^j)$ be an instance of* PP *such that, for $t = 0, \ldots, T-1$, $r_t^j = 1$ if and only if $r_t \geq j$. Then there exists an optimum solution for I which is the union of the respective optimum solutions of* PP *for instances $I^1, I^2, \ldots, I^R$.*

Proof sketch. Let S^* be an optimum solution of I, and let S^{j*} be an optimum solution of I^j, for $j \in [R]$. Suppose by contradiction that $\mathrm{cost}(S^*) \neq \sum_{j=1}^{R} \mathrm{cost}(S^{j*})$. Since $\bigcup_{j=1}^{R} S^{j*}$ is a feasible solution for I, then $\mathrm{cost}(S^*) < \sum_{j=1}^{R} \mathrm{cost}(S^{j*})$. Sort the permits in S^* in non-increasing order of permit type, breaking ties arbitrarily, and assign demands to permits such that no demand can be moved to a previous permit in the sorting. (See Figure 1.) Due to IM, S^* can be partitioned into feasible solutions of $I^1, \ldots, I^R$, a contradiction. □



(a) (b) (c)

Fig. 1. HTP applied to an instance with $T = 6$, $K = 3$, $\gamma = (1, 5/2, 4)$, $\delta = (1, 3, 6)$ and $r = (3, 1, 2, 0, 1, 2)$. (a) Optimum solution $\{(2, 0), (1, 0), (3, 0), (1, 5)\}$. (b) Optimum solution after reordering permits. (c) Optimum solution after reassigning demands; permits correspond to optimum solutions for PP instances.

So under IM, by HTP, MPP reduces to solve R instances of PP. Thus, given an α-approximation for PP, we obtain an α-approximation for MPP. Similarly, given an α-competitive online algorithm for PP, we obtain an α-competitive online algorithm for MPP.[4] By Lemma 2.2, we have a 4-approximation algorithm, as well as deterministic $\mathrm{O}(K)$-competitive and randomized $\mathrm{O}(\lg K)$-competitive online algorithms for MPP.[5] This reduction is pseudo-polynomial,

[4] Since an online algorithm does not know the value of R in advance, we must run new instances of the online algorithm for PP as the maximum demand increases.

[5] Lemma 2.3 is not valid if IM is not assumed, and we conjecture that MPP is NP-hard.

since the input size is proportional to $\mathrm{O}(\lg R)$; we overcome this as follows. Let $L := \lfloor \lg R \rfloor$; we run $L+1$ instances $\hat{I}^0, \hat{I}^1, \ldots, \hat{I}^L$ of PP. For each instant t, let $\ell := \lfloor \lg r_t \rfloor$. For $j = 0, \ldots, \ell$, let $\hat{r}^j_t := 1$ for $\hat{I}^j$. We buy 2^j copies of the permits bought by PP on $\hat{I}^j$. This is feasible since $\sum_{j=0}^{\ell} 2^j = 2^{\ell+1} - 1$ and $r_t < 2^{\ell+1}$. We call this online algorithm MppReduction. This idea is inspired on the online algorithm for SN [12]. We omit the proof of the following results.

Lemma 2.4 *Given an α-competitive algorithm for* PP*, algorithm* MppReduction *leads to a strictly polynomial-time 2α-competitive algorithm for* MPP*.*

Theorem 2.5 *There is an 8-approximation algorithm, as well as deterministic $\mathrm{O}(K)$-competitive and randomized $\mathrm{O}(\lg K)$-competitive online algorithms for* MPP*, all of which run in polynomial time.*

The offline result can be improved to a 4-approximation using a dynamic programming algorithm that performs a binary search on the demand levels, which finds an optimum solution under IM. We omit the description of the algorithm, but it relies on HTP. This algorithm can be combined with the deterministic online algorithm for 2DPP [6] to obtain a deterministic $\mathrm{O}(K)$-competitive online algorithm for MPP with smaller hidden constant factor.

Theorem 2.6 *There is a polynomial 4-approximation algorithm for* MPP*.*

Steiner Network Leasing

The input for SNLe consists of a graph $G = (V, E)$, a distance function $d : V \times V \mapsto \mathbb{R}_+$ satisfying symmetry and triangle inequality, K leasing types with scaling costs $\gamma_1, \ldots, \gamma_K$ and lengths in time $\delta_1, \ldots, \delta_K$, and a sequence $(u_0, v_0, r_0), \ldots, (u_{T-1}, v_{T-1}, r_{T-1})$ in which $u_t, v_t \in V$ and $r_t \in \mathbb{Z}_+$ for every t. The goal is to find a multiset of edge leases $S \subseteq E \times [K] \times \{0, \ldots, T-1\}$ such that, for $t = 0, \ldots, T-1$, the multiset $\{e \in E : (e, k, \hat{t}) \in S \text{ and } t \in [\hat{t}, \hat{t} + \delta_k)\}$ contains r_t edge-disjoint (u_t, v_t)-paths, and minimizes $\sum_{(e,k,\hat{t}) \in S} d_e \, \gamma_k$.

If G is a tree, then there is a unique path between each pair of vertices, so SNLe reduces to solve MPP in each edge, in order to decide how many copies and which leasing types to use. Thus, we obtain a solution to a generic graph by approximating (V, d) by a tree metric. In the offline setting, we can build a tree metric on the metric restricted to the n requested pairs. In the online setting, requested pairs are given in an online manner, edges must be leased to connect pairs as they arrive, and no edge leases may be removed. Thus, we cannot build a tree metric only on the requested pairs. Instead, we have to approximate the entire metric (V, d). Since we have a constant-approximation algorithm and an online $O(\lg K)$-competitive algorithm for MPP, we obtain the following result.

Theorem 2.7 *There are a $O(\lg n)$-approximation algorithm and an online $O(\lg K \lg |V|)$-competitive algorithm for* SNLe.

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