Real Semigroups, Real Spectra and Quadratic Forms over Rings

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Preface

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The present extended monograph contains most of the results obtained in joint research work carried out by the authors, along the period 2000 - 2014 in Paris, France, and Buenos Aires, Argentina. Parts of this work have appeared in print, [DP1] - [DP3] (proofs are omitted in [DP2]).

Both the theory of real semigroups, presented here, and its ancestor, that of **abstract real spectra** originating with Bröcker and Marshall (see [M], Chs. 6 - 8) (Cite Bröcker here.), arose from the idea of setting up an axiomatic framework to investigate order and quadratic form theory in (commutative, unitary) rings, both in their own right and in view of applications to real algebraic geometry. It was Bröcker [Br] who took the first steps, motivated by questions of minimal representations of constructible sets in real geometry. His ideas were further developed and exposed in [ABR], Ch. III, under the name "spaces of signs". At about the same time (1994-96) Marshall gave a new, drastically simplified, axiom system for abstract real spectra.

By their construction, both these abstract theories apply, in the context of rings, under the following constraints:

(1) To (structures constructed from) rings satisfying a mild orderability requirement: namely, those having a non-empty real spectrum. This condition is equivalent to -1 not being a sum of squares; rings with this property are called *semi-real*[1].

(2) To *diagonal* quadratic forms. Note that quadratic forms over rings seldom are diagonalizable (ADD REF).

However,

(3) No restriction is imposed on the coefficients of the quadratic forms under consideration [2].

The orderability requirement (1) seems indispensable to get an organized pattern common to the gigantic variety of rings occurring in mathematical practice. In fact, it is surprising that a theory –even a quite rich one– could at all emerge at that level of generality and, further, that this is achieved on the basis of a rather simple axiom system.

In the sequel to this Preface we describe the plan of the present work in conceptual terms, following a genetic (rather than a lexicographic) order of presentation.

The notion of a real semigroup. Origins.

[1] Unless explicit mention to the contrary, the word "ring" stands in this text for *commutative, unitary and semi-real* ring.

[2] Quadratic forms over preordered rings *with invertible coefficients* are investigated in [DM6], employing techniques from the theory special groups.

The axioms for abstract real spectra mix topology and algebra. In its original form the theory cannot be recast as a set of first-order statements in a suitable language, a very convenient format for many purposes. However, based on the previously known duality between reduced special groups and abstract order spaces ([DM1], Ch. 3), we aimed at devising an axiom system in a natural first-order language, *functorially dual* to the theory of abstract real spectra.

After some work at the early stages of our joint research we succeeded in obtaining a natural axiom system fulfilling these requirements, whose models we baptized "real semigroups" (abbreviated RS); these axioms appear in § I.2. They are formulated in a language $\mathcal{L}_{\mathrm{RS}}$ comprising a binary product operation "$\cdot$", constants $1, 0, -1$ and a ternary relation "D" (*representation by,* also called the *value set of*, binary forms). The duality real semigroups/abstract real spectra was proved in [DP1], Thm. 4.1; cf. § I.5 below.

As stated in § I.2, the axioms for real semigroups involve another ternary relation –called *transversal representation*, denoted by D^t–, definable in terms of D without involving quantifiers. Transversal representation is conceptually important: in the example of rings it reflects what is left from the latter's addition, an operation not compatible with ordinary representation D in passing from a ring to its associated real semigroup (product, however, is compatible). For a discussion of this point, see [M], p. 96.

Post algebras; the Post hull of a real semigroup.

Motivated by the existence of the Boolean hull of reduced special groups ([DM1], Ch. 4) our next goal was to search for a "hull" of a RS having "reasonable" functorial properties and capable of yielding some information about the given RS. Our search was guided by the following considerations:

- Every Boolean algebra has a natural structure of reduced special group ([DM1], Ch. 4, § 1).
- Every reduced special group has a (canonical) "hull" which is a Boolean algebra, i.e., an algebraic model of the classical (two-valued) propositional calculus ([DM1], Ch. 4, § 2).
- Real semigroups (and semi-real rings) can be conceived as essentially 3-valued objects. For example, an element a of a (semi-real) ring A can be thought of as a function $\overline{a}$: $\mathrm{Sper}\,(A) \longrightarrow \mathbf{3} = \{1, 0, -1\}$, where, for $\alpha \in \mathrm{Sper}\,(A)$,

  $\overline{a}(\alpha) :=$ sign of $a/\mathrm{supp}(\alpha)$ in the total order of the ring $A/\mathrm{supp}(\alpha)$ determined by α.

  ($\mathrm{supp}(\alpha)$ is the prime ideal $\alpha \cap \alpha$.)

For a more detailed discussion of this approach, see [DP2], pp. 50-51.

It was natural, then, to search for a "hull" amongst the algebraic counterparts of three-valued propositional logic. Of the various versions that had been studied since the early 1920's (and, in algebraic form, since the 1940's), the fact that RSs have an absorbent element 0 lead us to the choice of the formulation proposed by Post, namely Lukasiewicz's 3-valued propositional logic with a "center" having properties resembling the zero of the RSs.

We proved that any Post algebra carries a natural structure of real semigroup (§ IV.2), and that the Post algebra $\mathcal{C}(X_G, \mathbf{3})$ of continuous functions of the character space [3] X_G of G (with the (Boolean) topology induced from $\mathbf{3}^G$) into $\mathbf{3} = \{1, 0, -1\}$ with the discrete topology, is a "hull" (in the usual categorical sense) in which G embeds naturally by evaluation. From a

[3] A *character* of a RS is a $\mathcal{L}_{\mathrm{RS}}$-homomorphism $G \longrightarrow \mathbf{3} = \{1, 0, -1\}$; $\mathbf{3}$ has a *unique* RS structure; cf. I.2.5.

heuristic viewpoint this "Post hull" of a RS has a role parallel to that of the "Boolean hull" of a reduced special group [4].

Concerning the RS structure of Post algebras, we mention the following:

(A) A quite useful characterization of transversal representation involving the "modal" operations present in Post algebras, exhibiting a remarkable symmetry (Theorem IV.2.7 (i)).

(B) RS-morphisms between Post algebras coincide with Post-algebra morphisms (IV.2.11).

(C) The outstanding functorial properties of the Post hull construction (§ IV.4).

(D) Quotients of RSs commute with the formation of the Post hull (Theorem IV.4.10).

In § IV.6 we prove some model-theoretic results concerning Post algebras (viewed as RSs), and in § IV.7 we characterize the rings whose associated real semigroup is a Post algebra. We also prove that Post algebras are "realized" by rings, i.e., any Post algebra (viewed as a RS) is isomorphic to the RS associated to some ring.

In § IV.5 we characterize representation and transversal representation of forms of arbitrary dimension over Post algebras in terms of their order and their lattice and "modal" operations. Applied in the case of the Post hull of a RS, these characterizations make it possible to "read off" certain properties of the value sets and transversal value sets of Pfister forms over the given RS akin to those well known in the field case (cf. IV.5.6, IV.5.9).

The representation partial order.

An important by-product of the Post hull construction is that the (distributive lattice) partial order of the hull restricts to a partial order on the given RS. This order is not compatible with product in any standard algebraic sense; in fact, the relations between it and product are rather subtle (see Propositions I.6.4 and I.6.5). At any rate, this order is definable from the representation relation of the given RS (I.6.2).

This, hitherto unnoticed, partial order —that we call the **representation partial order**— has a structural role in the theory of RSs in many senses similar to that of the homonymous partial order in the theory of reduced special groups ([DMM], § 1.1, pp. 29-31). This is why we present it at an early stage of development of the theory (§ I.6). The following results substantiate the importance of this concept:

(E) The representation partial order of any RS, G, induces a natural bounded distributive lattice structure on the set $\mathrm{Id}(G) = \{x^2 | x \in G\}$ of idempotent elements of G (Proposition I.6.8).

(F) For spectral RSs (see below and Chapter V) the representation partial order endows G itself with a bounded distributive lattice structure, and conversely (Theorem V.6.6).

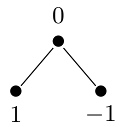
Spectral real semigroups; the spectral hull of a real semigroup.

In Chapter V we study in detail a class of RSs that we call spectral. The definition of the members of this class is done in terms of the spectral topology of the character space X_G of a RS, G; their role in relation to this topology is parallel to that of the Post algebras in relation to the constructible topology of X_G (see above).

Given a spectral space, X, the set $\mathrm{Sp}(X)$ consists of all spectral maps $X \longrightarrow \mathbf{3}$, with $\mathbf{3}$

[4] This analogy should be taken only as a guideline: Boolean algebras are never Post algebras.

endowed with the spectral topology whose specialization order is



Product, the constants $1, 0, -1$, and representation in $\mathrm{Sp}(()X)$ are pointwise defined.

Equipped with this structure, $\mathrm{Sp}(X)$ verifies all axioms for RSs with the possible exception of axiom [RS3b], asserting that $D^t(\cdot,\cdot) \neq \emptyset$ (Theorem V.1.4), and this axiom holds if and only if the space X is hereditarily normal, i.e., for every $x \in X$, the set $\overline{\{x\}} = \{y \in X | x \rightsquigarrow y\}$ is linearly ordered under $\rightsquigarrow$ (Theorem V.1.5).

Among our most significant results about spectral RSs, we mention:

(G) Existence of a functorial duality (anti-equivalence) between the category of hereditarily normal spectral spaces with spectral morphisms and that of spectral RSs with RS-homomorphisms (Theorem V.5.4).

(H) Spectral RSs are exactly those RSs for which the representation partial order is a distributive lattice (Theorem V.6.6).

(I) Every RS, G, has a "spectral hull" with the required functorial properties; namely, $\mathrm{Sp}(X_G)$, where the character space X_G is now endowed with its *spectral* topology. The spectral hull of a RS is included in its Post hull, but it is much smaller; in fact, the spectral hull of a RS is generated by the given RS *as a lattice* (Theorem V.6.2). Formation of the spectral hull is an idempotent operation (Theorem V.4.5).

(J) The class of spectral RSs is first-order axiomatizable in the language $\mathcal{L}_{\mathrm{RS}}$ (Theorems V.2.1 and V.7.4), and the specific form of the axioms guarantees closure of the class under a number of algebraic and model-theoretic constructions, amongst others the formation of quotients modulo arbitrary RS-congruences (see below).

(K) The character space of the spectral hull of a RS, G, is canonically homeomorphic to X_G, and the quotients of G are determined by the proconstructible subsets of X_G (V.8.2 and V.8.4).

In § V.7 we look at spectral RSs from the perspective of the so-called **Kleene algebras**, namely distributive lattices with a non-classical "negation" (corresponding, in RSs, to multiplication by -1). We characterize spectral RSs as Kleene algebras verifying some natural additional conditions (Theorem V.7.2).

Spectral RSs occur in profusion among rings; in § V.10 we prove that the RS associated to any lattice-ordered ring is a spectral RS. Further, the spectral hull of the RS associated to any semi-real ring is canonically isomorphic to the RS associated to its real closure in the sense of Prestel-Schwartz [PS].

Description of the next important class of RSs, the **RS-fans**, that we introduce and study in Chapter VI, requires a prior detour through the "pointed" semigroup structures (no representation relation) underlying the real semigroups, structures that we call **ternary semigroups** and examine in § I.1.

Ternary semigroups.

Ternary semigroups (abbreviated TS) are commutative, unitary semigroups with two extra constants $-1, 0$, such that $-1 \cdot -1 = 1$, and for all $x, x^3 = x, x \cdot 0 = 0$ and $x = -1 \cdot x \Rightarrow x = 0$.

These structures play, in the theory of RSs, a role comparable to that played by the groups of exponent 2 with a distinguished element -1 in the theory of reduced special groups. But while the latter have a rather trivial structure (vector spaces over the two-element field with a distinguished non-zero element), ternary semigroups have, in general, a far more complex structure.

In § I.1 we pay due attention to the construction of TS-characters (i.e., homomorphisms into $\mathbf{3}$ preserving the TS-structure); the techniques at work here foreshadow those, taken up in § I.4, employed to construct real semigroup characters, and used throughout the text. We also examine the quotients of TSs, and show how TS-congruences can be constructed from those of simpler structures (essentially, *bounded join semilattices*); we exhibit some examples of TS-congruences that will occur time and again.

Section § I.3 is devoted to examine a fairly general method of construction of RSs from TSs. Given a ternary semigroup and a set of its (TS-)characters, a ternary relation is defined that satisfies most of the axioms for real semigroups –indeed, all of them with the possible exception of the strong associativity axiom (axiom [RS3] in I.2.1). [5] By means of this construction, many proofs that some structures are RS's, are reduced to verifying the validity of the sole strong associativity axiom. We give examples showing that this axiom may, in general, fail, but also that it is satisfied by some finite sets of characters of small cardinality. Further, by showing that the closure of any set of TS-characters in the constructible topology defines the same ternary relation, we obtain an additional topological instrument facilitating many proofs.

RS-fans.

A class of field preorders called *fans*, was introduced by Bröcker in the 1970's and extensively studied thereafter (see [L2], Ch. 5). Fans turned out to be veritable "building blocks" in the algebraic theory of quadratic forms over fields. A no less important role have their generalizations to the settings of abstract order spaces ([M], Ch. 3) and reduced special groups ([DM1], pp. 8-9 and 89-90).

However, a suitable notion of a fan does not hitherto exist in the categories of real semigroups and abstract real spectra. In Chapter VI we introduce and study a natural notion of a fan in these categories, here dubbed **RS-fans** and **ARS-fans**, respectively. The results described below underline, in our opinion, the naturality and the relevance of these notions.

Among their many characterizations, fans in the category of reduced special groups (and its dual of abstract order spaces [6]) are those objects, F, such that every **group** character $F \longrightarrow \{\pm 1\}$ sending -1 to -1 preserves the representation relation D_F. Alternatively, this amounts to saying that D_F is the "smallest possible" relation compatible with the axioms; in fact,

$$D_F(a,b) = \begin{cases} \{a,b\} & \text{if } a \neq -b \\ F & \text{if } a = -b \end{cases}$$

We proceed by analogy in the case of RSs. Having observed that ternary semigroups are the structures underlying the RSs, we define a RS, G, to be a (RS-)fan if and only if its character space X_G consists of <u>all</u> TS-characters $G \longrightarrow \mathbf{3}$. It turns out that this is equivalent to require that G is a RS and the product of <u>any three</u> characters is again a character.

A necessary condition for any of these requirements to hold is that the zero-sets $Z(a) =$

[5] Particular cases of this general construction were used by Marshall ([M], § 6.6); these and other cases of importance in real algebraic geometry are treated in § II.3 below.

[6] Fans correspond to each other under this duality.

$h \in X_G | h(a) = 0$ of elements of G be totally ordered under inclusion.

For TSs, T, satisfying this necessary condition, we characterize in Theorem VI.2.1 the representation and transversal representation relations that, added to T, will result in a RS-fan. The actual result is far more involved than the above characterization of D_F for reduced special groups, but the intuitive guideline follows a similar path: the requirement that all TS-characters become RS-characters gives, of course, the biggest possible (RS-)character set and, dually, the "smallest possible" representation and transversal representation relations. Theorem VI.2.1 gives a precise formulation to this intuition and shows that it is consistent with the axioms. The proof is long and delicate.

Once this is accomplished, many consequences follow; to give a taste:

(L) Any TS-homomorphism of a RS-fan into any RS is a RS-homomorphism.

(M) The various definitions of "RS-fan" are equivalent.

(N) Every TS-ideal of a RS-fan is a saturated prime ideal.

(O) A TS-subsemigroup S of a RS-fan is saturated iff it contains all idempotents and $S \cap -S$ is an ideal.

In § VI.3 we describe in detail some examples of RS-fans among TSs with ≤ 3 generators. In each case we determine the representation partial order and the specialization order of the character space. We also show that RS-fans are (non-distributive) lattices under the representation partial order.

Section § VI.4 is essentially devoted to prove a characterization of RS-fans in terms of product, the specialization relation and quotients at saturated prime ideals (Theorem VI.4.2). As corollaries we obtain abstract analogs of the notion of a trivial fan, a basic concept in the theory of preorders on fields.

This abstract characterization of RS-fans is interpreted, in § VI.5, in the context of the real semigroups associated to preordered rings, yielding an algebraic characterization of the preordered rings whose associated real semigroup is a RS-fan (Corollary VI.5.13); this characterization constitutes a non-trivial generalization of classical results from the theory of preordered fields to the far vaster realm of preordered rings. It follows that the real semigroups associated to several outstanding classes of preordered rings are RS-fans (Corollary VI.5.14, Theorem VI.5.25).

The study of the fine structure of the character spaces of RS-fans —equivalently, ARS-fans— begins in § VI.7. A central notion here is that of a **level set** in a ARS-fan X_F, i.e., the set of $h \in X_F$ whose zero-set $Z(h)$ is a fixed ideal of F (necessarily prime and saturated); certain subsets of level sets are also important in this connexion. Any level set is, itself, an abstract space of orders.

We show that ARS-fans have a rich supply of involutions of their level sets which, in addition, are automorphisms of abstract order spaces; specifically, given an ideal I of F , any pair of elements $g_1, g_2 \in X_F$ such that $I \supseteq Z(g_i)(i = 1, 2)$ determine an involution of the level set corresponding to I, having many good properties. We also prove that these involutions move certain subsets of X_F (having a rather technical definition) in specific ways, resulting in strong constraints to the possible configurations of the specialization order of X_F (VI.7.11 - VI.7.18).

In § VI.8 we study the specialization root-system of *finite* ARS-fans. Our main result, the isomorphism theorem VI.8.9, shows that the order of specialization determines the isomorphism type of finite ARS-fans (isomorphism in the sense of [M], Def. p. 103). The proof relies on a notion of *standard generating system*.

A comparison with the case of abstract order spaces may help putting in focus the difficulties in the present situation. Abstract order spaces possess a natural structure of a combinatorial geometry (matroid) which, moreover, is isomorphic to that of linear dependence of a set of vectors inside some vector space over the 2-element field; this was proved in [D1] for the field case and generalized to abstract order spaces in [Li]. By use of this tool, the analog of the foregoing isomorphism theorem in the context of abstract order spaces is straightforward (cf. [D1], § 5). For ARSs the situation gets significantly more involved, owing to:

- The absence of a combinatorial geometric structure.
- The presence of the specialization order, that ought to be brought into the picture.

The notion of a standard generating system is an ersatz for that of a matroid basis, sufficient, however, to yield the stated isomorphism theorem.

As a by-product we get that the isomorphism type of finite ARSs is determined by a finite system of numerical invariants. This is done in § VI.9, while in § VI.10 we prove that these invariants form a complete system; the proof requires, of course, constructing finite ARSs having a prescribed system of invariants of the appropriate type.

Section VI.11 is devoted to determine the quotients of RS-fans. We prove in VI.11.2 that any such quotient is again a RS-fan, and that any RS-congruence of a RS-fan is determined by a proconstructible subset satisfying a certain closure condition (already considered in § II.2).

Quotients of real semigroups.

Chapter II is devoted to the rather delicate question of congruences of real semigroups and their quotients, and their application to the real semigroups arising from rings.

Observe at the outset that, since the class of RSs is not algebraic (the representation relation is not a function), there is not a ready-made notion of congruence to be used; however, there is such a notion for the algebraic class of ternary semigroups.

We have chosen a notion of congruence that seems natural in the present context: an equivalence relation $\equiv$ on a RS, G, which is a congruence for the TS structure underlying G, such that the quotient TS, $G/\!\equiv$, is equipped with a ternary relation $D_{G/\equiv}$ under which $(G/\!\equiv, D_{G/\equiv})$ becomes a RS; further, we require the quotient map $\pi : G \longrightarrow G/\!\equiv$ to be a RS-morphism verifying the following universal property: every RS-morphism $f : G \longrightarrow H$ into a RS H such that $a \equiv b \Rightarrow f(a) = f(b)$ holds for all $a, b \in G$, induces a RS-morphism (necessarily unique) $\widehat{f} : G/\!\equiv\ \longrightarrow H$ such that $\widehat{f} \circ \pi = f$.

Every such (RS-)congruence gives rise to a set of characters $\mathcal{H}_{\equiv} \subseteq X_G$ which is proconstructible and, equipped with the spectral topology induced from X_G, is homeomorphic to $X_{G/\equiv}$ as spectral spaces (Proposition II.2.8).

Any subset $\mathcal{H} \subseteq X_G$ defines in an obvious way a TS-congruence $\equiv_{\mathcal{H}}$ on G and a ternary relation $D_{\mathcal{H}}$ on the quotient ternary semigroup $G/\!\equiv_{\mathcal{H}}$ $(:= G/\mathcal{H})$ (I.3.2). We prove, without additional assumptions on the set $\mathcal{H}$, that the quotient structure $(G/\mathcal{H}, D_{\mathcal{H}}, \cdot, 1, 0, -1)$ verifies all axioms for RSs except, possibly, associativity of the ternary relation $D_{\mathcal{H}}$. This quotient structure is the same as that defined by the closure of $\mathcal{H}$ in the constructible topology of X_G. However, there are examples showing that the associativity axiom may fail without additional requirements on the set $\mathcal{H}$. In Theorem II.2.9 we give a necessary closure condition on $\mathcal{H}$ for $(G/\equiv, D_{G/\equiv})$ to be a RS, but a manageable sufficient condition is still lacking.

In § II.3 we deal with quotients of RSs determined by specific sets of characters frequently occurring in quadratic form theory and real algebraic geometry: localizations, quotients by saturated sets, by saturated subsemigroups, by transversally saturated subsemigroups, and residue spaces at saturated prime ideals. All these families of characters produce quotients in our sense; some of them have been previously considered by Marshall ([M], §§ 6.5, 6.6), who proved, in the dual terminology of ARSs, that, indeed, they are RSs. Our work in this section considerably extends Marshall's: for each of these families of characters we characterize the equivalence relation, as well as both representation relations in the quotient, in terms of the constants, operation and relations of the initial RS.

All the examples mentioned in the preceding paragraph are determined by sets of characters *convex* for the specialization partial order. However, among RS-fans and spectral RSs one finds examples where convexity fails, whence not determined by any set of characters of the above mentioned types.

In section § II.4 we examine the quotients considered in § II.3 in the important case of the RSs G_A (resp., $G_{A,T}$) associated to a ring A (resp., with a preorder T). We show that in each case the corresponding RS-quotient is the RS associated to a ring obtained from the given ring A by suitably combining the standard operations of forming rings of fractions and quotients modulo ideals (resp., using also preorders constructed from T).

A classical theme in commutative algebra and algebraic geometry is the representation of algebraic structures, e.g., rings, by means of algebraic structures consisting of continuous global sections of sheaves of other algebraic structures —usually with better properties— over topological spaces. Archetypal of results of this kind (and the most famous of them) is Grothendieck's representation of any ring (commutative, unitary) by continuous sections of a sheaf of local rings over its Zariski spectrum. The list is long: Hofmann [Ho] contains a survey of results of this type (up to the early 1970's).

In § III.1 we prove a representation result of this type: <u>any</u> RS is isomorphic to one consisting of continuous global sections of (quasi) reduced special groups over the (spectral) space of its saturated prime ideals.

An interesting application of the foregoing sheaf representation theorem is given in § III.2; we prove that, whenever the (spectral) space of saturated prime ideals of a RS, G, is <u>normal</u>, the stalks of the sheaf of (quasi) reduced special groups form a projective system —not necessarily along a right-directed index set— whose projective limit is a RS in which G embeds completely. Note that the category of RS's <u>is not</u>, in general, closed under the formation of projective limits. In fact, this result is a particular case of a far more general (and new) model-theoretic result about the preservation of Horn-geometric sentences by projective (not necessarily directed) limits of structures over index sets having an order property akin to normality.

In § III.3 we introduce a class of maps between $\mathcal{L}_{\mathrm{RS}}$-structures that we call **transversally 2-regular**, having rather strong properties. For example, when the quotient map $G \longrightarrow G/\equiv$ modulo an equivalence relation $\equiv$ in a RS, G, is transversally 2-regular, the congruence is automatically a RS-congruence of G, i.e., satisfies the requirements laid dawn above (Proposition III.3.2). Among our current examples, localizations and residue spaces at saturated prime ideals have this property, as well as arbitrary quotients of Post algebras (IV.4.12) and of RS-fans (VI.11.3). However, in III.3.5 we give an example showing that quotients modulo saturated subsemigroups may not be transversally 2-regular.

Paris, Buenos Aires, April 2011.

Chapter I

Real Semigroups

Important note. Most of the material in Sections 1 – 4 of this chapter has been published in [DP1], including full proofs of results. Since this material is constantly used in the present text, omitting it completely would have definitely impaired readability. In order to avoid unduly increasing the length of this paper, we have decided to include all the material from [DP1] necessary to the understanding of the mathematical content —motivations, definitions, notations and the statement of results—, but omit the proofs already given there. Proofs of those results not appearing in [DP1] are, of course, included here.

I.1 Ternary semigroups

In this section we introduce the notion of *ternary semigroup*. This class of (commutative, unitary) semigroups with additional individual constants, underlies the notion of real semigroup —to be introduced in §2 below— in a sense parallel to which the groups of exponent 2 with a distinguished element -1 underlie the notion of special group.

In spite of the similarity of their roles, these classes of structures differ in very significant ways. Firstly, while the groups of exponent 2 have a rather trivial algebraic structure — they are just vector spaces over the two-element field—, that of the ternary semigroups is far more complex. Secondly, while the set of characters of groups of exponent 2 carry only one natural topology —that of a Boolean space—, the set of characters of a ternary semigroup (into $\{1, 0, -1\}$) is naturally endowed with a spectral topology with a non-trivial specialization order, carrying also an associated constructible (Boolean) topology.

Definition I.1.1 A **ternary semigroup** (abbreviated TS) is a structure $\langle S, \cdot, 1, 0, -1\rangle$ with individual constants $1, 0, -1$, and a binary operation " $\cdot$ " such that:

[TS1] $\langle S, \cdot, 1\rangle$ is a commutative semigroup with unit.

[TS2] $x^3 = x$ for all $x \in S$.

[TS3] $-1 \neq 1$ and $(-1)(-1) = 1$.

[TS4] $x \cdot 0 = 0$ for all $x \in S$.

[TS5] For all $x \in S$, $x = -1 \cdot x \;\Rightarrow\; x = 0$.

We shall write $-x$ for $-1\cdot x$. The semigroups verifying conditions [TS1] and [TS2] (no constants other than 1) will be called **3-semigroups**. $\square$

Remark. Note that the invertible elements of a 3-semigroup —in particular, those of a ternary semigroup— are exactly the elements a such that $a^2 = 1$ [Proof: if $ab = 1$ for some b, scaling by a^2 gives $a^2 = a^2(ab) = a^3b = ab = 1$.]] $\square$

Examples I.1.2 (a) The three-element structure $\mathbf{3} = \{1, 0, -1\}$ has an obvious ternary semigroup structure.

(b) For any set X, the set $\mathbf{3}^X$ under pointwise operation and constant functions with values $1, 0, -1$, is a TS. More generally:

(c) The class of ternary semigroups is closed under direct product and substructures (but not under homomorphic images). In particular, if Δ is a subsemigroup of a TS containing 0 and 1, then $\Delta \cup -\Delta$ is a TS. Further, since the axioms [TS1]–[TS5] are Horn sentences of the (natural) language $\mathcal{L}_{\mathrm{TS}} = \{\cdot, 0, 1, -1\}$ for *ternary* semigroups, the class is also closed under reduced products (cf. [CK], § 6.2).

(d) Any group of exponent 2 obviously is a 3-semigroup; the pointed group of exponent 2 with a distinguished element $-1 \neq 1$ underlying a reduced special group (henceforth abbreviated RSG [1]) also verifies [TS3]. Any such group, G, becomes a ternary semigroup by adding a new absorbent element 0, i.e., extending the operation by $x \cdot 0 = 0$, for $x \in G \cup \{0\}$. Note that the set of invertible elements of a 3-semigroup is a group of exponent 2 (see Remark above).

I.1.3 Reminder. (The real spectrum of a ring)

We shall assume familiarity with the construction and basic properties of the real spectrum of (commutative, unitary) rings A, denoted $\mathrm{Sper}\,(A)$. The basic theory of the real spectrum is expounded in [BCR], § 7.1, [DST], § 23, or [M], Ch. 5, pp. 83 ff.

For the reader's benefit we briefly summarize the basics of this construction, leaving it to him finding in the given references the (geometric) motivation that led to the choice of objects described hereafter.

Objects of $\mathrm{Sper}\,(A)$. These are the *prime cones* of A (also called *orderings* in [M]), i.e., the preorders T of A [$T \dotplus T \subseteq T$ and $A^2 \subseteq T$] such that, in addition, $T \cup -T = A$ and $T \cap -T$ is a (proper) prime ideal, called the **support** of T and denoted by $\mathrm{supp}(T)$. For further details and information, see [BCR], §§ 4.2, 4.3 or [DST], § 23.1. We shall denote the elements of $\mathrm{Sper}\,(A)$ by Greek letters. Note that, for $\alpha \in \mathrm{Sper}\,(A)$, the set $\{a/\mathrm{supp}(\alpha) \mid a \in \alpha\}$ is a (total) ring order of the quotient domain $A/\mathrm{supp}(\alpha)$, denoted $\leq_\alpha$. The canonical quotient map from A to $A/\mathrm{supp}(\alpha)$ is denoted by π_α. A ring is called **semi-real** if $\mathrm{Sper}\,(A) \neq \emptyset$; this condition is equivalent to $-1 \notin \sum A^2$.

Topology of $\mathrm{Sper}\,(A)$. A subbasis for the (spectral, also called Harrison) topology of $\mathrm{Sper}\,(A)$ is given by the family of sets $H(a) = \{\alpha \in \mathrm{Sper}\,(A) \mid \pi_\alpha(a) > 0\}$, for $a \in A$. Thus, a basis consists of the family of sets $H(a_1, \ldots, a_n) = \{\alpha \in \mathrm{Sper}\,(A) \mid \pi_\alpha(a_1) > 0 \wedge \ldots \wedge \pi_\alpha(a_n) > 0\}$ for all finite sequences $a_1, \ldots, a_n$ of elements of A. $\square$

(e) For any semi-real ring A, let the set G_A consist of all functions $\overline{a} : \mathrm{Sper}\,(A) \to \mathbf{3}$, for $a \in A$, where

$$\overline{a}(\alpha) = \begin{cases} 1 & \text{if } a \in \alpha \setminus (-\alpha) \\ 0 & \text{if } a \in \alpha \cap -\alpha \\ -1 & \text{if } a \in (-\alpha) \setminus \alpha. \end{cases}$$

[1] For the notion of *special group* the reader is referred to [DM1]; see especially Chapter 1, for basic definitions.

with the operation induced by product in A, is a TS. More generally, given a (proper) preorder T of a ring A one can relativize the definition above to T, by considering functions $\overline{a}$ defined on $\mathrm{Sper}\,(A,T) = \{\alpha \in \mathrm{Sper}\,(A) \,|\, \alpha \supseteq T\}$, instead of $\mathrm{Sper}\,(A)$; the corresponding ternary semigroup will be denoted $G_{A,T}$. The case above is obtained for $T = \sum A^2$. □

Notation I.1.4 (a) By a **subsemigroup** of a unitary semigroup we mean a subset closed under the operation $\cdot$ and containing 1. Thus, a subsemigroup of a TS may not contain 0 or -1 and hence may not be a substructure for the language $\mathcal{L}_{\mathrm{TS}}$ for *ternary* semigroups (cf. I.1.2 (c)).

(b) The definition of **ternary semigroup (TS-) homomorphism** is standard: preservation of product and the constants $1, 0, -1$ is required. A **TS-character** is a TS-homomorphism into $\mathbf{3}$. If $h : T \longrightarrow \mathbf{3}$ is a TS-character we write $P(h)$ (set of "positive" elements) for the set $h^{-1}[0,1]$ and $Z(h)$ (set of "zeros") for $h^{-1}[0]$. The set of TS-characters of T will be denoted by $\mathrm{Hom}_{\mathrm{TS}}(T, \mathbf{3})$ or, alternatively, by X_T.

(c) The product of two TS-homomorphisms $f, g : T_1 \longrightarrow T_2$ between ternary semigroups T_1, T_2 is pointwise defined: for $t \in T_1$, $(f \cdot g)(t) := f(t)g(t)$. Obviously, $f \cdot g$ is a homomorphism of unitary semigroups sending 0 to 0, but it is not a TS-homomorphism, since -1 is sent to 1. However, the product of any three (or any odd number of) TS-homomorphisms is a TS-homomorphism. This closure property, will play a crucial role throughout the present text, especially in Chapter VI.

(d) An **ideal** of a semigroup S is a non-empty subset $I \subseteq S$ such that $I \cdot S \subseteq I$. An ideal is **prime** if it is proper and $ab \in I \Rightarrow a \in I$ or $b \in I$, for all $a, b \in S$. □

We shall frequently use the following Fact, whose proof is standard and left to the reader:

Fact I.1.5 *Let I be an ideal in a TS, T, and let Δ be a subsemigroup of T such that $I \cap \Delta = \emptyset$. Let J be an ideal of G containing I and maximal with respect to being disjoint from Δ. Then, J is prime. In particular, if $a \notin I$ (by setting $\Delta = \{1, a^2\}$) it follows that an ideal maximal for not containing a is prime.* □

If T is a ternary semigroup, then $\mathrm{Id}(T)$ will denote the set of *idempotents* of T, i.e., $\mathrm{Id}(T) = \{x \in T \mid x = x^2\}$. Clearly, $\mathrm{Id}(T)$ is a subsemigroup of T containing 0.

Remark I.1.6 If $h : T \to \mathbf{3}$ is a TS-character, the set $S = h^{-1}[\{0,1\}]$ verifies:

(i) S is a subsemigroup of T containing $\mathrm{Id}(T)$.

(ii) $S \cap -S$ $(= h^{-1}[0])$ is a prime ideal.

(iii) $S \cup -S = T$.

A subset verifying these properties will be called a **prime subsemigroup** of T. A prime subsemigroup contains 0. □

The prime subsemigroups S of T are in one-one correspondence with the TS-characters of T; indeed, S defines a TS-character upon setting, for $x \in T$:

$$h_S(x) = \begin{cases} 1 & \text{if } x \in S \setminus (-S) \\ 0 & \text{if } x \in S \cap -S \\ -1 & \text{if } x \in (-S) \setminus S. \end{cases}$$

The following Lemma and Proposition give the tools used in practice to construct TS-characters:

Lemma I.1.7 *Let T be a TS and let I be a prime ideal of T. Let S be a subsemigroup of T such that:*

(1) $\mathrm{Id}(T) \cup I \subseteq S$.

(2) *S is maximal such that $S \cap -S = I$.*

Then, S is a prime subsemigroup, i.e., $S \cup -S = T$. The TS-character h_S defined by S (as above) verifies $I = h_S^{-1}[0]$ and $S = h_S^{-1}[\{0,1\}]$.

Proof. See [DP1], Lemma 1.5, p. 102. □

Notation I.1.8 Given a subsemigroup Δ of a TS, T, containing $\mathrm{Id}(T)$, the set $I[\Delta] = \{x \in T \mid -x^2 \in \Delta\}$ is a (possibly improper) ideal of T containing $\Delta \cap -\Delta$. In this situation, $\Gamma = \Delta \cup I[\Delta]$ is a subsemigroup of T, and $I[\Gamma] \subseteq \Gamma$. The easy proof of this assertion is left as an exercise. □

Theorem I.1.9 (Separation theorem for subsemigroups.) *Let T be a TS and let $\Delta \subseteq T$ be a subsemigroup such that $\mathrm{Id}(T) \cup I[\Delta] \subseteq \Delta$. Then, for every $a \in T \setminus \Delta$ there is a character $h \in X_T$ such that $\Delta \subseteq P(h)$ and $h(a) = -1$.*

Proof. By Zorn's lemma there is a subsemigroup S of T such that $\Delta \cup I[S] \subseteq S$ and S is maximal for $a \notin S$. We prove that, for such an S the following hold:

(a) The ideal $I[S]$ is prime.

Proof of (a). Otherwise, there are $p, q \in T$ so that $p, q \notin I[S]$ but $pq \in I[S]$, that is, $-p^2, -q^2 \notin S$ but $-p^2q^2 \in S$. Let $S_p = S \cup -p^2S \cup I[S \cup -p^2S]$. The remark in I.1.8 implies that S_p is a subsemigroup of T and $I[S_p] \subseteq S_p$; obviously $S_p \supset S$. By maximality of S, $a \in S_p$. Since $a \notin S$, we have $a \in -p^2S$ or $a \in I[S \cup -p^2S]$, i.e., $-a^2 \in S \cup -p^2S$. But $-a^2 \in S$ entails $a \in I[S] \subseteq S$, contradiction; hence, either $a \in -p^2S$ or $-a^2 \in -p^2S$. A similar argument gives $a \in -q^2S$ or $-a^2 \in -q^2S$. Thus, we have the following cases:

(1) $a \in -p^2S \cap -q^2S$. Then, $a = -p^2s_1 = -q^2s_2$ with $s_1, s_2 \in S$, whence $-a^2 = -p^2q^2s_1s_2$; but $-p^2q^2 \in S$ yields, then, $-a^2 \in S$, i.e., $a \in I[S] \subseteq S$, contradiction.

(2) $a \in -p^2S$ and $-a^2 \in -q^2S$. Then, $a = -p^2s_1$ and $-a^2 = -q^2s_2$, with $s_1, s_2 \in S$. Since $-p^2q^2 \in S$, we get $a = (-a)(-a^2) = -p^2q^2s_1s_2 \in S$, contradiction. The case $a \in -q^2S$, $-a^2 \in -p^2S$ is similar.

(3) $-a^2 \in -p^2S \cap -q^2S$. Then, $-a^2 = -(-a^2)(-a^2) = -(-p^2s_1)(-q^2s_2) = -p^2q^2s_1s_2 \in S$, whence $a \in I[S] \subseteq S$, absurd.

These contradictions prove item (a).

(b) $S \cup -S = T$.

Proof of (b). Otherwise, let $p \in T$ be so that $p, -p \notin S$, and set $S'_p = S \cup pS \cup I[S \cup pS]$. Then S'_p is a subsemigroup of T containing $\mathrm{Id}(T)$ and $I[S'_p] \subseteq S'_p$ (I.1.8); clearly $S'_p \supset S$. Obviously, the same assertions hold for $S'_{-p} = S \cup -pS \cup I[S \cup -pS]$. By maximality of S, $a \in S'_p \cap S'_{-p}$, and since $a \notin S$, either $a \in pS$ or $-a^2 \in pS$, and $a \in -pS$ or $-a^2 \in -pS$. This gives four cases to consider:

(1) $a \in pS \cap -pS$. Thus, $a = ps_1 = -ps_2$, $s_1, s_2 \in S$, whence $-a^2 = p^2s_1s_2 \in S$, i.e., $a \in I[S] \subseteq S$, contradiction.

(2) $a \in pS$ and $a^2 \in pS$. In this case, $a = a \cdot a^2 \in p^2S \subseteq S$, contradiction. The case $-a^2 \in pS,\ a \in -pS$ is similar.

(3) $-a^2 \in pS \cap -pS$. Then, $-a^2 = ps_1 = -ps_2,\ s_1, s_2 \in S$, whence $-a^2 = (-a^2)(a^2) = p^2 s_1 s_2 \in S$, contradiction.

All cases being contradictory proves item (b). Items (a) and (b) together with $\mathrm{Id}(T) \subseteq S$ show that S is a prime subsemigroup of T. The character h_S induced by S verifies $\Delta \subseteq S = P(h_S)$ and, since $a \notin S$, $h_S(a) = -1$, as required. □

The following separation theorem —proved in [DP1], Thm. 1.6, p. 103— is also a corollary of the foregoing result; we sketch the proof using I.1.9.

Theorem I.1.10 (Separation theorem for ideals.) *Let T be a TS, I be an ideal of T, and $a \in T \setminus I$. Then:*

(a) *There is a TS-character h of T such that $h\lceil I = 0$ and $h(a) \neq 0$.*

(b) *If, in addition, $-a \cdot \mathrm{Id}(T) \cap \mathrm{Id}(T) \subseteq I$, then there is a character h so that $h\lceil I = 0$ and $h(a) = 1$.*

If I is prime, in both (a) *and* (b) *the character h can be chosen so that $h^{-1}[0] = I$.*

Sketch of proof. (a) Apply Theorem I.1.9 to the subsemigroup $\Delta = I \cup \mathrm{Id}(T)$. Note that if $x \in I[\Delta]$, i.e., $-x^2 \in \Delta$, then either $-x^2 \in \mathrm{Id}(T)$, i.e., $-x^2 = x^2$, and then $x = 0 \in I \subseteq \Delta$, or $-x^2 \in I$, which implies $x = (-x)(-x^2) \in I \subseteq \Delta$. This also shows that $I[\Delta] \subseteq I$. Conversely, $I \subseteq I[\Delta]$, for $x \in I \Rightarrow -x^2 = x(-x) \in I \subseteq \Delta \Rightarrow x \in I[\Delta]$. So, $I[\Delta] = I$. Since $a \notin I$, we get $-a^2 \notin \Delta$. By I.1.9 there is $h \in X_T$ such that $\Delta \subseteq P(h)$ and $h(-a^2) = -1$, whence $h(a) \neq 0$; we also have $I = I[\Delta] = \Delta \cap -\Delta \subseteq Z(h)$.

(b) Here we apply I.1.9 to the subsemigroup $\Delta = \mathrm{Id}(T) \cup a \cdot \mathrm{Id}\ (T) \cup I$. To check $I[\Delta] \subseteq \Delta$, suppose that $-x^2 \in \Delta$. If $-x^2 \in \mathrm{Id}(T)$, then $-x^2 = x^2$, and hence $x = 0 \in I \subseteq \Delta$. If $-x^2 \in a \cdot \mathrm{Id}(T)$, then $x^2 \in -a \cdot \mathrm{Id}\ (T) \cap \mathrm{Id}(T)$, and $x^2 \in I$ by assumption, whence $x \in I \subseteq \Delta$. Finally, if $-x^2 \in I$, then $x \in I \subseteq \Delta$.

Note also that $-a \notin \Delta$. Otherwise, since $a \notin I$, we either have:

(i) $-a \in \mathrm{Id}(T)$, whence $-a \in -a \cdot \mathrm{Id}(T) \cap \mathrm{Id}(T) \subseteq I$, and $a \in I$, absurd, or

(ii) $-a \in a \cdot \mathrm{Id}(T)$, i.e., $-a = az^2$, whence $-a^2 = a^2z^2 = (-a)^2 = a^2$, and then $a = 0 \in I$, absurd again.

By I.1.9 there is a character $h \in X_T$ so that $\Delta \subseteq P(h)$ —which yields $I \subseteq Z(h)$—, and $h(-a) = -1$, i.e., $h(a) = 1$. The last assertion is left to the reader. □

Definition I.1.11 For $c \in T$, let $I_c = \{x \in T \mid c^2x = x\}$. □

It is easily checked that I_c is the ideal of T generated by c (possibly improper).

Theorem I.1.12 (Separation theorem for ternary semigroups.) *Let T be a TS and let $a, b \in T$, $a \neq b$. Then, there is a TS-character h of T so that $h(a) \neq h(b)$. In other words, the set X_T of TS-characters separates points (in T). Equivalently, the evaluation map from T to $\mathbf{3}^{X_T}$ is an injective TS-homomorphism.*

Proof. See [DP1], Theorem 1.9, pp. 103–104. □

Together with Proposition I.1.14 below, Theorem I.1.12 implies a similar separation result for 3-semigroups; namely:

Theorem I.1.13 (Separation theorem for 3-semigroups.) *Let Δ be a 3-semigroup and let $a \neq b$ be in Δ. Then, there is a 3-semigroup character $h : \Delta \longrightarrow \mathbf{3}$ so that $h(a) \neq h(b)$. In particular, the set $\chi(\Delta)$ of 3-semigroup characters of Δ with values in $\mathbf{3}$ separates points (in Δ). Equivalently, the evaluation map from Δ to $\mathbf{3}^{\chi(\Delta)}$ is an injective homomorphism of 3-semigroups.* $\square$

Proposition I.1.14 *Every 3-semigroup Δ can be embedded (as a unitary semigroup) into a ternary semigroup $\widehat{\Delta}$. Hence, the restriction to Δ of any TS-character of $\widehat{\Delta}$ to Δ is a unitary semigroup character.*

Proof. Adding, if necessary, an absorbent element 0 to Δ, we can assume, without loss of generality, that Δ possesses such an element $0 \neq 1$. Let Δ' be a set disjoint from Δ, of the same cardinality as $\Delta \setminus \{0\}$, and let μ be a bijection from $\Delta \setminus \{0\}$ onto Δ'. With $\cdot$ denoting the product in Δ, we endow the set $\widehat{\Delta} := \Delta \cup \Delta'$ with a product operation $*$ defined by: for $x, y \in \widehat{\Delta}$,

(i) $x * 0 = 0 * x = 0$.

(ii) If $x, y \in \Delta \setminus \{0\}$, then $x * y = x \cdot y$.

(iii) If $x \in \Delta \setminus \{0\}$ and $y \in \Delta'$ with, say, $y = \mu(\delta)$, $\delta \in \Delta \setminus \{0\}$, then $x * y = \mu(x \cdot \delta)$.

(iv) If $x, y \in \Delta'$ with, say, $x = \mu(\delta), y = \mu(\delta')$, $\delta, \delta' \in \Delta \setminus \{0\}$, then $x * y = \delta \cdot \delta'$.

Claim. With $-1 := \mu(1)$, $\langle \widehat{\Delta}, *, 0, 1, -1 \rangle$ is a ternary semigroup.

Proof of Claim. Verification that $\widehat{\Delta}$ is a commutative semigroup with 0 as an absorbent element is straightforward and left to the reader.

— $x * 1 = x$ for all $x \in \widehat{\Delta}$.
By (i) and (ii) this is clear if $x \in \Delta$. If $x \in \Delta'$ and $x = \mu(\delta)$ with $\delta \in \Delta \setminus \{0\}$, by (iii) we have $x * 1 = \mu(\delta \cdot 1) = \mu(\delta) = x$.

— $\widehat{\Delta}$ satisfies axiom [TS3] in I.1.1.
$1 \neq -1$ since $1 \in \Delta \setminus \{0\}, -1 = \mu(1) \in \Delta'$ and $\Delta \cap \Delta' = \emptyset$. By (iv) we have $(-1)*(-1) = 1 \cdot 1 = 1$.

— $x^3 = x$ for all $x \in \widehat{\Delta}$ (axiom [TS2]).
This is clear if $x \in \Delta$. If $x \in \Delta'$ with, say, $x = \mu(\delta), \delta \in \Delta$, by (iv) we have $x^2 = x * x = \delta^2$ and, by (iii), $x^3 = x^2 * x = \mu(\delta^2 \cdot \delta) = \mu(\delta^3) = \mu(\delta) = x$.

— $\widehat{\Delta}$ satisfies axiom [TS5].
We prove the contrapositive. Let $x \in \widehat{\Delta} \setminus \{0\}$. If $x \in \Delta \setminus \{0\}$, since $-1 = \mu(1)$, from (iii) we get $-1 * x = \mu(1 \cdot x) = \mu(x) \in \Delta'$; hence, $-1 * x \neq x$, as Δ and Δ' are disjoint. If $x \in \Delta'$ with, say, $x = \mu(\delta), \delta \in \Delta$, from (iv) we get $-1 * x = 1 \cdot \delta = \delta \in \Delta$ and we conclude $-1 * x \neq x$.

Clearly, $\widehat{\Delta}$ extends Δ as a unitary semigroup. Note that the Claim and Theorem I.1.12 imply Theorem I.1.13. Note also that (iii) implies $-1 * \delta = \mu(\delta)$ for $\delta \in \Delta$. $\square$

Remark. The unitary semigroup character $\mathbb{1}$ sending all of Δ to 1 cannot be extended to a TS-character of $\widehat{\Delta}$. $\square$

Omit next Theorem.

Another offshoot of this technique is the following duality-type result (not used in the sequel).

Theorem I.1.15 *Let T be a TS. Let $\chi(T)$ denote the set of 3-semigroup homomorphisms from T to* $\mathbf{3}$ *(no preservation of* 0 *or* -1 *required). Under pointwise product, $\chi(T)$ is a 3-semigroup. With the topology induced by the product topology on* $\mathbf{3}^T$ *(discrete topology on* $\mathbf{3}$*), $\chi(T)$ is a compact, Hausdorff topological semigroup. The set $\mathcal{C}(\chi(T))$ of continuous semigroup homomorphisms from $\chi(T)$ to* $\mathbf{3}$ *(no preservation of* 1 *required) admits a structure of ternary semigroup, with the distinguished elements* $-1, 0, 1$ *represented by the respective constant functions on $\chi(T)$. Then, the evaluation map from T into $\mathcal{C}(\chi(T))$ is a TS-isomorphism.* □

I.1.16 Reminder. (Spectral spaces)

(a) A **spectral space** is a T_0, quasi-compact topological space [2] having a basis of quasi-compact open sets closed under finite intersections, and such that every closed irreducible set is the closure of a (necessarily unique) point.

(b) In any spectral space, the binary relation between points defined by:

$$x \rightsquigarrow y \quad \text{if and only if} \quad y \in \overline{\{x\}} \ (= \text{closure of } \{x\}),$$

is a partial order, called the **specialization** partial order (we say, y *specializes* x).

(c) If $\mathcal{B}$ is a basis of quasi-compact open sets of a spectral space, X, the family $\{U \setminus V \mid U, V \in \mathcal{B}\}$ forms the basis of another topology on X called the **constructible topology** and denoted by X_{con}. With this topology, X becomes a Boolean space (compact, Hausdorff, totally disconnected). The closed subsets of X_{con} are called **proconstructible**.

(d) A spectral space X is called **hereditarily normal** [3] iff for every $x \in X$ the set $\{y \in X \mid x \rightsquigarrow y\}$ is totally ordered under specialization.

A thorough development of the theory of spectral spaces will appear in [DST]; we adopt here the notation and terminology used in this monograph. Further references containing basic information on spectral spaces are [KS], Chapter 3, and [M], §6.3, pp. 111-114.

It is well-known that the real spectrum of a ring is a hereditarily normal spectral space; cf. [BCR], Prop. 7.1.22, p. 117; this is also the case of abstract real spectra, [M], Prop. 6.4.1, p. 114. □

I.1.17 Topologies on $X_T = \mathbf{Hom}(T, \mathbf{3})$, T a ternary semigroup.

(a) Let X_T denote the set of TS-characters of T, a subset of $\mathbf{3}^T$. The set X_T carries a spectral topology given by the sets

$$H(t_1, \ldots, t_n) = \bigcap\nolimits_{i=1}^{n} [\![\, t_i = 1 \,]\!] \qquad (t_i \in T),$$

as a basis of quasi-compact opens, where, for $t \in T$ and $i \in \{-1, 0, 1\}$, $[\![\, t = i \,]\!] = \{f \in X_T \mid f(t) = i\}$.

Shall we include a proof that this topology is spectral; or leave it as exercise?.

Lemma I.1.18 below gives several algebraic characterizations of the specialization partial order. Note that the specialization order in an arbitrary ternary semigroup may not be hereditarily normal, as shown in Example I.1.20 below.

(b) The associated constructible topology on the set X_T has as basis of clopens the sets

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[2] That is, compact in the usual sense but not necessarily Hausdorff.
[3] Also called *completely normal.*

$$\bigcap_{i=1}^{n} [\![t_i = 1]\!] \cap \bigcap_{j=1}^{m} [\![t'_j = 0]\!] \qquad\qquad (t_i, t'_j \in T)$$

The set X_T is a closed subset of $\mathbf{3}^T$, endowed with the product topology (discrete topology on $\mathbf{3}$); further, the latter induces precisely the constructible topology on X_T (see also [M], §6.3, pp. 110-112), denoted in the remainder of this text by $(X_T)_{\text{con}}$.

(c) Sets of type $[\![t = 1]\!]$ and $[\![t = 0]\!]$ are sometimes denoted by $U(t)$ and $Z(t)$, respectively; see, for example, [M], pp. 102-103. We shall use either of these notations. □

Added Nov. 2011 **Warning.** Throughout this monograph *the default topology* on all character spaces *is the spectral topology*. Whenever the associated constructible topology is used, the modifier $(.)_{\text{con}}$ will be attached to the name of the space. □

The next Lemma gives several characterizations of the specialization order in ternary semigroups that will be repeatedly used throughout this text.

Lemma I.1.18 *Let T be a TS, and let $g, h \in X_T$. The following are equivalent:*

(1) $g \rightsquigarrow h$ *(i.e., h is an specialization of g).*

(2) $h^{-1}[1] \subseteq g^{-1}[1]$ *(equivalently, $h^{-1}[-1] \subseteq g^{-1}[-1]$).*

(3) $g^{-1}[\{0,1\}] \subseteq h^{-1}[\{0,1\}]$.

(4) $Z(g) \subseteq Z(h)$ *and* $\forall\, a \in G\, (a \notin Z(h) \;\Rightarrow\; g(a) = h(a))$.

(5) $h = h^2 g$ *(equivalently, $h^2 = hg$).*

Proof. By definition, in any spectral space we have:

$$g \rightsquigarrow h \quad \text{iff} \quad h \in \overline{\{g\}} \quad \text{iff} \quad \text{for every subbasic open } U,\ \ h \in U \Rightarrow g \in U.$$

(cf. I.1.16). Since the subbasic opens of X_T, are the sets $\{h \in X_T \,|\, h(a) = 1\}$ for $a \in G$ (I.1.17), we get at once the equivalence of (1) and (2). By taking complements and replacing a by $-a$, (3) is equivalent to (2).

(1)/(3) $\Rightarrow$ (4). For the first assertion, if $g(a) = 0$, (3) gives $h(a) \in \{0,1\}$, but (2) precludes $h(a) = 1$. For the second, (2) gives $h(a) = 1 \Rightarrow g(a) = 1$; if $h(a) = -1$, just replace a by $-a$.

(4) $\Rightarrow$ (5). The identity $h = h^2 g$ obviously holds if $h(a) = 0$; if $h(a) \neq 0$, it follows from the second assertion in (4) and $h^2(a) = 1$.

(5) $\Rightarrow$ (2). $h = h^2 g$ and $h(a) = 1$ clearly imply $g(a) = 1$. □

We also register the following algebraic characterizations of inclusion and equality between the zero-sets of elements of X_T.

Lemma I.1.19 *Let T be a TS, and let $u, g, h \in X_T$. Then:*

(1) $Z(g) = Z(h) \Leftrightarrow g^2 = h^2$.

(2) $Z(g) \subseteq Z(h) \Leftrightarrow h^2 = h^2 g^2$.

(3) *If $u \rightsquigarrow g, h$, then $Z(g) \subseteq Z(h)$ if and only if $g \rightsquigarrow h$.*

Proof. (1) follows at once from (2).

(2) The implication ($\Leftarrow$) is trivial, and the reverse implication ($\Rightarrow$) is easily verified: if $g(x) \neq 0$, then $g^2(x) = 1$ and the equality holds; if $g(x) = 0$, then $h(x) = 0$, and both sides of the equality are 0.

(3) Obviously $u \rightsquigarrow g, h$ implies $Z(u) \subseteq Z(g), Z(h)$. The implication ($\Leftarrow$) is clear.

($\Rightarrow$) Assuming $Z(g) \subseteq Z(h)$, it suffices to verify the second clause of I.1.18(4). Let $a \notin Z(h)$. Then, $a \notin Z(g)$, and the equivalence of items (1) and (4) in I.1.18, together with $u \rightsquigarrow g$ and $u \rightsquigarrow h$ yields $u(a) = g(a)$ and $u(a) = h(a)$, respectively. Thus, $g(a) = h(a)$, as required. □

Example I.1.20 Let $T = \{-1, 0, 1, x_1, x_2, x_1 x_2, -x_1, -x_2, -x_1 x_2\}$ be the ternary semigroup on two generators x_1, x_2, with relations $x_i^2 = x_i$ $(i = 1, 2)$, The specialization order of X_T is not hereditarily normal.

Proof. Let $g, h_1, h_2 \in X_T$ be defined on generators by:

$$g: \ x_i \mapsto 1 \ (i = 1, 2); \qquad h_1 : \begin{cases} x_1 \mapsto 0 \\ x_2 \mapsto 1, \end{cases} \qquad h_2 : \begin{cases} x_1 \mapsto 1 \\ x_2 \mapsto 0 . \end{cases}$$

Using the equivalence of (1) and (5) in I.1.18 one easily verifies that $g \rightsquigarrow h_1$ and $g \rightsquigarrow h_2$, but $h_1 \not\rightsquigarrow h_2$, $h_2 \not\rightsquigarrow h_1$. □

However, below we show that the character space of a ternary semigroups is normal in the standard topological meaning of this notion: a topological space is **normal** if disjoint closed sets have disjoint open neighborhoods. (**Warning.** The space is not assumed to be Hausdorff nor T_1.) In the context of spectral spaces, normality admits the following characterization in terms of the specialization order:

Fact I.1.21 ([DST], Thm. 7.4.5) *A spectral space X is normal if and only every element has a unique specialization maximal in the poset $(X, \rightsquigarrow)$; i.e., for all $x \in X$ there is a unique $y \in X^{\max}$ such that $x \rightsquigarrow y$.* □

Proposition I.1.22 *The spectral space X_T, T a ternary semigroup (with topology as in I.1.17(a)), is normal.*

Proof. Using the characterization in I.1.21, assume, towards a contradiction, that there are $g \in X_T$ and $h_1 \neq h_2$ in $(X_T)^{\max}$ such that $g \rightsquigarrow h_1, h_2$.

Check whether refs. to [DP1] in this proof can be replaced by internal refs.

Claim 1. There is $t \in T$ such that $h_1(t) \neq 0$ and $h_2(t) = 0$.
Proof of Claim 1. Otherwise, for all $t \in T$, $h_2(t) = 0 \Rightarrow h_1(t) = 0$, i.e., $Z(h_2) \subseteq Z(h_1)$. Recall (Lemma I.1.18 (4)) that $g \rightsquigarrow h_2$ implies $Z(g) \subseteq Z(h_2)$. Let $x \notin Z(h_1)$. From $g \rightsquigarrow h_1$ and I.1.18 (4) we get $g(x) = h_1(x)$; likewise, $g \rightsquigarrow h_2$ entails $g(x) = h_2(x)$; hence, $h_1(x) = h_2(x)$. By I.1.18 (4) again, $h_2 \rightsquigarrow h_1$ and, from $h_2 \in (X_T)^{\max}$ we get $h_1 = h_2$, contradiction. □

Reversing the roles of h_1 and h_2 we get an element $t' \in T$ such that $h_2(t') \neq 0$ and $h_1(t') = 0$.

Let $I_t = \{x \in T \mid t^2 x = x\}$ be the ideal of T generated by t (cf. [DP1], Def. 1.8, p. 103). We have $t' \notin I_t$; else, $t^2 t' = t'$, whence $h_2(t') = h_2(t)^2 h_2(t') = 0$, contradiction.

Let I be an ideal of T containing I_t and maximal for not containing t'; any such ideal is prime, cf. [DP1], Fact 1.3, p. 101. Let $h \in \mathbf{3}^T$ be defined by:

$$h \lceil I = 0 \quad \text{and} \quad h \lceil (T \setminus I) = h_1 \lceil (T \setminus I).$$

Claim 2. $h \in X_T$.

Proof of Claim 2. Clearly, $h(i) = i$ for $i \in \mathbf{3}$. It remains to prove that $h(t_1t_2) = h(t_1)h(t_2)$ for $t_1, t_2 \in T$. There are two cases:

— $t_1t_2 \in I$.

Since I is prime, either t_1 or t_2 is in I. Hence, both terms of the required equality are 0.

— $t_1t_2 \notin I$.

Then, $t_1, t_2 \notin I$, and we have:

$$h(t_1t_2) = h_1(t_1t_2) =^{h_1 \in X_T} h_1(t_1)h_1(t_2) =^{t_i \notin I} h(t_1)h(t_2). \qquad \square$$

Further, we have

Claim 3. $h_1 \rightsquigarrow h$.

Proof of Claim 3. We check by cases the equality $h = h^2h_1$ in I.1.18 (4). If $x \in I$, the value of both sides at x is 0. If $x \notin I$, then $h(x) = h_1(x)$ and either both these terms are 0, or else $h^2(x) = 1$; in either case the equality holds. $\square$

Since $h_1 \in (X_T)^{\max}$, we get $h_1 = h$. But $t \in I$; so, $h(t) = 0$, while $h_1(t) \neq 0$, contradiction. $\square$

Remark. Given a TS-homomorphism $f : T_1 \longrightarrow T_2$, the dual map $f^* : X_{T_2} \longrightarrow X_{T_1}$ defined by $f^*(h) := h \circ f$ for $h \in X_{T_2}$, is spectral: routine verification shows that, for $t_1, \ldots, t_n \in T$ $(n \geq 1)$:

$$(f^*)^{-1}(H(t_1, \ldots, t_n) = H(f(t_1), \ldots, f(t_n)). \qquad \square$$

The Proposition that follows —a generalization of Lemma I.1.18— gives a structural relationship between non-negativity, product and specialization of finitely many ternary semigroup characters at a time.

Proposition I.1.23 *Let T be a TS, and let $h, h_1, \ldots, h_n \in X_T$ be TS-characters. The following are equivalent:*

(i) $\bigcap_{i=1}^n P(h_i) \subseteq P(h)$;

(ii) There is $R \subseteq \{1, \ldots, n\}$ of odd cardinality such that $h = h^2 \cdot \prod_{i \in R} h_i$;

(iii) There is $R \subseteq \{1, \ldots, n\}$ of odd cardinality such that $(\prod_{i \in R} h_i) \rightsquigarrow h$.

Proof. (ii) $\Leftrightarrow$ (iii) is the equivalence (1) $\Leftrightarrow$ (5) in Lemma I.1.18 (since card(R) is odd, $\prod_{i \in R} h_i \in X_T$), and (iii) $\Rightarrow$ (i) follows from (1) $\Rightarrow$ (3) in that Lemma.

(i) $\Rightarrow$ (ii). Induction on n. The case $n = 1$ is (3) $\Rightarrow$ (5) in I.1.18. We assume, then, that $n \geq 2$ and the implication holds for all $k < n$. If there is $j \in \{1, \ldots n\}$ such that $\bigcap_{i=1, i \neq j}^n P(h_i) \subseteq P(h)$, the induction hypothesis applies and the conclusion follows. So, we also assume

(I) For all $j \in \{1, \ldots n\}$, $\bigcap_{i=1, i \neq j}^n P(h_i) \not\subseteq P(h)$.

For each $j \in \{1, \ldots n\}$ choose an element $x_j \in T$ such that

(II) $h_i(x_j) \geq 0$ for all $i \in \{1, \ldots n\}, i \neq j$, and $h(x_j) = -1$.

From $\bigcap_{i=1}^n P(h_i) \subseteq P(h)$ follows

(III) $h_j(x_j) = -1$ $(j = 1, \ldots, n)$.

Since $Z(g) = P(g) \cap P(-g)$ $(g \in X_T)$, our assumption implies

(IV) $\bigcap_{i=1}^n Z(h_i) \subseteq Z(h)$.

Since $Z(h)$ is a prime ideal, (IV) entails the existence of $m \in \{1, \ldots n\}$ such that $Z(h_m) \subseteq Z(h)$ [otherwise, for each $m \in \{1, \ldots n\}$ there is $y_m \in T$ such that $h_m(y_m) = 0$ and $h(y_m) \neq 0$;

setting $y := \prod_m y_m$ we have $h(y) \neq 0$ ($Z(h)$ prime) but $h_m(y) = 0$ for all m, contrary to (IV)].

We set,

$$R := \{j \in \{1, \ldots, n\} \mid Z(h_j) \subseteq Z(h)\}\ (\neq \emptyset).$$

If $j \in \{1, \ldots, n\} \setminus R$, then $Z(h_j) \not\subseteq Z(h)$, and there is $a_j \in T$ such that

(V) $h_j(a_j) = 0$ and $h(a_j) \neq 0$ $(j \in \{1, \ldots, n\} \setminus R)$.

Let $a := \prod_{j \notin R} a_j$ and $x := \prod_{j \in R} x_j$. Clearly, $h(a) \neq 0$, whence $h(a^2) = 1$.

Assuming card(R) even, by (II) we have $h(x) = 1$, whence

(VI) $h(a^2 x) = 1$.

On the other hand we have

(VII) $h_j(a^2 x) \in \{0, -1\}$ for $j = 1, \ldots, n$.

In fact, if $j \notin R$, (V) entails $h_j(a) = 0$, and hence $h_j(a^2 x) = 0$. If $j \in R$, we have $h_j(x) = \prod_{k \in R} h_j(x_k) = h_j(x_j) \cdot \prod_{k \in R, k \neq j} h_j(x_k)$. From (IV) and $h(x) = 1$ comes $h_j(x) \neq 0$, and then $h_j(x_k) \neq 0$ which, by (II), yields $h_j(x_k) = 1$ for $k \in R \setminus \{j\}$. Altogether this gives $h_j(x) = -1 \cdot 1 = -1$ for $j \in R$, implying $h_j(a^2 x) = -h_j(a^2) \in \{0, -1\}$ for $j \in R$, as asserted.

From (VII) we get $-a^2 x \in \bigcap_{i=1}^n P(h_i) \subseteq P(h)$, i.e., $h(a^2 x) \leq 0$, contradicting (VI).

Conclusion: card(R) is odd.

Claim. $h = h^2 \cdot \prod_{j \in R} h_j$.

Proof of Claim. Fix $y \in T$. The equality is clear if $h(y) = 0$. Assume $h(y) = 1$ and the right-hand side $\neq 1$. From $y \notin Z(h)$ we get $y \notin Z(h_j)$ for $j \in R$, and hence $\prod_{j \in R} h_j(x) = -1$.

Let $S := \{j \in R \mid h_j(y) = -1\}$; then card($S$) is odd. Let $z := a^2 y \prod_{j \in S} x_j$. Since $h(y) = 1$, $h(a^2) = 1$, $h(x_j) = -1$ $(j \in S)$ (by (II)), and card(S) is odd, we get $h(z) = -1$.

Let $j \in \{1, \ldots, n\}$. We consider three cases.

(i) $j \notin R$. By (V) we have $h_j(a_j) = 0$, whence $h_j(a) = 0$, and $h_j(z) = 0$.

(ii) $j \in S$. Then (invoking (III) for the last equality) we have:

(†) $$h_j\left(\prod_{k \in S} x_k\right) = \prod_{k \in S} h_j(x_k) = h_j(x_j) \cdot \prod_{k \in S \setminus \{j\}} h_j(x_k) = -\prod_{k \in S \setminus \{j\}} h_j(x_k).$$

By (II), $h_j(x_k) \geq 0$ for $k \in S \setminus \{j\}$. Since $j \in R$, if $h_j(x_k) = 0$ we would have $h(x_k) = 0$, contradicting (II). So, $h_j(x_k) = 1$ for all $k \in S \setminus \{j\}$, whence (from (†)), $h_j(\prod_{k \in S} x_k) = -1$. Since $h_j(y) = -1$, we get $h_j(y \cdot \prod_{k \in S} x_k) = 1$, and then $h_j(z) = h_j(a^2) \geq 0$.

(iii) $j \in R \setminus S$. In this case $h_j(y) = 1$ and, since $j \neq k$ for $k \in S$, from (II) we get $h_j(x_k) = 1$ for all $k \in S$, whence $h_j(\prod_{k \in S} x_k) = \prod_{k \in S} h_j(x_k) = 1$. It follows that $h_j(z) = h_j(a^2) \geq 0$.

In all three cases we have $z \in P(h_j)$, whence $z \in \bigcap_{i=1}^n P(h_i) \subseteq P(h)$, contradicting $h(z) = -1$. This proves that the product in the right-hand side of the statement is 1, i.e., the Claim, whenever $h(y) = 1$.

If $h(y) = -1$, replacing y by $-y$ the same argument proves the required equality, showing that the Claim holds, and completing the proof of the Proposition. □

Corollary I.1.24 *Let T be a TS, and let $h, h_1, \ldots, h_n \in X_T$ be TS-characters. If $\bigcap_{i=1}^n P(h_i) \subseteq P(h)$, there is a set $R \subseteq \{1, \ldots, n\}$ of odd cardinality such that $\bigcap_{i\in R} Z(h_i) \subseteq Z(h)$. In particular, $Z(h_i) \subseteq Z(h)$ for some $i \in \{1, \ldots, n\}$.*

Proof. By the preceding Proposition, $h = h^2 \cdot \prod_{i\in R} h_i$, which clearly implies the conclusion. $\square$

Quotients of ternary semigroups. In the remainder of this section we develop a general theory of quotients in the category of ternary semigroups. Since the class of ternary semigroups is not an equational class, we define explicitly the notions of *congruence* and of *quotient* in this category.

Definition I.1.25 A congruence of ternary semigroups (abbreviated **TS-congruence**) is an equivalence relation $\equiv$ on a TS, G, compatible with the semigroup operation and such that the induced quotient structure $G/\!\equiv$ is a ternary semigroup. [This is equivalent to require $\equiv$ to be proper, i.e. $\equiv\, \subset G \times G$, and for $x \in G$, $\;x \equiv -x \;\Rightarrow\; x \equiv 0$.]

$Con(G)$ denotes the set of all TS-congruences of G. $\square$

Remarks I.1.26 (a) The condition that $\equiv$ is proper ensures that $1 \not\equiv 0$, and hence, by the last requirement, $1 \not\equiv -1$.

(b) Since the axioms for TSs are universal, the quotient map $\pi_{\equiv} : G \longrightarrow G/\!\equiv$ is automatically a TS-homomorphism.

(c) For each non-empty set $\mathcal{H} \subseteq X_G$, the relation

$$a \equiv_{\mathcal{H}} b \quad\Leftrightarrow\quad \text{For all } h \in \mathcal{H},\ h(a) = h(b),$$

$(a, b \in G)$ defines a TS-congruence of G (straightforward checking). We shall write $G/\mathcal{H}$ for the quotient TS $G/\!\equiv_{\mathcal{H}}$. $\square$

Our main theorem I.1.27 below shows that every TS-congruence of a ternary semigroup G is of the form $\equiv_{\mathcal{H}}$ for a suitable set $\mathcal{H}$ of TS-characters. More precisely, with X_G endowed with the constructible topology (cf. I.1.17), let $C(G)$ denote the family of all closed (i.e., proconstructible) subsets of X_G which are also closed under products of any three of its elements. We have:

Theorem I.1.27 *Let G be a ternary semigroup. Then the map $\mu : C(G) \longrightarrow Con(G)$ defined by $\mu(\mathcal{H}) := \,\equiv_{\mathcal{H}}$ is an order-isomorphism of the poset $C(G)$ (under inclusion) onto the dual poset of $Con(G)$. Moreover, the map $\theta : X_{G/\mathcal{H}} \longrightarrow \mathcal{H}$ given by $\theta(g) = g \circ \pi$ $(g \in X_{G/\mathcal{H}})$ is well-defined and establishes a homeomorphism between the (spectral) spaces $X_{G/\mathcal{H}}$ and $\mathcal{H}$ (hence also a homeomorphism between the Boolean spaces $(X_{G/\equiv})_{\mathrm{con}}$ and $\mathcal{H}_{\mathrm{con}}$, endowed with their constructible topologies).*

Proof. (1) μ is injective. Let $\mathcal{H}_1, \mathcal{H}_2 \in C(G)$ be such that $\mu(\mathcal{H}_1) = \mu(\mathcal{H}_2)$, i.e., $\equiv_{\mathcal{H}_1} = \equiv_{\mathcal{H}_2}$, and assume, towards a contradiction, that $\mathcal{H}_1 \neq \mathcal{H}_2$. Suppose without loss of generality that $\mathcal{H}_1 \not\subseteq \mathcal{H}_2$ and take $h_1 \in \mathcal{H}_1 \setminus \mathcal{H}_2$. Since $\mathcal{H}_2$ is closed, recalling the shape of the sets forming a basis for the constructible topology of X_G (see I.1.17 above), there are elements $a_1, \ldots, a_n, b_1, \ldots, b_k$ in G such that $h_1 \in U(a_1) \cap \ldots \cap U(a_n) \cap Z(b_1) \cap \ldots \cap Z(b_k)$ and $U(a_1) \cap \ldots \cap U(a_n) \cap Z(b_1) \cap \ldots \cap Z(b_k) \cap \mathcal{H}_2 = \emptyset$. Let $c = \prod_{i=1}^n a_i^2$. It is plain that

$$(*) \qquad \begin{aligned} &h_1 \in U(c\,a_1) \cap \ldots \cap U(c\,a_n) \cap Z(b_1) \cap \ldots \cap Z(b_k) \quad \text{and} \\ &U(c\,a_1) \cap \ldots \cap U(c\,a_n) \cap Z(b_1) \cap \ldots \cap Z(b_k) \cap \mathcal{H}_2 = \emptyset. \end{aligned}$$

First we claim that $n \neq 0$. Assuming otherwise, we consider the following cases:

Case 1) $n = 0$ and $k = 1$. Hence $Z(b_1) \cap \mathcal{H}_2 = \emptyset$, which means $b_1^2 \equiv_{\mathcal{H}_2} 1$. From our assumptions we get $b_1^2 \equiv_{\mathcal{H}_1} 1$, contradicting that $h_1(b_1) = 0$.

Case 2) $n = 0$ and $k \geq 2$. Let k be the smallest natural number such that there are elements $b_1, \dots, b_k$ so that $h_1 \in Z(b_1) \cap \dots \cap Z(b_k)$ and $Z(b_1) \cap \dots \cap Z(b_k) \cap \mathcal{H}_2 = \emptyset$. Then, $Z(b_1) \cap \dots \cap Z(b_{k-1}) \cap \mathcal{H}_2 \neq \emptyset$ and $Z(b_2) \cap \dots \cap Z(b_k) \cap \mathcal{H}_2 \neq \emptyset$, i.e., there are $h_2, h_3 \in \mathcal{H}_2$ such that $h_2(b_1) = \dots = h_2(b_{k-1}) = 0$ and $h_3(b_2) = \dots = h_3(b_k) = 0$. Since $\mathcal{H}_2$ is closed under products of any three elements, $h_2^2 h_3 \in \mathcal{H}_2$, and since $h_2^2 h_3(b_i) = 0$ for $1 \leq i \leq k$, we get $h_2^2 h_3 \in \mathcal{H}_2 \cap Z(b_1) \cap \dots \cap Z(b_k)$, contradiction. Hence, $n \neq 0$.

Suppose that $n = 1$. We have the following cases:

Case 3) $n = 1$ and $k = 0$. Then $U(a_1) \cap \mathcal{H}_2 = \emptyset$, which implies $a_1 \equiv_{\mathcal{H}_2} -a_1^2$ and hence $a_1 \equiv_{\mathcal{H}_1} -a_1^2$, contradicting $h_1(a_1) = 1$.

Case 4) $n = 1$ and $k = 1$. Hence $U(a_1) \cap Z(b_1) \cap \mathcal{H}_2 = \emptyset$. Since $h_1(a_1) = 1, h_1(b_1) = 0$ and $h_1 \in \mathcal{H}_1$, it follows that $a_1^2 \not\equiv_{\mathcal{H}_1} a_1^2 b_1^2$, and then $a_1^2 \not\equiv_{\mathcal{H}_2} a_1^2 b_1^2$. Therefore, there is $h_2 \in \mathcal{H}_2$ such that $h_2(b_1) = 0$ and $h_2(a_1) \neq 0$. From $U(a_1) \cap Z(b_1) \cap \mathcal{H}_2 = \emptyset$ comes $h_2(a_1) = -1$. On the other hand, by case 3) there is $h_3 \in \mathcal{H}_2$ such that $h_3(a_1) = 1$. Then, $h_3 h_2^2 \in \mathcal{H}_2$, $h_3 h_2^2(a_1) = 1$ and $h_3 h_2^2(b_1) = 0$, absurd, since $U(a_1) \cap Z(b_1) \cap \mathcal{H}_2 = \emptyset$.

Case 5) $n = 1$ and $k \geq 2$. Let k be the least integer such that there are $a_1, b_1, \dots, b_k \in G$ with $U(a_1) \cap Z(b_1) \cap \dots \cap Z(b_k) \cap \mathcal{H}_2 = \emptyset$. Then, there are $h_2, h_3 \in \mathcal{H}_2$ satisfying $h_2(a_1) = h_3(a_1) = 1$, $h_2(b_1) = \dots = h_2(b_{k-1}) = 0$ and $h_3(b_2) = \dots = h_3(b_k) = 0$. It follows that $h_2^2 h_3 \in \mathcal{H}_2 \cap U(a_1) \cap Z(b_1) \cap \dots \cap Z(b_k)$, contradiction.

Thus, we have shown that $n > 1$. Let n be the smallest natural number such that (*) holds for some $a_1, \dots, a_n, b_1, \dots, b_k \in G$, $k \geq 0$. By minimality of n, for each index $i \in \{1, \dots, n\}$ there is $g_i \in \mathcal{H}_2$ such that $g_i \in U(c\,a_1) \cap \dots \cap U(c\,a_{i-1}) \cap U(c\,a_{i+1}) \cap \dots \cap U(c\,a_n) \cap Z(b_1) \cap \dots \cap Z(b_k)$. Since $g_i(c) = 1$, we get $g_i(a_i) \neq 0$, and since $U(c\,a_1) \cap \dots \cap U(c\,a_n) \cap Z(b_1) \cap \dots \cap Z(b_k) \cap \mathcal{H}_2 = \emptyset$ we get $g_i(a_i) = -1$. Note also that $g_i(a_j) = 1$ for $i \neq j$. On the other hand, since $h_1 \in U(a_1 \cdot a_2 \cdot \dots \cdot a_n) \cap Z(b_1) \cap \dots \cap Z(b_k)$, from cases 4), 5) we get $U(a_1 \cdot a_2 \cdot \dots \cdot a_n) \cap Z(b_1) \cap \dots \cap Z(b_k) \cap \mathcal{H}_2 \neq \emptyset$. Let g_{n+1} belong to this intersection. In particular, $g_{n+1}(a_1 \cdot \dots \cdot a_n) = 1$. Since $U(c\,a_1) \cap \dots \cap U(c\,a_n) \cap Z(b_1) \cap \dots \cap Z(b_k) \cap \mathcal{H}_2 = \emptyset$, we must have $g_{n+1}(a_i) = -1$ for some index i. It is immediate to see that the set $\{i \in \{1, \dots, n\} \mid g_{n+1}(a_i) = -1\}$ has even cardinality; let $\{i_1, \dots, i_{2k}\}$ be an enumeration of it. Let $g = g_{n+1} \cdot \prod_{j=1}^{2k} g_{i_j}$. Since g is the product of an odd number of elements of $\mathcal{H}_2$, we get $g \in \mathcal{H}_2$. Clearly $g \in Z(b_1) \cap \dots \cap Z(b_k)$. Let $i \in \{1, \dots, n\}$. If $i \notin \{i_1, \dots, i_{2k}\}$, then $g_{i_j}(a_i) = 1$ for every index j and $g_{n+1}(a_i) = 1$, implying $g(a_i) = 1$. If $i = i_j$ for some index $j \in \{1, \dots, 2k\}$, then $g_{i_j}(a_{i_j}) = -1$, $g_{i_\ell}(a_{i_j}) = 1$ for $\ell \neq j$, and $g_{n+1}(a_{i_j}) = -1$. So we conclude again that $g(a_i) = 1$. Therefore $g \in U(a_1) \cap \dots \cap U(a_n) \cap Z(b_1) \cap \dots \cap Z(b_k) \cap \mathcal{H}_2$ and then $g \in U(c\,a_1) \cap \dots \cap U(c\,a_n) \cap Z(b_1) \cap \dots \cap Z(b_k) \cap \mathcal{H}_2$, a contradiction. These contradictions show that $\mathcal{H}_1 \subseteq \mathcal{H}_2$. A similar argument proves the other inclusion, so $\mathcal{H}_1 = \mathcal{H}_2$, and then μ is injective.

Remark. The proof above (cf. cases (1), (3) and (4)) also shows that $\equiv_{\mathcal{H}_2} \subseteq \equiv_{\mathcal{H}_1} \Rightarrow \mathcal{H}_1 \subseteq \mathcal{H}_2$ (a fact that we will use later).

(2) μ is surjective. Let $\equiv$ be a congruence of G. We define the following subset of X_G:

$$\mathcal{H} = \{h \in X_G \,|\, a \equiv b \text{ implies } h(a) = h(b) \text{ for all } a, b \in G\}.$$

It is easily checked that $\mathcal{H} \in C(G)$. We claim that $\equiv_{\mathcal{H}}$ is identical to $\equiv$. To prove this, let $a, b \in G$ be such that $a \equiv_{\mathcal{H}} b$ and suppose $a \not\equiv b$; then $\pi(a) \neq \pi(b)$, where $\pi : G \longrightarrow G/{\equiv}$ is the canonical projection. By the separation theorem for ternary semigroups (I.1.12), there exists $h \in X_{G/\equiv}$ such that $h(\pi(a)) \neq h(\pi(b))$. But $h \circ \pi \in \mathcal{H}$, contradicting $a \equiv_{\mathcal{H}} b$. Hence $\equiv_{\mathcal{H}} \subseteq \equiv$. The inclusion $\equiv \subseteq \equiv_{\mathcal{H}}$ follows at once from the definition of $\mathcal{H}$. Therefore μ is surjective.

It is clear that $\mathcal{H} \subseteq \mathcal{H}'$ implies $\mu(\mathcal{H}) \supseteq \mu(\mathcal{H}')$ for $\mathcal{H}, \mathcal{H}' \in C(G)$. To complete the proof we must show:

(3) The map $\theta : X_{G/\mathcal{H}} \longrightarrow X_G$ given by $\theta(g) = g \circ \pi$ is a well-defined homeomorphism between the spectral spaces $X_{G/\mathcal{H}}$ and $\mathcal{H}$.

(3.i) We first show that $g \circ \pi \in \mathcal{H}$ for every $g \in X_{G/\mathcal{H}}$, and $\mathcal{H} = \mathrm{Im}(\theta)$.

To ease notation we set $\widehat{\mathcal{H}} := \mathrm{Im}(\theta))$. Routine argument shows that, for $h \in \mathcal{H}$ the map $g : X_{G/\mathcal{H}} \longrightarrow \mathbf{3}$ given by $g \circ \pi = h$ is a well-defined TS-character. Hence, $\mathcal{H} \subseteq \widehat{\mathcal{H}}$. It is also clear that $\widehat{\mathcal{H}}$ is closed under the product of any three of its elements. We claim:

(*) $\widehat{\mathcal{H}}$ is a proconstructible subset of X_G.

Proof of (*). Let $q \in C\ell(\widehat{\mathcal{H}})$ (= closure of $\widehat{\mathcal{H}}$ in the constructible topology of X_G). Let $g : G/\mathcal{H} \longrightarrow \mathbf{3}$ be the map $g(\pi(a)) = q(a)$. To show that g is well-defined assume, towards a contradiction, that $\pi(a) = \pi(b)$, i.e., $a \equiv_{\mathcal{H}} b$, but $q(a) \neq q(b)$, for some $a, b \in G$. Then the set $U = \{p \in X_G \,|\, p(a) \neq p(b)\}$ is a neighborhood of q, and since $q \in C\ell(\widehat{\mathcal{H}})$ there is $r \in \widehat{\mathcal{H}}$ such that $r(a) \neq r(b)$. Then, there is $k \in X_{G/\mathcal{H}}$ such that $k \circ \pi = r$. Since $\pi(a) = \pi(b)$, we get $r(a) = r(b)$, contradiction.

Clearly g is a TS-character, i.e., $g \in X_{G/\mathcal{H}}$. It follows that $q \in \widehat{\mathcal{H}}$, proving $C\ell(\widehat{\mathcal{H}}) = \widehat{\mathcal{H}}$, as asserted. In particular, we have $\widehat{\mathcal{H}} \in C(G)$.

Next we observe:

(**) $\equiv_{\mathcal{H}} = \equiv_{\widehat{\mathcal{H}}}$.

The inclusion $\supseteq$ is clear from $\mathcal{H} \subseteq \widehat{\mathcal{H}}$. Conversely, if $a \equiv_{\mathcal{H}} b$, then $\pi(a) = \pi(b)$, and $g(\pi(a)) = g(\pi(b))$ for all $g \in X_{G/\mathcal{H}}$. By the definition of $\widehat{\mathcal{H}}$ this proves $a \equiv_{\widehat{\mathcal{H}}} b$, as required.

In other words, (**) proves $\mu(\mathcal{H}) = \mu(\widehat{\mathcal{H}})$. Since μ is injective, we conclude $\mathcal{H} = \widehat{\mathcal{H}}$, i.e., $\mathcal{H} = \mathrm{Im}(\theta)$, proving (3.i). By its own definition it is quite clear that θ is injective.

(3.ii) θ is a homeomorphism between $X_{G/\mathcal{H}}$ and $\mathcal{H}$ (spectral topologies).

Taking into account the shape of the basic opens for the spectral topologies of $X_{G/\mathcal{H}}$ and $\mathcal{H}$ (cf. I.1.17), this is an immediate consequence of the following identities which are checked without difficulty: for $a \in G$,

$$(***) \qquad \theta^{-1}[U(a) \cap \mathcal{H}] = U(\pi(a)), \qquad \theta[U(\pi(a))] = U(a) \cap \mathcal{H}.$$

Thus, θ and θ^{-1} are injective spectral maps, i.e., θ is a spectral isomorphism. Since the equalities (***) hold as well with U replaced by Z, the map θ is also a homeomorphism for the corresponding constructible topologies. $\square$

Next we shall give a different characterization of congruences of ternary semigroups. We will show that every such congruence in G, say, arises from a congruence in the subsemigroup $\mathrm{Id}(G)$ of idempotents of G —an equational class—, together with another subsemigroup of G, satifying some mild compatibility conditions (and conversely). We shall first deal briefly with congruences in the algebraic structures corresponding to the set $\mathrm{Id}(G)$.

Definition I.1.28 A **2-semigroup** is a structure $(G, \cdot, 1, 0)$ with individual constants $0, 1$, and a binary operation "$\cdot$" such that:

[2S.1] $(G, \cdot, 1)$ is a commutative semigroup with unit 1.

[2S.2] $x^2 = x$ for all $x \in G$.

[2S.3] $x \cdot 0 = 0$ for all $x \in G$.

Remarks I.1.29 (i) Scaling the identity [2S.2] by x yields that a 2-semigroup is automatically a 3-semigroup. If G is a ternary semigroup, the set $\mathrm{Id}(G)$ of idempotents of G is a 2-semigroup with the induced multiplication. The set $-\mathrm{Id}(G)$ is also a 2-semigroup under the operation $x \odot y = -(x \cdot y)$, with constants -1 and 0.

(ii) The class of 2-semigroups is clearly an equational class. Moreover, 2-semigroups are in essence the same thing as *bounded join-semilattices*. Indeed, if G is a 2-semigroup, it is obvious that the binary relation:

$$a \leq b \text{ if and only if } b = ab, \qquad (a, b \in G)$$

is a partial order which makes G into a bounded join-semilattice, where join is product, 1 is the first element, and 0 is the last element. Conversely, if L is a bounded join-semilattice with first element $\perp$ and last element $\top$, then L is a 2-semigroup where the product of two elements a, b is the join $a \vee b$, the unit is $\perp$, and the absorbent element 0 is $\top$. □

We denote by $\mathbf{2} = \{0, 1\}$ the 2-semigroup with two elements and by $\mathrm{Hom}(G, \mathbf{2})$ the set of all 2-semigroup homomorphisms of G into $\mathbf{2}$ (also called *characters*). Note that an ideal of a 2-semigroup contains 0 (cf. I.1.4). A standard argument shows that the prime ideals of a 2-semigroup are exactly the kernels of its characters. Further, the characters of a 2-semigroup separate points:

Proposition I.1.30 *Let G be a 2-semigroup and let $a, b \in G$. If $a \neq b$, there exists $h \in \mathrm{Hom}(G, \mathbf{2})$ such that $h(a) \neq h(b)$.*

Proof. If $a \neq b$, then either $a \neq ab$ or $b \neq ab$. Suppose, without loss of generality, that $a \neq ab$. Then $a \notin I_b$, where $I_b = \{bx \mid x \in G\}$ is the ideal generated by b: for if $a = bx$ for some $x \in G$, then $ab = b^2x = bx = a$, contradiction. By Zorn's lemma pick an ideal I of G containing I_b, maximal for $a \notin I$. I is prime, for if $xy \in I$ but $x \notin I$, then $a \in I[x]$, where $I[x] = I \cup \{xz \mid z \in G\}$ is the ideal generated by $I \cup \{x\}$. Hence $a = xz$ for some $z \in G$. Likewise, if $y \notin I$, then $a = yw$ for some $w \in G$. It follows that $a = a^2 = xywz$, and the assumption $xy \in I$ yields $a \in I$, contradiction. The map into $\mathbf{2}$ with kernel I is the desired character. □

The congruences of 2-semigroups admit a characterization similar to that of Theorem I.1.27. The arguments are about the same, replacing "$\mathbf{3}$" by "$\mathbf{2}$" everywhere. Let G be a 2-semigroup. With notation as in I.1.27, $\mathrm{Hom}(G, \mathbf{2})$ is a closed subset of $\mathbf{2}^G$ (product of discrete topology in $\mathbf{2}$). Note also that, under pointwise defined product, $\mathrm{Hom}(G, \mathbf{2})$ is a subsemigroup of $\mathbf{2}^G$. For $\mathcal{H} \subseteq \mathrm{Hom}(G, \mathbf{2})$ define an equivalence relation $\equiv_{\mathcal{H}}$ on G exactly as in I.1.26 (c). *Con*(G) denotes the set of congruences of G. Then we have:

Theorem I.1.31 *Let G be a 2-semigroup. Then,*

(i) For each subset $\mathcal{H}$ of $\mathrm{Hom}(G, \mathbf{2})$*, the relation $\equiv_{\mathcal{H}}$ is a congruence of G.*

(ii) If $C(G)$ denotes the poset (under inclusion) of all closed subsets of $\mathrm{Hom}(G, \mathbf{2})$ *that are closed under product, then the map $\mu : C(G) \longrightarrow Con(G)$ given by $\mu(H) = \equiv_{\mathcal{H}}$ is an order isomorphism between $C(G)$ and the dual poset of $Con(G)$.*

Remarks and notation. Notation will be as in I.1.17 and Theorem I.1.27. Since 2-semigroup characters are $\mathbf{2}$-valued, we have $U(a)^c = Z(a)$ for $a \in G$. Note also that $U(a) \cap U(b) = U(ab)$; hence the sets $U(a) \cap Z(b_1) \cap \ldots \cap Z(b_k)$, $a, b_1, \ldots, b_k \in G$ $(k \geq 0)$, form a basis for the constructible topology of $\mathrm{Hom}(G, \mathbf{2})$.

Sketch of proof. (i) is immediate. To prove (ii), let $\mathcal{H}_1, \mathcal{H}_2$ be sets in $C(G)$ such that $\mu(\mathcal{H}_1) = \mu(\mathcal{H}_2)$. Assuming, as in I.1.27, that $\mathcal{H}_1 \neq \mathcal{H}_2$ and letting $h_1 \in \mathcal{H}_1 \setminus \mathcal{H}_2$, since $\mathcal{H}_2$ is closed, there are $a, b_1, \ldots, b_k \in G$ such that

(*) $\qquad h_1 \in U(a) \cap Z(b_1) \cap \ldots \cap Z(b_k)$ and $U(a) \cap Z(b_1) \cap \ldots \cap Z(b_k) \cap \mathcal{H}_2 = \emptyset$.

If $U(a) \cap \mathcal{H}_2 = \emptyset$, we get $a \equiv_{\mathcal{H}_2} 0$, whence $a \equiv_{\mathcal{H}_1} 0$, contrary to $h_1 \in U(a)$. So $U(a) \cap \mathcal{H}_2 \neq \emptyset$, implying $k \geq 1$. Taking k minimal so that (*) holds, we observe that $k > 1$.

Indeed, if $k = 1$, i.e., $U(a) \cap Z(b_1) \cap \mathcal{H}_2 = \emptyset$, we claim that $a \equiv_{\mathcal{H}_2} ab_1$. In fact, $h(a) = 1$ and $h(b_1) = 0$ for some $h \in \mathcal{H}_2$, would imply $h \in U(a) \cap Z(b_1) \cap \mathcal{H}_2$, contradiction; hence, $h(a) = h(ab_1)$ for all $h \in \mathcal{H}_2$, yielding the asserted congruence. From the assumption $\equiv_{\mathcal{H}_1} = \equiv_{\mathcal{H}_2}$ follows, then, $a \equiv_{\mathcal{H}_1} ab_1$, and hence $h_1(a) = h_1(ab_1)$, contradicting that $h_1(a) = 1$ and $h_1(b_1) = 0$ (cf. (*)).

By minimality of k there are $g_1, \ldots, g_k \in \mathcal{H}_2$ so that for $1 \leq i, j \leq k$, $g_i(a) = g_i(b_i) = 1$ and $g_i(b_j) = 0$ whenever $i \neq j$. Since $\mathcal{H}_2$ is closed under product, $g := g_1 g_2 \cdots g_k \in \mathcal{H}_2$. On the other hand it is obvious that $g \in U(a) \cap Z(b_1) \cap \ldots \cap Z(b_k)$, contradicting (*). This shows that $\mathcal{H}_1 \subseteq \mathcal{H}_2$, and by symmetry we also get $\mathcal{H}_2 \subseteq \mathcal{H}_1$, proving that μ is injective.

Note that we have shown $\equiv_{\mathcal{H}_1} \subseteq \equiv_{\mathcal{H}_2} \;\Rightarrow\; \mathcal{H}_1 \subseteq \mathcal{H}_2$.

To show that μ is surjective, let $\equiv \in Con(G)$ and let $\pi : G \longrightarrow G/\!\equiv$ be the quotient map. Setting $\mathcal{H} = \{h \in Hom(G, \mathbf{2}) \mid a \equiv b \text{ implies } h(a) = h(b) \text{ for all } a, b \in G\}$, we get $\mathcal{H} \in C(G)$ and $\equiv \subseteq \equiv_{\mathcal{H}}$. If $a \not\equiv b$, then $\pi(a) \neq \pi(b)$, and by Proposition I.1.30 there is $\widehat{h} \in \mathrm{Hom}(G/\!\equiv, \mathbf{2})$ so that $\widehat{h}(\pi(a)) \neq \widehat{h}(\pi(b))$. Since $\widehat{h} \circ \pi \in \mathcal{H}$, it follows that $a \not\equiv_H b$ and then $\equiv = \equiv_H$, as required. The remaining assertions are left as an easy exercise. $\square$

Let G be a ternary semigroup and let $\equiv$ be a congruence of G. It is clear that the restriction $\equiv_{Id(G)}$ of $\equiv$ to $\mathrm{Id}(G)$ is a congruence of 2-semigroups. Further, it is easy to check that the set $\Delta_{\equiv} = \{x \in G \mid x \equiv x^2\}$ is a subsemigroup of G verifying:

(i) $\mathrm{Id}(G) \subseteq \Delta_{\equiv}$.

(ii) If $a^2 b \in \Delta_{\equiv}$ and $a^2 \equiv_{Id(G)} b^2$, then $b \in \Delta_{\equiv}$.

(iii) For all $x \in G$, $-x^2 \in \Delta_{\equiv}$ if and only if $x^2 \equiv_{Id(G)} 0$.

Using that in every ternary semigroup H, $a = b \Leftrightarrow a^2 = b^2$ and $ab \in \mathrm{Id}(H)$, it follows that the correspondence assigning to every congruence $\equiv$ of G the pair $(\equiv_{Id(G)}, \Delta_{\equiv})$ is one–one. Conversely, if $\sim$ is a congruence of 2-semigroups on $\mathrm{Id}(G)$ and Δ is a subsemigroup of G satisfying conditions (i)–(iii) above (with $\equiv_{Id(G)}$ replaced by $\sim$, and $\Delta_{\equiv}$ by Δ), the binary relation $\equiv$ in G defined by

$$a \equiv b \text{ if and only if } a^2 \sim b^2 \text{ and } ab \in \Delta,$$

is a TS-congruence of G. Indeed, condition (i) implies that $\equiv$ is reflexive, and this relation is obviously symmetric. Suppose that $a \equiv b$ and $b \equiv c$, with $a, b, c \in G$. Since $\sim$ is transitive, it follows that $a^2 \sim c^2$. On the other hand, $ab \in \Delta$ and $bc \in \Delta$ imply that $b^2ac \in \Delta$. From $b^2 \sim a^2$ and $b^2 \sim c^2$ we obtain $b^2 \sim a^2c^2$ and, by condition (ii), we conclude that $ac \in \Delta$; this shows that $\equiv$ is transitive and hence an equivalence relation. A straightforward argument shows that $\equiv$ is compatible with the semigroup operation. Finally, suppose that $a \equiv -a$. Then, $-a^2 \in \Delta$, and from condition (iii) we have $a^2 \sim 0$, and hence $a^2 \equiv 0$. Scaling by a we obtain $a \equiv 0$. Thus, we have shown that $\equiv$ is a TS-congruence. Further, $\mathrm{Id}(G) \subseteq \Delta$, implies that the restriction of $\equiv$ to $\mathrm{Id}(G)$ is $\sim$. We also have $\Delta_{\equiv} = \Delta$. To see this, let $x \in \Delta_{\equiv}$. Hence $x \equiv x^2$, and by the definition of $\equiv$ we get $x = xx^2 \in \Delta$. So $\Delta_{\equiv} \subseteq \Delta$. The reverse inclusion is obvious. We have shown:

Proposition I.1.32 *There is a bijective correspondence between the set of TS-congruences of a ternary semigroup G and the set of pairs $(\sim, \Delta)$ consisting of a congruence $\sim$ of 2-semigroups on* $\mathrm{Id}(G)$ *and a subsemigroup Δ of G satisfying conditions (i) – (iii) above.* □

We will call *compatible* the members of a pair $(\sim, \Delta)$ satisfying conditions (i)–(iii) above.

Remark I.1.33 Given a ternary semigroup G and a congruence $\sim$ of 2-semigroups in $\mathrm{Id}(G)$, there is always a subsemigroup Δ of G compatible with $\sim$. Let $\overline{\Delta} = \mathrm{Id}(G) \cup \{-a^2 \,|\, a^2 \sim 0\}$. Obviously $\overline{\Delta}$ verifies condition (i) and it is easily checked that $\overline{\Delta}$ is a subsemigroup of G; $\overline{\Delta}$ also verifies (iii). Indeed, if $-x^2 \in \overline{\Delta}$, either $-x^2 \in \mathrm{Id}(G)$, whence $-x^2 = x^2$ and then $x^2 = 0$, or $-x^2 = -a^2$ with $a^2 \sim 0$, and then $x^2 \sim 0$; in both cases we have $x^2 \sim 0$. Conversely, by the definition of $\overline{\Delta}$, $x^2 \sim 0 \Rightarrow -x^2 \in \overline{\Delta}$. Let

$$\Delta = \{x \in G \,|\, \text{There is } y \in G \text{ such that } y^2x \in \overline{\Delta} \text{ and } x^2 \sim y^2\}.$$

Clearly Δ is a subsemigroup of G containing $\overline{\Delta}$. We claim that Δ is compatible with $\sim$. Since $\mathrm{Id}(G) \subseteq \overline{\Delta} \subseteq \Delta$, Δ satisfies condition (i). To check condition (ii), let $x, y \in G$ be such that $x^2y \in \Delta$ and $x^2 \sim y^2$. Then, there is $z \in G$ such that $z^2x^2y \in \overline{\Delta}$ and $z^2 \sim (x^2y)^2 = x^2y^2$. Since $x^2 \sim y^2$, it follows that $z^2x^2 \sim x^2y^2 \sim y^2$, whence $y \in \Delta$. Finally, to verify condition (iii), let $-x^2 \in \Delta$. Then, there exists $y \in G$ such that $y^2 \sim (-x^2)^2 = x^2$ and $-x^2y^2 \in \overline{\Delta}$. Since $\overline{\Delta}$ satisfies condition (iii) we obtain $x^2y^2 \sim 0$; from $x^2 \sim y^2$ we conclude $x^2 \sim 0$. Conversely, $x^2 \sim 0$, implies $-x^2 \in \overline{\Delta} \subseteq \Delta$.

This argument shows, in fact, that for any subsemigroup $\overline{\Delta}$ of a ternary semigroup G satisfying conditions (i) and (iii), the subsemigroup Δ defined above is compatible with every congruence of 2-semigroups on $\mathrm{Id}(G)$. □

The examples of quotients of ternary semigroups that follow will appear time and again in the rest of this monograph.

Examples I.1.34 (1) Let G be a ternary semigroup and let $I \subseteq G$ be a prime ideal of G. We associate to I the set of characters:

$$X_I = \{h \in X_G \,|\, Z(h) = I\}.$$

Straightforward verification shows that X_I is a proconstructible subset of X_G and is closed under the product of any three of its members. We write $\equiv_I$ for $\equiv_{X_I}$, and denote by G/I the quotient of G by the congruence $\equiv_I$.

Claim. *For $a, b \in G$, $a \equiv_I b$ if and only if either (i) $a, b \in I$, or (ii) $a, b \notin I$ and there is $x \in G \setminus I$ such that $ax = bx$.*

Proof of Claim. ($\Leftarrow$) If $a, b \in I$, then $h(a) = h(b) = 0$ for all $h \in X_I$, whence $a \equiv_I b$. If $ax = bx$ with $a, b, x \in G \setminus I$, then $h(a)h(x) = h(b)h(x)$ for all $h \in X_I$ and, since $h(x) \neq 0$, we get $h(a) = h(b)$; again we have $a \equiv_I b$.

($\Rightarrow$) Assume $a \equiv_I b$. Clearly, $a \in I \Leftrightarrow b \in I$. Suppose, by contradiction, that $a, b \notin I$ but $\{x \in G \mid ax = bx\} \subseteq I$. Let $\Delta = \mathrm{Id}(G) \cup I \cup (-ab)\cdot\mathrm{Id}(G)$. It is easy to check that Δ is a subsemigroup of G. Let $x \in \Delta \cap -\Delta$. Then $-x^2 \in \Delta$. Clearly, $-x^2 \in \mathrm{Id}(G) \cup I$ implies $x \in I$. If $-x^2 = -abz^2$ for some $z \in G$, then $x^2 = abz^2$. Scaling by x^2z^2 we obtain

$(*)\quad x^2z^2 = a\,b\,x^2z^2.$

Hence $a^2x^2z^2 = a^2(abx^2z^2) = abx^2z^2 = x^2z^2$. Scaling $(*)$ by a we have $ax^2z^2 = ba^2x^2z^2 = bx^2z^2$. Therefore, $a(x^2z^2) = b(x^2z^2)$ and, by our assumption, $x^2z^2 \in I$. Since I is prime, either $x^2 \in I$ or $z^2 \in I$ and, since $x^2 = abz^2$, in both cases we come to $x^2 \in I$, and hence $x \in I$. Thus, $\Delta \cap -\Delta \subseteq I$, and hence $\Delta \cap -\Delta = I$. By Lemma I.1.7 any subsemigroup S of G containing Δ and maximal for $S \cap -S = I$ determines a character $h \in X_G$ such that $\Delta \subseteq P(h)$ and $Z(h) = I$. In particular, $h(-ab) = 1$ because $ab \notin I$, and then $h(a) \neq h(b)$, contradicting $a \equiv_I b$ and proving the claim. □

An important remark is that $(G/I)\setminus\{\pi_I(0)\}$ is a group of exponent 2, where $\pi_I : G \longrightarrow G/I$ is the canonical projection. In fact, since $x \equiv_I 0 \Leftrightarrow x \in I$, for $x \notin I$ and $h \in X_I$ we have $h(x) \neq 0$, and then $h(x^2) = 1$. Therefore $x^2 \equiv_I 1$. As a corollary we obtain that every ternary semigroup is embeddable in a direct product (in fact, a *subdirect* product) of ternary semigroups of the form $G \cup \{0\}$, where G is a group of exponent 2.

Corollary I.1.35 *Let G be a ternary semigroup and let $\mathcal{P}(G)$ be the set of all prime ideals of G. Then the map $\mu : G \to \prod_{I \in \mathcal{P}(G)} G/I$ defined by $\mu(a) = \langle \pi_I(a) \mid I \in \mathcal{P}(G) \rangle$, for $a \in G$, is an injective homomorphism of ternary semigroups.*

Proof. It is clear that μ is a TS-homomorphism. To show μ injective, let $a, b \in G$ be such that $a \neq b$. If $a^2 \neq b^2$, there exists $h \in X_G$ such that $h(a^2) \neq h(b^2)$ (I.1.12). Assume, without loss of generality, $h(a^2) = 0$ and $h(b^2) = 1$, and let $I = Z(h)$. Clearly $I \in \mathcal{P}(G)$, $a \in I$ and $b \notin I$; hence $\mu(a)_I = 0$ and $\mu(b)_I \neq 0$. If $a^2 = b^2$, then $a \neq b$ implies $ab \notin \mathrm{Id}(G)$. Then, there exists $f \in X_G$ such that $f(ab) = -1$ (Theorem I.1.9), and hence $f(a) \neq f(b)$. With $J = Z(f)$, we thus have $\pi_J(a) \neq \pi_J(b)$, showing that $\mu(a)_J \neq \mu(b)_J$, as required. □

(2) Let G be a ternary semigroup and let I be an ideal of G (not necessarily prime). We associate to I the set of characters:

$$X(I) = \{h \in X_G \mid I \subseteq Z(h)\}.$$

Again, $X(I)$ is a closed subset of X_G, and $h_1h_2h_3 \in X(I)$ whenever one of h_1, h_2 or h_3 is in $X(I)$; in particular, $X(I)$ is closed under the product of three elements.

Claim. *$a \equiv_{X(I)} b$ if and only either (i) $a, b \in I$ or (ii) $a, b \notin I$ and $a = b$.*

Proof of Claim. The implication ($\Leftarrow$) is obvious.

($\Rightarrow$) Suppose that $a \equiv_{X(I)} b$, and let $\Delta = I \cup \mathrm{Id}(G)$. It is plain that Δ is a subsemigroup of G satisfying $I = \Delta \cap -\Delta$; further, $I[\Delta] \subseteq \Delta$, as it is easily verified (cf. I.1.8). Assume $ab \notin \Delta$. By the separation theorem I.1.9 there is a character $h \in X_G$ such that $h(ab) = -1$ and $\Delta \subseteq P(h)$. Since $I = \Delta \cap -\Delta$, we have $I \subseteq Z(h)$, and hence $h \in X(I)$. Therefore $h(a) = h(b)$, contradicting $h(ab) = -1$. So $ab \in I \cup \mathrm{Id}(G)$.

Direct application of Theorem I.1.10 (a) shows that a belongs to the ideal $I \cup b \cdot G$ generated by I and b, and that b belongs to the ideal $I \cup a \cdot G$. Together, these conditions imply that $a \in I \Leftrightarrow b \in I$, and show that if one of a, b is not in I, then $a^2 = b^2$ [in fact, $a = bx$ and $b = ay$ imply $ba^2 = b\,b^2x^2 = bx^2$ and $ab^2 = ay^2$; squaring these equalities gives $a^2b^2 = b^2x^2 = a^2$ and $a^2b^2 = a^2y^2 = b^2$].

If $a, b \notin I$, then $a \in bG$ and $b \in aG$, whence $a^2, b^2 \in ab\,G$, which implies $ab \notin I$. Hence $ab \in \mathrm{Id}(G)$. But in any TS, $a^2 = b^2$ and $ab \in \mathrm{Id}(G)$ are equivalent to $a = b$, proving (ii). $\square$

(3) Let G be a ternary semigroup and let Δ be a subsemigroup of G. Let us consider the following set of characters:

$$\mathcal{H}_{\Delta} = \{h \in X_G \,|\, \Delta \subseteq P(h)\}.$$

It is easy to verify that $\mathcal{H}_{\Delta}$ is a closed subset of X_G and is closed under the product of any three of its members. We write $\equiv_{\Delta}$ for $\equiv_{\mathcal{H}_{\Delta}}$, and denote by G/Δ the corresponding quotient set.

Claim. *For $a, b \in G$, $a \equiv_{\Delta} b$ if and only if either (i) $-a^2, -b^2 \in \Delta$, or (ii) $-a^2, -b^2 \notin \Delta$, $a^2 = b^2$, and $ab \in \Delta$.*

Proof of Claim. The implication $(\Leftarrow)$ is easy.

$(\Rightarrow)$ Assume $a \equiv_{\Delta} b$. Let us first see that $-a^2 \in \Delta \Leftrightarrow -b^2 \in \Delta$. Otherwise, say, $-a^2 \in \Delta$ and $-b^2 \notin \Delta$. Observing that $-b^2 \notin \Delta$ implies $-b^2 \notin \Delta \cup I[\Delta]\,(= \Gamma$, say), we have $I[\Gamma] \subseteq \Gamma$, and the separation theorem I.1.9 can be applied, yielding a character $h \in X_G$ such that $\Delta \subseteq \Gamma \subseteq P(h)$ and $h(-b^2) = -1$, which means $h(b) \neq 0$. Since $-a^2 \in \Delta$, we have $h(-a^2) \geq 0$, and then $h(a) = 0$. Since $h \in \mathcal{H}_{\Delta}$, it follows that $a \not\equiv_{\Delta} b$, contradiction.

Suppose now that $-a^2, -b^2 \notin \Delta$. If $a^2 \neq b^2$, there exists $g \in X_G$ such that, say, $g(a) \neq 0$ and $g(b) = 0$ (see I.1.12; the case $g(a) = 0$ and $g(b) \neq 0$ is similar). As in the preceding paragraph, $-a^2 \notin \Delta$ implies the existence of a character $h \in X_G$ such that $\Delta \subseteq P(h)$ and $h(a) \neq 0$ (I.1.9). Then $h \in \mathcal{H}_{\Delta}$, and it is clear that g^2h also lies in $\mathcal{H}_{\Delta}$. Since $a \equiv_{\Delta} b$, then $g^2h(a) = g^2h(b)$, contradicting $g^2h(a) \neq 0$ and $g^2h(b) = 0$; hence, $a^2 = b^2$. Finally, assume $ab \notin \Delta$; if $ab \in I[\Delta]$, i.e., $-a^2b^2 \in \Delta$, from $a^2 = b^2$ we get $-b^2 \in \Delta$, contrary to assumption. Thus, $ab \notin \Delta \cup I[\Delta]\,(= \Gamma$, say); since $I[\Gamma] \subseteq \Gamma$ (cf. I.1.8), theorem I.1.9 yields a character $h \in X_G$ such that $\Delta \subseteq \Gamma \subseteq P(h)$ and $h(ab) = -1$, whence $h \in \mathcal{H}_{\Delta}$ and $h(a) \neq h(b)$, contradicting $a \equiv_{\Delta} b$. This proves $ab \in \Delta$, and hence item (ii) of the Claim. $\square$

I.2 Real semigroups

In this section we introduce the notion of *real semigroup*, the central notion in this monograph. To this end we enrich the language $\{\cdot, 1, 0, -1\}$ of ternary semigroups with a ternary relation D. The resulting language, $\{\cdot, D, 0, 1, -1\}$, will be denoted $\mathcal{L}_{\mathrm{RS}}$.

In agreement with standard notation (cf. [M], p. 99 ff.), we shall write $a \in D(b, c)$ instead of $D(a, b, c)$. We also set:

[t-rep] $$a \in D^t(b, c) \Leftrightarrow a \in D(b, c) \wedge -b \in D(-a, c) \wedge -c \in D(b, -a).$$

The relations D and D^t are called **representation** and **transversal representation**, respectively.

Definition I.2.1 A **real semigroup** (abbreviated RS) is a ternary semigroup together with a ternary relation D satisfying the following axioms:

[RS0] $c \in D(a,b)$ if and only if $c \in D(b,a)$.

[RS1] $a \in D(a,b)$.

[RS2] $a \in D(b,c)$ implies $ad \in D(bd,cd)$.

[RS3] (*Strong associativity*) If $a \in D^t(b,c)$ and $c \in D^t(d,e)$, then there exists $x \in D^t(b,d)$ such that $a \in D^t(x,e)$.

[RS4] $e \in D(c^2a, d^2b)$ implies $e \in D(a,b)$.

[RS5] If $ad = bd, ae = be$, and $c \in D(d,e)$, then $ac = bc$.

[RS6] $c \in D(a,b)$ implies $c \in D^t(c^2a, c^2b)$.

[RS7] (*Reduction*) $D^t(a,-b) \cap D^t(b,-a) \neq \emptyset$ implies $a = b$.

[RS8] $a \in D(b,c)$ implies $a^2 \in D(b^2,c^2)$. $\square$

I.2.2 Remarks and examples. (1) The axioms for RSs will insure the validity of the Duality Theorem I.5.1 below.

(2) The ternary semigroup G_A of I.1.2(e) endowed with the representation and transversal representation relations given by:

[R] $\overline{c} \in D_A(\overline{a}, \overline{b}) \Leftrightarrow \forall \alpha \in \mathrm{Sper}\,(A)\, [\overline{c}(\alpha) = 0 \vee \overline{a}(\alpha)\overline{c}(\alpha) = 1 \vee \overline{b}(\alpha)\overline{c}(\alpha) = 1]$,

[TR] $\overline{c} \in D^t_A(\overline{a}, \overline{b}) \Leftrightarrow \forall \alpha \in \mathrm{Sper}(A)[(\overline{c}(\alpha) = 0 \wedge \overline{a}(\alpha) = \overline{-b}(\alpha)) \vee \overline{a}(\alpha)\overline{c}(\alpha) = 1 \vee \overline{b}(\alpha)\overline{c}(\alpha) = 1]$.

for $a, b, c \in A$, is a real semigroup. A similar definition with $\mathrm{Spec}_R(A)$ replaced by $\mathrm{Spec}_R(A,T)$ —T a proper preorder of A— also endows the ternary semigroup $G_{A,T}$, defined in I.1.2 (e), with a structure of real semigroup; see also [M], p. 92.

(3) The notion of a RS generalizes that of a reduced special group, [DM1]. We have already remarked that, adding an absorbent element 0 to a RSG, G, gives raise to a ternary semigroup $G^* = G \cup \{0\}$ (see I.1.2 (d)). Extending the representation relation of G to G^* by

$$D_{G^*}(a,b) = \begin{cases} \{a,b\} & \text{if } a = 0 \text{ or } b = 0 \\ D_G(a,b) \cup \{0\} & \text{if } a,b \in G, \end{cases}$$

gives a representation relation verifying the axioms for RSs, as shown by straightforward checking. Since in an RSG we have:

$$a \in D(b,c) \;\Rightarrow\; -b \in D(-a,c),$$

(see [DM1], pp. 2, 3) it follows from (†) above that the value sets D and D^t coincide on binary forms with entries in G. If one of the entries is 0 we have $D_{G^*}(0,b) = \{0,b\}$ and, from I.2.3(11),(14) below, $D^t_{G^*}(0,b) = \{b\}$. Real semigroups obtained by adding a zero to a reduced special group will repeatedly occur in this text; we will call them **quasi reduced special groups** (abbreviated **quasi-RSG** or **QRSG**). $\square$

The next Proposition summarizes some consequences of the axioms [RS0]–[RS8] concerning binary representation and transversal representation frequently used in the sequel:

Proposition I.2.3 *The properties below hold in any RS, G, for arbitrary $a, b, c, d, e, x, y \in G$:*

(0) $a \in D^t(b,c) \Rightarrow -b \in D^t(-a,c)$. (1) $0 \in D(a,b)$.

(2) $a \in D^t(b,c) \Rightarrow ad \in D^t(bd, cd)$.
(3) $a \in D(0,1) \cup D(1,1) \Rightarrow a = a^2$.

(4) $d \in D(ca, cb) \Rightarrow d = c^2 d$.

(5) $a^2 \in D(1,b)$. *Hence* (*by* (3)), $\mathrm{Id}(G) = D(1,1)$.

(6) $a \in D^t(b,b) \Leftrightarrow a = b$.
(7) $a \in D(0,0) \Leftrightarrow a = 0$.
(8) $1 \in D^t(1,a)$.

(9) $D^t(1,-1) = G$.
(10) $ab \in D(1,-a^2)$.
(11) $0 \in D^t(a,b) \Leftrightarrow a = -b$.

(12) $a \in D(b,c) \wedge b, c \in D(x,y) \Rightarrow a \in D(x,y)$.

(13) $a \in D(b,c) \Leftrightarrow ab \in D(1,bc) \wedge ac \in D(1,bc) \wedge a^2 \in D(b^2,c^2)$.

(14) $D^t(a,b) \neq \emptyset$.

(15) (Weak associativity) $a \in D(b,c) \wedge c \in D(d,e) \Rightarrow \exists x[x \in D(b,d) \wedge a \in D(x,e)]$.

NOTE. Add (numbering?)

$D(a,-a) = a^2 \cdot G$.

$a \in D^t(b,c) \Leftrightarrow a^2 \in D(b^2,c^2) \wedge ab \in D^t(b^2,bc) \wedge ac \in D^t(c^2,bc)$.

Proof. See [DP1], Proposition 2.3, pp. 107–109. □

Remarks and Notation I.2.4 In [M], Prop. 6.1.1, p. 100, and Thm. 6.2.4, pp. 107-108, Marshall proves, in the context of abstract real spectra, that items (14) and (15) of the preceding Proposition together are equivalent to the strong associativity axiom [RS3]. His proof remains valid in the present context of real semigroups. These statements will be used separately in several parts of this text. Paraphrasing the terminology used in [M], Ch. 6, pp. 99-100 we will use the name [RS3a] for *weak associativity* (item (15)), and [RS3b] for item (14). □

Corollary I.2.5 *The ternary semigroup* $\mathbf{3} = \{1, 0, -1\}$ *has a unique structure of real semigroup, with representation given by:*

$D_{\mathbf{3}}(0,0) = \{0\}$; $\quad D_{\mathbf{3}}(0,1) = D_{\mathbf{3}}(1,0) = D_{\mathbf{3}}(1,1) = \{0,1\}$;

$D_{\mathbf{3}}(0,-1) = D_{\mathbf{3}}(-1,0) = D_{\mathbf{3}}(-1,-1) = \{0,-1\}$; $\quad D_{\mathbf{3}}(1,-1) = D_{\mathbf{3}}(-1,1) = \mathbf{3}$;

and transversal representation given by:

$D^t_{\mathbf{3}}(0,0) = \{0\}$; $\quad D^t_{\mathbf{3}}(0,1) = D^t_{\mathbf{3}}(1,0) = D^t_{\mathbf{3}}(1,1) = \{1\}$;

$D^t_{\mathbf{3}}(0,-1) = D^t_{\mathbf{3}}(-1,0) = D^t_{\mathbf{3}}(-1,-1) = \{-1\}$; $\quad D^t_{\mathbf{3}}(1,-1) = D^t_{\mathbf{3}}(-1,1) = \mathbf{3}$. □

Proof. See [DP1], Corollary 2.4, p. 109. □

Another consequence of Proposition I.2.3 is the following converse to Remark I.2.2(3), which amounts to a characterization of the reduced special groups amongst the real semigroups.

Corollary I.2.6 *Let G be a RS in which the representation and transversal representation relations coincide, up to 0, on non-zero entries, i.e., for all $a, b \in G \setminus \{0\}$, $D_G(a,b) = D^t_G(a,b)$ if $a = -b$, and $D^t_G(a,b) = D_G(a,b) \setminus \{0\}$ if $a \neq -b$. Then,*

$$\forall x \in G\ (x \neq 0\ \Rightarrow\ x^2 = 1).$$

Hence, $\overline{G} = G \setminus \{0\}$ with representation induced by that of G is a reduced special group (and, of course, $\overline{G}^* = G$).

Proof. See [DP1], Corollary 2.5, pp. 109–110. □

I.2.7 Notation. We shall use, among others, the basic concepts and notation from quadratic form theory as introduced in [M], Ch. 6 (cf. p. 105); these apply verbatim in our context. We explicitly mention the following:

(a) Representation by forms of dimension $n \geq 3$ is inductively defined by:

$$D(\langle a_1, \ldots, a_n \rangle) = \bigcup \{D(a_1, b) \mid b \in D(\langle a_2, \ldots, a_n \rangle)\},$$

and similarly for transversal representation (for $n = 1$, $D(\langle a \rangle) = \{b^2 a \mid b \in G\}$, $D^t(\langle a \rangle) = \{a\}$).

(b) A **Pfister form (of degree *n*)** over an RS, G, is a form of the shape $\bigotimes_{i=1}^{n} \langle 1, a_i \rangle$, with the a_i's elements of G; Pfister forms as above will, as usual, be denoted $\langle\langle a_1, \ldots, a_n \rangle\rangle$.

(c) We shall use a suitable version of Witt-equivalence: if $\varphi = \langle a_1, \ldots, a_n \rangle$, $\psi = \langle b_1, \ldots, b_m \rangle$, are forms over a Rs, G (possibly of different dimensions), we set:

$$\varphi \cong_G \psi \;\Leftrightarrow\; \text{For all } h \in X_G,\ \sum_{i=1}^{n} h(a_i) = \sum_{j=1}^{m} h(b_j) \quad (\text{sum in } \mathbb{Z}).$$

(d) For forms φ, ψ over a RS, G, we set:

$$\varphi \sim \psi \;\Leftrightarrow\; D_G(\varphi) = D_G(\psi) \quad \textit{and} \quad \varphi \sim^t \psi \;\Leftrightarrow\; D^t_G(\varphi) = D^t_G(\psi).$$

(with a subscript G, if necessary). $\square$

The following result states some of the basic properties of representation and transversal representation by forms of arbitrary dimension needed in the sequel. Most (if not all) of these results appear in Chapter 6 of [M], where they are derived from the axioms for ARSs. The point of the proof given in [DP1] is to make sure that these properties follow from the axioms [RS0]–[RS8] for RSs, as they will frequently be used throughout this monograph.

Proposition I.2.8 *Let G be a RS and let φ, ψ be forms with entries in G. Then:*

(1) *$D(\varphi)$ and $D^t(\varphi)$ do not depend on the order of the entries of φ, i.e., for any permutation σ of those entries, $\varphi \sim \varphi^\sigma$ and $\varphi \sim^t \varphi^\sigma$.*

(2) *For $a, c \in G$,*

$$a \in D(\varphi) \Rightarrow ac \in D(c\varphi) \quad \textit{and} \quad a \in D^t(\varphi) \Rightarrow ac \in D^t(c\varphi).$$

(3) *$a \in D(c\varphi) \Rightarrow a = c^2 a$ and $a \in D(\varphi) \Rightarrow a \in D^t(a^2\varphi)$.*

(4) *If $\varphi = \langle a_1, \ldots, a_n \rangle$ and $c_1, \ldots, c_n \in G$, then $D(\langle c_1^2 a_1, \ldots, c_n^2 a_n \rangle) \subseteq D(\varphi)$.*

(5) *$a \in D(\varphi \oplus \psi) \Leftrightarrow$ There are $b \in D(\varphi)$, $c \in D(\psi)$ such that $a \in D(b, c)$.*

A similar statement holds replacing D by D^t.

(6) *If a is a coefficient of φ, then $a \in D(\varphi)$.*

(7) *The relations $\sim$ and $\sim^t$ are compatible with the sum of forms:*

$$\varphi_1 \sim \psi_1 \quad \textit{and} \quad \varphi_2 \sim \psi_2 \quad \Rightarrow \quad \varphi_1 \oplus \varphi_2 \sim \psi_1 \oplus \psi_2,$$

and similarly for $\sim^t$.

(8) *$\varphi \oplus \varphi \sim \varphi$ and $\varphi \oplus \varphi \sim^t \varphi$.*

(9) *$a \in D(\varphi) \wedge b \in D(\psi) \Rightarrow ab \in D(\varphi \otimes \psi)$.*

A similar statement holds replacing D by D^t.

(10) $a \in D^t(\langle a_1, \ldots, a_n \rangle) \;\Leftrightarrow$

$a \in D(\langle a_1, \ldots, a_n \rangle)$ *and* $-a_i \in D(\langle a_1, \ldots, a_{i-1}, -a, a_{i+1}, \ldots, a_n \rangle)$ *for* $i = 1, \ldots, n \;\Leftrightarrow$

$-a_i \in D^t(\langle a_1, \ldots, a_{i-1}, -a, a_{i+1}, \ldots, a_n \rangle)$ *for* $i = 1, \ldots, n$.

(11) *For* $b \in G$ *and* $n \geq 1$, $\; n\langle b \rangle = \langle b, \ldots, b \rangle \sim^t \langle b \rangle$.

Proof. See [DP1], Proposition 2.7, pp. 110–112. □

Remark. Many basic results concerning the behaviour of quadratic forms over real semigroups follow from Proposition I.2.8. Here is an example:

Corollary I.2.9 *Let G be a RS and let $\varphi = \langle a_1, \ldots, a_n \rangle$ be a form with entries in G. Suppose the non-empty sets $I_1, \ldots, I_k$ partition the index set $\{1, \ldots, n\}$. For each $j \in \{1, \ldots, n\}$ let φ_j denote the form having as entries the elements a_ℓ with $\ell \in I_j$ (in any order). Then,*

$$D(\varphi) = D(\textstyle\bigoplus_{j=1}^{k} \varphi_j), \qquad \textit{i.e.,} \quad \varphi \sim \textstyle\bigoplus_{j=1}^{k} \varphi_j.$$

A similar statement holds for D^t. □

The proof, which we leave as an exercise, is by induction on k, using items (1) and (5) of I.2.8.

A number of other elementary results known to hold for isometry and/or Witt-equivalence of quadratic forms over fields and over special groups are also valid for quadratic forms over RSs, upon replacing these relations by either $\cong$, $\sim$ or $\sim^t$. Whenever needed, these results will be used, implicitly or explicitly. □

We now give a reformulation of the strong associativity axiom [RS3] that will be relevant in the study of quotients of RSs.

Proposition I.2.10 *In the presence of axiom* [RS2]*, the following is equivalent to axiom* [RS3]*:*

[RS3′] $\;\forall a, b, c, d\, (D^t(a, b) \cap D^t(c, d) \neq \emptyset \;\Rightarrow\; D^t(a, -c) \cap D^t(-b, d) \neq \emptyset)$.

Proof. [RS3] $\Rightarrow$ [RS3′]. Let $x \in D^t(a, b) \cap D^t(c, d)$; the definition of D^t yields $-b \in D^t(a, -x)$ (see I.2.3 (0)), and scaling by -1 ([RS2]) gives $-x \in D^t(-c, -d)$. By [RS3] there is $y \in D^t(a, -c)$ so that $-b \in D^t(y, -d)$. Again, the definition of D^t and [RS2] yield $-y \in D^t(b, -d)$, and $y \in D^t(-b, d)$. Hence, $D^t(a, -c) \cap D^t(-b, d) \neq \emptyset$).

[RS3′] $\Rightarrow$ [RS3]. Assume [RS3′] and let $x \in D^t(a, b)$ with $b \in D^t(c, d)$. By the definition of D^t, $-b \in D^t(a, -x)$, and by [RS2], $b \in D^t(-a, x)$, i.e., $D^t(-a, x) \cap D^t(c, d) \neq \emptyset$. By [RS3′] there is $y \in D^t(-a, -c) \cap D^t(-x, d)$. By the same manipulation as above, we get $-y \in D^t(a, c)$ and $x \in D^t(-y, d)$. So, [RS3] is verified with witness $-y$. □

Remark. Note that, while the weak associativity axiom [RS3a] (I.2.4) obtained by replacing transversal representation by ordinary representation in [RS3] is a non-trivial property (in the sense that it does *not* follow from the remaining axioms), the corresponding weak version of [RS3′] *does follow* from the remaining axioms for RSs: $0 \in D(a, b)$ for all a, b, and hence $D(a, b) \cap D(c, d)$ always contains 0; cf. I.2.3 (1); the proof of this only uses [RS1] and [RS4] (see [DP1], Proposition 2.3, p. 107). □

I.2.11 The group of invertible elements of a real semigroup.

A natural and important question is to know the structure of the set $G^\times$ of invertible elements of a real semigroup, G, with induced product and representation. In particular, it is important to elucidate in which cases $G^\times$ is a reduced special group.

Using the axiomatization of RSGs in terms of the binary relation

$$a \preceq b \ :\Leftrightarrow \ a \in D_G(1,b) \qquad\qquad (a,b \in G^\times),$$

given in [DMM], Prop. 1.2, p. 30, it is a routine exercise to check that the following axioms hold under no restriction on G:

[R0] $(G^\times, \cdot, 1)$ is a group of exponent 2.

[R1] $\preceq$ is a partial order on $G^\times$ with first element 1 and last element -1.

[R2] For all $a,b \in G^\times$, $a \preceq b \Leftrightarrow -b \preceq -a$.

[R3] For all $b \in G^\times$, $\{x \in G^\times \mid x \preceq b\}$ is a subgroup of $G^\times$.

However, the validity of the *weak compatibility* axiom

[R4] $\forall\, a,b,c,d \in G^\times\, (a \preceq b \wedge bd \preceq cd \Rightarrow \exists\, e \in G^\times\, (e \preceq d \wedge ae \preceq ce))$,

is a far more delicate question. Note that [R4] is a slightly simplified version of the weak associativity axiom [RS3a] (see I.2.4). With notation as in [RS3a] (cf. I.2.3 (15)), the difficulty lies in obtaining an invertible witness x whenever the entries $a,b,c,d,e \in G$ are invertible.

Later in these notes we prove that $G^\times$ is a RSG in the following cases:

(1) G is a Post algebra (Fact IV.2.4).

(2) G is a spectral real semigroup (Corollary V.6.7).

In both these cases, $G^\times$ with the induced structure is a Boolean algebra.

(3) G is a RS-fan (Corollary VI.2.7).

In this case, $G^\times$ with the induced structure is a RSG-fan (i.e., a fan in the category of reduced special groups, cf. [DM1], Ex. 1.7, pp. 8–9).

In [M], Thm. 8.1.7, p. 154, using the dual terminology of abstract real spectra, Marshall adds to this list the case:

(4) G has *many units* (cf. [M], 8.1.1, p. 152) [4]. This includes the case where G is *semi-local*, i.e., has finitely many maximal ideals ([M], Prop. 8.1.2, p. 152).

In case $G = G_{A,T}$ is the real semigroup arising from a preordered ring (A,T), see I.2.2 (2) and I.1.2 (e), results from [DM6] show that $G^\times$ is a RSG in the following circumstances (for undefined notions, see [DM6]; for item (4), see also [M], p. 153):

(5) A is a ring with many units such that every residue field has at least 7 elements ([DM6], Thm. 5.5).

(6) (A,T) is a preordered, faithfully quadratic ring with T-bounded inversion (i.e., $1+T \subseteq A^\times$); [DM6], Cor. 8.19. Examples are reduced f-rings whose natural order is sums of squares, e.g., rings of real-valued continuous functions on a topological space, and real closed rings (in the sense of Prestel-Schwartz [PS]).

In cases (5) and (6), not only is $G^\times$ a RSG but, moreover, representation in $G^\times$ by forms of arbitrary dimension faithfully reflects representation by corresponding forms with invertible coefficients in the ring (A,T). $\square$

[4] Indeed, in case $n = 3$, [M], Lemma 8.1.6, p. 154, proves that axiom [R4] holds in $G^\times$.

I.3 From ternary semigroups to real semigroups

New section; added Jan. 2014. Generalizes results previously included in section I.2: II.2.6 – II.2.8 and II.2.11.

The purpose of this section is to explore and develop a natural, and fairly general method to construct, from a given ternary semigroup and some TS-character sets, ternary relations verifying as many as possible of the axioms for real semigroups.

Definition I.3.1 Given a ternary semigroup, G, and a set $\mathcal{H} \subseteq X_G = \mathrm{Hom}_{\mathrm{TS}}(G, \mathbf{3})$, we define a ternary relation $D_{G,\mathcal{H}}$ on G —abridged $D_{\mathcal{H}}$ if G is clear from context— as follows: for $a, b, c \in G$,

$[D]_{\mathcal{H}}$ $\quad a \in D_{G,\mathcal{H}}(b, c) \;\Leftrightarrow\;$ For all $h \in \mathcal{H}$, $h(a) \in D_{\mathbf{3}}(h(b), h(c))$.

To avoid triviality we assume $\mathcal{H} \neq \emptyset$; to get best results we also make the rather mild assumption that the set $\mathcal{H}$ separates points in G: given $a \neq b$ in G, there is $h \in \mathcal{H}$ such that $h(a) \neq h(b)$. $\square$

Remark I.3.2 Given a TS, G, and a set $X \subseteq \mathbf{3}^G$, Marshall [M], p. 99, defines representation relations on G, as follows: for $a, b, c \in G$,

[R] $\quad a \in D_X(b, c)$ iff $\forall\, h \in X\, [h(a) = 0 \vee (h(a) \neq 0 \wedge (h(a) = h(b) \vee h(a) = h(c)))]$.

[TR] $\quad a \in D^t_X(b, c)$ iff $\forall\, h \in X\, [(h(a) = 0 \wedge h(b) = -h(c)) \vee (h(a) \neq 0 \wedge$
$\wedge\, (h(a) = h(b) \vee h(a) = h(c)))]$.

When G is the real semigroup G_A associated to a semi-real ring A, we have already encountered these relations, cf. I.2.2 (2).

It turns out that the representation relation $D_{\mathcal{H}}$ defined by clause [R] is identical with the relation defined by clause $[D]_{\mathcal{H}}$ in I.3.1. This is obvious by the fact that the conditions $h(a) \in D_{\mathbf{3}}(h(b), h(c))$ and $h(a) = 0 \vee (h(a) \neq 0 \wedge (h(a) = h(b) \vee h(a) = h(c)))$ are equivalent for any $h \in X_G$ and all $a, b, c \in G$; this is straightforward checking using Corollary I.2.5.

Likewise, the transversal representation relation $D^t_{\mathcal{H}}$ defined by [TR] is identical to the transversal representation relation defined in terms of $D_{G,\mathcal{H}}$ by clause [t-rep], Section I.2, since the conditions $h(a) \in D^t_{\mathbf{3}}(h(b), h(c))$ and $(h(a) = 0 \wedge h(b) = -h(c)) \vee (h(a) \neq 0 \wedge (h(a) = h(b) \vee h(a) = h(c)))$ are equivalent, again by I.2.5. $\square$

Next, we show that, under the only assumptions on $\mathcal{H}$ set down in Definition I.3.1, the structure $(G, D_{\mathcal{H}}, \ldots)$ satisfies all axioms for real semigroups, *except*, possibly, axiom [RS3].

Theorem I.3.3 *Let G be a ternary semigroup and let $\mathcal{H}$ be a non-empty subset of X_G separating points in G. The representation relation $D_{\mathcal{H}}$ defined in I.3.1 satisfies all axioms for real semigroups except, possibly, the axiom* [RS3] *of strong associativity.*

Proof. The verification of axioms [RS0], [RS1], [RS2], [RS4] and [RS8] being straightforward, we deal only with the remaining axioms.

[RS5] Let $a, b, c, d, e \in G$ be such that $ad = bd$, $ae = be$ and $c \in D_{\mathcal{H}}(d, e)$. Let us prove that $ac = bc$. Since $\mathcal{H}$ separates points in G, this boils down to proving $h(ac) = h(bc)$ for all $h \in \mathcal{H}$. This is clear if $h(c) = 0$. Let $h(c) \neq 0$. Since $c \in D_{\mathcal{H}}(d, e)$, either $h(c) = h(d)$ or $h(c) = h(e)$. Since $ad = bd$ and $ae = be$, invoking Definition I.3.1, in both cases we get the equality $h(ac) = h(bc)$. By $[D]_{\mathcal{H}}$ once again, we conclude that $ac = bc$, as required.

[RS6] Let $a, b, c \in G$ be such that $c \in D_{\mathcal{H}}(a, b)$, and take $h \in \mathcal{H}$. Then, $h(c) \in D_{\mathbf{3}}(h(a), h(b))$. The real semigroup $\mathbf{3}$ verifies [RS6], and then $h(c) \in D^t_{\mathbf{3}}(h(c)^2 h(a), h(c)^2 h(b))$. From the definition of D^t (cf. I.2, [t-rep]), we have the following relations:

$(i)\quad h(c) \in D_{\mathbf{3}}(h(c^2a), h(c^2b)), \qquad (ii)\ -h(c^2a) \in D_{\mathbf{3}}(-h(c), h(c^2b)), \qquad$ and

$(iii) -h(c^2b) \in D_{\mathbf{3}}(-h(c), h(c^2a))$.

Since h is arbitrary, from (i), (ii), (iii) and I.3.1 $[D]_{\mathcal{H}}$ we get:

$(i')\quad c \in D_{\mathcal{H}}(c^2a, c^2b), \qquad (ii')\ -c^2a \in D_{\mathcal{H}}(-c, c^2b),$

$(iii')\ -c^2b \in D_{\mathcal{H}}(-c, c^2a),$

which, together, amount to $c \in D^t_{G/\mathcal{H}}(c^2a), c^2b)$.

[RS7] Let $a, b \in G$ be such that $D^t_{\mathcal{H}}(a, -b) \cap D^t_{\mathcal{H}}(b, -a) \neq \emptyset$. Take an element $c \in G$ in this intersection. We must prove that $a = b$. By I.3.1 $[D]_{\mathcal{H}}$ this boils down to showing that $h(a) = h(b)$ for all $h \in \mathcal{H}$. We consider the following cases:

(i) $h(c) = 0$. If either $h(a) \neq 0$ or $h(b) \neq 0$, from the relations $-a \in D_{\mathcal{H}}(-c, -b)$ and $-b \in D_{\mathcal{H}}(-c, -a)$ we obtain $h(-a) = h(-b)$, and then $h(a) = h(b)$. If $h(a) = h(b) = 0$, there is nothing to prove.

(ii) $h(c) \neq 0$. Since $c \in D_{\mathcal{H}}(a, -b) \cap D_{\mathcal{H}}(b, -a)$, we have $h(c) = h(a)$ or $h(c) = -h(b)$, and $h(c) = h(b)$ or $h(c) = -h(a)$. If $h(a) \neq h(b)$, these conditions yield either $h(c) = h(a) = -h(a)$ or $h(c) = h(b) = -h(b)$; in both cases we have $h(c) = 0$, a contradiction. Hence, $h(a) = h(b)$. □

The next two Propositions gives some simple examples —used later on— of finite sets $\mathcal{H}$ of low cardinality for which the structure $(G, D_{\mathcal{H}}, \ldots)$ verifies also axiom [RS3], and hence (if $\mathcal{H}$ separates points) is a real semigroup.

Proposition I.3.4 *Let G be a ternary semigroup and let $h_1, h_2 \in X_G$. With $\mathcal{H} = \{h_1, h_2\}$, $(G, D_{\mathcal{H}}, \ldots)$ verifies axiom* [RS3].
Proof. Let $a, b, c, d, e \in G$ be such that

$(*)\ \ a \in D_{\mathcal{H}}(b, c)$ and $c \in D_{\mathcal{H}}(d, e)$.

We must find an $x \in G$ so that $a \in D_{\mathcal{H}}(x, e)$ and $x \in D_{\mathcal{H}}(b, d)$.

The argument is by cases:

(i) $a \in D_{\mathcal{H}}(b, d)$. In this case it suffices to take $x = a$.

(ii) $a \in D_{\mathcal{H}}(b, e)$. In this case it suffices to take $x = b$.

(iii) $a \notin D_{\mathcal{H}}(b, d) \cup D_{\mathcal{H}}(b, e)$. In this case there are indices $i, j \in \{1, 2\}$ such that

$$(**) \qquad h_i(a) \notin D_{\mathbf{3}}(h_i(b), h_i(d)) \ \text{ and } \ h_j(a) \notin D_{\mathbf{3}}(h_j(b), h_j(e)).$$

Then, we have $h_i(a) \neq 0\,, h_i(a) \neq h_i(b)$ and $h_i(a) \neq h_i(d)$. It follows from $(*)$ and the definition of $D_{\mathcal{H}}$ that $h_i(a) = h_i(b)$ or $h_i(a) = h_i(c)$ —whence the latter—, and $h_i(c) = h_i(d)$ or $h_i(c) = h_i(e)$; therefore, $h_i(a) = h_i(e)$. Likewise, we get $h_j(a) = h_j(d)$. If $j = i$ we would have $h_j(a) = h_i(e) \in D_{\mathbf{3}}(h_i(b), h_i(e)) = D_{\mathbf{3}}(h_j(b), h_j(e))$, contradicting $(**)$. Hence, $i \neq j$, which means $\mathcal{H} = \{h_i, h_j\}$.

We claim that $a \in D_{\mathcal{H}}(d, e)$. Otherwise, we could find an index $k \in \{1, 2\}$ such that $h_k(a) \notin D_{\mathbf{3}}(h_k(d), h_k(e))$. In particular, $h_k(a) \neq 0$. From $(*)$ we infer that $h_k(a) = h_k(b)$. On the other hand, either $k = i$ or $k = j$. In the first case we get $h_i(a) \in D_{\mathbf{3}}(h_i(b), h_i(d))$, and in the second case $h_j(a) \in D_{\mathbf{3}}(h_j(b), h_i(e))$, contradicting $(**)$. So, $a \in D_{\mathcal{H}}(d, e)$, and it is clear that $x = d$ is as needed. □

Proposition I.3.5 *Let G be a ternary semigroup and let h_1, h_2, h_3 be three distinct elements of X_G such that $h_i \rightsquigarrow h_j$ for some $i \neq j \in \{1,2,3\}$. With $\mathcal{H} = \{h_1, h_2, h_3\}$, $(G, D_{\mathcal{H}}, \ldots)$ verifies axiom* [RS3].

Proof. It follows the same line of argument as the proof of the preceding Proposition I.3.4. With notation therein, we assume $(*)$; cases (i) and (ii) are as in I.3.4; we deal with the remaining case (iii). As above, there are indices $i, j \in \{1,2,3\}$ such that $(**)$ holds. From this, we have:

(a) $h_i(a) \neq 0\,, h_i(a) \neq h_i(b)\,, h_i(a) \neq h_i(d)$ and (b) $h_j(a) \neq 0\,, h_j(a) \neq h_j(b)\,, h_j(a) \neq h_j(e)$.

As in the proof of I.3.4 we get

(†) $h_i(a) = h_i(e)$ and $h_j(a) = h_j(d)$.

We also have $i \neq j$, for $i = j$ entails $h_j(a) = h_i(e) \in D_{\mathbf{3}}(h_i(b), h_i(e)) = D_{\mathbf{3}}(h_j(b), h_j(e))$, contrary to $(**)$.

Claim. $a \in D_{\mathcal{H}}(d, e)$. (Note that this yields the desired conclusion upon taking $x = d$.)

<u>Proof of Claim</u>. Assuming otherwise, there is $k \in \{1,2,3\}$ such that $h_k(a) \notin D_{\mathbf{3}}(h_k(d), h_k(e))$; this entails:

(c) $h_k(a) \neq 0\,, h_k(a) \neq h_k(d)\,, h_k(a) \neq h_k(e)$.

Note, again, that $(*)$ implies

(††) $h_k(a) = h_k(b)$.

Indeed, the first representation in $(*)$ yields $h_k(a) \in D_{\mathbf{3}}(h_k(b), h_k(c))$. Since $h_k(a) \neq 0$, we get $h_k(a) = h_k(b)$ or $h_k(a) = h_k(c)$. The latter, and the second representation in $(*)$ yield $h_k(a) = h_k(c) \in D_{\mathbf{3}}(h_k(d), h_k(e))$, contrary to the choice of h_k; hence, $h_k(a) = h_k(b)$.

Next, we observe that $k \notin \{i, j\}$, and hence $\{i, j, k\} = \{1,2,3\}$. If, e.g., $k = i$, (††) yields $h_i(a) = h_k(a) \in D_{\mathbf{3}}(h_k(b), h_k(d)) = D_{\mathbf{3}}(h_i(b), h_i(d))$, contradicting $(**)$. Likewise, $k \neq j$.

Now, we use items (a) – (c) and (†), (††) together with the characterization of $\rightsquigarrow$ in Lemma I.1.18 to show that $h_\ell \not\rightsquigarrow h_m$ for every pair of distinct indices $\ell, m \in \{1,2,3\}$, contrary to our assumption. The argument being similar for every pair of indices, we illustrate it in a couple of cases, leaving further details to the reader.

(1) $h_i \not\rightsquigarrow h_j$.

Otherwise, since $h_j(a) \neq 0$ ((b)), we have $h_i(a) = h_j(a)$. Since $h_j(a) = h_j(d)$ (see (†)), using $h_i \rightsquigarrow h_j$ again, we get $h_i(d) = h_j(d) = h_j(a) = h_i(a)$, contradicting (a).

(2) $h_k \not\rightsquigarrow h_i$.

Otherwise, since $h_i(a) \neq 0$ ((a)), Lemma I.1.18 (4) gives $h_k(a) = h_i(a)$. But $h_i(a) = h_i(e)$ (see (†)) and $h_k \rightsquigarrow h_i$ yields $h_i(e) = h_k(e)$; thus, $h_k(a) = h_k(e)$, contrary to (c). □

Here is an example of a RS, G, and a subset $\mathcal{H}$ of X_G separating points in G, such that axiom [RS3] does not hold in the structure $(G, D_{\mathcal{H}})$.

Example I.3.6 Let G be a finite group of exponent 2 and let b, c, d, e be elements of G such that $\{-1, b, c, d, e\}$ is a basis of G over the two-element field; here -1 is any element of G not in the linear span of $\{b, c, d, e\}$. We endow G with the reduced special group structure of a fan (cf. [DM1], Ex. 1.7, pp. 8-9). Let $\mathcal{H} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$ be the set of group characters of G into ± 1, where $\sigma_6 = \prod_{i=1}^{5} \sigma_i$ and for $i \in \{1,2,3,4,5\}$ the values of σ_i on generators are given by the following table (all five characters send -1 to -1):

	σ_1	σ_2	σ_3	σ_4	σ_5
b	-1	1	1	1	1
c	1	-1	1	-1	1
d	1	-1	1	1	1
e	-1	-1	1	-1	-1

Straightforward checking using this table shows that, with $G^* = G \cup \{0\}$ and $\sigma_i(0) = 0$ for $i = 1, \ldots, 6$:

(*) $1 \in D_{G^*}(b, c)$ and $c \in D_{G^*}(d, e)$.

Recall that in the present case we have $D_{G^*} = D^t_{G^*} \cup \{0\}$ (cf. I.2.2 (3)). If [RS3] holds in G^*, there is $x \in G$ such that

(**) $1 \in D_{G^*}(x, e)$ and $x \in D_{G^*}(b, d)$.

From the last line in the table we have $e \neq 1$, whence $x \neq 0$, i.e., $x \in G$.

Checking with the table above (and using I.2.5) the first representation in (**) shows that x must verify:

$$\sigma_1(x) = \sigma_2(x) = \sigma_4(x) = \sigma_5(x) = 1,$$

while the second clause in (**) yields $\sigma_3(x) = 1$. Therefore $\sigma_i(x) = 1$ for all $i \in \{1, 2, 3, 4, 5\}$, and hence $\sigma_6(x) = 1$. On the other hand, checking with the table we have $\sigma_6(b) = \sigma_6(d) = -1$, and using again the second representation in (**) we get $\sigma_6(x) = -1$, contradiction. Hence [RS3] (and [RS3a]) fails in G^*.

In order to check that $\mathcal{H}$ separates points in G —so that $(G, D_{\mathcal{H}})$ verifies the remaining axioms for RS, I.3.3— it suffices to compute, using the table above, the value of the characters σ_i $(i = 1, \ldots, 5)$ at each of the fifteen products of the generators b, c, d, e, to see that none of these five values is either 1 or -1. This straightforward checking is left to the reader. □

Remark. The fact that axiom [RS3] fails in $(G, D_{\mathcal{H}})$ follows, alternatively, from [M], Cor. 3.3.7, p. 46. Note also that the necessary condition for G^* to be a RS in Theorem II.2.9 below fails in this example: $\sigma_i \in \mathcal{H}$ for $i = 1, 2, 3$, and $\sigma_1\sigma_2\sigma_3 \in X_G$, but this product is not in $\mathcal{H}$. □

The next Proposition gives additional information concerning the $\mathcal{L}_{\mathrm{RS}}$-structure $(G, D_{\mathcal{H}})$, where G is a ternary semigroup endowed with the ternary relation $D_{\mathcal{H}}$ defined by clause $[D]_{\mathcal{H}}$ in I.3.1, and $\mathcal{H} \subseteq X_G$ is a non-empty set of TS-characters separating points in G. Note that $(G, D_{\mathcal{H}})$ *is not* required to be a real semigroup.

Proposition I.3.7 *Let G be a ternary semigroup and let $\mathcal{H}$ be a subset of X_G. Then,*

(1) *The closure $\overline{\mathcal{H}}$ of $\mathcal{H}$ in $(X_G)_{\mathrm{con}}$ defines on G the same representation relation as $\mathcal{H}$.*

(2) *Let $\widetilde{\mathcal{H}}$ be the set of all $p \in X_G$ such that for all $a, b, c \in G$,*

$(*)$ $$a \in D_G(b, c) \;\Rightarrow\; p(a) \in D_{\mathbf{3}}(p(b), p(c)).$$

Then,

(i) $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$.

(ii) $\mathcal{H}$ *and* $\widetilde{\mathcal{H}}$ *define the same ternary relation on G.*

(iii) $\widetilde{\mathcal{H}}$ *is maximal satisfying conditions (i) and (ii): if $\mathcal{H} \subseteq \mathcal{G} \subseteq X_G$ and $D_{\mathcal{G}}$ is identical to $D_{\mathcal{H}}$, then $\mathcal{G} \subseteq \widetilde{\mathcal{H}}$.*

In particular,

(iv) $\widetilde{\mathcal{H}}$ *is a proconstructible subset of X_G.*

Proof. Note first that, for fixed $a, b, c \in G$, the sets $\{g \in X_G \,|\, g(a) = g(b)\}$, $\{g \in X_G \,|\, g(a) \in D_{\mathbf{3}}(g(b), g(c))\}$ (and their complements) are open in $(X_G)_{\mathrm{con}}$.

(1) The inclusion $D_{\overline{\mathcal{H}}} \subseteq D_{\mathcal{H}}$ follows from $\mathcal{H} \subseteq \overline{\mathcal{H}}$. Conversely, let $a \in D_{\mathcal{H}}(b,c)$ and assume $h'(a) \notin D_{\mathbf{3}}(h'(b), h'(c))$ for some $h' \in \overline{\mathcal{H}}$. Then, the (clopen) set $\{g \in X_G \mid g(a) \notin D_{\mathbf{3}}(g(b), g(c))\}$ is a neighborhood of h' in the constructible topology, which implies $\mathcal{H} \cap \{g \in X_G \mid g(a) \notin D_{\mathbf{3}}(g(b), g(c))\} \neq \emptyset$, i.e., $h(a) \notin D_{\mathbf{3}}(h(b), h(c))$ for some $h \in \mathcal{H}$, contradicting the assumption $a \in D_{\mathcal{H}}(b,c)$.

(2). (i) is clear.

(ii) Inclusion $D_{\widetilde{\mathcal{H}}} \subseteq D_{\mathcal{H}}$ follows from $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$. The reverse inclusion follows from $(*)$ in the definition of $\widetilde{\mathcal{H}}$ by use of clause $[D]_{\widetilde{\mathcal{H}}}$ in Definition I.3.1.

(iii) Straightforward checking, using the assumptions in (iii) and I.3.1, $[D]_{\mathcal{G}}$, shows that any $g \in \mathcal{G}$ verifies clause $(*)$ in the definition of $\widetilde{\mathcal{H}}$, whence, $g \in \widetilde{\mathcal{H}}$.

(iv) is an immediate consequence of (1) (applied with $\widetilde{\mathcal{H}}$), and (2.iii). □

The next example shows that the inclusion $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$ in Proposition I.3.7 (2;i) may be strict.

Example I.3.8 Let G be a group of exponent 2 with a distinguished element $-1 \neq 1$. Pick five group characters $\sigma_1, \ldots, \sigma_5 : G \longrightarrow \{\pm 1\}$ sending -1 to -1, which separate points (see Example I.3.6). Let $\mathcal{H}$ consist of $\sigma_1, \ldots, \sigma_5$ plus all products of three of them, except $\sigma_1\sigma_2\sigma_3$. With $D_{\mathcal{H}}$ denoting the ternary relation defined by $\mathcal{H}$ (I.3.1), we prove:

Claim. For $x, y \in G, x \in D_{\mathcal{H}}(1,y)$ implies $x = 1, x = y$ or $y = -1$. In other words, $D_{\mathcal{H}}$ is the fan representation relation on G (cf. [DM1], Ex. 1.7, pp. 8-9).

Proof of Claim. Assume, towards a contradiction, that $x \neq 1, y$, and $y \neq -1$. Since $\{\sigma_1, \ldots, \sigma_5\}$ separates points, there are indices $1 \leq i, j, k \leq 5$ so that

(†) $\sigma_i(y) = 1, \sigma_j(x) = 1, \sigma_k(x) \neq \sigma_k(y)$.

From $x \in D_{\mathcal{H}}(1,y)$ follows $\sigma_i(x) = 1, \sigma_j(y) = -1$. These values show that the indices i, j, k are distinct.

Suppose next that $\{i,j,k\} \neq \{1,2,3\}$. Then, $\sigma := \sigma_i\sigma_j\sigma_k \in \mathcal{H}$, and

(††) $\sigma(x) \in D_{\mathbb{Z}_2}(1, \sigma(y))$.

From the values in (†) we get $\sigma(x) = -\sigma_k(x)$ and $\sigma(y) = -\sigma_k(y)$. From $\sigma_k(x) \neq \sigma_k(y)$ follows $\sigma_k(x) = -\sigma_k(y)$, and hence $\sigma(x) = -\sigma(y)$. Thus, (††) forces $\sigma(x) = 1$ and $\sigma(y) = -1$, whence $\sigma_k(x) = -1$ and $\sigma_k(y) = 1$, contradicting $\sigma_k(x) \in D_{\mathbb{Z}_2}(1, \sigma_k(y))$. Hence $\{i,j,k\} = \{1,2,3\}$.

From (†) and what was just proven comes $\sigma_4(y) = \sigma_5(y) = -1$ and $\sigma_4(x) = \sigma_5(x) = 1$.

Since $\sigma_i\sigma_j\sigma_4(x) = -1$ and $\sigma_i\sigma_j\sigma_5(x) = 1$, these characters do not preserve the representation $x \in D_{\mathcal{H}}(1,y)$, contradicting that they belong to $\mathcal{H}$, and proving the Claim.

Since $D_{\mathcal{H}}$ defines the fan structure on G, every group character $G \longrightarrow \{\pm 1\}$ sending -1 to -1 preserves it, i.e., is in $\widetilde{\mathcal{H}}$. Thus, $\sigma_1\sigma_2\sigma_3 \in \widetilde{\mathcal{H}} \setminus \mathcal{H}$, as asserted. □

Note. Of course, this example can be thought of as a counterexample of real semigroups, by adding a zero to G and to $\mathbb{Z}_2$ (note that $\mathbb{Z}_2 \cup \{0\} = \mathbf{3}$), and stipulating that all characters send 0 to 0. □

I.4 Saturation. Constructing RS-characters

Next we shall proceed to the construction of RS-characters with specific properties; these constructions, needed, e.g., in the proof of the Separation Theorems I.5.2 – I.5.4, and of the Duality Theorem I.5.1, will also play a crucial role throughout this monograph. As usual, a **real semigroup** (RS-) **homomorphism** is a homomorphism for the language $\mathcal{L}_{\mathrm{RS}} =$

$\{\cdot, 1, 0, -1, D\}$ for real semigroups, and a **RS-character** is a RS-homomorphism onto **3**, endowed with the unique RS-structure given by I.2.5. The set of all RS-characters of a RS, G, is denoted by X_G. Thus, RS-homomorphisms are TS-homomorphisms preserving representation (equivalently, transversal representation), and hence the remarks and constructions from § I.1 apply to the present case as well; the additional property needed to extend these constructions to real semigroups, is given in the next definition.

A. Saturated sets.

Definition I.4.1 Let G be a RS. A subset $S \subseteq G$ is **saturated** (resp., **transversally saturated**) iff for all $a, b \in S, D_G(a, b) \subseteq S$ (resp., $D^t_G(a, b) \subseteq S$). □

Remarks I.4.2 (a) An easy induction on dimension proves: If I (resp., S) is a saturated ideal (resp., subsemigroup) of G and $a_1, \ldots, a_n \in I$ (resp., S), then $D_G(\langle a_1, \ldots, a_n \rangle) \subseteq I$ (resp., S).

(b) For any saturated subsemigroup S of G, $\mathrm{Id}(G) = D_G(1, 1) \subseteq S$. Thus,

(c) Our "saturated subsemigroups" are the same thing as Marshall's "preorderings"; cf. [M], §6.6, p. 121.

(d) Note that S saturated and $1 \in S \Rightarrow \mathrm{Id}(G) \cdot S \subseteq S$.

In fact, given $x \in G$ and $s \in S$, we have $x^2 \in D(1, 1)$ (I.2.3 (5)). By axiom [RS2], $x^2 s \in D(s, s) \subseteq S$, as asserted.

(e) S saturated $\Rightarrow$ S transversally saturated, since $D^t(\cdot, \cdot) \subseteq D(\cdot, \cdot)$ (cf. [t-rep] in § I.2).

(f) The converse implication fails frequently. For example, $S = \{1\}$ is transversally saturated by I.2.3 (8), but doesn't contain 0, hence is not saturated (cf. I.2.3 (1)). However, we have:

(*) S transversally saturated and $\mathrm{Id}(G) \cdot S \subseteq S$ imply S saturated.

[Proof. Let $a, b \in S$ and $c \in D(a, b)$. By the second assumption, $c^2 a, c^2 b \in S$, and by transversal saturatedness, $D^t(c^2 a, c^2 b) \subseteq S$. Axiom [RS6] guarantees $c \in D^t(c^2 a, c^2 b)$, whence $c \in S$, showing that $D(a, b) \subseteq S$.]

In particular, both saturatedness notions are equivalent for subsemigroups of G containing $\mathrm{Id}(G)$. □

Examples I.4.3 (Saturated sets.) Proposition I.2.3 (12) shows that the value sets of binary forms are saturated. We will now show that this property holds for the value sets of arbitrary forms as well.

Proposition I.4.4 *Let G be a RS and let φ be a form with entries in G. Then,*

(1) *$D(\varphi)$ is saturated.*

(2) *$D^t(\varphi)$ is transversally saturated.*

Proof. (1) We prove first:

(1.i) Let $\varphi_1, \ldots, \varphi_n$ be forms over G such that $D(\varphi_i)$ is saturated for all $i \in \{1, \ldots, n\}$. Then $D(\bigoplus_{i=1}^n \varphi_i)$ is saturated.

We do the proof for $n = 2$. A straightforward induction proves it for all $n \geq 3$; for $n = 1$ there is nothing to prove.

Let $y, z \in D(\varphi_1 \oplus \varphi_2)$ and $x \in D(y, z)$. By I.2.8 (5) there are y_i, z_i ($i = 1, 2$) such that $y_i, z_i \in D(\varphi_i)$ and $y \in D(y_1, y_2)$, $z \in D(z_1, z_2)$. Then (by I.2.8 (5),(1)),

$$x \in D(y,z) \subseteq D(y_1, y_2, z_1, z_2) = D(\langle y_1, z_1\rangle \oplus \langle y_2, z_2\rangle)\,.$$

By I.2.8 (5) there are $a_i \in D(y_i, z_i)$ such that $x \in D(a_1, a_2)$. Since, by assumption, $D(\varphi_i)$ is saturated, $a_i \in D(\varphi_i)$. By I.2.8 (5) again, $x \in D(\varphi_1 \oplus \varphi_2)$.

Next, we have:

(1.ii) For all $a \in G$, $D(\langle a\rangle)$ is saturated.

We know (I.2.7 (a)) that $D(\langle a\rangle) = \{b^2 a \mid b \in G\}$. Let $y, z \in D(\langle a\rangle)$ and $x \in D(y,z)$. Then, $y = b_1^2 a$, $z = b_2^2 a$ for some $b_1, b_2 \in G$; thus, $x \in D(b_1^2 a, b_2^2 a)$. By I.2.3 (4), $x = a^2 x = a(ax)$. Now, by axiom [RS6], $x \in D^t(b_1^2 x^2 a, b_2^2 x^2 a)$, whence $ax \in D^t((b_1 xa)^2, (b_2 xa)^2)$. By [M], Prop. 6.1.5 (see also Corollary IV.5.3 (i)) ax is the unique element in this transversal value set, and $ax = (ax)^2$. It follows that $x = a(ax)^2 \in D(\langle a\rangle)$, as required.

Let $\varphi = \langle a_1, \dots, a_k\rangle$. From (i) and (ii) we get that $D(\langle a_1\rangle \oplus \dots \oplus \langle a_k\rangle)$ is saturated. But $D(\varphi) = D(\langle a_1\rangle \oplus \dots \oplus \langle a_k\rangle)$, by I.2.9.

(2) The analog of item (1.i) above for transversal representation is proved by an entirely similar argument, replacing I.2.8 (5),(1) by the corresponding statements for transversal representation. It only remains to show:

(2.ii) For $a \in G$, $D^t(\langle a\rangle) = \{a\}$ is transversally saturated.

Let $y, z \in D^t(\langle a\rangle)$ and $x \in D^t(y,z)$. Then, $y = z = a$, whence $x \in D^t(a,a)$; I.2.3 (6) gives $x = a$, as required. □

Further examples of saturated sets are obtained by taking (directed) unions:

Fact I.4.5 *Let $\mathcal{S}$ be a family of saturated subsets of a RS, directed under inclusion (i.e., for all $S_1, S_2 \in \mathcal{S}$ there is $S_3 \in \mathcal{S}$ such that $S_1, S_2 \subseteq S_3$). Then $\bigcup \mathcal{S}$ is saturated.*

Proof. Straightforward. □

Remarks. (a) Value sets of quadratic forms are not stable under product in general. A simple counterexample is the one-generator ternary semigroup $F_1 = \{1, 0, -1, x, -x, x^2, -x^2\}$ of VI.3.2 (A), endowed with the (fan) representation relation given by Theorem VI.2.1. In this example we have $x \in D_{F_1}(x,x) = x \cdot D_{F_1}(1,1) = \{0, x\}$, but $x^2 \notin D_{F_1}(x,x)$.

In Corollary IV.5.8 we will show that $D(\varphi)$ is a subsemigroup of G, whenever φ is a Pfister form. For illustration, here is a proof in the simplest case, when $\varphi = \langle 1, b\rangle$. Let $x, y \in D(1,b)$. By I.2.8(9), $xy \in D(1, b, b, b^2)$. Hence there are elements $p \in D(1,b), q \in D(b, b^2)$ such that $xy \in D(p,q)$ (I.2.8(5)). From [RS4] we get $q \in D(b, b^2) = D(1^2 \cdot b, b^2 \cdot 1) \subseteq D(1,b)$. Then, saturatedness entails $xy \in D(1,b)$.

For a related result, see Proposition I.5.8.

(b) An obvious counterexample to the saturatedness of $D^t_G(\varphi)$ is obtained by taking $G = \mathbf{3}$ and $\varphi = \langle 1, 1\rangle$, since $0 \notin D^t_{\mathbf{3}}(\varphi) = \{1\}$ but $0 \in D_{\mathbf{3}}(1,1)$. □

Important remark. The bijective correspondence between TS-characters and prime subsemigroups of TSs pointed out in Remark I.1.6 ff. extends to RSs upon using the adjective "saturated subsemigroup" instead of "prime subsemigroup".

Indeed, if $h : G \longrightarrow \mathbf{3}$ is a RS-character, I.2.5 shows that $h^{-1}[0,1]$ is a saturated subsemigroup of G. Conversely, given a saturated subsemigroup S of G, the map $h_S : G \longrightarrow \mathbf{3}$ defined in I.1.6 preserves representation: $a \in D_G(b,c) \Rightarrow h_S(a) \in D^t_{\mathbf{3}}(h_S(b), h_S(c))$. In fact, straightforward checking using Corollary I.2.5 shows that condition $h_S(a) \in D_{\mathbf{3}}(h_S(b), h_S(c))$ is equivalent to $h_S(b), h_S(c) \in \{0,1\} \Rightarrow h_S(a) \in \{0,1\}$. Since $S = h_S^{-1}[0,1]$, from $b, c \in S$, $a \in D_G(b,c)$ and saturatedness of S comes $a \in S$, i.e., $h_S(a) \in \{0,1\}$, as required. □

The following Proposition gives an explicit description of the saturated ideal (resp., subsemigroup) generated by an ideal (resp., subsemigroup, or subset) of a RS:

Proposition I.4.6 *Let G be a RS.*

(1) If $I \subseteq G$ is an ideal, then $[I] = \bigcup\{D_G(\varphi) \mid \varphi$ a form with entries in $I\}$ is the smallest saturated ideal of G containing I.

(2) If $S \subseteq G$ is a subsemigroup, then $[S] = \bigcup\{D_G(\varphi) \mid \varphi$ a form with entries in $S\}$ is the smallest saturated subsemigroup of G containing S.

(3) Let A be a non-empty subset of G and let $\prod A$ be the set of all finite products of elements of A; then $[A] = \bigcup\{D_G(n\langle 1\rangle \oplus \psi) \mid n \in \mathbb{N}$ and ψ a form with entries in $\prod A\}$ is the smallest saturated subsemigroup of G containing A.

In particular,

($1'$) Let I be a saturated ideal and $x \in G$. Then, $[I \cup x \cdot G] = \bigcup\{D_G(\langle i, x\,g\rangle) \mid i \in I$ and $g \in G\}$.

($2'$) Let S be a saturated subsemigroup and $x \in G$. Then, $[S \cup x \cdot S] = \bigcup\{D_G(s, xt) \mid s, t \in S\}$.

Remarks. We shall write $I(x)$ for $I \cup x \cdot G$, and $S(x)$ for $S \cup x \cdot S$. For $A \subseteq G$, the expression "form over A" means a form with entries in A. Item ($2'$) is Prop. 6.6.1(1) of [M].

Proof. Items (1), (2), ($1'$) and ($2'$) have been proved in [DP1], Proposition 3.3, pp. 113–114. For (3), just apply (2) with $S = \{1\} \cup \prod A =$ the semigroup generated by A, and write any form φ occurring in the right-hand side of the equality in (2) as $\varphi = n\langle 1\rangle \oplus \psi$, with $n \in \mathbb{N}$ and ψ a form with entries in $\prod A$. □

Corollary I.4.7 *Let M be a multiplicative subset of a RS, G, and let I be a saturated ideal disjoint from M. Let J be a saturated ideal containing I and maximal for being disjoint from M. Then, J is prime. In particular, a saturated ideal maximal for not containing a given element is prime.*

Proof. See [DP1], Corollary 3.4, p. 114. □

B. Constructing RS-characters. The following lemma is an analog of Lemma I.1.7 for RSs. This result, together with Lemmas I.4.10 and I.4.12, and Corollary I.4.11 below, are the main tools in constructing RS-characters.

Lemma I.4.8 *Let G be a RS. Let $I \subseteq G$ be a saturated prime ideal. Let $S \subseteq G$ be a saturated subsemigroup maximal for the condition $S \cap -S = I$. Then, $S \cup -S = G$. Such an S determines a RS-character $h : G \to \mathbf{3}$ such that $h^{-1}[0] = I$ and $h^{-1}[0,1] = S$.*

Proof. See [DP1], Lemma 3.5, p. 114. □

As an example of application of this Lemma, we have:

Corollary I.4.9 *Every saturated prime ideal of a RS is the zero-set of some character* (*and conversely*).

Proof. Let G be a RS. Clearly, the zero-set of a character of G is a (proper) saturated prime ideal (exercise).

For the interesting direction, let I be a (proper) saturated prime ideal of G, and let S_I denote the saturated subsemigroup generated by I.

Claim. $S_I \cap -S_I = I$.

Proof of Claim. The inclusion $\supseteq$ is obvious. Conversely, let $x \in S_I \cap -S_I$, whence $-x^2 \in S_I$. By Proposition I.4.6 (3), there are $n \in \mathbb{N}$ and a form ψ with entries in $\prod I = I$ such that $-x^2 \in D_G(n\langle 1\rangle \oplus \psi)$. If $n = 0$, then $-x^2 \in D_G(\psi) \subseteq I$ (I saturated), and hence $x \in I$. Assume $n \geq 1$. Invoking I.2.8 (5) we get an element $i \in D_G(\psi) \subseteq I$ such that $-x^2 \in D_G(1, i)$ and, by [RS6], $-x^2 \in D^t_G(x^2, j)$, with $j = ix^2 \in I$. Hence, $j \in D^t_G(x^2, x^2)$ which, by I.2.3 (6), yields $x^2 = j \in I$, and $x \in I$, as claimed.

By the Claim and Zorn's lemma, there is a saturated subsemigroup S of G maximal for $S \cap -S = I$. Lemma I.4.8 gives a character $h \in X_G$ such that $Z(h) = I$, as asserted. □

Lemma I.4.10 *Let G be a real semigroup, and let I be a saturated ideal, Δ a saturated subsemigroup, and T a multiplicative subset of G, respectively. Define:*

$$I[\Delta] = \{x \in G \mid -x^2 \in D(i, d) \text{ for some } i \in I \text{ and some } d \in \Delta\}.$$

Then:

(a) $I[\Delta]$ is a saturated ideal of G containing $I \cup (\Delta \cap -\Delta)$.

(b) If $I[\Delta] \cap T = \emptyset$, there exists a saturated prime ideal J of G containing I, such that $J = J[\Delta]$ and $J \cap T = \emptyset$. Moreover, J, Δ and T induce a character $h \in X_G$ such that $Z(h) = J$, $\Delta \subseteq P(h)$ and $h(t^2) = 1$ for all $t \in T$.

Proof. (a) $I \subseteq I[\Delta]$ is clear since $-x^2 \in D(-x^2, 1)$, with $1 \in \Delta$ and $-x^2 \in I$, whenever $x \in I$. Likewise, using $-x^2 \in D(0, -x^2)$ with $x \in \Delta \cap -\Delta$, we get $-x^2 \in \Delta$, and then $x \in I[\Delta]$, whence $\Delta \cap -\Delta \subseteq I[\Delta]$.

To check that $I[\Delta]$ is an ideal, let $x \in I[\Delta]$ and $g \in G$; then $-x^2 \in D(i, d)$, with $i \in I$, $d \in \Delta$. Scaling by g^2 ([RS2]) gives $-g^2x^2 \in D(ig^2, dg^2)$, with $ig^2 \in I$ and $dg^2 \in \Delta$, whence $gx \in I[\Delta]$.

To prove that $I[\Delta]$ is saturated, let $x, y \in I[\Delta]$ and $z \in D(x, y)$. Then, there are elements $i, j \in I, d, e \in \Delta$ such that $-x^2 \in D(i, d)$ and $-y^2 \in D(j, e)$. From $z \in D(x, y)$ we get $z^2 \in D(x^2, y^2)$ ([RS8]), and then $-z^2 \in D(-x^2, -y^2) \subseteq D(i, d, j, e) = D(i, j, d, e)$ ([RS2], I.2.8 (5)). By I.2.8 (5) again, $-z^2 \in D(k, f)$ for some $k \in D(i, j)$ and some $f \in D(d, e)$. Since I and Δ are saturated sets, we get $k \in I$ and $f \in \Delta$, and hence $z \in I[\Delta]$.

(b) By Zorn's lemma, the family

$$\mathcal{F} = \{\widehat{I} \subset G \mid \widehat{I} \text{ is a saturated ideal containing } I \text{ and } \widehat{I}[\Delta] \cap T = \emptyset\}$$

has a maximal element, J. We claim:

(i) J is a prime ideal. Assume, towards a contradiction, that there are $p, q \in G$ such that $pq \in J$ but $p, q \notin J$. Let J_p be the saturated ideal generated by $J \cup \{p\}$. By the maximality of J we have $J_p[\Delta] \cap T \neq \emptyset$; hence, there are elements $t \in T$, $j \in J$, $d \in \Delta$, $g \in G$ such that $-t^2 \in D(j, gp, d)$ (I.4.6 (1′)). Then $-t^2 \in D^t(jt^2, gpt^2, dt^2) = D^t(j', t^2gp, d')$ with $j' = jt^2 \in J$ and $d' = t^2d \in \Delta$ (I.2.8 (3)). Hence $-t^2gp \in D^t(j', t^2, d')$ (I.2.3 (0)). Similarly, there are elements $s \in T$, $h \in G$, $j'' \in J$, $e' \in \Delta$ so that $-s^2hq \in D^t(j'', s^2, e')$. Then, $z := (-t^2gp)(-s^2hq) = t^2s^2ghpq$ is in J, and $z \in D^t(j'j'', j's^2, j'e', j''t^2, j''d', t^2s^2, t^2e', s^2d', d'e')$ (I.2.8 (9)). Hence, $z \in D^t(k, t^2s^2, t^2e', s^2d', d'e')$ for some $k \in D^t(j'j'', j's^2, j'e', j''t^2, j''d')$ (I.2.8 (5)). Since the entries of this form are in J, and J is saturated, $k \in J$. Similarly, since Δ is saturated and t^2e', s^2d', $d'e' \in \Delta$, we obtain $z \in D^t(k, t^2s^2, f)$ with $f \in D^t(t^2e', s^2d', d'e') \subseteq \Delta$. It follows that $-t^2s^2 \in D^t(k, -z, f)$ (I.2.3 (0)). But $x = st \in T$ and $k, -z \in J$ imply $-x^2 \in D^t(l, f) \subseteq D(l, f)$ with $l \in D^t(k, -z) \subseteq J$, whence $x \in J[\Delta] \cap T$, contradiction. This proves (i).

(ii) $J[\Delta] = J$. Assume $J \subset J[\Delta]$. By (a) we know that $J[\Delta][\Delta]$ is an ideal, and by the maximality of J we have $J[\Delta][\Delta] \cap T \neq \emptyset$. Then, there are elements $t \in T$, $x \in J[\Delta]$, $d \in \Delta$ such that $-t^2 \in D(x, d)$, and elements $j \in J$, $e \in \Delta$ such that $-x^2 \in D(j, e)$. From $-t^2 \in D(x, d)$ we have $-t^2 \in D^t(t^2x, t^2d)$ ([RS6]), which implies $-t^2x \in D^t(t^2, t^2d)$ (I.2.3 (0)). From $-x^2 \in D(j, e)$ we obtain $-x^2 \in D^t(x^2j, x^2e)$ ([RS6]). Let $d' = t^2d$, $e' = x^2e$ and $j' = x^2j$. Clearly $d', e' \in \Delta$, $j' \in J$, $-t^2x \in D^t(t^2, d')$ and $-x^2 \in D^t(j', e')$. Scaling by t^2 we obtain $-x^2t^2 \in D^t(t^2j', t^2e') = D^t(j'', e'')$ with $j'' = t^2j' \in J$ and $e'' = t^2e' \in \Delta$. On the other hand, using [RS1] and [RS6] gives $-x^2t^2 \in D^t(-xt^2, -x^2t^2)$, and by (I.2.8 (5)) and the above we have $-x^2t^2 \in D^t(t^2, d', j'', e'')$. Since Δ is saturated and $d', e'' \in \Delta$, we have $-x^2t^2 \in D^t(j'', t^2, f)$ with $f \in \Delta$. Therefore $-t^2 \in D^t(j'', x^2t^2, f)$ (I.2.3 (0)), and using again that Δ is saturated, we arrive to $-t^2 \in D^t(j'', g)$ with $g \in D^t(x^2t^2, f) \subseteq \Delta$. Hence $t \in T \cap J[\Delta]$, which is impossible, proving (ii).

Let $\hat{\Delta}$ be the saturated subsemigroup generated by $\Delta \cup J$ and let $x \in \hat{\Delta} \cap -\hat{\Delta}$. Then $-x^2 \in \hat{\Delta}$, i.e., $-x^2 \in D(j, d)$ for some $j \in J$ and some $d \in \Delta$. Then $x \in J[\Delta] = J$, proving $\hat{\Delta} \cap -\hat{\Delta} = J$. Lemma I.4.8 gives a character $h \in X_G$ such that $Z(h) = J$, $\Delta \subseteq \hat{\Delta} \subseteq P(h)$. Since $J \cap T = \emptyset$, we have $h(t^2) = 1$ for all $t \in T$. □

Remark. Lemma I.4.10 is about the strongest result on the construction of RS-characters under those general assumptions. Indeed, given a RS-character $h \in X_G$, it is immediately verified that the conclusions of the Lemma hold for $I = J = Z(h)$, $\Delta = P(h)$ and $T = h^{-1}[\{1, -1\}]$ □

Corollary I.4.11 *Let G be a real semigroup and let $a \in G$. Let Δ be a saturated subsemigroup of G and T be a non-empty multiplicative subset of G such that $aT \cap \Delta = \emptyset$, where $aT = \{at \,|\, t \in T\}$. Then there exists $h \in X_G$ such that $\Delta \subseteq P(h)$ and $h(at^2) = -1$ for all $t \in T$.*

Proof. Let $\Delta[-a]$ be the saturated subsemigroup of G generated by $\Delta \cup \{-a\}$. Let $\hat{T} = a^2T$ and let $I = \{0\}$. Clearly, $\hat{T}$ is multiplicative and I is a saturated ideal of G. We claim that $I[\Delta[-a]] \cap \hat{T} = \emptyset$. Otherwise, we would have elements $t \in T$, $d_1, d_2 \in \Delta$ such that $-a^2t^2 \in D(d_1, -d_2a)$. In particular we have $-a^2t^2 \in D(1, d_1, -a, -d_2a)$, and then $-a^2t^2 \in D^t(a^2t^2, d_1a^2t^2, -at^2, -d_2at^2)$. Hence (by I.2.3 (0)),

$$at^2 \in D^t(a^2t^2, d_1a^2t^2, a^2t^2, -d_2at^2) = D^t(a^2t^2, d_1a^2t^2, -d_2at^2).$$

It follows that $at^2 \in D^t(q, -d_2at^2)$ for some $q \in D^t(a^2t^2, d_1a^2t^2)$. From the first condition we obtain $q \in D^t(at^2, d_2at^2)$ (I.2.3 (0) and [RS2]). By I.2.8 (9),

$$q^2 \in D^t(\langle\, at^2, d_2at^2 \,\rangle \otimes \langle\, a^2t^2, d_1a^2t^2 \,\rangle) = D^t(at^2, d_1at^2, d_2at^2, d_1d_2at^2).$$

Hence $q^2 \in D^t(at^2, p)$ for some $p \in D^t(d_1at^2, d_2at^2, d_1d_2at^2)$. From this condition we get $p = a^2t^2p$ and $ap \in D^t(d_1a^2t^2, d_2a^2t^2, d_1d_2a^2t^2)$ (I.2.8 (3)). Since the entries of this form are in Δ, we get $ap \in \Delta$, i.e., $p = at^2d$ with $d = ap \in \Delta$. It follows that $q^2 \in D^t(at^2, at^2d)$, and then $-at^2 \in D^t(-q^2, at^2d) \subseteq D(-1, at^2d)$. Then $at^2 \in D(1, -at^2d)$, which implies $at^2 \in D^t(a^2t^2, -at^2d)$, and we obtain $-a^2t^2 \in D^t(-at^2, -at^2d)$. Scaling by $-a$ gives $at^2 \in D^t(a^2t^2, a^2t^2d)$ and, since this entries are in Δ, we have $at^2 \in \Delta$, contradicting $aT \cap \Delta = \emptyset$ and proving our claim.

Lemma I.4.10 gives a character $h \in X_G$ such that $\Delta[-a] \subseteq P(h)$ and $h(a^2t^2) = 1$ for all $t \in T$. Since $h(-a) \geq 0$, it follows that $h(a) = -1$, and hence $h(at^2) = -1$ for all $t \in T$. □

As a particular case of the preceding Corollary for $T = \{1\}$, we obtain:

Corollary I.4.12 ([M], Lemma 6.6.3, pp. 122-124) *Let G be a RS and let $a \in G$. If S is a saturated subsemigroup of G maximal for the condition $a \notin S$, then S is a prime subsemigroup. Such an S determines a RS-character $h : G \longrightarrow \mathbf{3}$ such that $h^{-1}[0,1] = S$ and $h(a) = -1$.* □

Corollary I.4.13 *If G is a RS, then $X_G \neq \emptyset$.*

Proof. Apply Corollary I.4.12 with $a = -1$. Since $-1 \notin \mathrm{Id}(G)$, by Zorn's lemma there is a saturated subsemigroup of G maximal for $-1 \notin S$. [If $-1 = x^2$, then $(-1)x^2 = -x^2 = x^2$ which (by [TS5]) implies $x^2 = 0$, and then $-1 = 0$, absurd.] □

I.5 Separation theorems and duality

Now we come to one of our principal results; namely:

Theorem I.5.1 (The Duality Theorem) *There is a functorial duality between the category* **RS** *of real semigroups (with RS-homomorphisms), and the category* **ARS** *of abstract real spectra (with ARS-morphisms). Moreover, this duality establishes an isomorphism between the categories* **RS** *and* $\mathbf{ARS}^{op}$*, the opposite category of* **ARS**. [5]

As usual in these matters, the proof of a duality result of this kind rests on a separation theorem. The result presently needed follows from Lemmas I.4.8 and I.4.12:

Theorem I.5.2 *Let G be a RS and let $a \in G$. Then*

(a) *If I is a saturated ideal of G not containing the element a, then there exists a RS-character h such that $h(a) \neq 0$ and $h(x) = 0$ for all $x \in I$.*

(b) *If S is a saturated subsemigroup of G not containing the element a, then there exists a RS-character h such that $h(a) = -1$ and $h(x) \in \{0, 1\}$ for all $x \in S$.*

Proof. See [DP1], Theorem 4.2, p. 115. □

Theorem I.5.2 implies in turn:

Theorem I.5.3 *Let G be a RS, and let $a, b \in G$. Then:*

(1) *If $a \notin D_G(1, b)$, then there is a RS-character $h \in X_G$ such that $h(b) \in \{0, 1\}$ and $h(a) = -1$.*

(2) *If $a^2 \notin D_G(b^2, c^2)$, then there is a RS-character $h \in X_G$ such that $h(b^2) = h(c^2) = 0$ and $h(a^2) = 1$.*

Proof. See [DP1], Theorem 4.3, p. 116. □

Finally, the separation result actually used in the proof of Theorem I.5.1 is a consequence of the foregoing theorem, and takes the following form:

Theorem I.5.4 (Separation Theorem) *Let G be a RS, and let $a, b, c \in G$. Then:*

(1) *$a \in D_G(b, c)$ if and only if for all $h \in X_G$, $h(a) \in D_{\mathbf{3}}(h(b), h(c))$.*

(2) *$a \in D^t_G(b, c)$ if and only if for all $h \in X_G$, $h(a) \in D^t_{\mathbf{3}}(h(b), h(c))$.*

(3) *If $a \neq b$, there is $h \in X_G$ such that $h(a) \neq h(b)$.*

[5] The Duality Theorem was proved in [DP1], Thm. 4.1. Below we present a sketch of the proof.

Proof. See [DP1], Theorem 4.4, pp. 116–117. □

Now we give a hint of the

Proof of the Duality Theorem I.5.1. For later reference we register the definition of the contravariant functors, Φ, from the category **RS** into the category **ARS**, and Ψ, in the opposite direction that, together, establish the isomorphism of categories asserted in the statement of the theorem. For the remaining details of the proof, the reader is referred to [DP1], pp. 117–118.

(I) To a given RS, G, the functor Φ assigns the pair $(X_G, \overline{G})$, where:

— X_G is the set of RS-characters of G, and

— $\overline{G}$ is the image of G in $\mathbf{3}^{X_G}$ under the evaluation map: $\overline{G} = \{\, \overline{a} \mid a \in G\}$, where $\overline{a} \in \mathbf{3}^{X_G}$ denotes the evaluation at a, i.e., for $\sigma \in X_G$, $\quad \overline{a}\,(\sigma) = \sigma(a)$.

The facts that:

(a) The map $a \mapsto \overline{a}$ $(a \in G)$ is injective,

(b) The pair $(X_G, \overline{G}) = \Phi(G)$ is an ARS,

follow, respectively, from items (3) and (1) of Theorem I.5.4, via establishing that the (axiomatically given) relation D_G coincides with the representation relation D_{X_G} defined by condition [R] in I.2.2(2), with Sper$\,(A)$ replaced by X_G (see also [M], § 6.1, p. 99).

The functor Φ is defined on morphisms as follows. Given a RS-homomorphism $f : G \longrightarrow H$, its dual $\Phi(f) = f^*$ is defined by composition: given $\sigma \in X_H$, we set,

$$f^*(\sigma) = \sigma \circ f.$$

(II) The functor Ψ assigns to each ARS, (X, G), the semigroup $(G, \cdot, 1, 0, -1)$ endowed with the representation relation D_X defined by the analog of the stipulation [R] in I.2.2(2) (with Spec$_R(A)$ replaced by X), namely,

$$\bar{c} \in D_A(\bar{a}, \bar{b}) \Leftrightarrow \forall x \in X\, [\bar{c}(x) = 0 \vee \bar{a}(x)\bar{c}(x) = 1 \vee \bar{b}(x)\bar{c}(x) = 1].$$

Routine checking shows that this structure verifies axioms [RS0]–[RS8] (see also [M], §6.2, pp. 105–110).

In order to define the functor Ψ on morphisms recall ([M], Def., p. 103) that a **morphism of ARSs**, $g : (Y, H) \longrightarrow (X, G)$, is a map $g : Y \longrightarrow X$ such that,

[ARS-mor] For each $a \in G$ the composite mapping $a \circ g : Y \longrightarrow \mathbf{3}$ belongs to Y.

It follows that every ARS-morphism induces a RS-homomorphism $g^* : (G, D_X) \longrightarrow (H, D_Y)$ by setting, for $a \in G$,

$$g^*(a) = \text{the unique } b \in H \text{ such that } a \circ g = b.$$

Then, Ψ is defined on ARS-morphisms $g : (Y, H) \longrightarrow (X, G)$ by: $\Psi(g) = g^*$. □

Remark I.5.5 In the important case where G is G_A, the real semigroup associated to a semi-real ring A, see Examples I.1.2 (e) and I.2.2 (2), the Duality Theorem proves that the character space X_{G_A} is isomorphic (in the category **ARS**) to Sper$\,(A)$, the real spectrum of A.

Explicitly, the isomorphism is:

— To each $\alpha \in$ Sper$\,(A)$ there corresponds a character $h_\alpha \in X_{G_A}$ defined, for $a \in A$, by:

$$h_\alpha(\overline{a}) = sgn_\alpha(\pi_\alpha(a)),$$

i.e., the sign $(1, 0$ or $-1)$ of $\pi_\alpha(a)$ in the total order $\leq_\alpha$ of $A/\mathrm{supp}(\alpha)$ determined by α ($\pi_\alpha : A \longrightarrow A/\mathrm{supp}(\alpha)$ canonical); see I.1.3. It is clear that $h_\alpha : G_A \longrightarrow \mathbf{3}$ is a well defined homomorphism of ternary semigroups. That h_α preserves representation follows, e.g., from the characterization of representation in G_A given in [M], Prop. 5.5.1 (5), pp. 95–96,[6] and the fact that π_α is a ring homomorphism.

— Conversely, to each RS-character $h \in X_{G_A}$ there corresponds a set

$$\alpha_h = \{a \in A \mid h(\overline{a}) \in \{0, 1\}\}.$$

As an exercise the reader can check without difficulty that α_h is, indeed, a prime cone of A (cf. [BCR], 4.2.1, p. 86 and 4.3.1, p. 88), i.e., $\alpha_h \in \mathrm{Sper}\,(A)$.

A similar argument applies in the case $G = G_{A,T}$, where T is a preorder of A, showing that $X_{G_{A,T}}$ is isomorphic, as an ARS, to $\mathrm{Sper}\,(A, T) = \{\alpha \in \mathrm{Sper}\,(A) \mid T \subseteq A\}$. □

The following result is just one application, among many, of the Duality Theorem I.5.1. Further examples will occur later in this monograph.

Proposition I.5.6 (1) *The category of abstract real spectra is closed under* (*filtering*) *projective limits over right-directed index sets.*

(2) *Every ARS is a* (*filtering*) *projective limit of ARSs whose dual RSs are countable* (*even finitely generated*).

Proof. See [DP1], Proposition 4.5, p. 119. □

It follows from Corollary I.4.12 and results to be proved in Chapter IV (§ IV.5) that the separation property of Theorem I.5.4 (1) extends to Pfister forms.

Corollary I.5.7 *Let φ be a Pfister form over an RS, G, and let $a \in G$. Then,*

$$a \in D_G(\varphi) \quad \textit{if and only if} \quad \forall\, h \in X_G\, (h(a) \in D_{\mathbf{3}}(h * \varphi)).$$

Proof. The implication ($\Rightarrow$) is obvious since every $h \in X_G$ is a RS-morphism.

($\Leftarrow$) Corollary IV.5.8 (1),(2) proves that $D_G(\varphi)$ is a saturated subsemigroup of G. Assuming $a \notin D_G(\varphi)$, pick a saturated subsemigroup S of G containing $D_G(\varphi)$ and maximal for $a \notin S$. Then, Corollary I.4.12 gives a character $h \in X_G$ such that $h(a) = -1$ and $h\lceil S \subseteq \{0, 1\}$. In particular, all entries of $h * \varphi$ are 0 or 1, hence $D_{\mathbf{3}}(h * \varphi) \subseteq \{0, 1\}$, and $h(a) \notin D_{\mathbf{3}}(h * \varphi)$. □

The separation theorem I.5.4 implies the following *tri-semigroup* property of representation and transversal representation by binary forms:

Proposition I.5.8 *Let G be a real semigroup and let $a, b \in G$. Then, $D_G(a, b)$ and $D^t_G(a, b)$ are closed under the product* (<u>*in G*</u>) *of any three of its elements: for $c_1, c_2, c_3 \in G$,*

$$\textit{If } c_i \in D_G(a, b) \textit{ for } i = 1, 2, 3, \textit{ then } c_1c_2c_3 \in D_G(a, b),$$

and similarly for D^t.

---

[6] Namely, for $a, b, c \in A$,

$$\overline{a} \in D_{G_A}(\overline{b}, \overline{c}) \Leftrightarrow \exists\, t_0, t_1, t_2 \in \textstyle\sum A^2\, (t_0 a = t_1 b + t_2 c \text{ and } \overline{t_0 a} = \overline{a}).$$

Proof. We do the proof for D^t; that of D is even simpler. We omit the index G. By Theorem I.5.4 (2) it suffices to prove that, for every $h \in X_G$, $h(c_1c_2c_3) \in D^t_{\mathbf{3}}(h(a), h(b))$. By assumption (and I.5.4 (2)) we have

(*) $h(c_i) \in D^t_{\mathbf{3}}(h(a), h(b))$ for $i = 1, 2, 3$.

<u>Case 1.</u> $h(c_1c_2c_3) = 0$.

Then, $h(c_i) = 0$ for some i. By Corollary I.2.5, (*) entails that either $h(a) = h(b) = 0$ or $h(a) = -h(b) \neq 0$; by I.2.5 again, this implies $h(c_1c_2c_3) = 0 \in D^t_{\mathbf{3}}(h(a), h(b))$.

<u>Case 2.</u> $h(c_1c_2c_3) \neq 0$.

Then, $h(c_i) \neq 0$ for all indices $i = 1, 2, 3$. Hence, at least two of the $h(c_i)$ are equal, say $h(c_1) = h(c_2)$. Then, $h(c_1c_2) = 1$, and $h(c_1c_2c_3) = h(c_3)$. Then, the representation (*) with $i = 3$ yields $h(c_1c_2c_3) = h(c_3) \in D^t_{\mathbf{3}}(h(a), h(b))$. □

Remark. V. I. Arnold observed (in [A], Thm. 1) that the set of integers represented by a binary quadratic form over $\mathbb{Z}$, <u>whether diagonal or not</u>, is closed under the product of any three of its elements. He called this the *"tri-group"* property (though it would be more appropriate to call it "tri-semigroup" property). The preceding Proposition shows, in particular, that this property holds for <u>diagonal</u> forms over <u>any</u> semi-real ring (Example I.2.2 (2)). □

I.6 The representation partial order

In this section we address the question whether real semigroups carry a partial order induced in a natural way by its representation relations. This question is motivated by the fact that, for reduced special groups representation itself is a partial order for which the operation "–" is an involution.

However, in the context of RS's, none of the binary relations $a \in D(1, b)$ or $a \in D^t(1, b)$ defines a partial order for which the operation "–" is an involution:

— $a \in D(1, b)$ is reflexive (axioms [RS0], [RS1]) and transitive (Proposition I.2.3(12)), but neither antisymmetric nor involutive in general.

— $a \in D^t(1, b)$ is transitive and antisymmetric (these follow, respectively, from axioms [RS3] and [RS7], using items (0) and (6) in Proposition I.2.3), and involutive (Proposition I.2.3(0)), but not reflexive in general.

A closer examination of the case of RSG's reveals, however, another approach to our question based on the connection between representation in an RSG, G, and the natural order of its Boolean hull B_G. The facts known in this case are as follows.

Proposition I.6.1 *Let G be a RSG. Then,*

(a) The binary relation $a \leq b :\Leftrightarrow a \in D_G(1, b)$ defines a partial order on G such that $a \leq b$ $\Leftrightarrow -b \leq -a$.

(b) 1 and -1 are, respectively, the least and greatest elements for $\leq$.

(c) The relation $\leq$ is not compatible with the group operation (i.e., $a \leq b$ does not imply $ac \leq bc$; take $a < b$ and $c = -1$). Only the following "weak compatibility" relation holds:

$$a \leq b \wedge bd \leq cd \Rightarrow \exists e \leq d\,(ae \leq ce).$$

[This law is the equivalent for RSGs of the associativity of the representation relation, expressed

by axiom [RS3] *above; see also Proposition* I.2.3(15).]

(*d*) *The natural order of the Boolean hull B_G of G induces the order $\leq$ on G.*

(*e*) $a \in D_G(1, b) \Leftrightarrow \forall \sigma \in X_G\ (\sigma(b) = \perp\ \Rightarrow \sigma(a) = \perp) \Leftrightarrow \forall \sigma \in X_G\ (\sigma(a) \leq \sigma(b))$. □

For details and proofs, see [DM1], Cor. 4.4 and Prop. A, p. 53; for item (c), cf. [DMM], §1.1, pp. 29-32.

As mentioned in the Preface, every RS has a canonical "hull" which is a Post algebra. Every such algebra carries an order that makes it into a distributive lattice and, as we shall prove in the next chapter, a representation relation that makes it into a RS; the relationship between these structures is given in the next definition (cf. Proposition IV.2.10 (i)). In analogy to I.6.1(d), one should expect the "natural" partial order in a RS to be induced by the order of its Post hull, rather than by representation alone. This leads to:

Definition I.6.2 Let G be a RS, and let $a, b \in G$. We set:

$$a \leq_G b \quad \text{iff} \quad a \in D_G(1, b) \text{ and } -b \in D_G(1, -a).$$

[Unless necessary we omit the subscript in $\leq_G$.] □

Remark I.6.3 Note that when $G = \mathbf{3}$ this definition gives $1 <_{\mathbf{3}} 0 <_{\mathbf{3}} -1$, the <u>opposite</u> of the order of these elements as integers. □

With the notion of order just defined we get an analog of Proposition I.6.1 for RSs:

Proposition I.6.4 *Let G be a RS. Then:*

(*a*) *The relation $\leq$ is a partial order on G such that $a \leq b \Leftrightarrow -b \leq -a$.*

(*b*) *For all $a \in G$, $1 \leq a \leq -1$.*

(*c*) $a \leq 0 \ \Leftrightarrow\ a = a^2 \in \mathrm{Id}(G)$,

$0 \leq a \ \Leftrightarrow\ a = -a^2 \in -\mathrm{Id}(G)$.

(*d*) *Let X_G be the character space of G. For $a, b \in G$,*

$$a \leq_G b \ \Leftrightarrow\ \forall h \in X_G\ (h(a) \leq_{\mathbf{3}} h(b)) \ \Leftrightarrow$$

$$\Leftrightarrow\ \forall h \in X_G\ [(h(b) = 1 \Rightarrow h(a) = 1) \wedge (h(b) = 0 \Rightarrow h(a) \in \{0, 1\})].$$

Proof. First we prove item (d) and then derive the other assertions by means of the Separation Theorem I.5.4.

(d) In view of Remark I.6.3 the last condition is just a restatement of the second. As for the first equivalence, Definition I.6.2 implies that $\forall h \in X_G\ (h(a) \leq_{\mathbf{3}} h(b))$ is equivalent to $\forall h \in X_G\ (h(a) \in D_{\mathbf{3}}(1, h(b)) \wedge -h(b) \in D_{\mathbf{3}}(1, -h(a)))$, which, by Theorem I.5.4 (1), is in turn equivalent to $a \in D_G(1, b) \wedge -b \in D_G(1, -a)$, i.e., to $a \leq_G b$.

(a) $\leq$ is obviously reflexive.

Antisymmetry. Assume $a \leq_G b$ and $b \leq_G a$. From (d) we get $\forall h \in X_G\ (h(a) \leq_{\mathbf{3}} h(b) \wedge h(b) \leq_{\mathbf{3}} h(a))$. Since $\leq_{\mathbf{3}}$ is a total order —hence antisymmetric— we have $\forall h \in X_G\ (h(a) = h(b))$ which, by I.5.4 (3), entails $a = b$.

A similar argument, using item (d), proves transitivity. The last assertion of (a) is obvious.

(b) Clear, by direct inspection in Definition I.6.2.

(c) With $b = 0$, condition (d) reduces to

$$a \leq 0 \;\Leftrightarrow\; \forall h \in X_G\; (h(a) \in \{0, 1\}).$$

By I.5.4(3) the right-hand side is equivalent to $a = a^2$.

The second assertion follows from the first and item (a). □

The order defined in I.6.2 will be called the **representation partial order** of G.

The following Proposition summarizes the main properties of the representation partial order.

Proposition I.6.5 (Properties of the representation partial order.) *Let G be a RS. For $a, b, x, y \in G$ we have:*

(1) *The following are equivalent:*

(i) $a^2 \leq b \leq -a^2$; (ii) $Z(a) \subseteq Z(b)$; (iii) $b = a^2 b$.

In particular,

(2) $a^2 \leq ab \leq -a^2$ *and* $b^2 \leq ab \leq -b^2$ (*hence* $a^2 \leq \pm\, a \leq -a^2$).

(3) *If* $a^2 \leq b \leq -a^2$ *and* b *is invertible, then* a *is invertible.*

(4) $a \leq x, y \Rightarrow a \leq -xy$. *Hence,* $x, y \leq a \Rightarrow xy \leq a$.

(5) *For* $a \in G$, *the set* $a^{\downarrow} = \{x \in G \,|\, x \leq a\}$ *is a transversally saturated subsemigroup of* G.

More generally,

(6) *Let D be a non-empty subset of G (right) directed under the order $\leq$ (i.e., for all $x, y \in D$ there is $z \in D$ such that $x, y \leq z$). Then, the set $D^{\downarrow} = \{x \in G \,|\, \text{There is } y \in D \text{ such that } x \leq y\}$ is a transversally saturated subsemigroup of G.*

(7) *For all $a \in G$, the infimum and the supremum of a and $-a$ for the representation partial order $\leq$ exist, and $a \wedge -a = a^2$, $a \vee -a = -a^2$. In particular,*

(8) $a \wedge -a \leq 0 \leq b \vee -b$ *for all* $a, b \in G$.[7]

(9) *With notation as in I.1.11,* $I_a = \{x \in G \mid a^2 \leq x \leq -a^2\}$.

Note. Here $Z(a) = \{h \in X_G \mid h(a) = 0\}$, and $I_a = \{x \in G \mid a^2 x = x\}$ is the principal ideal of G generated by a $(a \in G)$.

Proof. The characterization of $\leq$ given by Proposition I.6.4 (d) is repeatedly used in this proof.

(1) (i) $\Leftrightarrow$ (ii). By I.6.4 (d), condition (i) is equivalent to

(i′) $\quad \forall\, h \in X_G\; [\, h(a)^2 \leq_{\mathbf{3}} h(b) \leq_{\mathbf{3}} -h(a)^2].$

Clearly, this condition implies (ii). Conversely, (ii) implies (i′); this is evident if $h(a) = 0$; if $h(a) \neq 0$, then $h(a)^2 = 1$, and (i′) holds by I.6.4(b).

Obviously, (iii) $\Rightarrow$ (ii). For (ii) $\Rightarrow$ (iii), by the separation theorem for RSs (Theorem

[7] Called the *Kleene inequality*; cf. IV.1.2 (b).

I.5.4 (3)), it suffices to prove that $h(a^2b) = h(b)$ for all $h \in X_G$. This is clear if $h(b) = 0$; if $h(b) \neq 0$, by (ii), $h(a) \neq 0$, whence $h(a)^2 = 1$, and we get $h(a^2b) = h(a)^2h(b) = h(b)$.

(2) follows from (1.ii), as $Z(a), Z(b) \subseteq Z(a) \cup Z(b) = Z(ab)$.

(3) follows from (1.ii) or (1.iii) (note that b is invertible iff $Z(b) = \emptyset$).

(4) For the first assertion, assume $a \leq x, y$ and $h(-xy) = 1$, with $h \in X_G$; then, one of $h(x)$ or $h(y)$ is 1, and the assumption, together with I.6.4 (d), entails $h(a) = 1$. If $h(-xy) = 0$, one of $h(x)$ or $h(y)$ is 0, and we get $h(a) \in \{0, 1\}$. For the second assertion, use the first together with I.6.4 (a).

(5) By (4), $a^{\downarrow}$ is closed under product, and by I.6.4 (a) it contains 1.

To show it is closed under transversal representation, let $x, y \in a^{\downarrow}$ and $z \in D^t_G(x, y)$. Again, we invoke I.6.4 (d) to prove $z \leq a$. Let $h \in X_G$ be such that $h(a) = 1$; since $x, y \leq a$, we get $h(x) = h(y) = 1$, and hence $h(z) \in D^t_{\mathbf{3}}(h(x), h(y)) = D^t_{\mathbf{3}}(1, 1) = \{1\}$. Suppose next $h(a) = 0$; the assumption and I.6.4 (d) give $h(x), h(y) \in \{0, 1\}$, whence, by Corollary I.2.5, $h(z) \in D^t_{\mathbf{3}}(h(x), h(y)) \subseteq \{0, 1\}$, as required.

(6) This follows easily form (5). In detail: let $y \in D$ ($\neq \emptyset$); since $1 \leq y$ (I.6.4 (a)), we have $1 \in D^{\downarrow}$.

— $D^{\downarrow}$ is multiplicative.

For $i = 1, 2$, let $x_i \in D^{\downarrow}$, and let $y_i \in D$ be such that $x_i \leq y_i$. Since D is directed, there is $y_3 \in D$ such that $y_1, y_2 \leq y_3$. Hence, $x_1, x_2 \leq y_3$ and (5) yields $x_1x_2 \leq y_3$, i.e., $x_1x_2 \in D^{\downarrow}$.

— $D^{\downarrow}$ is closed under tranversal representation.

For $i = 1, 2$, let $x_i \in D^{\downarrow}$, $z \in D^t_G(x_1, x_2)$, and $y_i \in D$ be such that $x_i \leq y_i$. Since D is directed, there is $y_3 \in D$ so that $x_i \leq y_i \leq y_3$ $(i = 1, 2)$. By (5), $z \leq y_3$, whence $z \in D^{\downarrow}$.

(7) By (2) it only remains to prove:

$$\text{For all } x \in G,\ \ x \leq a \ \text{ and } \ x \leq -a \ \text{ imply } \ x \leq a^2,$$

and the dual condition for the supremum; these follow at once from (4).

(8) follows from (7) and $a^2 \leq 0 \leq -b^2$ (I.6.4 (c)).

(9) is clear from the definition of I_a and (1.iii). □

Remark. A set of the form $a^{\downarrow}$ may not be saturated; if, e.g., $0 \not\leq a$, we have $0 \in D_G(a, a)$ (I.2.3 (1)), but $0 \notin a^{\downarrow}$. □

Added December 2011.

The next Proposition gives an internal characterization of the representation partial order of the real semigroup associated to a preordered ring.

I.6.6 Reminder. (i) Given a ring A and a preorder T of A, in the proof below we shall use the identification of the character space of the real semigroup $G_{A,T}$ with Sper (A, T) established by the bijection defined in I.5.5:

$$\alpha \in \text{Sper}\,(A, T) \quad \longleftrightarrow \quad h_\alpha \in X_{G_{A,T}},$$

where $h_\alpha(\overline{a}) = \mathrm{sgn}_\alpha(\pi_\alpha(a))$, with $\pi_\alpha : A \longrightarrow A/\mathrm{supp}(\alpha)$ canonical.

The characterization of the representation partial order in terms of characters given in Proposition I.6.4 (d) will also be used.

(ii) For $a \in A$ we denote by $\overline{a}$ the map $\mathrm{Sper}\,(A,T) \longrightarrow \mathbf{3}$ defined in I.1.2 (e). □

Proposition I.6.7 *Let A be a ring, T a preorder of A and $G_{A,T}$ be the RS associated to (A,T); $\leq$ will denote the representation partial order of $G_{A,T}$. For $a, b \in A$, the following are equivalent:*

(1) $\overline{a} \leq \overline{b}$.

(2) *There are $s_0, s_1, s_2 \in T$ such that*

 (*i*) $s_0 a = s_1 + s_2 b$;

 (*ii*) *For all $\alpha \in \mathrm{Sper}\,(A,T)$, $\pi_\alpha(b) >_\alpha 0 \Rightarrow \pi_\alpha(s_2) >_\alpha 0$ and $\pi_\alpha(a) <_\alpha 0 \Rightarrow \pi_\alpha(s_0) >_\alpha 0$.*

(3) *There are $s_0, s_1, s_2 \in T$ such that*

 (*i*) $s_0 a = s_1 + s_2 b$;

 (*ii*) $\overline{a} \leq \overline{s_0 a}$ *and* $\overline{s_0 b} \leq \overline{b}$.

Notes. (a) Since $s_i \in T$, the conditions $\pi_\alpha(s_i) >_\alpha 0$ in 2.(ii) are equivalent to $\pi_\alpha(s_i) \neq 0$.

(b) For an analog of item (3) for binary representation (instead of the order $\leq$), see Proposition 5.5.1 (5), p. 95, of [M].

Proof. To abridge, we write D for $D_{G_{A,T}}$.

(1) $\Rightarrow$ (2). By Definition I.6.2, (1) means $\overline{a} \in D(1, \overline{b})$ and $-\overline{b} \in D(1, -\overline{a})$. Using [M], Prop. 5.5.1 (5) on each of these representations, there are $t_i, t'_i \in T$ $(i = 1, 2)$ such that

$$(\dagger) \qquad t_0 a = t_1 + t_2 b\,, \; -t'_0 b = t'_1 - t'_2 a \;\text{ and }\; \overline{t_0 a} = \overline{a}\,, \; \overline{t'_0 b} = \overline{b}\,.$$

From this comes: $(t_0 + t'_2)\, a = (t_1 + t'_1) + (t'_0 + t_2)\, b$. Setting $s_0 = t_0 + t'_2$, $s_1 = t_1 + t'_1$, $s_2 = t'_0 + t_2$ (all in T), gives at once (2.i).

Now we use the last two equalities in ($\dagger$) to get (2.ii). Let $\alpha \in \mathrm{Sper}\,(A,T)$.

— Let $\pi_\alpha(b) >_\alpha 0$. The identification in I.6.6 (i) yields $h_\alpha(\overline{b}) = 1$ and (from $\overline{t'_0 b} = \overline{b}$), $h_\alpha(\overline{t'_0 b}) = 1$; it follows that $\pi_\alpha(t'_0 b) >_\alpha 0$. This implies $\pi_\alpha(t'_0) \neq 0$, and (since $t'_0 \in T \subseteq \alpha$) $\pi_\alpha(t'_0) >_\alpha 0$. We conclude: $\pi_\alpha(s_2) = \pi_\alpha(t'_0) + \pi_\alpha(t_2) >_\alpha 0$.

— Likewise, if $\pi_\alpha(b) <_\alpha 0$ we get $h_\alpha(\overline{a}) = -1$ and, from $\overline{t_0 a} = \overline{a}$ follows $h_\alpha(\overline{t_0 a}) = -1$, i.e., $\pi_\alpha(t_0 a) <_\alpha 0$. This entails $\pi_\alpha(t_0) \neq 0$, whence $\pi_\alpha(t_0) >_\alpha 0$ and, finally, $\pi_\alpha(s_0) = \pi_\alpha(t_0) + \pi_\alpha(t'_2) >_\alpha 0$.

(2) $\Rightarrow$ (3). Now we reinterpret conditions (2.ii) in terms of the representation partial order to get (3.ii), and conversely.

(2.ii) $\Rightarrow$ (3.ii). Using Proposition I.6.4 (d) with the characters h_α, we must show, for $\alpha \in \mathrm{Sper}\,(A,T)$,

$$h_\alpha(s_0 a) = 1 \;\Rightarrow\; h_\alpha(a) = 1 \quad\text{and}\quad h_\alpha(a) = -1 \;\Rightarrow\; h_\alpha(s_0 a) = -1\,,$$

or, in terms of the maps π_α,

(*) $\qquad \pi_\alpha(s_0 a) >_\alpha 0 \;\Rightarrow\; \pi_\alpha(a) >_\alpha 0 \quad \text{and} \quad \pi_\alpha(a) <_\alpha 0 \;\Rightarrow\; \pi_\alpha(s_0 a) <_\alpha 0.$

Likewise,

(**) $\qquad \pi_\alpha(b) >_\alpha 0 \;\Rightarrow\; \pi_\alpha(s_2 b) >_\alpha 0 \quad \text{and} \quad \pi_\alpha(s_2 b) <_\alpha 0 \;\Rightarrow\; \pi_\alpha(b) <_\alpha 0.$

Proof of (*). Since $\pi_\alpha(s_0) \geq_\alpha 0$ ($s_0 \in T$), we have $\pi_\alpha(s_0 a) >_\alpha 0 \Rightarrow \pi_\alpha(s_0) \neq 0 \Rightarrow \pi_\alpha(s_0) >_\alpha 0$. Also, $\pi_\alpha(a) \neq 0$, and $\pi_\alpha(s_0 a) >_\alpha 0$ together with $\pi_\alpha(s_0) >_\alpha 0$ give $\pi_\alpha(a) >_\alpha 0$. [Note this *does not* use (2.ii).]

By (2.ii), $\pi_\alpha(a) <_\alpha 0$ implies $\pi_\alpha(s_0) >_\alpha 0$; these clearly yield $\pi_\alpha(s_0 a) <_\alpha 0$.

Proof of (**). Similar to that of (*). From (2.ii) follows:

$$\pi_\alpha(b) >_\alpha 0 \;\Rightarrow\; \pi_\alpha(s_2 b) >_\alpha 0 \;\Rightarrow\; \pi_\alpha(s_2 b) >_\alpha 0.$$

Also,

$$\pi_\alpha(s_2 b) <_\alpha 0 \;\Rightarrow\; \pi_\alpha(b) \neq 0 \wedge \pi_\alpha(s_2) \neq 0 \;\Rightarrow\; \pi_\alpha(b) \neq 0 \wedge \pi_\alpha(s_2) >_\alpha 0 \;\Rightarrow\; \pi_\alpha(b) <_\alpha 0\,.$$

(3.ii) $\Rightarrow$ (2.ii). This is clear, reversing the preceding reasoning; in detail:

— Let $\pi_\alpha(b) >_\alpha 0$. From the second inequality in (3.ii), using I.6.4 (d), $\pi_\alpha(s_2 b) >_\alpha 0$, which entails $\pi_\alpha(s_2) \neq 0$ (equivalent to $\pi_\alpha(s_2) >_\alpha 0$).

— Let $\pi_\alpha(a) <_\alpha 0$. The first inequality in (3.ii) and I.6.4 (d) yield $\pi_\alpha(s_0 a) <_\alpha 0$ which, in turn, entails $\pi_\alpha(s_0) \neq 0$.

(2) $\Rightarrow$ (1). By I.6.4 (d) applied with the characters h_α, and I.6.6 (i), we must show, for $\alpha \in \operatorname{Sper}(A, T)$:

$$\pi_\alpha(b) >_\alpha 0 \;\Rightarrow\; \pi_\alpha(a) >_\alpha 0 \quad \text{and} \quad \pi_\alpha(a) <_\alpha 0 \;\Rightarrow\; \pi_\alpha(b) <_\alpha 0\,.$$

For the first implication: from $\pi_\alpha(b) >_\alpha 0$, by (2.ii), $\pi_\alpha(s_2) >_\alpha 0$, and hence $\pi_\alpha(s_2 b) >_\alpha 0$. Since $\pi_\alpha(s_1) \geq_\alpha 0$ from (2.i) comes $\pi_\alpha(s_0 a) = \pi_\alpha(s_1) + \pi_\alpha(s_2 b) >_\alpha 0$. This implies $\pi_\alpha(s_0), \pi_\alpha(a) \neq 0$, whence $\pi_\alpha(s_0) >_\alpha 0$, and then $\pi_\alpha(a) >_\alpha 0$, as required.

For the second implication: by (2.ii), $\pi_\alpha(a) <_\alpha 0 \Rightarrow \pi_\alpha(s_0) >_\alpha 0 \Rightarrow \pi_\alpha(s_0 a) <_\alpha 0$. From (2.i) we have $s_2 b = s_0 a - s_1$; since $\pi_\alpha(-s_1) \leq_\alpha 0$, we get $\pi_\alpha(s_2 b) = \pi_\alpha(s_0 a) - \pi_\alpha(s_1) <_\alpha 0$. Then, $\pi_\alpha(s_2) \neq 0$, whence $\pi_\alpha(s_2) >_\alpha 0$; altogether, this gives $\pi_\alpha(b) <_\alpha 0$, as needed. □

We shall consider the restriction of the representation partial order to the set of idempotents of a RS, G. Our next result proves, firstly, that this order coincides with the order given by the 2-semigroup structure of $\operatorname{Id}(G)$, cf. I.1.29 (i),(ii). More remarkably, it proves that under this order the set $\operatorname{Id}(G)$, is a bounded *distributive* lattice where the join and meet operations have a natural meaning.

Proposition I.6.8 *Let G be a RS.*

(1) *The restriction of the representation partial order of G to $\operatorname{Id}(G)$ coincides with the order of the 2-semigroup structure of $\operatorname{Id}(G)$, cf. I.1.29.*

(2) *With join and meet defined by*

$$\begin{aligned} a \vee b &= a \cdot b\,, \\ a \wedge b &= \textit{the unique element } c \in D^t(a,b)\,. \end{aligned}$$

for $a, b \in \operatorname{Id}(G)$, $\langle \operatorname{Id}(G), \wedge, \vee, 1, 0 \rangle$ is a distributive lattice with first element 1 and last element 0, where the (lattice) order is the (restriction of the) representation partial order.

Remark. [M], Prop. 6.1.5(1), p. 103, proves that $D^t(a,b)$ is a singleton whenever a, b are idempotents; see Corollary IV.5.3 (i) below.

Proof. (1) By I.1.29 (ii) and Proposition I.6.4, this amounts to showing:

$$b = b \cdot a \;\Leftrightarrow\; \forall\, h \in X_G\ (h(a) \leq_{\mathbf{3}} h(b)).$$

By Theorem I.5.4 (3) the left-hand side is equivalent to $\forall\, h \in X_G\ (h(b) = h(b)h(a))$. Hence it suffices to check that

$$x = x \cdot y \;\Leftrightarrow\; y \leq_{\mathbf{3}} x,$$

holds for idempotents x, y of $\mathbf{3}$, i.e., whenever $x, y \in \{0, 1\}$, a routine verification.

Note that for elements $a \in \mathrm{Id}(G)$ the characterization of $\leq$ in I.6.4 (d) boils down to:

$$(*) \qquad \begin{cases} a^2 \leq b & \Leftrightarrow\ \forall\, h \in X_G\ [h(b) = 1 \Rightarrow h(a^2) = 1], \\ b \leq -a^2 & \Leftrightarrow\ \forall\, h \in X_G\ [(h(a) = 0 \Rightarrow h(b) \in \{0, 1\})]\,. \end{cases}$$

(2) a) For $a, b \in \mathrm{Id}(G)$, $a \wedge b$ is the glb of a and b for the representation partial order.

Let $c = a \wedge b$. We prove:

— $c \leq a, b$.

As noted in (*) it suffices to prove $\forall\, h \in X_G\ (h(a) = 1 \Rightarrow h(c) = 1)$. Since any $h \in X_G$ preserves transversal representation, from $c \in D^t_G(a, b)$ we get $h(c) \in D^t_{\mathbf{3}}(h(a), h(b)) = D^t_{\mathbf{3}}(1, h(b))$. Since $h(b) \in \{0, 1\}$, inspection of the explicit definition of $D^t_{\mathbf{3}}$ (I.2.5) shows that in either case $h(c) = 1$. Similar proof for $c \leq b$.

— $\forall d \in \mathrm{Id}(G)\ (d \leq a \wedge d \leq b \Rightarrow d \leq c)$.

Suppose $d \leq a, b$, and let $h(c) = 1$. Since $c \in D^t_G(a, b) = D^t_{X_G}(a, b)$, we have $h(c)h(a) = 1$ or $h(c)h(b) = 1$ (see [TR] in §1), whence $h(a) = 1$ or $h(b) = 1$. From $d \leq a, b$ we obtain $h(d) = 1$, as required.

b) $(\mathrm{Id}(G), \wedge, \vee)$ is a distributive lattice.

We only check the distributive law, $c \vee (a \wedge b) = (c \vee a) \wedge (c \vee b)$, leaving further details to the reader.

Since $c \vee a = c \cdot a$, $c \vee b = c \cdot b$, setting $e = (c \vee a) \wedge (c \vee b) = ca \wedge cb$, we have $e \in D^t(ca, cb)$. Let $d = a \wedge b$, so that $d \in D^t(a, b)$. Scaling by c we have $cd \in D^t(ca, cb)$ (I.2.3 (2)). Since this set is a singleton, we conclude that $cd = e$, as desired. $\square$

Added December 2011.

The Proposition that follows characterizes binary representation in terms of the representation partial order and (implicitly) the lattice structure of the idempotents of a RS.

Proposition I.6.9 *Let G be a real semigroup and let $\leq$ denote its representation partial order. For $a, b, c \in G$ the following are equivalent:*

(1) $a \in D_G(b, c)$.

(2) $ab \leq a^2bc$, $ac \leq a^2bc$ *and* $z^2 \leq a^2$, *where* $z = z^2$ *is the unique element in* $D^t_G(b^2, c^2)$.

Proof. (1) $\Rightarrow$ (2). We use I.6.4 (d) to check the inequalities in (2). Let $h \in X_G$.

— If $h(a^2bc) = 1$, then $h(a) \neq 0$, whence $h(a^2) = 1$, and $h(b) = h(c) \neq 0$. From (1), it follows

$h(a) = h(b)$, or $h(a) = h(c)$; in either case, $h(a) = h(b)$, i.e., $h(ab) = 1$.

— Assume $h(a^2bc) = 0$. Then, either $h(a) = 0$ or $h(b) = 0$, whence $h(ab) = 0$, or both these values are $\neq 0$, and $h(c) = 0$. In this case, (1) implies $h(a) \in D_{\mathbf{3}}(h(b), h(c)) = D_{\mathbf{3}}(h(b), 0) = \{h(b), 0\}$, and hence $h(a) = h(b)$, i.e., $h(ab) = 1$.

The second inequality in (2) follows from the first by interchanging b and c.

Since $h(z^2)$ only takes on the values $0, 1$, to prove $z^2 \leq a^2$, assume $h(a^2) = 1$, i.e., $h(a) \neq 0$. By (1), one of $h(b)$ or $h(c)$ is $\neq 0$, i.e., either $h(b^2) = 1$ or $h(c^2) = 1$. Then, $z^2 \in D^t_G(b^2, c^2)$ and I.2.5 yield $h(z^2) = 1$, as required.

(2) $\Rightarrow$ (1). We must show:

$$\forall\, h \in X_G\, (h(a) \neq 0 \Rightarrow h(a) = h(b) \vee h(a) = h(c))\,.$$

Note that any RS-character of G is, by I.6.2, monotone with respect to $\leq$.

— If $h(a) = 1$, the first two inequalities in (2) give $h(b) \leq_{\mathbf{3}} h(b)h(c)$ and $h(c) \leq_{\mathbf{3}} h(b)h(c)$. The inequality $z^2 \leq a^2$ gives $h(z^2) = 1$ and, by $z^2 \in D^t_G(b^2, c^2)$, either $h(b^2) = 1$ or $h(c^2) = 1$, i.e., $h(b) \neq 0$ or $h(c) \neq 0$. If $h(c) = 0$, then $h(b) \leq_{\mathbf{3}} 0$ and, since $h(b) \neq 0$, we get $h(b) = 1$. If $h(c) = -1$, then $-1 = h(c) \leq_{\mathbf{3}} -h(b)$, whence $h(b) \leq_{\mathbf{3}} 1$, i.e., $h(b) = 1$. In either case, $h(b) = 1 = h(a)$.

— If $h(a) = -1$, by I.2.5 one of $h(b)$ or $h(c)$ is $\neq 0$. The first two inequalities in (2) give $-h(b), -h(c) \leq_{\mathbf{3}} h(b)h(c)$. If $h(b) = 1$ then, $h(c) = -1$, and $h(a) = h(c)$. If $h(b) = 0$, then $h(c) \neq 0$ and $-h(c) \leq_{\mathbf{3}} 0$, whence $0 \leq_{\mathbf{3}} h(c)$ and $h(c) = -1 (= h(a))$. The remaining case is $h(b) = -1 = h(a)$, as required. $\square$

A first, interesting, corollary of Proposition I.6.9 is:

Corollary I.6.10 *In the presence of product and the constants* $1, 0, -1$, *the binary representation relation of a RS is first-order mutually interdefinable with its representation partial order.*

Proof. The representation partial order is, by Definition I.6.2, quantifier-free definable from binary representation. Conversely, Propositions I.6.9 and I.6.8 show that the ternary relation $D(\cdot,\cdot)$ is first-order (but *not* quantifier-free) definable in the language $\{1, 0, -1, \cdot, \leq\}$. Indeed, since the unique element z such that $z^2 \in D^t_G(b^2, c^2)$ is the infimum of b^2 and c^2 for the representation partial order *retricted to idempotents* (I.6.8), it suffices to substitute the last requirement in I.6.9 (2) by the usual definition of the infimum, namely:

$$\forall z\, [\, z^2 \leq b^2 \wedge z^2 \leq c^2 \wedge \forall x\, (x^2 \leq b^2 \wedge x^2 \leq c^2 \rightarrow x^2 \leq z^2) \rightarrow z^2 \leq a^2\,]\,. \qquad \square$$

Corollary I.6.11 *Let G be a RS and let $h : G \longrightarrow \mathbf{3}$ be a character of ternary semigroups. The following are equivalent:*

(1) $h \in X_G$.

(2) *i)* *h is monotone for the representation partial order.*

ii) $h\lceil \mathrm{Id}(G) : \mathrm{Id}(G) \longrightarrow \{1, 0\}$ *is a homomorphism of bounded lattices* $(1 < 0)$.

Remark I.6.12 Condition (2.ii) is equivalent to either:

(2.iii) h preserves infima of the distributive lattice $\mathrm{Id}(G)$.

(2.iv) For $a, b, c \in G$, $a^2 \in D^t_G(b^2, c^2)$ and $h(b) = h(c) = 0$ imply $h(a) = 0$.

The proof of these equivalences is straightforward upon observing that, since h is assumed to preserve product, it automatically preserves suprema of $\mathrm{Id}(G)$; cf. I.6.8 (2). □

Proof. (1) ⇒ (2). This implication is clear, as both the representation partial order and the lattice operations in $\mathrm{Id}(G)$ are quantifier-free definable in terms of constants, product and binary representation (I.6.2, I.6.8 (2)).

The implication (2) ⇒ (1) follows from (2) ⇒ (1) in Proposition I.6.9. □

Corollary I.6.11 entails a criterion for a product of three RS-characters to be a RS-character:

Corollary I.6.13 *Let G be a RS and let $h_i \in X_G$ $(i = 1, 2, 3)$ be RS-characters of G.*

(1) *Assume that the zero set of one of the h_i's contains the zero sets of the others. Then, $(h_1 h_2 h_3)\lceil \mathrm{Id}(G)$ is a homomorphism of bounded lattices onto $\{1, 0\}$.*

In particular,

(2) *$h_1 h_2 h_3 \in X_G$ if and only if*

- (*i*) *The zero set of one of the h_i's contains the zero sets of the others.*
- (*ii*) *$h_1 h_2 h_3$ is monotone for the representation partial order of G.*

Proof. (1) Without loss of generality we suppose $Z(h_2), Z(h_3) \subseteq Z(h_1)$. By I.6.12 (2.iv) it suffices to show

$$a^2 \in D^t_G(b^2, c^2) \text{ and } h_1 h_2 h_3(b) = h_1 h_2 h_3(c) = 0 \text{ imply } h_1 h_2 h_3(a) = 0\,.$$

By the assumption on the zero sets of the h_i's, $h_1 h_2 h_3(b) = 0$ implies $h_1(b) = 0$; likewise, $h_1(c) = 0$. Since $h_1 \in X_G$, from $a^2 \in D^t_G(b^2, c^2)$ comes $h_1(a) = 0$, and hence $h_1 h_2 h_3(a) = 0$.

(2) For the implication (⇒), assertion (2.i) comes from Lemma II.2.11 (1), and (2.ii) comes from (1) ⇒ (2.i) in Corollary I.6.11.

Conversely, assumptions (2.i), (2.ii) and item (1) show that the two conditions in I.6.11 (2) are verified with $h = h_1 h_2 h_3$, implying $h_1 h_2 h_3 \in X_G$. □

For products of the form $h^2 g$, we can give a more precise and manageable necessary and sufficient condition for $h^2 g$ to be a RS-character.

Proposition I.6.14 *Let G be a RS and let $h, g \in X_G$ be such that $Z(g) \subseteq Z(h)$. The following are equivalent:*

(1) *$h^2 g \in X_G$.*

(2) *The set $Z(h) \cup g^{-1}[1]$ is decreasing under the representation partial order of G.*

Proof. (1) ⇒ (2). Let $b \in Z(h) \cup g^{-1}[1]$ and $a \leq b$. If $g(b) = 1$, then $a \leq b$ and $g \in X_G$ entail $g(a) = 1$ (I.6.4 (d)). So, assume $g(b) \neq 1$ and, hence, $h(b) = 0$; then, $h^2 g(b) = 0$. Together with (1) and $a \leq b$ this implies $h^2 g(a) \in \{0, 1\}$ (I.6.4 (d)).

— If $h^2 g(a) = 1$, then $g(a) = 1$, and $a \in Z(h) \cup g^{-1}[1]$.

— If $h^2 g(a) = 0$, then $a \in Z(g) \cup Z(h) = Z(h) \subseteq Z(h) \cup g^{-1}[1]$, as required.

(2) ⇒ (1). Assume (2). By Corollary I.6.13 (2) it suffices to show that $h^2 g$ is monotone for $\leq$.

Let $a, b \in G\,, a \leq b$. We must show

$$(*)\ \ h^2g(a) = 0 \Rightarrow h^2g(b) \in \{0, -1\} \quad \text{and} \quad (**)\ \ h^2g(a) = -1 \Rightarrow h^2g(b) = -1\,.$$

Proof of (*). Assume $h^2g(a) = 0$. Then (by $Z(g) \subseteq Z(h)$), $h(a) = 0$. From $a \leq b$ and $h \in X_G$ follows $h(b) \in \{0, -1\}$. If $h(b) = -1$, then (from $Z(g) \subseteq Z(h)$), $g(b) \neq 0$. If $g(b) = -1$, we get $h^2g(b) = -1$.

So, suppose $g(b) = 1$. Then (by $a \leq b$), $g(a) = 1$. From $h(a) = 0$ we get $-a \in Z(h)$. Since $-b \leq -a$ (cf. I.6.4 (a)), by assumption (2) we get $-b \in Z(h) \cup g^{-1}[1]$, in contradiction to $g(b) = 1\,, h(b) = -1$.

Proof of (**). From $h^2g(a) = -1$ comes $h(a) \neq 0$ and $g(a) = -1$ and hence (by $g \in X_G$ and $a \leq b$), $g(b) = -1$. Since $a \notin Z(h) \cup g^{-1}[1]$, assumption (2) yields $b \notin Z(h) \cup g^{-1}[1]$, whence $h(b) \neq 0$, and $h^2g(b) = -1$, as required. □

Remark. By Lemma II.2.11 (1), if $h^2g \in X_G$, the zero sets $Z(g)$ and $Z(h)$ are comparable under inclusion. The alternative $Z(h) \subseteq Z(g)$ implies $g^2 = h^2g^2$ (Lemma I.1.19 (2)), which in turn gives $g = h^2g$. Note also that Lemma I.1.18 (4) yields, in any case, $g \rightsquigarrow h^2g$. □

The foregoing results, especially Corollary I.6.11, raise the natural question whether, for TS-characters, monotonicity with respect to the representation partial order implies preservation of binary representation (i.e., is equivalent to being a RS-character). In the sequel we exhibit a counterexample showing that the answer is negative. The counterexample will be constructed in two steps; the first is:

Lemma I.6.15 *Let G be a real semigroup whose only invertible elements are 1 and -1. The map $h : G \longrightarrow \mathbf{3}$ defined by*

$$h(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \text{ and } x \neq -1 \\ -1 & \text{if } x = -1, \end{cases}$$

is a character of ternary semigroups, monotone for the representation partial order.

Proof. (1) h is a TS-character.

Obviously h preserves constants: $h(i) = i$ for $i \in \{1, 0, -1\}$.

h is multiplicative. This is also straightforward arguing according the values of $h(xy)$, for $x, y \in G$. In detail:

— If $h(xy) = 1$, then $xy = 1$, i.e., both x and y are invertible; by assumption, $x, y \in \{\pm 1\}$, and hence $x = y = 1$ or $x = y = -1$, which clearly yields $h(x)h(y) = 1 = h(xy)$.

— If $h(xy) = 0$, then $x, y \notin \{\pm 1\}$, which implies $x \notin \{\pm 1\}$ or $y \notin \{\pm 1\}$. By the definition of h, $h(x) = 0$ or $h(y) = 0$, whence $h(x)h(y) = 0 = h(xy)$.

— If $h(xy) = -1$, then $xy = -1$, and both x and y are invertible; by assumption, $x, y \in \{\pm 1\}$ and, since $xy = -1$, we have either $x = 1$ and $y = -1$, or $x = -1$ and $y = 1$, implying $h(x)h(y) = -1 = h(xy)$.

(2) h is monotone for the representation partial order $\leq$.

Let $a, b \in G$, $a \leq b$. If $h(b) = 1$, then $b = 1$, which entails $a = 1$ (I.6.4 (b)), and hence $h(a) = 1 \leq h(b)$. If $h(b) = 0$, then $b \notin \{\pm 1\}$. From $a \leq b$ comes $a \neq -1$, hence $h(a) \in \{0, 1\}$, and $h(a) \leq h(b)$. □

The bulk of the construction is summarized in:

Proposition I.6.16 *There exists a real semigroup G having ± 1 as only invertible elements and such that the map $h : G \longrightarrow \mathbf{3}$ defined in Lemma I.6.15* does not *preserve binary representation.*

Warning. The proof uses the contruction, in IV.1.10, of the centered Kleene algebra $K(L)$ associated to a bounded distributive lattice L, as well as the proof, in V.7.2, that, under a certain requirement (therein called [cn]) the Kleene algebra $K(L)$ is a (spectral) real semigroup. For unexplained notation and details, see these references. $\square$

I.6.17 Reminder. We recall a number of known notions and facts from general topology.

Definition. (Cf. [Mun], Ch. 4, p. 234.) A topological space is called **completely normal** if every subspace is normal. $\square$

Theorem. (Cf. [Mun], Ch. 4, p. 234.) A space X is completely normal if and only if for every pair of subsets $A, B \subseteq X$ such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ there are disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. $\square$

A somewhat less known restatement of the latter condition (coming from [Mon]) is:

Proposition A. A space X is completely normal if and only if for every pair of open sets $U, V \subseteq X$ there are open sets W_1, W_2 such that $U \cup W_1 \supseteq V$, $V \cup W_2 \supseteq U$ and $W_1 \cap W_2 = \emptyset$.

Proof. ($\Rightarrow$) Let X be completely normal and let $U, V \subseteq X$ be open. With $A = U \cap (X \setminus V)$ and $B = V \cap (X \setminus U)$ it is easily checked that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. The preceding characterization gives disjoint open sets W_1, W_2 such that $A \subseteq W_1$ and $B \subseteq W_2$ which, in turn, imply $U \cup W_1 \supseteq V$ and $V \cup W_2 \supseteq U$.

($\Leftarrow$) Let $A, B \subseteq X$ be such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Applying the stated condition with $U = X \setminus \overline{A}$, $V = X \setminus \overline{B}$, we get disjoint open sets W_1, W_2 such that $U \cup W_1 \supseteq V$, $V \cup W_2 \supseteq U$; this, in turn, gives $W_1 \supseteq (X \setminus \overline{B}) \cap \overline{A}$ and $W_2 \supseteq (X \setminus \overline{A}) \cap \overline{B}$. But $A \cap \overline{B} = \emptyset$ implies $(X \setminus \overline{B}) \cap \overline{A} \supseteq (X \setminus \overline{B}) \cap A = A$, and so $W_1 \supseteq A$. A similar argument proves $W_2 \supseteq B$. $\square$

Finally, we recall:

Proposition B. (Cf. [Mun], Ch. 4, pp. 234, 243.) Any metric space is completely normal. $\square$

Proof of Proposition I.6.16. We choose X to be a connected metric space having at least two points, and take $L = \Omega(X)$ to be the bounded distributive lattice of all opens of X ordered under inclusion. Let $K(L) = \{(U, V) \in L \times L^{\mathrm{inv}} \,|\, U \cap V = \emptyset\}$ denote the centered Kleene algebra constructed in IV.1.10 from the lattice L. Note that the order $\leq$ of $K(L)$ is inclusion ($\subseteq$) in the "L-coordinate" and reverse inclusion in the "L^{inv}-coordinate"; the center $\underline{\mathbf{c}}$ is $(\emptyset, \emptyset)$, and the opposite ("negation") of (U, V) is (V, U).

In order to show that $K(L)$ (under product given by symmetric difference) is a real semigroup it suffices to verify property [cn] in Theorem V.7.2 (2):

[cn] For all $a, b \in K(L)$ such that $a, b \leq \underline{\mathbf{c}}$, there are $x, y \in K(L)$ such that $a \wedge x \leq b$, $b \wedge y \leq a$ and $x \vee y = \underline{\mathbf{c}}$.

From $a, b \leq \underline{\mathbf{c}} = (\emptyset, \emptyset)$ we infer that a, b are of the form $a = (\emptyset, U), b = (\emptyset, V)$. Since X is completely normal, from Proposition A in I.6.17 we get disjoint open sets W_1, W_2 such that $U \cup W_1 \supseteq V$ and $V \cup W_2 \supseteq U$. Letting $x := (\emptyset, W_1), y := (\emptyset, W_2)$, from the first of these inclusions we get

$$a \wedge x = (\emptyset, U) \wedge (\emptyset, W_1) = (\emptyset, U \cup W_1) \leq (\emptyset, V) = b,$$

and, from the second inclusion, $b \wedge y \leq a$. Also,

$$x \vee y = (\emptyset, W_1) \vee (\emptyset, W_2) = (\emptyset, W_1 \cap W_2) = (\emptyset, \emptyset) = \underline{\mathbf{c}}.$$

Thus, $K(L)$ is a real semigroup which, in addition, is spectral, that we call G. Connectivity of X implies that the only invertible elements of G are $1 = (\emptyset, X)$ and $-1 = (X, \emptyset)$. In fact, recalling that product in $K(L)$ is symmetric difference, an element $(U, V) \in K(L)$ is invertible if and only if $(U, V) \wedge \neg(U, V) = (U, V) \wedge (V, U) = 1 = (\emptyset, X)$, and this is equivalent to $U \cap V = \emptyset$ and $U \cup V = X$, i.e., the sets U, V form a clopen partition of X. From the connectivity assumption follows, then, that one of U or V is empty and the other is X, proving that $G^\times = \{1, -1\}$.

Consider now the monotone TS-character h of G constructed in I.6.15.

Claim. h *does not* preserve the binary representation relation of G.

Proof of Claim. Consider the pairs $a = (\emptyset, X \setminus \{x\}), b = (\emptyset, X \setminus \{y\})$, where x, y are *distinct* points of X; clearly (since points of X are closed) $X \setminus \{x\}, X \setminus \{y\} \in \Omega(X) = L$.

We assert that $1 \in D_G(a, b)$. Since $a, b \leq \underline{\mathbf{c}}$ we have $a = a^2$ and $b = b^2$ (cf. I.6.4 (c)) and the stated representation boils down to

$$(*) \qquad a^2 \wedge b^2 \wedge \underline{\mathbf{c}} = a^2 \wedge b^2 \leq 1 \leq a^2 \vee b^2 \vee \underline{\mathbf{c}} = \underline{\mathbf{c}},$$

or, equivalently, $a^2 \wedge b^2 = 1 = (\emptyset, X)$. Since $a^2 \wedge b^2 = a \wedge b = (\emptyset, (X \setminus \{x\}) \cup (X \setminus \{y\})) = (\emptyset, X) = 1$, (*) is proved.

On the other hand, since $a, b \notin \{\pm 1\}$, we have $h(a) = h(b) = 0$ and then obtain $h(1) = 1 \notin D_{\mathbf{3}}(h(a), h(b) = D_{\mathbf{3}}(0, 0) = \{0\}$, proving the Claim, and also Proposition I.6.16. □

Next we prove that the filters of the distributive lattice $(\mathrm{Id}(G), \wedge, \vee, 1, 0)$ are in one-one correspondence with the saturated ideals of G.

Proposition I.6.18 *Let G be a RS. The assignment $I \longmapsto I \cap \mathrm{Id}(G)$ establishes a bijective correspondence between the saturated ideals of G and the filters of the distributive lattice* $\mathrm{Id}(G)$. *Under this correspondence,*

(i) *Prime saturated ideals of G correspond bijectively to prime filters of* $\mathrm{Id}(G)$.

(ii) *Principal ideals of G correspond bijectively to principal filters of* $\mathrm{Id}(G)$.

The correspondence preserves inclusion and proper inclusion.

Proof. a) If I is a saturated ideal of G, $I \cap \mathrm{Id}(G)$ is a filter of $\mathrm{Id}(G)$.

(a.i) $a \in I \cap \mathrm{Id}(G) \ \wedge \ b \in \mathrm{Id}(G) \ \wedge \ a \leq b \Rightarrow b \in I$.

From $a \leq b$ we have $b = a \vee b = a \cdot b \in I$ (I.6.8 (2)).

(a.ii) $a, b \in I \cap \mathrm{Id}(G) \Rightarrow a \wedge b \in I \cap \mathrm{Id}(G)$.

Let $c = a \wedge b$. Since I is saturated and $a, b \in I$, we get $\{c\} = D^t_G(a, b) \subseteq I$.

b) I is a prime ideal iff $I \cap \mathrm{Id}(G)$ is a prime filter.

Immediate from $a \vee b = a \cdot b \quad (a, b \in \mathrm{Id}(G))$.

c) For every filter F of $\mathrm{Id}(G)$, the set $I(F) = \{a \in G \mid a^2 \in F\}$ is a saturated ideal of G, and $I(F) \cap \mathrm{Id}(G) = F$. Further, $I(F)$ is the saturated ideal generated by F.

(c.i) $I(F)$ is an ideal.

Let $a \in I(F)$ and $b \in G$. Since $a^2b^2 = a^2 \vee b^2 \geq a^2$ and $a^2 \in F$, we get $a^2b^2 \in F$, and $ab \in I(F)$.

(c.ii) $I(F) \cap \mathrm{Id}(G) = F$.

Let $a \in I(F) \cap \mathrm{Id}(G)$; then $a^2 \in F$; since $a = a^2$, we get $a \in F$. The other inclusion is obvious.

(c.iii) $I(F)$ is the ideal generated by F.

Let J be an ideal of G containing F, and let $a \in I(F)$. Then, $a^2 \in F \subseteq J$. Since J is radical, $a \in J$. Hence $J \supseteq I(F)$.

(c.iv) $I(F)$ is saturated.

Let $a, b \in I(F)$ and $c \in D_G(a, b)$; we must prove that $c \in I(F)$. Axiom [RS8] gives $c^2 \in D_G(a^2, b^2)$, and, by [RS6], $c^2 \in D^t_G(a^2c^2, b^2c^2)$. Using the distributive law in $\mathrm{Id}(G)$ we have $c^2 = (a^2c^2) \wedge (b^2c^2) = (a^2 \vee c^2) \wedge (b^2 \vee c^2) = (a^2 \wedge b^2) \vee c^2$, whence $a^2 \wedge b^2 \leq c^2$. Since F is a filter and $a^2, b^2 \in F$, we get $c^2 \in F$, and hence $c \in I(F)$.

For $a \in \mathrm{Id}(G)$, $F_a = \{b \in \mathrm{Id}(G) \mid a \leq b\}$ denotes the principal filter generated by a.

d) For $a \in G$, $I_a \cap \mathrm{Id}(G) = F_{a^2}$. Conversely, if $b \in \mathrm{Id}(G)$, then $I(F_b) = I_b$.

(d.i) For the first assertion, let $x \in I_a \cap \mathrm{Id}(G)$. From Proposition I.6.5 (9), $a^2 \leq x = x^2 \leq -a^2$, which shows that $x \in F_{a^2}$. For the other inclusion use Proposition I.6.4(c).

(d.ii) For the second assertion, Proposition I.6.5 (9) gives, for $x \in G$:

$$x \in I_b \Leftrightarrow x^2 \in I_b \Leftrightarrow b^2 \leq x^2 \leq -b^2 \Leftrightarrow b = b^2 \leq x^2 \Leftrightarrow x^2 \in F_b \Leftrightarrow x \in I(F_b).$$

Proposition I.6.18 follows from (a) – (d). (For the last assertion, recall that a bijective, inclusion-preserving map also preserves proper inclusion.) □

Corollary I.6.19 *Let G be a RS. The correspondence $P \longmapsto P \cap \mathrm{Id}(G)$ is a homeomorphism of the set $\mathrm{Spec}_{\mathrm{sat}}(G)$ of saturated prime ideals of G with the topology having as a basis of opens the sets $D(a) = \{P \in \mathrm{Spec}_{\mathrm{sat}}(G) \mid a \notin P\}$ $(a \in G)$, onto the spectrum $\mathrm{Spec}\,(\mathrm{Id}(G))$ of the distributive lattice $\mathrm{Id}(G)$, i.e., the set of its prime filters with its canonical spectral topology generated by the sets $D_{\mathrm{Id}(G)}(x) = \{\mathfrak{p} \in \mathrm{Spec}\,(\mathrm{Id}(G)) \mid x \notin \mathfrak{p}\}$ as a basis of quasi-compact opens (cf. [DST], 1.4.3). Hence, $\mathrm{Spec}_{\mathrm{sat}}(G)$ with the described topology is a spectral space.*

Proof. By I.6.18 (i) (and its proof), the given correspondence, say $f(P) = P \cap \mathrm{Id}(G)$ $(P \in \mathrm{Spec}_{\mathrm{sat}}(G))$, is a bijection whose inverse is $\mathfrak{p} \longmapsto I(\mathfrak{p}) = \{a \in G \mid a^2 \in \mathfrak{p}\}$ $(\mathfrak{p} \in \mathrm{Spec}\,(\mathrm{Id}(G))$. It is clear that, for $x \in \mathrm{Id}(G)$, we have $f^{-1}[D_{\mathrm{Id}(G)}(x)] = D(x)$, i.e., f^{-1} maps bijectively the basis of $\mathrm{Spec}_{\mathrm{sat}}(G))$ onto the basis of $\mathrm{Spec}\,(\mathrm{Id}(G))$; hence, f is a homeomorphism. In particular, the basic sets of $\mathrm{Spec}_{\mathrm{sat}}(G))$ are quasi-compact open and, therefore, $\mathrm{Spec}_{\mathrm{sat}}(G))$ is a spectral space. □

Corollary I.6.20 *With notation as above, we have:*

(1) *Any ideal I of a RS, G, is the union of a family of saturated principal ideals which, in particular, are intervals for the representation partial order: $I = \bigcup_{a \in I} I_a$.*

(2) *Any saturated ideal in a finite RS is principal.*

(3) *The saturated ideals of a RS ordered under inclusion form a distributive lattice.*

Proof. (1) Clear, from $a \in I \Rightarrow I_a = a \cdot G \subseteq I$.

(2) $\mathrm{Id}(G)$ is a finite distributive lattice. Every filter in a finite lattice is principal. Conclude by Proposition I.6.18(ii).

(3) Follows from [B], Ch. IX, Thm. 7, p. 141, and Proposition I.6.18. □

I.7 Semilattice structure of abstract real spectra

New section: Jan. 2015

Definition I.7.1 Let $(X, \preceq)$ be a root-system, and let $g_1, g_2 \in X$. Define:

$$g_1 \equiv_C g_2 \ \text{ iff } \ g_1, g_2 \text{ have a common } \preceq\text{-upper bound.}$$

$\equiv_C$ is an equivalence relation; its classes are called **connected components** of $(X, \preceq)$. □

In [DST], Prop. 7.5.4 (iii) Check ref. it is proved that any spectral space whose order of specialization is a root-system is a **conditionally complete join-semilattice**: every non-empty set having an upper bound has a *unique* least upper bound (**lub**), i.e., a supremum. In particular, this applies to the character space of *any* RS under the specialization order. In this case we can give an explicit description of the lub.

Theorem I.7.2 *Let G be a RS, and let X_G be its character space. Let $\{h_i \,|\, i \in I\} \subseteq X_G$ be a non-empty family of characters belonging to a single connected component of X_G (i.e., having a common upper bound under $\rightsquigarrow$). Then, the lub $\bigvee_{i\in I} h_i$ is the RS-character f defined by:*

- *$Z(f) = \bigcap\{Z(g) \,|\, g \in X_G$ is a $\rightsquigarrow$- upper bound of $\bigvee_{i\in I} h_i\}$.*
- *If $a \notin Z(f)$, then $f(a) = g(a)$, where $g \in X_G$ is any $\rightsquigarrow$- upper bound of $\bigvee_{i\in I} h_i$.*

Proof. We shall repeatedly use the obvious fact that the set of $\rightsquigarrow$- upper bounds of any non-empty set in a spectral root-system is totally ordered under specialization. To abridge, call Ub the set of upper bounds of $\{h_i \,|\, i \in I\}$ under $\rightsquigarrow$; by assumption, $\mathrm{Ub} \neq \emptyset$.

(i) f is well defined.

Given $a \notin Z(f)$ and $g, g' \in \mathrm{Ub}$ such that $a \notin Z(g) \cup Z(g')$, by Lemma I.1.18 (4) we have $g(a) = g'(a)$ if either $g \rightsquigarrow g'$ or $g' \rightsquigarrow g$; thus, the value $f(a)$ is independent of the choice of the upper bound g in the second clause of the definition of f.

(ii) f is a TS-character of G.

Inspection of the definition of f shows that $f(i = i$ for $i \in \{0, \pm 1\}$, and that $f(ab) = f(a)f(b)$ when either one of $f(a)$ or $f(b)$ is 0. If $a, b \notin Z(f)$, then $ab \notin Z(f)$: taking $g \in \mathrm{Ub}$ such that $f(x) = g(x)$ for $x \in \{a, b, ab\}$, since $g \in X_G$, we get $f(ab) = g(ab) = g(a)g(b) = f(a)f(b)$, as required.

(iii) f is a $\rightsquigarrow$- upper bound of $\bigvee_{i\in I} h_i$..

For any $g \in \mathrm{Ub}$ and any $i \in I$, $h_i \rightsquigarrow g$ implies $Z(h_i) \subseteq Z(g)$, whence $Z(h_i) \subseteq Z(f)$. Further, if $a \notin Z(f)$ and g is such that $a \notin Z(g)$, we have both $f(a) = g(a)$ (by the definition of f) and $h_i(a) = g(a)$ (by $h_i \rightsquigarrow g$, I.1.18 (4)). Then, $h_i(a) = f(a)$, proving $h_i \rightsquigarrow f$ for all $i \in I$, as asserted.

(iv) f is the lub of $\{h_i \,|\, i \in I\}$ under $\rightsquigarrow$.

This clearly follows from Lemma I.1.18 (4) and the definition of f.

Finally, we show:

(v) f preserves representation, i.e., $f \in X_G$.

Suppose $a \in D_G(b, c)$ $(a, b, c \in G)$. We can assume $a \notin Z(f)$. If, e.g., $f(b) = 0$, then $g(b) = 0$ for all $g \in \mathrm{Ub}$. If, in addition, $a \notin Z(g)$, then $g(a) = f(a)$; from $g \in X_G$ we get $g(a) \in D_{\mathbf{3}}(g(b), g(c)) = D_{\mathbf{3}}(0, g(c)) = \{g(c)\}$, whence $g(c) \neq 0$ and $f(c) = g(c)$. Hence, $f(a) \in D_{\mathbf{3}}(f(b), f(c))$. Same argument if $f(c) = 0$.

In case that $a, b, c \notin Z(f)$, let $g_1, g_2, g_3 \in \mathrm{Ub}$ be such that $a \notin Z(g_1), b \notin Z(g_2), c \notin Z(g_3)$. Since Ub is totally ordered by $\rightsquigarrow$, we may assume $g_1 \rightsquigarrow g_2 \rightsquigarrow g_3$; then, $Z(g_1) \subseteq Z(g_2) \subseteq Z(g_3)$, whence $a, b, c \notin Z(g_1)$ and $f(x) = g_1(x)$ for $x \in \{a, b, c\}$. From $g_1 \in X_G$ follows $g_1(a) \in D_{\mathbf{3}}(g_1(b), g_1(c))$, i.e., $f(a) \in D_{\mathbf{3}}(f(b), f(c))$, as required. $\square$

Remark. Under the hypotheses of Theorem I.7.2, if $x \in G$ is such that $(\bigvee_{i \in I} h_i)(x) \neq 0$, then $h_j(x) = (\bigvee_{i \in I} h_i)(x)$ for all $j \in I$: since $h_j \underset{G}{\rightsquigarrow} \bigvee_{i \in I} h_i$ and $(\bigvee_{i \in I} h_i)(x) \neq 0$, Lemma I.1.18 (4) entails the asserted equality.

Proposition I.7.3 *Let $f : G \longrightarrow H$ be a morphism of RSs and let $f^* : X_H \longrightarrow X_G$ be the ARS morphism dual to f. If $\{h_i \,|\, i \in I\} \subseteq X_H$ is a non-empty family of characters belonging to a single connected component of X_H, then $f^*(\bigvee_{i \in I} h_i) = \bigvee_{i \in I} f^*(h_i)$; i.e., f^* preserves arbitrary suprema (inside a connected component).*

Proof. Since $f^*(\sigma) = \sigma \circ f$ for $\sigma \in X_H$, the statement to be proved translates as:

$$(\textstyle\bigvee_{i \in I} h_i) \circ f = \bigvee_{i \in I} (h_i \circ f).$$

Claim. $\bigvee_{i \in I} (h_i \circ f)$ exists.

Proof of Claim. It suffices to observe that $\{h_i \circ f \,|\, i \in I\}$ has a $\underset{G}{\rightsquigarrow}$-upper bound in X_G. Indeed, if $g \in X_H$ is a $\underset{H}{\rightsquigarrow}$-upper bound of $\{h_i \,|\, i \in I\}$, then $g \circ f$ is a $\underset{G}{\rightsquigarrow}$-upper bound of $\{h_i \circ f \,|\, i \in I\}$: by Lemma I.1.18 (5), $h_i \underset{H}{\rightsquigarrow} g$ is equivalent to $g = g^2 h_i$; composing with f gives $g \circ f = (g \circ f)^2 (h_i \circ f)$, whence $h_i \circ f \underset{G}{\rightsquigarrow} g \circ f$, for $i \in I$.

Let $\sigma := \bigvee_{i \in I} h_i$ and $\gamma := \bigvee_{i \in I} (h_i \circ f)$. In order to prove $\gamma = \sigma \circ f$, it suffices to show that $Z(\sigma \circ f) \subseteq Z(\gamma)$ (I.1.18 (4)). Let $a \in G \setminus Z(\gamma)$; then, $\gamma(a) \neq 0$, i.e., $(\bigvee_{i \in I} h_i)(f(a)) \neq 0$, i.e., $\sigma(f(a)) \neq 0$, as required. $\square$

Chapter II

RS-congruences

II.1 Convex ideals in rings and saturated ideals in real semigroups

Setup. Throughout this section A stands for a commutative ring with unit, and T for a proper preorder of A (that is, $T \dotplus T, A^2 \subseteq T, -1 \notin T$). Note that:

(i) The existence of a (proper) preorder entails that A is semi-real, i.e., $-1 \notin \sum A^2$. Recall that A is semi-real iff $\mathrm{Sper}\,(A) \neq \emptyset$.

(ii) The condition $-1 \notin T$ is equivalent to $T \neq A$ only if 2 is invertible in A, or if $T - T = A$.

(iii) $G_{A,T}$ stands for the real semigroup associated to T and A (see I.1.2 (e)). □

A. The basic correspondences. (1) There is a correspondence associating to each ideal I of A a saturated ideal $\overline{I}$ of $G_{A,T}$, namely, $\overline{I} = \{\overline{a} \mid a \in I\}$. We show:

i) $\overline{I}$ is an ideal of $G_{A,T}$ $(= G)$.

Let $g \in G$ and $a \in I$. Then, $g = \overline{b}$ for some $b \in A$. Since $ab \in I$, we have $g \cdot \overline{a} = \overline{ab} \in \overline{I}$.

ii) $\overline{I}$ is saturated.

Let $a, b \in I$ and $c \in A$ be so that $\overline{c} \in D_G(\overline{a}, \overline{b})$. By [M], Prop. 5.5.1 (5), p. 95, there are $t_0, t_1, t_2 \in T$ so that $t_0 c = t_1 a + t_2 b$ and $\overline{t_0 c} = \overline{c}$. From $a, b \in I$ follows $t_0 c \in I$, so that $\overline{c} = \overline{t_0 c} \in \overline{I}$. □

Note. We <u>do not</u> claim that $\overline{I}$ is proper if I is, nor that the correspondence $I \longmapsto \overline{I}$ is injective (both contentions are false). Below we give necessary and sufficient conditions for this to happen.

(2) Conversely, given a saturated ideal J of G $(= G_{A,T})$, the set $\widehat{J} = \{a \in A \mid \overline{a} \in J\}$ is an ideal of A. Further, (i) $\overline{\widehat{J}} = J$ and (ii) J proper $\Rightarrow$ $\widehat{J}$ proper.

Proof. — $a, b \in \widehat{J} \Rightarrow a + b \in \widehat{J}$.

Easy consequence of the facts that J is saturated and $\overline{a+b} \in D_G(\overline{a}, \overline{b})$ (cf. [M], Prop. 5.5.1 (5)).

— $a \in \widehat{J}$ and $b \in A \Rightarrow ab \in \widehat{J}$.

Clear, since $\overline{a} \in J$, $\overline{b} \in G$, and J is an ideal of G.

(i) Since $a \in \widehat{J} \Rightarrow \overline{a} \in J$, we have $\overline{\widehat{J}} \subseteq J$. Conversely, if $\overline{a} \in J$, then $a \in \widehat{J}$, and from the definition of $\overline{I}$ follows $a \in \overline{\widehat{J}}$; thus, $J \subseteq \overline{\widehat{J}}$.

(ii) If $1 \in \widehat{J}$, then $\overline{1} = 1 \in J$, and J is improper. □

We also note:

(iii) I ideal of $A \Rightarrow I \subseteq \widehat{\overline{I}}$.

Further, we have:

Fact II.1.1 *For J, J_1, J_2 ideals of G,*

(iv) *J prime $\Leftrightarrow$ $\widehat{J}$ prime.*

(v) $J_1 \subseteq J_2 \Leftrightarrow \widehat{J}_1 \subseteq \widehat{J}_2$.

(vi) *The map $J \longmapsto \widehat{J}$ is injective.*

Proof. (vi) follows at once from (v).

(iv) ($\Rightarrow$) $ab \in \widehat{J} \Rightarrow \overline{a}\overline{b} = \overline{ab} \in J \Rightarrow \overline{a} \in J$ or $\overline{b} \in J \Rightarrow a \in \widehat{J}$ or $b \in \widehat{J}$.

($\Leftarrow$) $\overline{ab} \in J \Rightarrow ab \in \widehat{J} \Rightarrow a \in \widehat{J}$ or $b \in \widehat{J} \Rightarrow \overline{a} \in J$ or $\overline{b} \in J$.

(v) ($\Rightarrow$) For $a \in A$: $a \in \widehat{J}_1 \Rightarrow \overline{a} \in J_1 \Rightarrow \overline{a} \in J_2 \Rightarrow a \in \widehat{J}_2$.

($\Leftarrow$) Let $a \in A$ be such that $\overline{a} \in J_1$; then $a \in \widehat{J}_1$, whence $a \in \widehat{J}_2$, implying $\overline{a} \in J_2$. □

B. Compatibility and convexity. For ready reference we include in this paragraph a summary of known notions and results —used below— concerning the relationship between ideals and preorders in a ring; for proofs and further details the reader is referred to [BCR], §4.2.

Definition II.1.2 Let I be an ideal and T be a (proper) preorder of a ring A.

(1) I is **T-compatible** iff for all $t \in T$, $1 + t \notin I$.

(2) I is **T-radical** iff for all $a \in A$ and $t \in T$,

$$a^2 + t \in I \Rightarrow a \in I \text{ (and hence } t \in I).$$

(3) I is **T-convex** iff for $t_1, t_2 \in T$, $t_1 + t_2 \in I \Rightarrow t_1, t_2 \in I$. □

Remarks II.1.3 (a) I is T-convex and radical $\Rightarrow$ I is T-radical $\Rightarrow$ I is radical and T-compatible. (Take, respectively, $t_1 = a^2$, $t_2 = t$; $t = 0$ and $a = 1$ in (1) and (2) of the definition above.).

(b) I is T-convex iff $-(T/I) \cap T/I = \{0\}$ iff the quotient preorder T/I is a partial order.

(c) I is T-radical iff $-(A/I)^2 \cap T/I = \{0\}$.

(d) I is T-compatible iff quotient preorder T/I is proper; in particular, I is proper. □

The relationship between these notions is given in the following:

Proposition II.1.4 ([BCR], Prop. 4.2.5, p. 87) *For an ideal I and a (proper) preorder T in a ring A, the following are equivalent:*

(1) *I is T-convex and radical.*

(2) *I is T-radical.*

(3) *I is radical and, setting $a \leq_T b$ iff $b - a \in T$, we have:*

$$0 \leq_T a \leq_T b \text{ and } b \in I \Rightarrow a \in I. \qquad \square$$

Remark. Condition (3) gives our notion the look of convexity familiar when T is a total or a partial order. Note, however, that $\leq_T$ is not a partial order (antisymmetry fails, though it is reflexive and transitive). $\square$

In case the preorder T is $\sum A^2$, we get:

Fact II.1.5 *An ideal I of A is $\sum A^2$-convex if and only if it is real (i.e., $\sum_{i=1}^n a_i^2 \in I \Rightarrow a_1, \ldots, a_n \in I$, for all $a_1, \ldots, a_n \in A$ ($n \geq 1$)).* $\square$

Proposition II.1.6 ([BCR], Prop. 4.2.6, p. 87) *Let I be an ideal and T be a (proper) preorder in a ring A. Then:*

$$\sqrt[T]{I} = \{a \in A \mid \exists\, k \geq 0\ \exists\, t \in T \text{ such that } a^{2k} + t \in I\}$$

is the smallest T-convex ideal of A containing I (possibly improper). It is the intersection of all T-convex prime ideals containing I. I is T-convex iff $I = \sqrt[T]{I}$. $\square$

$\sqrt[T]{I}$ is called the **T-convex hull** or the **T-radical** of I.

C. The correspondence $I \longmapsto \overline{I}$ under convexity assumptions. We begin by noting:

Proposition II.1.7 *An ideal I of A is T-compatible iff $\overline{I}$ is a proper ideal of $G = G_{A,T}$.*

Proof. ($\Rightarrow$) Assume $1_G \in \overline{I}$. Then, there is $a \in I$ such that $\overline{a} = 1_G = \overline{1}$ (cf. A.1 above). By [M], Prop. 5.4.2 (2), p. 93, there are $s, t \in T$ such that $(1+s)\,a = 1 + t$. Then $1 + t \in I$, and I is not T-compatible.

($\Leftarrow$) Assume $\overline{I}$ is a proper ideal of G, i.e., $1_G = \overline{1} \notin I$. Let $t \in T$; we must show that $1+t \notin I$. Otherwise, $\overline{1+t} \in \overline{I}$. But we prove next that $\overline{1} = \overline{1+t}$, contradiction.

Indeed, we show that $(\overline{1+t})\,(\alpha) = 1$, i.e., $1 + t \in \alpha \setminus (-\alpha)$ for every $\alpha \in \mathrm{Sper}\,(A,T)$, that is, for every $\alpha \in \mathrm{Sper}\,(A)$ such that $\alpha \supseteq T$. Since $1, t \in T$, we have $1 + t \in T \subseteq \alpha$. Suppose $1 + t \in -\alpha$; since $t \in \alpha$, we get $-(1+t) + t = -1 \in \alpha$, contradiction. $\square$

Lemma II.1.8 (a) *For any ideal I of A, $I \subseteq \widehat{\overline{I}} \subseteq \sqrt[T]{I}$. In particular,*

(b) *If I is a T-convex ideal of A, then all three ideals in* (a) *are equal.*

Proof. (b) is immediate from (a) and Proposition II.1.6.

(a) By item A.2 (iii), only the second inclusion needs proof. Let $a \in \widehat{\overline{I}}$; then $\overline{a} \in \overline{I}$, i.e., there is $b \in I$ so that $\overline{a} = \overline{b}$. By [M], Cor. 5.4.3, p. 94, there are $s, t \in T$ and $k \geq 0$ so that $sab = (a^2 + b^2)^k + t$; then, $(a^2 + b^2)^k + t \in I$. If $k = 0$, then $1 + t \in I$, contradicting T-compatibility. If $k \geq 1$, then $(a^2 + b^2)^k + t = a^{2k} + t + b^2 \cdot r$, where $r = \sum_{j=0}^{k-1} \binom{k}{j}\, a^{2j}\, b^{2(k-j-1)}$. Since $b \in I$ and $(a^2 + b^2)^k + t \in I$, it follows $a^{2k} + t \in I$. Hence, $a \in \sqrt[T]{I}$. $\square$

Lemma II.1.9 *Let I_0, I_1 be ideals of A. Then,*

(*i*) $I_0 \subseteq I_1 \Rightarrow \overline{I_0} \subseteq \overline{I_1}$.

(*ii*) *If I_1 is T-convex, then* $\overline{I_0} \subseteq \overline{I_1} \Rightarrow I_0 \subseteq I_1$.

Proof. (i) Let $\overline{a} \in \overline{I_0}$; then, there is $b \in I_0$ so that $\overline{a} = \overline{b}$; by assumption $b \in I_1$, hence $\overline{a} = \overline{b} \in \overline{I_1}$.

(ii) Let $a \in I_0$; then, $\overline{a} \in \overline{I_0} \subseteq \overline{I_1}$. By definition of the correspondence $\widehat{\ }$ (cf. A.2), $a \in \widehat{\overline{I_1}}$, and by Lemma II.1.8, $a \in I_1$. □

Corollary II.1.10 *The correspondence $I \longmapsto \overline{I}$ is a bijection between T-convex ideals of the ring A and saturated ideals of the real semigroup $G_{A,T}$. It maps prime ideals onto prime ideals.*

Proof. Injectivity. Let I_0, I_1 be T-convex ideals of A such that $\overline{I_0} = \overline{I_1}$. Lemma II.1.9 (ii) applied to the inclusions $\subseteq$ and $\supseteq$ gives $I_0 = I_1$.

Surjectivity. Given a saturated ideal J of $G_{A,T}$, item A.2 (i) shows that $\overline{\widehat{J}} = J$. The fact that $\widehat{J}$ is T-convex follows from Theorem II.1.12 below.

As for the assertion about prime ideals, it suffices to prove:

Fact II.1.11 *Let I be a T-convex ideal of A. Then, I is prime if and only if $\overline{I}$ is prime.*

Proof. ($\Leftarrow$) Assume $ab \in I$; then, $\overline{ab} = \overline{a}\overline{b} \in \overline{I}$. Since $\overline{I}$ is prime, either $\overline{a} \in \overline{I}$ or $\overline{b} \in \overline{I}$, whence either $a \in \widehat{\overline{I}}$ or $b \in \widehat{\overline{I}}$. But, by Lemma II.1.8 (b) we have $\widehat{\overline{I}} = I$.

($\Rightarrow$) Assume $\overline{ab} \in \overline{I}$; then $ab \in \widehat{\overline{I}}$, which equals I by Lemma II.1.8 (b). Since I is assumed prime, $a \in I$ or $b \in I$, whence $\overline{a} \in \overline{I}$ or $\overline{b} \in \overline{I}$. □

Now we prove our main result in this section:

Theorem II.1.12 *Let J be a saturated ideal of $G = G_{A,T}$. Then $\widehat{J}$ is a T-radical ideal of A (hence, by II.1.4, T-convex and radical).*

Proof. Assume $a^2 + t \in \widehat{J}$, where $a \in A$, $t \in T$; we must show that $a \in \widehat{J}$. Write $j = a^2 + t$; then $\overline{j} \in J$ (definition of $\widehat{\ }$), and also $\overline{j} \in D^t_G(\overline{a}^{\,2}, \overline{t})$ (cf. [M], p. 96). Recall that $X_G = \mathrm{Sper}\,(A, T)$.

Let $\alpha \in \mathrm{Sper}\,(A, T)$ be such that $\overline{j}(\alpha) = 0$; then, $\overline{a}^{\,2}(\alpha) = -\overline{t}(\alpha)$. Since $t \in T \subseteq \alpha$, we have $-\overline{t}(\alpha) \in \{0, -1\}$. On the other hand, $\overline{a}^{\,2}(\alpha) \in \{0, 1\}$, since $\overline{a}^{\,2}$ is a square. Thus, the equality above forces $\overline{a}(\alpha) = \overline{t}(\alpha) = 0$. This proves that $Z(\overline{j}) \subseteq Z(\overline{a}) \cap Z(\overline{t}) \subseteq Z(\overline{a})$. This inclusion is equivalent to $\overline{a}^{\,2} = \overline{a}^{\,2} \cdot \overline{j}^{\,2}$ (see I.1.19 (2)). Then, $\overline{a}^{\,2} \in J$, whence $\overline{a} \in J$, which proves $a \in \widehat{J}$. □

A number of corollaries follow from this result.

Corollary II.1.13 *Let I be an ideal of A. Then:*

(*i*) *The ideal $\widehat{\overline{I}}$ is T-convex.*

(*ii*) (Converse to Lemma II.1.8.) $\widehat{\overline{I}} = I \Rightarrow I$ *is T-convex.*

Proof. (ii) is immediate from (i). Since $\overline{I}$ is saturated (A.1 (ii)), II.1.12 proves (i). □

Corollary II.1.14 *Let I be an ideal of A. Then the ideal $\widehat{\overline{I}}$ has the following properties:*

(*i*) *It is the smallest T-convex ideal containing I; hence $\widehat{\overline{I}} = \sqrt[T]{I}$.*

(*ii*) $\overline{\sqrt[T]{I}} = \overline{\widehat{\overline{I}}} = \overline{I}$.

Proof. (i) $\widehat{\overline{I}}$ is T-convex by (i) of the previous Corollary, and $I \subseteq \widehat{\overline{I}}$ is clear. Let I' be a T-convex ideal of A containing I. By Lemma II.1.9 (i), $\overline{I} \subseteq \overline{I'}$, and by Fact II.1.1 (v), $\widehat{\overline{I}} \subseteq \widehat{\overline{I'}}$. Since $\overline{I'}$ is T-convex, Lemma II.1.8 (b) gives $\widehat{\overline{I'}} = I'$; so, $I' \supseteq \widehat{\overline{I}}$.

(ii) By (i) it suffices to prove the second equality. Since $I \subseteq \widehat{\overline{I}}$, from II.1.9 (i) follows $\overline{I} \subseteq \overline{\widehat{\overline{I}}}$. Conversely, let $\bar{a} \in \overline{\widehat{\overline{I}}}$; then, there is $b \in \widehat{\overline{I}}$ such that $\bar{a} = \bar{b}$. By the definition of $\widehat{\ }$ we have $\bar{a} = \bar{b} \in \overline{I}$, showing $\overline{\widehat{\overline{I}}} \subseteq \overline{I}$. $\square$

Remark. Corollary II.1.14 (ii) shows that the map $I \longmapsto \overline{I}$ may not be injective on arbitrary ideals of A (it is on T-convex ideals, Corollary II.1.10). $\square$

Corollary II.1.15 (cf. [BCR], Prop. 4.2.7, p. 87) *Let T be a preorder of a ring A. Then T is proper if and only if there is a proper T-convex ideal in A.*

Proof. The implication ($\Leftarrow$) is Proposition 4.2.7 of [BCR]: if $-1 \in T$, then $1 + (-1) = 0 \in I$, and hence $1 \in I$ by T-convexity.

($\Rightarrow$) The required ideal is $\sqrt[T]{\{0\}}$. If it is not proper, by Proposition II.1.6 $1 + t = 0$ for some $t \in T$, whence $-1 = t \in T$, absurd. $\square$

Remark II.1.16 Note that if I is a T-convex ideal, then $T \cap -T \subseteq I$, since $t \in T \cap -T$ implies $t - t = 0 \in I$, and hence $t \in I$ by Proposition II.1.4 (3). It follows that $T \cap -T \subseteq \sqrt[T]{\{0\}}$, and that $T \cap -T$ is an ideal iff $T \cap -T = \sqrt[T]{\{0\}}$ (if $T \cap -T$ is an ideal, both $T \cap -T$ and $\sqrt[T]{\{0\}}$ are the smallest T-convex ideal of A). $\square$

Concluding this section we register, without proof, the following result which shows that the construction of the real semigroup associated to a ring produces a fair amount of collapsing:

Theorem. ([DM7]) *Given a ring A and a proper preorder T of A, there are a ring B and a proper preorder S of B —canonically and functorially constructed from A and T— such that*

(i) $G_{A,T} \simeq G_{B,S}$.

(ii) B is reduced (no non-zero nilpotent element).

(iii) B has S-bounded inversion: every element of the form $1 + s$ ($s \in S$) is invertible in B. $\square$

II.2 Congruences in real semigroups

Changes to be made in this section: restate II.2.6 – II.2.11 for quotients, as corollaries of results now in section I.3; proofs to be shortened or eliminated.

In this section we introduce the notion of quotient in the theory of real semigroups. Since the language for real semigroups contains a ternary relation, the category **RS** is not algebraic and hence there is not a ready-made notion of quotient to be used. On the other hand, any quotient defined in a real semigroup G must induce a congruence of ternary semigroups in the sense of Definition I.1.25. To mention a simplest example, when G is a reduced special group with a 0 added (as in Example I.2.2 (3)), quotients exist and are induced by the saturated subgroups of G (see [DM1], Prop. 2.28, p. 45), and all quotients are obtained in this way (see Example II.2.3 below).

However, as we shall see, the theory of quotients in real semigroups is far more complicated. We begin by introducing (Definition II.2.1) what seems to us the most natural choice for that notion. The congruences thus defined give raise to proconstructible subsets of the associated ARS (i.e., closed subsets of its constructible topology; Proposition **??** (ii)). The converse, however, is not quite true: in Theorem II.2.16 we prove that if G is a real semigroup, then any subset $\mathcal{H}$ of its character space X_G naturally determines an equivalence relation on G and a representation relation such that the quotient structure $G/\mathcal{H}$ verifies all axioms for real semigroups *with the possible exception of* [RS3] (even only the weak associativity axiom [RS3a], cf. I.2.4). To avoid misunderstandings, note that the quotient $\mathcal{L}_{\mathrm{RS}}$-structure $G/\mathcal{H}$ induced by a set $\mathcal{H} \subseteq X_G$ is the same as that induced by its closure in the constructible topology of X_G (Proposition I.3.7 (1)). A further necessary condition for a set of characters to define a RS-quotient is given in Theorem II.2.9.

For the most natural choices of sets $\mathcal{H} \subseteq X_G$ —notably those occurring in real algebra and geometry— the quotient structure $G/\mathcal{H}$ is, indeed, a real semigroup. These include, among others, the "saturated sets" and the "localizations" first studied by Marshall in [M] (ADD REF.). Investigation of these and other related examples is pursued, beyond Marshall's initial results, in § II.3. Further, in later chapters we will show that when G is a Post algebra, a spectral real semigroup or a RS-fan, then $G/\mathcal{H}$ is a RS for *any* proconstructible subset $\mathcal{H} \subseteq X_G$ (in the case of fans the necessary condition of II.2.9 is also required to hold). However, there are examples where [RS3] fails, as shown in I.3.6 below.

A. Definition of RS-congruences. Examples.

Definition II.2.1 A **(RS-)congruence** of a real semigroup G is an equivalence relation $\equiv$ satisfying the following requirements:

(i) $\equiv$ is a congruence of ternary semigroups (I.1.25).

(ii) There is a ternary relation $D_{G/\equiv}$ in the quotient *ternary semigroup* $(G/\equiv, \cdot, -1, 0, 1)$ so that $(G/\equiv, \cdot, D_{G/\equiv}, -1, 0, 1)$ is a real semigroup, and the canonical projection $\pi : G \longrightarrow G/\equiv$ is a RS-morphism.

(iii) (Factoring through π.) For every RS-morphism $f : G \longrightarrow H$ into a real semigroup H such that $a \equiv b$ implies $f(a) = f(b)$ for all $a, b \in G$, there exists a RS-morphism (necessarily unique), $\widehat{f} : G/\equiv \longrightarrow H$, such that $\widehat{f} \circ \pi = f$, i.e. the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/\equiv \arrow[ur, "\widehat{f}"'] & \end{tikzcd}$$

We shall denote by $Con(G)$ the set of congruences of G. The real semigroup $(G/\equiv, \cdot\, , D_{G/\equiv}, -1, 0, 1)$ will be called the **RS-quotient** of G **modulo** $\equiv$.

Remark. Condition (iii) amounts to saying that the TS-morphism $\widehat{f}$ (well-)defined by the functional equation $\widehat{f} \circ \pi = f$ preserves the representation relation $D_{G/\equiv}$. □

The following examples show that already known constructions conform to the general notion of congruence (or quotient) of real semigroups just defined.

Example II.2.2 Let A be a commutative ring with unit 1, let G_A be the real semigroup associated to A (I.2.2 (2)), and let $T \subseteq A$ be a proper preorder of A (the existence of T ensures A is semi-real and hence that G_A exists). The real semigroup $G_{A,T}$ is a quotient of G_A in the sense of the preceding definition. Recall (I.1.2 (e) and I.2.2 (2)) that the elements of G_A are functions $\overline{a} : \mathrm{Sper}\,(A) \longrightarrow \mathbf{3}$ ($a \in A$), and those of $G_{A,T}$ are their restrictions to Sper (A, T) which, to keep notation straight, in this example we denote by $\overline{a_T} = \overline{a} \lceil \mathrm{Sper}\,(A,T)$. G_A and $G_{A,T}$ carry the representation relations defined in I.2.2 (2) [R].

To present $G_{A,T}$ as a RS-quotient of G_A it suffices to define the following equivalence relation on G_A; for $a, b \in A$,

$$\overline{a} \equiv_T \overline{b} \;\Leftrightarrow\; \overline{a_T} = \overline{b_T} \;\Leftrightarrow\; \overline{a}(\alpha) = \overline{b}(\alpha) \text{ for all } \alpha \in \mathrm{Sper}\,(A,T).$$

It is quite obvious that $\equiv_T$ is a congruence of ternary semigroups, and $G_{A,T}$ is isomorphic to $G_A/\equiv_T$ (as TSs), via the map $\overline{a_T} \mapsto \pi_T(\overline{a}) = \overline{a}/_{\equiv_T}$ ($a \in A$). To turn this map into an isomorphism of RSs it suffices to transport to $G_A/\equiv_T$ the representation relation already existing in $G_{A,T}$; for $a, b, c \in A$ (and with $\pi = \pi_T$),

$$(\dagger)\;\; \pi(\overline{a}) \in D_T(\pi(\overline{b}), \pi(\overline{c})) \;\Leftrightarrow\; \overline{a_T} \in D_{G_{A,T}}(\overline{b_T}, \overline{c_T}) \;\Leftrightarrow\; \forall \alpha \in \mathrm{Sper}\,(A,T)\, (\overline{a}(\alpha) \in D_{\mathbf{3}}(\overline{b}(\alpha), \overline{c}(\alpha)).$$

Obviously, $\overline{a_T} \mapsto \pi_T(\overline{a})$ ($a \in A$) is now an isomorphism of $\mathcal{L}_{\mathrm{RS}}$-structures and, since $G_{A,T}$ is a RS, so is $G_A/\equiv_T$.

It remains to show that $G_A/\equiv_T$ also satisfies the factoring condition II.2.1 (iii). For this we recall that the representation ($\dagger$) is equivalent to

$$(\dagger\dagger)\quad \pi(\overline{a}) \in D_T(\pi(\overline{b}), \pi(\overline{c})) \;\Leftrightarrow\; \exists\, t_0, t_1, t_2 \in T \text{ such that } \pi(\overline{t_0 a}) = \pi(\overline{a}) \text{ and } t_0 a = t_1 b + t_2 c\,.$$

[This equivalence follows from items (2) and (5) of [M], Prop. 5.5.1, p. 95.]

Now, assume that H is a RS and $f : G_A \longrightarrow H$ is a morphism of real semigroups such that $\overline{a} \equiv_T \overline{b}$ implies $f(\overline{a}) = f(\overline{b})$ for all $a, b \in A$. Let $\widehat{f} : G_A/\equiv_T \longrightarrow H$ be defined by $\widehat{f}(\pi(\overline{a})) = f(\overline{a})$, for $a \in A$. We must check that $\widehat{f}$ preserves representation. Let $a, b, c \in A$ be such that $\pi(\overline{a}) \in D_T(\pi(\overline{b}), \pi(\overline{c}))$. By ($\dagger\dagger$) there are $t_0, t_1, t_2 \in T$ such that $\pi(\overline{t_0 a}) = \pi(\overline{a})$ and $t_0 a = t_1 b + t_2 c$. Since $\overline{t_0 a} \in D_{G_A}(\overline{t_1 b}, \overline{t_2 c})$, it follows that $f(\overline{t_0 a}) \in D_H(f(\overline{t_1 b}), f(\overline{t_2 c}))$, and from $\pi(\overline{t_0 a}) = \pi(\overline{a})$, follows $f(\overline{t_0 a}) = f(\overline{a})$. Likewise, since $\overline{t_i} \equiv_T \overline{t_i^2}$ ($i = 1, 2$) we get $f(\overline{t_1 b}) = f(\overline{t_1^2 b})$ and $f(\overline{t_2 c}) = f(\overline{t_2^2 c})$ which implies $f(\overline{a}) \in D_H(f(\overline{b}), f(\overline{c}))$ by use of axiom [RS4] in H. $\square$

Example II.2.3 Let G^* be a reduced special group and let $G = G^* \cup \{0\}$, i.e. G is the real semigroup determined by G^* with a zero added, see I.2.2 (3). Let $\Delta \subseteq G^*$ be a saturated subgroup of G. It is easy to check that the equivalence relation $\equiv_\Delta$ on G given by:

$$a \equiv_\Delta b \;\Leftrightarrow\; a = b = 0 \,\vee\, (a \neq 0 \wedge b \neq 0 \wedge ab \in \Delta)$$

is a congruence of ternary semigroups. Proposition II.2.4 below shows that every congruence of TS defined in G is of this form. As in the case of reduced special groups, we denote by G/Δ the quotient induced by Δ. Moreover, by [DM1] [REF] and the definition of D_{G^*} given in I.2.2 (3), it is immediate that the representation relation

$(*)$ $\pi_\Delta(a) \in D_{G/\Delta}(\pi_\Delta(b), \pi_\Delta(c))$ if and only if there are $t_1, t_2 \in \Delta$ such that $a \in D_G(t_1 b, t_2 c)$,

($\pi_\Delta : G \longrightarrow G/\Delta$ canonical), turns G/Δ into a real semigroup; hence, $\equiv_\Delta$ is a congruence of real semigroups in the sense of II.2.1. In fact, every congruence of G is of this form:

Proposition II.2.4 *Let G^* be a reduced special group and let $G = G^* \cup \{0\}$. Then,*
(i) An equivalence relation $\equiv$ on G is a congruence of ternary semigroups iff there is a subgroup Δ of G^* such that $\equiv\; = \;\equiv_\Delta$.
(ii) $\Delta = \{x \in G^* \,|\, x \equiv 1\}$ is a saturated subgroup of G^*, and any representation relation on $G/\equiv$ verifying II.2.1 (ii) is identical to the relation $D_{G/\Delta}$ defined by $(*)$ above.

Proof. (i) The implication ($\Leftarrow$) has already been noted.
($\Rightarrow$) Given a TS-congruence on $G = G^* \cup \{0\}$, set $\Delta = \{x \in G^* \mid x \equiv 1\}$. Let $a, b \in G$.
(a) $a \equiv b \;\Rightarrow\; a \equiv_\Delta b$.
Note that $0 \equiv b \Rightarrow b = 0$ (as $0 \equiv b \Rightarrow (-1) \cdot 0 \equiv (-1) \cdot b = -b$, whence $-b \equiv b$, and then $b = 0$ by the ternary semigroup axiom [TS5]). Hence, if $a \equiv b$ and one of a, b is 0, both are 0, and $a \equiv_\Delta b$. If $a, b \neq 0$, i.e., $a, b \in G^*$, since G^* is a group of exponent 2 and $\equiv$ is compatible with product, $a \equiv b$ implies $ab \equiv b^2 = 1$, i.e., $ab \in \Delta$, whence $a \equiv_\Delta b$.
(b) $a \equiv_\Delta b \;\Rightarrow\; a \equiv b$.
If $a = b = 0$, then, $a \equiv b$. Let $a, b \neq 0$; then, $a \equiv_\Delta b$ yields $ab \in \Delta$, i.e., $ab \equiv 1$. Since G^* is a group of exponent 2, scaling by a we have $b = a^2 b \equiv 1 \cdot a = a$.

(ii) Item (i) proves that, as ternary semigroups, $G/{\equiv} = G/\Delta$ where $G/\Delta = (G^*/\Delta) \cup \{0\}$. For (ii) we must show that, with $\pi : G \longrightarrow G/{\equiv}$ canonical, and $a, b, c \in G$,

$$\pi(c) \in D_{G/\equiv}(\pi(a), \pi(b)) \Leftrightarrow \text{ There are } t_1, t_2 \in \Delta \text{ such that } c \in D_G(t_1 a, t_2 b).$$

The implication ($\Leftarrow$) is clear, using that π is a RS-morphism (II.2.1 (ii)) and $t \equiv 1$ for $t \in \Delta$.

For the converse, let $\pi_\Delta : G \longrightarrow G/\Delta$ be the canonical projection. From (i), $a \equiv b \Rightarrow \pi_\Delta(a) = \pi_\Delta(b)$. By II.2.1 (iii), there is a RS-morphism $\widehat{\pi_\Delta} : G/{\equiv} \longrightarrow G/\Delta$ such that $\widehat{\pi_\Delta} \circ \pi = \pi_\Delta$. Since $\widehat{\pi_\Delta}$ preserves representation, we have

$$\pi(c) \in D_{G/\equiv}(\pi(a), \pi(b)) \;\Rightarrow\; \pi_\Delta(c) \in D_{G/\Delta}(\pi_\Delta(a), \pi_\Delta(b)),$$

that, by the definition of $D_{G/\Delta}$ (see (*) above), yields the required conclusion. □

Example II.2.5 (Retracts) Let $G \xrightarrow{g} H \xrightarrow{h} G$ be a retract scheme of RSs, i.e., $h \circ g = \mathrm{id}_G$; G is a *retract* of H. We claim that H is a quotient of G in the sense of Definition II.2.1, with

$$\ker(h) := \{\langle a, b \rangle \mid a, b \in H \text{ and } h(a) = h(b)\}$$

as RS-congruence. In fact, it is easily checked that $\ker(h)$ is a congruence of ternary semigroups on H. The map $\overline{h} : H/\ker(h) \longrightarrow G$ given by $\overline{h}(a/\ker(h)) = h(a)$ is a TS-isomorphism: $\overline{h}$ is well defined and injective because $\langle a, b \rangle \in \ker(h) \Leftrightarrow h(a) = h(b)$, and it is surjective because h is. To ease notation we set $\overline{a} := a/\ker(h)$ for $a \in H$.

Representation in $H/\ker(h)$ is defined as follows: for $a, b, c \in H$,
$\overline{a} \in D_{H/\ker(h)}(\overline{b}, \overline{c}) \Leftrightarrow$ There are $a', b', c' \in H$ such that $\overline{a} = \overline{a'}, \overline{b} = \overline{b'}, \overline{c} = \overline{c'}$ and $a' \in D_H(b', c')$.
We have
Fact. *For $a, b, c \in H$, $\overline{a} \in D_{H/\ker(h)}(\overline{b}, \overline{c}) \Leftrightarrow h(a) \in D_G(h(b), h(c))$; i.e., representation in $H/\ker(h)$ is obtained by "transporting" representation in G by the map $\overline{h}^{-1}$.*
Proof of Fact. ($\Rightarrow$) Assuming $\overline{a} \in D_{H/\ker(h)}(\overline{b}, \overline{c})$, the preceding definition yields $a' \in D_H(b', c')$ for some $a', b', c' \in H$ such that $\overline{a} = \overline{a'}, \overline{b} = \overline{b'}, \overline{c} = \overline{c'}$. Taking images by h (a RS-morphism) in this representation yields $h(a') \in D_G(h(b'), h(c'))$, whence $h(a) \in D_G(h(b), h(c))$, since $h(a') = h(a)$, etc.

($\Leftarrow$) Taking images under g in the representation $h(a) \in D_G(h(b), h(c))$ and taking into account that, for $x \in H$ we have $h(g(h(x))) = h(x)$, i.e., $\langle g(h(x)), x \rangle \in \ker(h)$, i.e., $\overline{g(h(x))} = \overline{x}$, setting $a' = g(h(a))$, etc., we get $a' \in D_H(b', c')$ and $\overline{a'} = \overline{a}$, etc., as required. □

According to Definition II.2.1, it only remains to check the factoring condition II.2.1 (iii). So, given a RS-morphism $k : H \longrightarrow K$, with $K \models$ RS, such that $\ker(h) \subseteq \ker(k)$, there is a RS-morphism $\widehat{k} : H/\ker(h) \longrightarrow K$ such that $\widehat{k} \circ \pi = k$, with $\pi : H \longrightarrow H/\ker(h)$ $(\pi(x) = \overline{x})$ canonical.

Set $\widehat{k} := k \circ g \circ \overline{h}$, with $\overline{h}$ the (RS-)isomorphism defined above. Since $\overline{h} \circ \pi = h$, for $a \in H$

we have $\widehat{k}(\pi(a)) = k(g(h(a)))$. From $\langle g(h(a)), a \rangle \in \ker(h)$ (see above) and $\ker(h) \subseteq \ker(k)$ we get $k(g(h(a))) = k(a)$, whence $(\widehat{k} \circ \pi)(a) = k(a)$, as required. $\square$

As concrete instances of retracts in the case of the RSs associated to rings, we consider the retract schemes of rings

$$A \longrightarrow A[\![X]\!] \longrightarrow A \quad \text{and} \quad A \longrightarrow A[X_1, \ldots, X_n] \longrightarrow A,$$

where $A[\![X]\!]$ denotes the ring of formal power series with coefficients in A, and $A[X_1, \ldots, X_n]$ the ring of polynomials in n indeterminates over A, and where the inclusions on the left are canonical, and the surjections on the right are obtained by evaluating at 0. Applying the functor $A \longrightarrow G_A$ gives retract schemes

$$G_A \longrightarrow G_{A[\![X]\!]} \longrightarrow G_A \quad \text{and} \quad G_A \longrightarrow G_{A[X_1,\ldots,X_n]} \longrightarrow G_A,$$

showing, by II.2.5, that G_A is a RS-quotient of both $G_{A[\![X]\!]}$ and $G_{A[X_1,\ldots,X_n]}$.

The preceding example admits considerable generalization, namely:

Proposition II.2.6 *Let $H \xrightarrow{h} G$ be a surjective morphism of RSs with the property that, for every RS, K, and every RS morphism $\alpha : H \longrightarrow K$ such that $\ker(h) \subseteq \ker(\alpha)$, there is a* unique *RS-morphism $\widehat{\alpha} : G \longrightarrow K$ such that $\widehat{\alpha} \circ h = \alpha$. Then, G is a quotient of H given by the RS-congruence $\ker(h)$.*

Proof. We begin by forming the quotient ternary semigroup $H/\ker(h)$; let $\pi : H \longrightarrow H/\ker(h)$ be the canonical quotient TS-morphism. With the same proof as in II.2.5 one shows that the map $\overline{h} : H/\ker(h) \longrightarrow G$ given by $\overline{h}(a/\ker(h)) = h(a)$ is an isomorphism of TSs. Now, using the map $\overline{h}^{-1}$ copy the representation relation of G onto $H/\ker(h)$, making it RS-isomorphic to G.

To show that $H/\ker(h)$ is a RS-quotient of H, as in II.2.1 (iii) let $\alpha : H \longrightarrow K$ be a RS-morphism so that $\ker(h) \subseteq \ker(\alpha)$. By assumption, there is a unique RS-morphism $\widehat{\alpha} : G \longrightarrow K$ such that $\widehat{\alpha} \circ h = \alpha$. The map $\widehat{\alpha} \circ \overline{h} : \ker(\alpha) \longrightarrow K$ verifies $(\widehat{\alpha} \circ \overline{h})(\pi(a)) = \alpha(a)$. $\square$

B. The space of characters of a RS-quotient.

New subsection started in January 2016. Intended to gather all properties of the set of characters of a quotient by incorporating results previously spread in section II.2.

Fix a quotient $G/\equiv$ of a RS, G. Recall that the canonical quotient RS-morphism $\pi_{\equiv} := \pi : G \longrightarrow G/\equiv$ induces a dual map $\pi^* : X_{G/\equiv}$ defined by $\pi^*(\sigma) := \sigma \circ \pi$, for $\sigma \in X_{G/\equiv}$. The aim of this paragraph is to study the space $X_{G/\equiv}$ and the map π^*.

Notation II.2.7 (i) We shall indistinctly write $\mathcal{H}_{\equiv}$ for $\mathrm{Im}[\pi^*]$.
(ii) In the sequel the default topology on the sets $X_{G/\equiv}$ and X_G will be the spectral topology. The modifier $(\cdot)_{\mathrm{con}}$ will be employed to denote the corresponding constructible topology; cf. I.1.16 and I.1.17. $\square$

The next Proposition summarizes the main basic properties of the map π^* and the set of characters $\mathrm{Im}[\pi^*]$.

Proposition II.2.8 *Let G be a RSand let $\equiv$ be a RS-congruence of G. With notation as in*

II.2.7, we have:
(i) The map π^* *is a spectral embedding of* $X_{G/\equiv}$ *into* X_G.
In particular,
(ii) $\mathcal{H}_\equiv$ *is a proconstructible subset of* X_G.
(iii) π^* *is a homeomorphism between the spectral spaces* $X_{G/\equiv}$ *and* $\mathrm{Im}[\pi^*]$. *In particular, these spaces are homeomorphic when endowed with the corresponding constructible topologies.*
(iv) $(\mathcal{H}_\equiv, \rightsquigarrow\lceil\mathcal{H}_\equiv)$ *is a sub-root system of* $(X_G, \rightsquigarrow)$.
(v) For $a, b, c \in G$,

$$\pi(a) \in D_{G/\equiv}(\pi(b), \pi(c)) \Leftrightarrow \text{ For all } p \in \mathcal{H}_\equiv, p(a) \in D_{\mathbf{3}}(p(b), p(c)).$$

In particular, we have

$$a \equiv b \Leftrightarrow \text{ For all } p \in \mathcal{H}_\equiv, p(a) = p(b).$$

Remark. A proconstructible subset of a spectral space with the topology induced from X is a spectral subspace. In fact, all subspaces of a spectral space are obtained in this manner; cf.[DST], Thm. 2.1.3. check ref.

Proof. (i) a) π^* is injective.
This follows at once from the fact that π is surjective: every $x \in G/\equiv$ is of the form $x = \pi(g)$ for some $g \in G$; then, if $\sigma_1, \sigma_2 \in X_{G/\equiv}$ are such that $\pi^*(\sigma_1) = \pi^*(\sigma_2)$, i.e., $\sigma_1 \circ \pi = \sigma_2 \circ \pi$, we have $\sigma_1(x) = \sigma_1(\pi(g)) = \sigma_2(\pi(g)) = \sigma_1(x)$, whence $\sigma_1 = \sigma_2$, since x is arbitrary.

b) π^* is spectral.
Recall (I.1.17) that the basic quasi-compact opens for the spectral topology on X_G are the sets

$$H(g_1, \ldots, g_n) = \bigcap_{i=1}^{n} [\![g_i = 1]\!] \quad (g_1, \ldots, g_n \in G),$$

and similarly for $X_{G/\equiv}$, replacing the g_i's by $\pi(g_i)$. Then, for $\sigma \in X_{G/\equiv}$ we have,

$$\sigma \in \pi^{*-1}[H(g_1, \ldots, g_n)] \Leftrightarrow \pi^*(\sigma) = \sigma \circ \pi \in H(g_1, \ldots, g_n) \Leftrightarrow$$
$$\sigma \circ \pi(g_i) = 1 \text{ for } i = 1, \ldots, n \Leftrightarrow \sigma \in H(\pi(g_1), \ldots, \pi(g_n)),$$

showing that $\pi^{*-1}[H(g_1, \ldots, g_n)] = H(\pi(g_1), \ldots, \pi(g_n))$, and hence that π^* is spectral.

(ii) follows from (i) by recalling that the image of a proconstructible set by any spectral map between spectral spaces is a proconstructible subset of the counterdomain ([DST], Cor. 2.1.4) Check ref.. However, to dispell any doubt we give a direct proof in the present case.

To ease notation we write $\mathcal{H} = \mathcal{H}_\equiv$ for the rest of this proof.

Let $q \in \overline{\mathcal{H}}^{\mathrm{con}}$ ($=$ closure of $\mathcal{H}$ in the constructible topology of X_G). Let $a, b \in G$ be such that $a \equiv b$. We claim that $q(a) = q(b)$. Indeed, $q(a) \neq q(b)$ implies that $U = \{\sigma \in X_G \,|\, \sigma(a) \neq \sigma(b)\}$ is a neighborhood of q (constructible topology). Then, $U \cap \mathcal{H} \neq \emptyset$; let $p \in U \cap \mathcal{H}$, and let $\sigma \in X_G$ be such that $p = \sigma \circ \pi$. Since $a \equiv b$, then $\pi(a) = \pi(b)$, and $p(a) = p(b)$ contradicting that $p \in U$. So $q(a) = q(b)$. Therefore, the map $\sigma : G/\equiv \longrightarrow \mathbf{3}$ defined by $\sigma(\pi(a)) = q(a)$ is a well defined character of ternary semigroups. To see that $\sigma \in X_{G/\equiv}$, we must show that σ preserves the representation relation. Let $a, b, c \in G$ be such that $\pi(a) \in D_{G/\equiv}(\pi(b), \pi(c))$, and assume that $q(a) \notin D_{\mathbf{3}}(q(b), q(c))$. It is easy to check that $U = \{\sigma \in X_G \,|\, \sigma(a) \notin D_{\mathbf{3}}(\sigma(b), \sigma(c))\}$ is an open set in the constructible topology, and hence a neighborhood of q. Letting $p \in U \cap \mathcal{H}$ and arguing as above we arrive in a similar way to a contradiction. Therefore, $\sigma \in X_{G/\equiv}$, and then $q \in \mathcal{H}$.

(iii) In order to prove that $\mathcal{H} = \mathrm{Im}(\pi^*)$ and $X_{G/\equiv}$ are spectrally homeomorphic it suffices to show that the map $\pi^{*-1}\lceil\mathcal{H} : \mathcal{H} \longrightarrow X_{G/\equiv}$ is spectral. To ease notation, set $f := \pi^{*-1}\lceil\mathcal{H}$; then, for $p \in \mathcal{H}$, $f(p)$ is the unique character $\sigma \in X_{G/\equiv}$ such that $p = \sigma \circ \pi$; thus, $p = f(p) \circ \pi$. Since $U(g) = [\![\, g = 1 \,]\!]$, we have:

$$p \in f^{-1}[U(\pi(g))] \Leftrightarrow f(p) \in U(\pi(g)) \Leftrightarrow (f(p) \circ \pi)(g) = 1 \Leftrightarrow p(g) = 1 \Leftrightarrow p \in U(g),$$

which shows $\mathcal{H} \cap f^{-1}[U(\pi(g))] = \mathcal{H} \cap U(g)$. Since the sets $\mathcal{H} \cap U(g)$ $(g \in G)$ form a subbasis for the spectral topology of $\mathcal{H}$ (and similarly for $X_{G/\equiv}$), it follows that the inverse map $\pi^{*-1}\lceil\mathcal{H}$ is spectral, as asserted.

Since in any ARS, (X, G), the sets of the form $U(a_1) \cap \ldots \cap U(a_n) \cap Z(b)$ are a basis for the constructible topology of X, and inverse images commute with set-theoretic operations, f is also a homeomorphism between $\mathcal{H}_{\mathrm{con}}$ and $(X_{G/\equiv})_{\mathrm{con}}$.

(iv) follows from (i), (iii) and the well known fact that any continuous map between spectral spaces preserves the specialization relation; if, in addition, the map is open and injective, it is an embedding of ordered sets, when the domain and counterdomain are endowed with the respective specialization relations.

(v) Suppose first that $\pi(a) \in D_{G/\equiv}(\pi(b), \pi(c))$ and let $p \in \mathcal{H}$. Then, $p = \sigma \circ \pi$ for some $\sigma \in X_{G/\equiv}$. Since σ is a RS-morphism, we have $p(a) = \sigma(\pi(a)) \in D_{\mathbf{3}}(\sigma(\pi(b)), \sigma(\pi(c))) = D_{\mathbf{3}}(p(b), p(c))$. Conversely, assume $\pi(a) \notin D_{G/\equiv}(\pi(b), \pi(c))$. By the separation theorem for RSs (I.5.4), there is a character $\sigma \in X_{G/\equiv}$ such that $\sigma(\pi(a)) \notin D_{\mathbf{3}}(\sigma(\pi(b)), \sigma(\pi(c)))$. Setting $p = \sigma \circ \pi \in \mathcal{H}_{\equiv}$, we have $p \in \mathcal{H}$ with $p(a)) \notin D_{\mathbf{3}}(p(b), p(c))$ as required to prove (*).

The equivalence (*) also implies $a \equiv b \Leftrightarrow p(a) = p(b)$ for all $p \in \mathcal{H}_{\equiv}$ $(a, b \in G)$. To see this it suffices to note that $a \equiv b$ if and only if $\pi(a) = \pi(b)$ which, in terms of the representation partial order, is equivalent to the conjunction of $\pi(a) \in D_{G/\equiv}(\pi(1), \pi(b)), \pi(b) \in D_{G/\equiv}(\pi(1), \pi(a)), -\pi(b) \in D_{G/\equiv}(\pi(1), \pi(-a))$ and $\pi(-a) \in D_{G/\equiv}(\pi(1), \pi(-b))$ (see Definition I.6.2). Applying (*) to these four representations yields corresponding statements in $\mathbf{3}$, $p(a) \in D_{\mathbf{3}}(p(b), p(c)), \ldots$ $(p \in \mathcal{H})$, which, altogether, are equivalent to $p(a) = p(b)$. □

In the remainder of this section we shall uncover *an additional constraint* that the set $\mathcal{H}_{\equiv}$ must satisfy for the quotient structure to be a real semigroup. Our result is:

Theorem II.2.9 *Let G be a RS and let $\equiv$ be a congruence of G. The set $\mathcal{H}_{\equiv} \subseteq X_G$ associated to $\equiv$ (see II.2.7(i)) is* **finitely closed***: for any any finite set of characters $h_1, \ldots, h_n \in X_G$ $(n \geq 1)$,*

$$h_1, \ldots, h_n \in \mathcal{H}_{\equiv} \text{ and } \prod_{i=1}^{n} h_i \in X_G \text{ imply } \prod_{i=1}^{n} h_i \in \mathcal{H}_{\equiv}\,.$$

Remarks. (i) Note that the requirement $\prod_{i=1}^{n} h_i \in X_G$ in the conclusion of Theorem II.2.9 entails that the number n of characters *is odd*, as $\prod_{i=1}^{n} h_i$ ought to map -1 to -1.

(ii) The proof of Theorem II.2.9 requires a result proved in the next section. However, since II.2.9 bears on the the structure of the set of characters determining a general congruence in arbitrary RSs, we include it in this section. Proposition II.2.10 and Lemma II.2.11 are also need in the proof of II.2.9. □

Short proof of Theorem II.2.9. Dec. 2011. Since $\mathcal{H}_{\equiv}$ is a proconstructible subset of X_G (**??** (ii)) and the sets

(*) $\quad \bigcap_{i=1}^{n} [\![a_i = 1]\!] \cap [\![b = 0]\!], \quad$ with $a_1, \ldots, a_n, b \in G$,

are a basis of clopens for the constructible topology of X_G (cf. [M], Note (1), p. 111), it follows that $\mathcal{H}_\equiv$ is an intersection of sets of the form (*). Since the condition to be proved is preserved under arbitrary intersections, it suffices to prove II.2.9 for sets of the form (*), which is immediate by direct inspection. □

Proposition II.2.10 *Let $\equiv$ be a congruence of a RS, G. With notation as in II.2.7, let $\Delta = \{x \in G \,|\, x \equiv x^2\} = \bigcap \{P(h) \,|\, h \in \mathcal{H}_\equiv\}$ and $\mathcal{I} = \{Z(h) \,|\, h \in \mathcal{H}_\equiv\}$. Then, for $h \in X_G$,*

$$h \in \mathcal{H}_\equiv \;\Leftrightarrow\; \Delta \subseteq P(h) \text{ and } Z(h) \in \mathcal{I}.$$

That is, every RS-congruence is determined by a set of characters of the form $\mathcal{H}_{\Delta,\mathcal{I}} = \{h \in X_G \,|\, \Delta \subseteq P(h)$ and $Z(h) \in \mathcal{I}\}$, for a suitable saturated subsemigroup Δ of G and a family $\mathcal{I}$ of saturated prime ideals.

Proof. The implication ($\Rightarrow$) is obvious.
($\Leftarrow$) Let $f \in X_G$ be as in the right-hand side of the statement; then, $Z(f) = Z(g)$ for some $g \in \mathcal{H}_\equiv$; call this ideal J. Let $\mathcal{H}_{\Delta,J} = \{h \in X_G \,|\, \Delta \subseteq P(h)$ and $Z(h) = J\}$, and let $\equiv_{\Delta,J}$ denote the congruence of G determined by $\mathcal{H}_{\Delta,J}$ as in $(\dagger)_\mathcal{H}$ of **??**. For $a, b \in G$ we have:
(*) $\quad a \equiv b \;\Rightarrow\; a \equiv_{\Delta,J} b.$
In fact, $a \equiv b$ implies $ab \equiv a^2b^2$, i.e., $ab \in \Delta$; then, $h(ab) \geq 0$ for $h \in \mathcal{H}_{\Delta,J}$. If $h(ab) = 1$, then $h(a) = h(b)$. If $h(a) = 0$, then $a \in J = Z(h) = Z(g)$; since $g \in \mathcal{H}_\equiv$ and $a \equiv b$, it follows $g(b) = 0$, i.e., $b \in J = Z(h)$. Interchanging a and b we get $h(a) = 0 \Leftrightarrow h(b) = 0$. Hence $h(a) = h(b)$ for all $h \in \mathcal{H}_{\Delta,J}$, i.e., $a \equiv_{\Delta,J} b$.

Theorem II.3.5 (d) proves that $G/{\equiv_{\Delta,J}}$ is a RS. Indeed, with notation therein, let Γ be the subsemigroup of G generated by $\Delta \cup J$, and let $T = G \setminus J$. First, we observe that $\Gamma \cap -\Gamma \subseteq J$: an element of $\Gamma \cap -\Gamma$ belongs either to $(\Delta \cap -\Delta) \cup J$ —hence to J—, or is a product jd with $j \in J, d \in \Delta$ and hence belongs to J. Since J is saturated, the saturated subsemigroup Γ' generated by Γ is also contained in J. By Fact II.3.4 (ii) there is a saturated subsemigroup $\widehat{\Gamma} \supseteq \Gamma'$ verifying the assumptions of II.3.5. Now, routine checking shows that $\mathcal{H}_{\Delta,J} = \mathcal{H}^T_{\widehat{\Gamma}}$, and hence that $G/{\equiv_{\Delta,J}}$ is a RS.

Let $\pi_{\Delta,J} : G \longrightarrow G/{\equiv_{\Delta,J}}$ and $\pi : G \longrightarrow G/{\equiv}$ be the quotient maps. The universal property II.2.1 (iii) and (*) imply that there exists a RS-morphism $\widehat{\pi} : G/{\equiv} \longrightarrow G/{\equiv_{\Delta,J}}$ so that $\widehat{\pi} \circ \pi = \pi_{\Delta,J}$. Since $f \in \mathcal{H}_{\Delta,J}$, (*) shows that the identity $\widehat{f} \circ \pi = f$ (well-) defines a map $\widehat{f}$ from $G/{\equiv}$ to **3**. Obviously $\widehat{f}$ is a semigroup homomorphism sending $0, 1, -1$ onto the corresponding elements in **3**.

We claim that $\widehat{f}$ preserves representation. Assume $\pi(a) \in D_{G/\equiv}(\pi(b), \pi(c))$. Since $\widehat{\pi}$ is a RS-morphism, we get $\pi_{\Delta,J}(a) \in D_{G/\equiv_{\Delta,J}}(\pi_{\Delta,J}(b), \pi_{\Delta,J}(c))$. Now we invoke item (b.iii) in Theorem II.3.5 (with $\widehat{\Gamma}, T$ as above) to get $a' \in G$ and $d_1, d_2 \in \widehat{\Gamma}$ so that $a \equiv_{\Delta,J} a'$ and $a' \in D_G(d_1 b, d_2 c)$. Since $f \in \mathcal{H}_{\Delta,J} = \mathcal{H}^T_{\widehat{\Gamma}}$, we have $f(a) = f(a')$ and $f(d_i) \in \{0, 1\}$, whence $f(d_i) = f(d_i)^2$, $i = 1, 2$. From $f \in X_G$ we get $f(a) = f(a') \in D_{\mathbf{3}}(f(d_1)^2 f(b), f(d_2)^2 f(c))$, and (by axiom [RS4]), $f(a) \in D_{\mathbf{3}}(f(b), f(c))$. This proves our claim, showing $\widehat{f} \in X_{G/\equiv}$, and hence $f \in \mathcal{H}_\equiv$, as required. □

Lemma II.2.11 *Let G be a RS. Let $h_1, \dots h_n \in X_G$ $(n \geq 2)$ be characters such that the product $h := \prod_{i=1}^n h_i$ also lies in X_G. Then,*
(1) *One of the zero-sets $\{Z(h_i) \,|\, i = 1, \dots, n\}$ includes all the others.*
In particular,
(2) *There is $j \in \{1, \dots, n\}$ so that $Z(h) = Z(h_j)$.*

As a preliminary step we note:

Fact. *Let $I, J_1, \dots, J_n$ be saturated ideals of a RS. Then,*
(*i*) $I \subseteq \bigcup_{i=1}^n J_i \;\Rightarrow\; I \subseteq J_k$ *for some $k \in \{1, \dots, n\}$.*
In particular,
(*ii*) *If $\bigcup_{i=1}^n J_i$ is a saturated ideal, then some J_k includes all the J_i's $(i \in \{1, \dots, n\})$.*

Proof of Fact. Item (ii) follows at once from (i) upon taking $I = \bigcup_{i=1}^n J_i$.

(i) Assume the conclusion false, i.e., for each $i = 1, \dots, n$ there is an element $a_i \in I \setminus J_i$. Let $c\,(= c^2)$ be the unique element of $D^t_G(a_1^2, \dots, a_n^2)$ (cf. Corollary IV.5.3 (i)); by saturatedness, $c \in I$. Recall that $Z(c) = \bigcap_{i=1}^n Z(a_i)$, which (by (*) in the proof of VI.1.2) implies $a_i^2 c^2 = a_i^2$ $(i = 1, \dots, n)$. Since $c \in I$, there is i such that $c \in J_i$; then $a_i^2 \in J_i$, whence $a_i \in J_i$, a contradiction. □

Proof of Lemma II.2.11. It suffices to prove (1). We set $h := \prod_{i=1}^n h_i$ (in X_G by assumption). First we show:
(a) There is $i \in \{1, \dots, n\}$ such that $Z(h_i) \subseteq \bigcup_{j \neq i} Z(h_j)$.
Otherwise, for each $i \in \{1, \dots, n\}$ there is an element $a_i \in G$ such that
(*) $\quad h_i(a_i) = 0$ and $h_j(a_i) \neq 0$ for all $j \neq i$.
Let c be the unique element of $D^t_G(a_1^2, \dots, a_n^2)$ (IV.5.3 (i)). We have
(**) $\quad Z(c) = \bigcap_{i=1}^n Z(a_i)$.
Since $h \in X_G$, $h(c) \in D^t_{\mathbf{3}}(h(a_1^2), \dots, h(a_n^2)) = D^t_{\mathbf{3}}(0, \dots, 0)$, and hence $h(c) = 0$. Then, $h_k(c) = 0$ for some $k \in \{1, \dots, n\}$. From (**) it follows that $h_k(a_j) = 0$ for all $j \in \{1, \dots, n\}$. Since $n \geq 2$, picking $j \neq k$ contradicts (*).

Up to a permutation of indices we can assume $Z(h_1) \subseteq \bigcup_{j=2}^n Z(h_j)$. By item (i) of the Fact there is $k \in \{2, \dots, n\}$ such that $Z(h_1) \subseteq Z(h_k)$, and hence $Z(h) = \bigcup_{j=2}^n Z(h_j)$. By item (ii) of the Fact there is $m \in \{2, \dots, n\}$ such that $Z(h_j) \subseteq Z(h_m)$ for all $j \in \{2, \dots, n\}$. Hence, $Z(h_1) \subseteq Z(h_k) \subseteq Z(h_m)$, i.e., $Z(h_i) \subseteq Z(h_m)$ for all $i \in \{1, \dots, n\}$, proving the Lemma. □

Proof of Theorem II.2.9. Let $h_1, \dots, h_n$ be as in the assumption, and let Δ and $\mathcal{I}$ be as in Proposition II.2.10, so that $\mathcal{H}_\equiv = \mathcal{H}_{\Delta,\mathcal{I}} = \{h \in X_G \,|\, \Delta \subseteq P(h) \text{ and } Z(h) \in \mathcal{I}\}$. Hence, $\Delta \subseteq P(h_i)$ and $Z(h_i) \in \mathcal{I}$ for $i = 1, \dots, n$. By Lemma II.2.11 (2), $Z(\prod_{i=1}^n h_i) = \bigcup_{i=1}^n Z(h_i) = Z(h_j) \in \mathcal{I}$, for some $j \in \{1, \dots, n\}$. Also, $\Delta \subseteq \bigcap_{i=1}^n P(h_i) \subseteq P(\prod_{i=1}^n h_i)$. By II.2.10 again, we have $\prod_{i=1}^n h_i \in \mathcal{H}_{\Delta,\mathcal{I}} = \mathcal{H}_\equiv$. □

C. Congruences and proconstructible character sets.

Proposition II.2.8 and Theorem II.2.9 give necessary conditions for a set of characters $\mathcal{H} \subseteq X_G$ to determine a RS-congruence of G. Proceeding in the reverse direction we investigate in this paragraph the extent to which a set of characters of a real semigroup determines a congruence. Most of the results that follow are direct application of those in section I.3.

II.2.12 Reminder. Let G be a ternary semigroup and let $\mathcal{H}$ be a non-empty subset of $X_G = \mathrm{Hom}_{\mathrm{TS}}(G, \mathbf{3})$. In I.1.26 (c) we defined a congruence of TSs by: for $a, b \in G$,

$(\dagger)_{\mathcal{H}}$ $\qquad\qquad a \equiv_{\mathcal{H}} b \quad\Leftrightarrow\quad \text{For all } h \in \mathcal{H},\ h(a) = h(b),$

and proved in I.1.27 that every congruence of TSs is of this form for a suitable (proconstructible) set $\mathcal{H}$. If $G/\mathcal{H}$ denotes the quotient set under $\equiv_{\mathcal{H}}$, and $\pi : G \longrightarrow G/\mathcal{H}$ is the canonical projection, in I.3.1 we defined a ternary relation, $D_{G/\mathcal{H}}$, on $G/\mathcal{H}$ as follows: for $a, b, c \in G$,

$(\dagger\dagger)_{\mathcal{H}}$ $\qquad \pi(a) \in D_{G/\mathcal{H}}(\pi(b), \pi(c)) \quad\Leftrightarrow\quad \text{For all } h \in \mathcal{H},\ h(a) \in D_{\mathbf{3}}(h(b), h(c)).$

Clearly, $D_{G/\mathcal{H}}$ is well-defined and, according to Proposition **??**, every RS-congruence is obtained in this way. To be precise:

Fact II.2.13 *Given a real semigroup G and a RS-congruence $\equiv$ of G, with $\mathcal{H}_{\equiv}$ denoting the set of characters defined in II.2.7(i), we have $(G/{\equiv}, D_{G/\equiv}) \cong (G/\mathcal{H}_{\equiv}, D_{G/\mathcal{H}_{\equiv}})$ as $\mathcal{L}_{\mathrm{RS}}$-structures; hence, $(G/\mathcal{H}_{\equiv}, D_{G/\mathcal{H}_{\equiv}})$ is a RS.*

Proof. To ease notation, set $\mathcal{H} := \mathcal{H}_{\equiv}$. Let $\pi : G \longrightarrow G/{\equiv}$ and $\pi' : G \longrightarrow G/\mathcal{H}$ be the quotient maps. The second equivalence in II.2.8 (v), together with II.2.12 $(\dagger)_{\mathcal{H}}$ give, for $a, b \in G$,

$$a \equiv b \Leftrightarrow \forall p \in \mathcal{H}\ (p(a) = p(b)) \Leftrightarrow a \equiv_{\mathcal{H}} b,$$

showing that the map $\pi(x) \overset{\varphi}{\longmapsto} \pi'(x)$ is bijective. Since π and π' are TS-homomorphisms, φ is an isomorphism of TSs. The first equivalence in II.2.8 (v) and II.2.12 $(\dagger\dagger)_{\mathcal{H}}$ show

$$\pi(a) \in D_{G/\equiv}(\pi(b), \pi(c)) \Leftrightarrow \forall p \in \mathcal{H}\ (p(a) \in D_{\mathbf{3}}(p(b), p(c)) \Leftrightarrow \pi'(a) \in D_{G/\mathcal{H}}(\pi'(b), \pi'(c)),$$

proving that φ is an isomorphism of $\mathcal{L}_{\mathrm{RS}}$-structures. $\square$

The set $\mathcal{H}$ can be identified with a subset $\widehat{\mathcal{H}} = \{\widehat{h} \mid h \in \mathcal{H}\}$ of $\mathbf{3}^{G/\mathcal{H}}$ by the map $h \mapsto \widehat{h}$, where $\widehat{h} : G/\mathcal{H} \longrightarrow \mathbf{3}$ is defined by the functional equation $\widehat{h} \circ \pi = h$. By clause $(\dagger)_{\mathcal{H}}$ above, $\widehat{h}$ is well-defined and the map $h \mapsto \widehat{h}$ is obviously injective.

Proposition I.3.7 yields:

Proposition II.2.14 *Let G be a RS and let $\mathcal{H}$ be a subset of X_G. Then,*

(1) The closure $\overline{\mathcal{H}}$ of $\mathcal{H}$ in $(X_G)_{\mathrm{con}}$ defines the same equivalence relation as $\mathcal{H}$ and the same representation relation on the quotient set $G/\mathcal{H} = G/\overline{\mathcal{H}}$.

(2) Let $\widetilde{\mathcal{H}}$ be the set of all $p \in X_G$ such that for all $a, b, c \in G$,

(a) $a \equiv_{\mathcal{H}} b \Rightarrow p(a) = p(b)$.

(b) $\pi(a) \in D_{G/\mathcal{H}}(\pi(b), \pi(c)) \Rightarrow p(a) \in D_{\mathbf{3}}(p(b), p(c))$.

Then,

(i) $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$.

(ii) $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ define the same equivalence relation on G, and the same representation relation on $G/\mathcal{H} = G/\widetilde{\mathcal{H}}$.

(iii) $\widetilde{\mathcal{H}}$ is maximal satisfying conditions (i) and (ii): if $\mathcal{H} \subseteq \mathcal{G} \subseteq X_G$, the equivalence relation $\equiv_{\mathcal{G}}$ is identical to $\equiv_{\mathcal{H}}$, and $D_{G/\mathcal{G}}$ is identical to $D_{G/\mathcal{H}}$, then $\mathcal{G} \subseteq \widetilde{\mathcal{H}}$.

In particular,

(iv) $\widetilde{\mathcal{H}}$ is proconstructible.

Proof. We only need check the assertions about the equivalence relations in (1), (2.ii) and (2.iii). Once this is done, the rest follows from Proposition I.3.7 applied with the TS $G/\mathcal{H}$ and the set $\widehat{\widetilde{\mathcal{H}}}$ of TS-characters. Note first that, for fixed $a, b \in G$, the set $\{g \in X_G \mid g(a) = g(b)\}$, is clopen in $(X_G)_{\mathrm{con}}$.

(1) We must show that $\equiv_{\overline{\mathcal{H}}} = \equiv_{\mathcal{H}}$. The inclusion $\equiv_{\overline{\mathcal{H}}} \subseteq \equiv_{\mathcal{H}}$ follows from $\mathcal{H} \subseteq \overline{\mathcal{H}}$. Conversely, let $a \equiv_{\mathcal{H}} b$ $(a, b \in G)$ and assume $h'(a) \neq h'(b)$ for some $h' \in \overline{\mathcal{H}}$. Then, $\{g \in X_G \mid g(a) \neq g(b)\}$ is a

neighborhood of h' in the constructible topology, which implies $\mathcal{H} \cap \{g \in X_G \mid g(a) \neq g(b)\} \neq \emptyset$, i.e., $h(a) \neq h(b)$ for some $h \in \mathcal{H}$, contrary to $a \equiv_{\mathcal{H}} b$.

(2.ii) The inclusion $\equiv_{\widetilde{\mathcal{H}}} \subseteq \equiv_{\mathcal{H}}$ follows from $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$. The reverse inclusion follows from clause (a) in the definition of $\widetilde{\mathcal{H}}$ by use of item $(\dagger)_{\widetilde{\mathcal{H}}}$ in II.2.12.

(2.iii) Straightforward checking, using the assumptions in (iii) and items $(\dagger)_{\mathcal{G}}$, $(\dagger\dagger)_{\mathcal{G}}$ in II.2.12. shows that any $q \in \mathcal{G}$ verifies clauses (a) and (b) in the definition of $\widetilde{\mathcal{H}}$, and hence, $q \in \widetilde{\mathcal{H}}$. □

Corollary II.2.15 *With notation as in II.2.12 and II.2.14 and assumptions therein, $\widehat{\widehat{\mathcal{H}}}$ is the set of all $\mathcal{L}_{\mathrm{RS}}$-homomorphisms from the $\mathcal{L}_{\mathrm{RS}}$-structure $(G/\mathcal{H}, D_{G/\mathcal{H}})$ onto* **3**.[1] *Further, $\widehat{\mathcal{H}}$ —and,* a fortiori *$\widehat{\widehat{\mathcal{H}}}$— separates points in $G/\mathcal{H}$.*

Proof. The elements of $\widehat{\widehat{\mathcal{H}}}$ are maps of the form $\widehat{p}$ with $p \in \widetilde{\mathcal{H}}$; clauses (2.a) and (2.b) in II.2.14 state precisely that $\widehat{p}$ is a $\mathcal{L}_{\mathrm{RS}}$-homomorphism from $(G/\mathcal{H}, D_{G/\mathcal{H}})$ into **3**.

Conversely, if $f : G/\mathcal{H} \longrightarrow \mathbf{3}$ is a TS-homomorphism preserving the relation $D_{G/\mathcal{H}}$, then $f \circ \pi \in \widetilde{\mathcal{H}}$ (II.2.14 (2.a.b)), whence $\widehat{f \circ \pi} \in \widehat{\widehat{\mathcal{H}}}$. Since $\widehat{f \circ \pi} = f$, we conclude that $f \in \widehat{\widehat{\mathcal{H}}}$, as required.

Separation of points is clear from $(\dagger)_{\mathcal{H}}$ in II.2.12 and the definition of $\widehat{\mathcal{H}}$: for $g, g' \in G$,

$$\begin{aligned} \pi(g) \neq \pi(g') &\Leftrightarrow \text{ There is } h \in \mathcal{H} \text{ such that } h(g) \neq h(g') \Leftrightarrow \\ &\Leftrightarrow \text{ There is } \widehat{h} \in \mathcal{H} \text{ such that } \widehat{h}(g) \neq \widehat{h}(g'). \end{aligned}$$

□

In a partial converse to Proposition II.2.14, our next result —a corollary to Theorem I.3.3—, shows that if G is a RS, *any* non-empty subset of X_G determines, via the notions defined in II.2.12, a quotient structure verifying all axioms for real semigroups *with the possible exception of the weak associativity axiom* [RS3a] (see **??**.

Theorem II.2.16 *Let G be a real semigroup and let $\mathcal{H}$ be a non-empty subset of X_G. Endowed with the ternary relation $D_{G/\mathcal{H}}$ defined in II.2.12 $(\dagger\dagger)_{\mathcal{H}}$, the $\mathcal{L}_{\mathrm{RS}}$-structure $(G/\mathcal{H}, D_{G/\mathcal{H}})$ satisfies all axioms for real semigroups except, possibly, the weak associativity axiom* [RS3a].

Proof. First note that the set $\widehat{\mathcal{H}}$ —and *a fortiori* $\widehat{\widehat{\mathcal{H}}}$— separates points in ternary semigroup $G/\mathcal{H}$: if $a, b \in G$ are such that $\pi(a) \neq \pi(b)$, i.e., $a \not\equiv_{\mathcal{H}} b$, by II.2.12 $(\dagger)_{\mathcal{H}}$ there is $h \in \mathcal{H}$ such that $h(a) \neq h(b)$, which implies $\widehat{h}(\pi(a)) \neq \widehat{h}(\pi(b))$.

Then, Theorem I.3.3 can be applied to this situation, implying that the $\mathcal{L}_{\mathrm{RS}}$-structure $(G/\mathcal{H}, D_{G/\mathcal{H}})$ satisfies all axioms for real semigroups except, possibly, the strong associativity axiom [RS3].

On the other hand, the validity of axiom [RS3b] (cf. I.2.4) is inherited from G: according to our definition of $D_{G/\mathcal{H}}$, the canonical projection $\pi : \geq \longrightarrow G/\mathcal{H}$ is a morphism of ternary semigroups and preserves transversal representation. The validity of $D^t_G(a, b) \neq \emptyset$ implies, then $D^t_{G/\mathcal{H}}(a, b) \neq \emptyset$, and hence of [RS3b], as π is surjective. □

Remark. The validity of the weak associativity axiom [RS3a] in G *does not* entail, in general, its validity in $G/\mathcal{H}$; a counterexample is obtained by easily adapting Example I.3.6, a task that we leave as an exercise to the reader. □

~~A straightforward adaptation of P~~ropositions I.3.4 and I.3.5, yields:

[1] I.e., the "$\mathcal{L}_{\mathrm{RS}}$-characters" of $(G/\mathcal{H}, D_{G/\mathcal{H}})$.

Proposition II.2.17 *Let G be a RS and let $h_1, h_2 \in X_G$. With $\mathcal{H} = \{h_1, h_2\}$, $G/\mathcal{H}$ verifies* [RS3]*, and then is a real semigroup.* □

Proposition II.2.18 *Let G be a RS and let h_1, h_2, h_3 be three distinct elements of X_G such that $h_i \rightsquigarrow h_j$ for some $i \neq j \in \{1, 2, 3\}$. With $\mathcal{H} = \{h_1, h_2, h_3\}$, $(G, D_{\mathcal{H}}, \ldots)$ verifies axiom* [RS3]*, and then is a real semigroup.* □

Proofs are omitted.

The next example shows that an equivalence relation $\equiv$ on a real semigroup G may verify conditions (i) and (ii) of Definition II.2.1 but not necessarily the universal property (iii).

Example II.2.19 Let G be the ternary semigroup: $G = \{-1, 0, 1, x, y, -x, -y\}$, subject to the conditions $x^2 = x$, $y^2 = 1$ and $xy = x$. It is immediate to see that either $a^2b^2 = a^2$ or $a^2b^2 = b^2$, for all $a, b \in G$. Hence, G admits a unique structure of RS-fan (cf. Chapter VI, §§ 1, 3). Moreover, straightforward verification shows that X_G has three characters h_1, h_2, h_3 defined by:

$$h_1(x) = 0,\ h_1(y) = 1;\ h_2(x) = 0,\ h_2(y) = -1;\ h_3(x) = h_3(y) = 1.$$

Therefore the specialization partial order is given by $h_3 \rightsquigarrow h_1$ (h_2 is not $\rightsquigarrow$-related to the other two characters), and we also have $h_1 = h_2^2 h_3$. Let $\mathcal{H} = \{h_2, h_3\}$, and let $\equiv_{\mathcal{H}}$ be the congruence of ternary semigroups induced by $\mathcal{H}$. The preceding Proposition shows that $\equiv_{\mathcal{H}}$ satisfies conditions II.2.1 (i),(ii). Moreover, $h_1 = h_2^2 h_3$ implies that the equivalence relation $\equiv_{\mathcal{H}}$ is just equality. Hence, $G = G/\mathcal{H}$, and the quotient map π is the identity on G. Assuming that $\equiv_{\mathcal{H}}$ verifies condition II.2.1 (iii), the identity map $id : G \longrightarrow G$ factors through π via a morphism $\widehat{id} : G/{\equiv_{\mathcal{H}}} \longrightarrow G$ which manifestly is also the identity (since $\widehat{id} \circ \pi = id$). On the other hand, checking values at h_2, h_3 it immediately comes that $-1 = \pi(-1) \in D_{G/\mathcal{H}}(\pi(-x), \pi(y))$, and hence $-1 \in D_G(-x, y)$. However, $h_1(-1) = -1 \notin D_{\mathbf{3}}(h_1(-x), h_1(y)) = D_{\mathbf{3}}(0, 1)$, contradiction. □

II.3 Congruences of real semigroups defined by saturated sets

This section is devoted to study the quotients of RSs determined by certain outstanding sets of characters: *localizations, saturated sets, saturated subsemigroups* (also called *subspaces*), *transversally saturated subsemigroups, residue spaces.* Quotients of these types occur under various guises in real algebraic geometry and real algebra; in the dual language of abstract real spectra some of these quotients have been considered by Marshall in [M], Ch. 6, whose work we considerably extend here.

Our main aim is to give explicit characterizations of the congruences generated by sets of these types and of both representation relations in the respective quotients, in terms of the semigroup operation, the constants and the corresponding relations in the initial RSs. Several useful remarks appear on the way, such as the convexity of saturated sets in the specialization order of the character space X_G (II.3.7).

We begin with the simplest case, that of localizations.

A. Localizations. Let G be a RS; let $T \subseteq G$ be a multiplicative subset of G containing 1 but not 0. Sets of this form give raise in an obvious way to RS-congruences of G: namely,

Definition II.3.1 With G and T as above we define, for $a, b, c \in G$:

(1) $a \sim_T b :\Leftrightarrow$ There is $t \in T$ such that $at = bt$.

(Equivalently, there is $t \in T$ such that $at^2 = bt^2$.) Clearly, this is a congruence of ternary semigroups, and $t^2 \sim_T 1$ for $t \in T$ (since $t^3 = t$).

(2) With $\pi = \pi_T : G \longrightarrow G/\sim_T$ canonical,

$$\pi(a) \in D_{G/\sim_T}(\pi(b), \pi(c)) \ :\Leftrightarrow \ \text{There is } t \in T \text{ such that } at \in D_G(bt, ct).$$

(Equivalently, there is $t \in T$ such that $at^2 \in D_G(bt^2, ct^2)$.) □

Proposition II.3.2 *Let G be a RS; let $T \subseteq G$ be a multiplicative subset of G containing 1 but not 0. With $\sim_T$ and $D_{G/\sim_T}$ as in the preceding definition:*

(1) *The canonical map π preserves representation.*

(2) *For $h \in X_G$,*

$$h \lceil T^2 = 1 \ \Leftrightarrow \ \textit{There is a } \mathcal{L}_{\mathrm{RS}}\textit{-character } \sigma : G/\sim_T \longrightarrow \mathbf{3} \textit{ such that } h = \sigma \circ \pi.$$

(3) *The $\mathcal{L}_{\mathrm{RS}}$-structure $(G/\sim_T, \cdot, D_{G/\sim_T}, -1, 0, 1)$ is a real semigroup.*

(4) *$\sim_T$ is a RS-congruence, i.e., verifies the factorization condition II.2.1 (iii).*

(5) *For $a, b, c \in G$,*

$$\pi(a) \in D^t_{G/\sim_T}(\pi(b), \pi(c)) \Leftrightarrow \ \textit{There is } t \in T \textit{ such that } at^2 \in D^t_G(bt^2, ct^2).$$

Proof. For (1) use "$\Leftarrow$" in item (2) of II.3.1 with $t = 1 \in T$.

(2) ($\Leftarrow$) Given a $\mathcal{L}_{\mathrm{RS}}$-character $\sigma : G/\sim_T \longrightarrow \mathbf{3}$ such that $h = \sigma \circ \pi$ and $t \in T$, $\pi(t^2) = 1$ implies $h(t^2) = \sigma(\pi(t^2)) = \sigma(1) = 1$.

($\Rightarrow$) Given $h \in X_G$ such that $h \lceil T^2 = 1$, define $\sigma : G/\sim_T \longrightarrow \mathbf{3}$ by $\sigma(\pi(a)) := h(a)$ ($a \in G$).

— σ is well defined: for $a, b \in G$,

$$\pi(a) = \pi(b) \ \Leftrightarrow \ a \sim_T b \ \Leftrightarrow \ \exists t \in T\, (at^2 = bt^2),$$

whence $h(a) = h(at^2) = h(bt^2) = h(b)$.

We leave as an exercise for the reader the routine checking that σ is a TS-character preserving $D_{G/\sim_T}$.

(3) With notation as in I.1.17, $U(T^2) = \{h \in X_G \mid h \lceil T^2 = 1\} = \bigcap_{t \in T} [\![\, t^2 = 1 \,]\!]$. Since $[\![\, t^2 = 1 \,]\!]$ is, by definition of the spectral topology (cf. I.1.17), quasi-compact open in X_G, $U(T^2)$ is proconstructible. By Theorem II.2.16 the $\mathcal{L}_{\mathrm{RS}}$-structure $(G/\sim_T, \cdot, D_{G/\sim_T}, -1, 0, 1)$ verifies all axioms for real semigroups except, possibly, weak associativity ([RS3a]), which we check now.

Let $a, b, c, d, e \in G$ be such that $\pi(a) \in D_{G/\sim_T}(\pi(b), \pi(c))$ and $\pi(c) \in D_{G/\sim_T}(\pi(d), \pi(e))$. By the definition of $D_{G/\sim_T}$ there are $t_1, t_2 \in T$ such that $at_1^2 \in D_G(bt_1^2, ct_1^2)$ and $ct_2^2 \in D_G(dt_2^2, et_2^2)$. Scaling these representations by $t_1^2 t_2^2$ and setting $t := t_1 t_2$, we have $at^2 \in D_G(bt^2, ct^2)$ and $ct^2 \in D_G(dt^2, et^2)$. Since D_G is weakly associative, there is $x \in G$ so that $x \in D_G(bt^2, dt^2)$ and $at^2 \in D_G(x, et^2)$. Scaling these representations by t^2 and using II.3.1 (2) again, shows that $\pi(x) \in D_{G/\sim_T}(\pi(b), \pi(d))$ and $\pi(a) \in D_{G/\sim_T}(\pi(x), \pi(e))$, as required.

(4) This is straightforward checking: let $f : G \longrightarrow H$ ($H \models \mathrm{RS}$) be a RS-morphism such that $a \sim_T b \Rightarrow f(a) = f(b)$ ($a, b \in G$). We must check that the map $\widehat{f} : G/\sim_T \longrightarrow H$ given

by $\widehat{f} \circ \pi = f$ preserves representation: if $\pi(a) \in D_{G/\sim_T}(\pi(b), \pi(c))$, there is $t \in T$ so that $at^2 \in D_G(bt^2, ct^2)$; taking images under f in this representation and using that $f(t^2) = 1$ for $t \in T$ (recall that $t^2 \sim_T 1$), yields $f(a) \in D_H(f(b), f(c))$, as required.

(5) The implication ($\Leftarrow$) is clear, since π is a RS-morphism and $\pi(t^2) = 1$. Conversely, assuming $\pi(a) \in D^t_{G/\sim_T}(\pi(b), \pi(c))$, we have $\pi(a) \in D_{G/\sim_T}(\pi(b), \pi(c))\,, -\pi(b) \in D_{G/\sim_T}(-\pi(a), \pi(c))$ and $-\pi(c) \in D_{G/\sim_T}(\pi(b), -\pi(a))$ (cf. [t-rep] in I.2). By II.3.1 (2) there are $t_1, t_2, t_3 \in T$ such that $at_1^2 \in D_G(bt_1^2, ct_1^2)\,, -bt_2^2 \in D_G(-at_2^2, ct_2^2)$ and $-ct_3^2 \in D_G(bt_3^2, -at_3^2)$. Setting $t := t_1t_2t_3$ and scaling these representations by t^2 we get $at^2 \in D_G(bt^2, ct^2)\,, -bt^2 \in D_G(-at^2, ct^2)$ and $-ct^2 \in D_G(bt^2, -at^2)$. By [t-rep] again, we get $at^2 \in D^t_G(bt^2, ct^2)$, as claimed. $\Box$

Remarks and Notation II.3.3 (i) The RS-congruences $\sim_T$ defined above, given by a subsemigroup T of G not containing 0, are called **localizations** (at T). Item (2) of II.3.2 shows that $U(T^2) = \{h \in X_G \,|\, h \lceil T^2 = 1\}$ coincides with the set of characters $\mathcal{H}_{\sim_T}$ associated to the localization at T, defined in Proposition **??**.

(ii) Note that the set $U(T^2)$ of characters associated to the localization at T is *downwards closed* under specialization: $h \in U(T^2)$, $g \in X_G$ and $g \rightsquigarrow h$ imply $g \in U(T^2)$. This follows from Lemma I.1.18: $g \rightsquigarrow h$ is equivalent to $h^{-1}[1] \subseteq g^{-1}[1]$ and $h \in U(T^2)$ means $T^2 \subseteq h^{-1}[1]$. Subsets of a spectral space downwards closed under specialization are called *generically closed*; cf. [DST], Def. 6.1.4. $\Box$

The examples of congruences of a real semigroup G considered in the sequel will be constructed from a family of characters of G; the corresponding equivalence relation, as well as both representation relations in the quotient set will then be characterized in terms of the operation and the representation relations of G (note that localizations were defined by giving directly the equivalence relation and representation).

B. Saturated sets. Let G be a RS; given a saturated subsemigroup Δ of G and a multiplicative subset $T \subseteq G$ containing 1, we define:

$$\mathcal{H}^T_\Delta = \{h \in X_G \,|\, \Delta \subseteq P(h) \text{ and } Z(h) \cap T = \emptyset\}.$$

Sets of this form are called **saturated sets** ([M], p. 126). To avoid trivialities we assume $\mathcal{H}^T_\Delta \neq \emptyset$. Under this assumption we have:

Fact II.3.4 *With notation as above, if $\mathcal{H}^T_\Delta \neq \emptyset$, then:*

(i) $\Delta \cap -\Delta \cap T = \emptyset$.

(ii) *There is a saturated subsemigroup of G, $\widehat{\Delta} \supseteq \Delta$, so that $\mathcal{H}^T_\Delta = \mathcal{H}^T_{\widehat{\Delta}}$, verifying in addition:*

(C) *If $a \in G$, $x \in T$ and $ax^2 \in \widehat{\Delta}$, then $a \in \widehat{\Delta}$.*

Proof. (i) Suppose $x \in \Delta \cap -\Delta \cap T$, and let $h \in \mathcal{H}^T_\Delta$. Since $x \in \Delta \cap -\Delta$, then $-x^2 \in \Delta$, which implies $h(-x^2) \geq 0$, and hence $h(x) = 0$. So, $x \in Z(h) \cap T$, contrary to our definition of $\mathcal{H}^T_\Delta$.

(ii) Let $\widehat{\Delta} = \bigcup_{x \in T} \{a \in G \,|\, ax^2 \in \Delta\}$. Clearly $\Delta \subseteq \widehat{\Delta}$. Let $a, b \in \widehat{\Delta}$. Then $ax^2 \in \Delta$ and $by^2 \in \Delta$ for some $x, y \in T$; hence, $abx^2y^2 \in \Delta$, and since $xy \in T$ we get $ab \in \widehat{\Delta}$. Therefore $\widehat{\Delta}$ is a subsemigroup of G.

To prove that $\widehat{\Delta}$ is saturated, take $p,q \in \widehat{\Delta}$, and let $z \in D(p,q)$. Then, there are $x,y \in T$ such that $px^2 \in \Delta$ and $qy^2 \in \Delta$. Hence $px^2y^2 \in \Delta$ and $qx^2y^2 \in \Delta$. On the other hand, $z \in D(p,q)$ implies $zx^2y^2 \in D(px^2y^2, qx^2y^2)$ and hence $zx^2y^2 \in \Delta$ because Δ is saturated and the entries of this form are in Δ. Since $xy \in T$ we conclude that $z \in \widehat{\Delta}$.

Next, let us see that $\mathcal{H}^T_\Delta = \mathcal{H}^T_{\widehat{\Delta}}$. Clearly, $\mathcal{H}^T_{\widehat{\Delta}} \subseteq \mathcal{H}^T_\Delta$. Conversely, let $h \in \mathcal{H}^T_\Delta$. Since $Z(h) \cap T = \emptyset$ it only remains to show that $\widehat{\Delta} \subseteq P(h)$. Let $a \in \widehat{\Delta}$, i.e., $ax^2 \in \Delta$ for some $x \in T$. Then $h(ax^2) \geq 0$; since $h(x) \neq 0$, we get $h(x^2) = 1$ and $h(a) = h(ax^2) \geq 0$. Hence $h \in \mathcal{H}^T_{\widehat{\Delta}}$.

Further, $\widehat{\Delta}$ and T satisfy condition II.3.4 (C). Indeed, let $a \in G$, $x \in T$ be such that $ax^2 \in \widehat{\Delta}$. Then there exists $y \in T$ such that $ax^2y^2 \in T$. Since $xy \in T$ and $a(xy)^2 = ax^2y^2$ we conclude that $a \in \widehat{\Delta}$. □

To ease notation, the equivalence $\equiv_{\mathcal{H}^T_\Delta}$ induced by $\mathcal{H}^T_\Delta$ will be denoted by $\sim_{\Delta,T}$.

Theorem II.3.5 *Let G be a RS. Suppose that Δ is a saturated subsemigroup of G and T is a multiplicative subset of G satisfying $\Delta \cap -\Delta \cap T = \emptyset$ and condition II.3.4 (C). Then, for $a,b,c \in G$:*

(a) *$a \sim_{\Delta,T} b$ if and only if $ab \in \Delta$ and there are elements $t \in T$ and $d_1, d_2 \in \Delta$ such that $a^2t^2 \in D^t_G(-d_1, a^2b^2t^2)$ and $b^2t^2 \in D^t_G(-d_2, a^2b^2t^2)$.*

(b) *The following are equivalent:*

- (i) $\pi(a) \in D_{G/\mathcal{H}^T_\Delta}(\pi(b), \pi(c))$.
- (ii) *There are $p,q,r \in G$, $t \in T$ such that $p \sim_{\Delta,T} q \sim_{\Delta,T} r \sim_{\Delta,T} a^2t^2$ and $ap \in D_G(bq, cr)$.*
- (iii) *There are $d_1, d_2 \in \Delta$ and $a' \in G$ such that $a \sim_{\Delta,T} a'$ and $a' \in D_G(d_1b, d_2c)$.*

(c) *The following are equivalent:*

- (i) $\pi(a) \in D^t_{G/\mathcal{H}^T_\Delta}(\pi(b), \pi(c))$.
- (ii) *There are $x,y,z \in G$ such that $x \sim_{\Delta,T} a^2$, $y \sim_{\Delta,T} b^2$, $z \sim_{\Delta,T} c^2$ and $ax \in D^t_G(by, cz)$.*

(d) *$D_{G/\mathcal{H}^T_\Delta}$ verifies axiom* [RS3] *and $G/\mathcal{H}^T_\Delta$ satisfies the universal property of Definition II.2.1 (iii); hence $\sim_{\Delta,T}$ is a RS-congruence of G.*

Proof. By Fact II.3.4 (ii) we may assume that Δ verifies condition II.3.4 (C).

Proof of (a). ($\Leftarrow$) Suppose that $a,b \in G$ verify $ab \in \Delta$, $a^2t^2 \in D^t(-d_1, a^2b^2t^2)$ and $b^2t^2 \in D^t(-d_2, a^2b^2t^2)$ for some elements $t \in T$, $d_1, d_2 \in \Delta$. Let $h \in \mathcal{H}^T_\Delta$. Since $\Delta \subseteq P(h)$ we have $h(ab) \geq 0$. If $h(a) = 0$, from $b^2t^2 \in D^t(-d_2, a^2b^2t^2)$ comes $h(b^2t^2) = h(b^2) \in D^t_{\mathbf{3}}(-h(d_2), 0) = \{-h(d_2)\}$. Note that $h(b^2t^2) = h(b^2)$ because $Z(h) \cap T = \emptyset$. Hence $h(b^2) = -h(d_2)$. Since $d_2 \in \Delta$, we get $-h(d_2) \leq 0$, and then $h(b^2) \leq 0$, whence $h(b) = 0$. A similar argument using $a^2t^2 \in D^t(-d_1, a^2b^2t^2)$, shows that $h(b) = 0$ implies $h(a) = 0$. Thus, $h(a) = 0 \Leftrightarrow h(b) = 0$. Since $h(ab) \geq 0$ we conclude $h(a) = h(b)$, and since $h \in \mathcal{H}^T_\Delta$ is arbitrary, $a \sim_{\Delta,T} b$.

($\Rightarrow$) Conversely, assume $a \sim_{\Delta,T} b$. Let us prove first that $ab \in \Delta$. Otherwise, condition II.3.4 (C) yields $abT^2 \cap \Delta = \emptyset$. Since T^2 is multiplicative, by Corollary I.4.11 there exists a character $h \in X_G$ such that $\Delta \subseteq P(h)$ and $h(abt^2) = -1$ for all $t \in T$. In particular, $Z(h) \cap T = \emptyset$, which means that $h \in \mathcal{H}^T_\Delta$. Hence $h(a) = h(b)$, contradicting $h(ab) = -1$. So $ab \in \Delta$.

In order to prove the remaining conditions in (a), let $I = \Delta \cap -\Delta$ and let I_{ab} be the saturated ideal generated by $I \cup \{ab\}$. Let $T_a = T \cup a^2T$; clearly T_a is a multiplicative subset of G. Suppose that $I_{ab}[\Delta] \cap T_a = \emptyset$. Lemma I.4.10 gives a character $h \in X_G$ such that $I_{ab} \subseteq Z(h)$, $\Delta \subseteq P(h)$ and $Z(h) \cap T_a = \emptyset$. Since $T \subseteq T_a$, we have $h \in \mathcal{H}^T_\Delta$. Hence $h(a) = h(b)$, and then $h(ab) = 0$ yields $h(a) = h(b) = 0$. On the other hand, if $t \in T$, we have $a^2t \in T_a$, and then $h(a^2t) \neq 0$, contradiction. Therefore, $I_{ab}[\Delta] \cap T_a \neq \emptyset$. Using the expression for the saturated ideal generated by $I \cup \{ab\}$ given by Proposition I.4.6 (1′), we get elements $x \in T_a$, $y \in G$, $i \in I$, $d \in \Delta$ such that $-x^2 \in D(i, aby, d)$. Then (by [RS6]), $-x^2 \in D^t(ix^2, abyx^2, dx^2) = D^t(j, abyx^2, e)$ with $j = ix^2 \in I$ and $e = dx^2 \in \Delta$. Therefore $-abyx^2 \in D^t(j, e, x^2)$ (I.2.3 (0)), which implies

$$a^2b^2y^2x^2 = (-abyx^2)^2 \in D^t(\langle j, e, x^2 \rangle \otimes \langle j, e, x^2 \rangle).$$

Since I and Δ are saturated sets, I is an ideal and Δ is multiplicative, this formula yields $a^2b^2y^2x^2 \in D^t(k, f, x^2)$ for some $k \in I$ and $f \in \Delta$. Hence $-x^2 \in D^t(k, f, -a^2b^2x^2y^2) \subseteq D(k, f, -a^2b^2x^2y^2)$, which, using [RS4], yields $-x^2 \in D(k, f, -a^2b^2x^2)$. Scaling by $-a^2$ we obtain $a^2x^2 \in D(-a^2k, -a^2f, a^2b^2x^2)$. By the definition of I, $a^2k \in \Delta$ and since a^2f is also in Δ, saturation of Δ gives $a^2x^2 \in D(-\widehat{d}, a^2b^2x^2)$ for some $\widehat{d} \in D(a^2k, a^2f) \subseteq \Delta$. On the other hand, $x \in T_a$ implies $a^2x^2 = a^2t^2$ for some $t \in T$. Therefore $a^2t^2 \in D(-\widehat{d}, a^2b^2t^2)$ and then $a^2t^2 \in D^t(-\widehat{d}a^2t^2, a^2b^2t^2) = D^t(-d', a^2b^2t^2)$ with $d' = \widehat{d}a^2t^2 \in \Delta$. By a similar argument (scaling by $-b^2$) we find elements $s \in T$, $e' \in \Delta$ such that $b^2s^2 \in D^t(-e', a^2b^2s^2)$. Since $st \in T$ we obtain

$$a^2z^2 \in D^t(-d's^2, a^2b^2z^2), \qquad b^2z^2 \in D^t(-e't^2, a^2b^2z^2),$$

with $z = st \in T$. Since $d's^2, e't^2 \in \Delta$, the proof of item (a) is complete.

Proof of (b). Note first that, $x \in G, t \in T$ imply $xt^2 \sim_{\Delta,T} x$ (since $h(t^2) = 1$ for all $h \in \mathcal{H}^T_\Delta$).

(ii) $\Rightarrow$ (iii). With p, q, r as in (ii) we have $ap^2 \in D_G(bpq, cpr)$. But $p \sim_{\Delta,T} q \sim_{\Delta,T} a^2$ implies $p^2 \sim_{\Delta,T} a^2$ and (by (a)) $pq \in \Delta$; likewise, $pr \in \Delta$. Setting $d_1 := pq$, $d_2 := pr$ and $a' := ap^2$, assertion (iii) follows.

(iii) $\Rightarrow$ (i). Let $a' \in G$, $d_1, d_2 \in \Delta$ be such that $a \sim_{\Delta,T} a'$ and $a' \in D(d_1b, d_2c)$, and let $h \in \mathcal{H}^T_\Delta$. Then $h(a) = h(a') \in D_{\mathbf{3}}(h(d_1)h(b), h(d_2)h(c))$. Since $\Delta \subseteq P(h)$ we have $h(d_i) = (h(d_i))^2$ $(i = 1, 2)$, which, by [RS4], implies $h(a) \in D(h(b), h(c))$. Since h is arbitrary, we conclude that $\pi(a) \in D_{G/\mathcal{H}^T_\Delta}(\pi(b), \pi(c))$.

(i) $\Rightarrow$ (ii). This is the delicate part of the proof. The argument is similar to (and generalizes) the proof of Lemma 6.6.6 in [M], p. 125. We just sketch it. Assume $\pi(a) \in D_{G/\mathcal{H}^T_\Delta}(\pi(b), \pi(c))$.

Claim 1. There are finitely many $d_1, \ldots, d_n \in \Delta$ and $t \in T$ such that, with $Y = \{h \in X_G \mid \{d_1, \ldots, d_n\} \subseteq P(h) \text{ and } h(t^2) = 1\}$ and $\pi_Y : G \longrightarrow G/Y$ canonical, we have $\pi_Y(a) \in D_{G/Y}(\pi_Y(b), \pi_Y(c))$.

Proof of Claim 1. This is a simple argument using compactness of the constructible topology of X_G. For $Z \subseteq X_G$ we set $W(Z) = \{h \in X_G \mid Z \subseteq P(h)\}$. To abridge, call *finitely generated in* $\{\Delta, T\}$ any saturated set of the form $W(\Delta') \cap U(T')$, where T' is a finite subsemigroup of T, and Δ' is a finitely generated subsemigroup of Δ.

Assume we had $\pi_{Y'}(a) \notin D_{G/Y'}(\pi_{Y'}(b), \pi_{Y'}(c))$ for each saturated set Y' finitely generated in $\{\Delta, T\}$. Then, for each such set Y' there is $h \in Y'$ so that $h(b), h(c) \geq 0$ and $h(a) = -1$; that is, $Y' \cap W(b,c) \cap U(-a) \neq \emptyset$. Thus, $\mathcal{F} = \{Y' \cap W(b,c) \cap U(-a) \,|\, Y' \text{ finitely generated in } \{\Delta, T\}\}$ is a family of proconstructible subsets of X_G with the finite intersection property (the intersection of two saturated sets of this form is again saturated and finitely generated in $\{\Delta, T\}$). By compactness, $\bigcap \mathcal{F} \neq \emptyset$. Since $\bigcap\{Y' \,|\, Y' \text{ finitely generated in } \{\Delta, T\}\} = \mathcal{H}^T_\Delta$, we get $\mathcal{H}^T_\Delta \cap W(b,c) \cap U(-a) \neq \emptyset$, contrary to assumption (i).

Hence, there are $d_1, \ldots, d_n \in \Delta$ and $t_1, \ldots, t_k \in T$ such that, with $Y = W(d_1, \ldots, d_n) \cap U(t_1^2, \ldots, t_k^2)$ we have $\pi_Y(a) \in D_{G/Y}(\pi_Y(b), \pi_Y(c))$. Setting $t := t_1 \cdot \ldots \cdot t_k$ proves Claim 1.

With Y as in Claim 1, since $\mathcal{H}^T_\Delta \subseteq Y$, we have $x \sim_Y y \Rightarrow x \sim_{\Delta,T} y$ for $x, y \in G$. Hence, it suffices to prove item (ii) for Y.

Claim 2. We may assume $a^2t^2 = 1$ (and hence $a^2 = t^2 = 1$).

Proof of Claim 2. It suffices to replace X_G by its localization at the multiplicative set $\{1, a^2t^2\}$, which is a RS by Proposition II.3.2 (3) (see also [M], Prop. 6.5.7, pp. 119-120). Note that $Y' := Y \cap U(a^2t^2) = W(d_1, \ldots, d_n) \cap U(a^2t^2)$.

Assuming the result valid for $\{1, a^2t^2\}$ and Y', there are $p', q', r' \in G$ such that $p' \sim_{Y'} q' \sim_{Y'} r' \sim_{Y'} a^2t^2$ and $ap' \in D_G(bq', cr')$. Setting $p := p'a^2t^2, q := q'a^2t^2, r := r'a^2t^2$ we get $ap \in D_G(bq, cr)$ and $p \sim_Y q \sim_Y r \sim_Y a^2t^2$. To show, e.g., that $p \sim_Y a^2t^2$, let $h \in Y$; if $h \in U(a^2t^2)$, then $h \in Y'$, and $h(p') = h(a^2t^2)$, which gives $h(p) = h(a^2t^2)$; if $h \notin U(a^2t^2)$, then $h(a^2t^2) = 0$, and hence also $h(p) = 0$, proving Claim 2.

Now, assuming $a^2t^2 = 1$ we have $a^2 = t^2 = 1$, $Y = W(d_1, \ldots, d_n)$, and the result follows exactly as in the proof of [M], Lemma 6.6.6.

Proof of (c). The implication (ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (ii). From $\pi(a) \in D^t_{G/\mathcal{H}^T_\Delta}(\pi(b), \pi(c))$ we get $\pi(ab) \in D^t_{G/\mathcal{H}^T_\Delta}(\pi(b^2), \pi(bc))$, whence $\pi(b^2) \in D^t_{G/\mathcal{H}^T_\Delta}(\pi(ab), -\pi(bc))$ and, in particular, $\pi(b^2) \in D_{G/\mathcal{H}^T_\Delta}(\pi(ab), -\pi(bc))$. By item (b) there are $p, q, r \in G$, $t \in T$ such that $p \sim_{\Delta,T} q \sim_{\Delta,T} r \sim_{\Delta,T} b^2t^2$ $(\sim_{\Delta,T} b^2)$ and $b^2p \in D_G(abq, -bcr)$. Therefore, $b^2p \in D^t_G(abp^2q, -bcp^2r)$, and $abp^2q \in D^t_G(b^2p, bcp^2r)$. Set $v := bp^2q$ and $a' := av$. Clearly, $v \sim_{\Delta,T} b$, whence

(*) $a' \sim_{\Delta,T} ab$, and $a' = av \in D^t_G(b^2p, bcp^2r)$.

A similar argument, using $\pi(ac) \in D^t_{G/\mathcal{H}^T_\Delta}(\pi(c^2), \pi(bc))$, shows that there are elements $p', q', r', w \in G$ such that $p' \sim_{\Delta,T} q' \sim_{\Delta,T} r' \sim_{\Delta,T} c^2$, $w \sim_{\Delta,T} c$ and $aw \in D^t_G(c^2p', bcp'^2r')$. Setting $a'' := aw$, we have:

(**) $a'' \sim_{\Delta,T} ac$, and $a'' = aw \in D^t_G(c^2p', bcp'^2r')$.

Now, let $x = x^2$ be the unique element of $D^t_G(a'^{\,2}, a''^{\,2})$ (cf. IV.5.3 (1)).

Claim 3. $x \sim_{\Delta,T} a^2$.

Proof of Claim 3. Let $h \in \mathcal{H}^T_\Delta$. If $h(x) = 0$ we have $h(a') = h(a'') = 0$, which, by the congruences in (*) and (**), entail $h(ab) = h(ac) = 0$. If $h(a) \neq 0$, this gives $h(b) = h(c) = 0$ which, by assumption (i), yields $h(a) = 0$, contradiction. So, $h(x) = 0 \Rightarrow h(a) = 0$. Conversely, by (*) and (**), $h(a) = 0$ implies $h(a') = h(a'') = 0$, and hence $h(x) = 0$.

We also have:

$$ax \in D^t_G(aa'^2, aa''^2) = D^t_G(a \cdot (av)^2, a \cdot (aw)^2 = D^t_G(av^2, aw^2).$$

From the representations in (*) and (**) we get $av^2 \in D^t_G(b^2pv, bcp^2rv)$ and $aw^2 \in D^t_G(c^2p'w, bcp'^2r'w)$, whence $ax \in D^t_G(b^2pv\,, bcp^2rv\,, c^2p'w\,, bcp'^2r'w)$. There are elements α_1, α_2 such that

(***) $\alpha_1 \in D^t_G(b^2pv, bcp'^2r'w)\,, \alpha_2 \in D^t_G(c^2p'w, cbp^2rv)$ and $ax \in D^t_G(\alpha_1, \alpha_2)$.

By I.2.3 (4), the first of these representations gives

(****) $\alpha_1 = b^2\alpha_1$ and $b\alpha_1 \in D^t_G(bpv, b^2cp'^2r'w)$.

Claim 4. $b\alpha_1 \sim_{\Delta,T} b^2$.

Proof of Claim 4. Taking images of the representation in (***) under π, and taking into account that $\pi(p) = \pi(b^2), \pi(v) = \pi(b), \pi(p') = \pi(r') = \pi(c^2)$, the equality $\pi(w) = \pi(c)$, gives:

$$\pi(b\alpha_1) \in D^t_{G/\mathcal{H}^T_\Delta}(\pi(bpv), \pi(b^2cp'^2r'w)) = D^t_{G/\mathcal{H}^T_\Delta}(\pi(b^2), \pi(b^2c^2) = \{\pi(b^2)\}.$$

A similar argument, using the second representation in (****) shows that $\alpha_2 = c^2\alpha_2$ and $c\alpha_2 \sim_{\Delta,T} c^2$.

From the last representation in (***) comes:

$$ax \in D^t_G(\alpha_1, \alpha_2) = D^t_G(b^2\alpha_1, c^2\alpha_2) = D^t_G(b(b\alpha_1), c(c\alpha_2)).$$

Setting $y := b\alpha_1$ and $z := c\alpha_2$, we have $y \sim_{\Delta,T} b^2\,, z \sim_{\Delta,T} c^2\,, x \sim_{\Delta,T} a^2$ and $ax \in D^t_G(by, cz)$, concluding the proof of (c).

Proof of (d). Let us first verify axiom [RS3]. We write D for D_G. By Remark **??** it suffices to prove that $D_{G/\mathcal{H}^T_\Delta}$ is weakly associative. Let $a, b, c, d, e \in G$ be such that $\pi(a) \in D_{G/\mathcal{H}^T_\Delta}(\pi(b), \pi(c))$ and $\pi(c) \in D_{G/\mathcal{H}^T_\Delta}(\pi(d), \pi(e))$. By condition (b), there are $c' \in G$, $d_1, d_2 \in \Delta$ such that $c \sim_{\Delta,T} c'$ and $c' \in D(d_1d, d_2e)$. Since $\pi(c) = \pi(c')$, we have $\pi(a) \in D_{G/\mathcal{H}^T_\Delta}(\pi(b), \pi(c'))$, and, by (b) again, there are $a' \sim_{\Delta,T} a$ and $d_3, d_4 \in \Delta$ such that $a' \in D(d_3b, d_4c')$. Since D is weakly associative, we have $a' \in D(x, d_4d_2e)$ for some $x \in D(d_3b, d_4d_1d)$. Hence $\pi(a) = \pi(a') \in D_{G/\mathcal{H}^T_\Delta}(\pi(x), \pi(d_4d_2)\pi(e))$. From $\pi(d_4d_2) = \pi(d_4^2d_2^2)$ comes $D_{G/\mathcal{H}^T_\Delta}(\pi(x), \pi(d_4d_2)\pi(e)) \subseteq D_{G/\mathcal{H}^T_\Delta}(\pi(x), \pi(e))$ ([RS4]), whence $\pi(a) \in D_{G/\mathcal{H}^T_\Delta}(\pi(x), \pi(e))$. A similar argument shows that $x \in D(d_3b, d_4d_1d)$ implies $\pi(x) \in D_{G/\mathcal{H}^T_\Delta}(\pi(b), \pi(d))$, thus proving that $D_{G/\mathcal{H}^T_\Delta}$ is weakly associative.

To complete the proof, only item (iii) in Definition II.2.1 remains to be proved. Let H be a RS and $f : G \longrightarrow H$ be a RS-morphism such that $a \sim_{\Delta,T} b$ implies $f(a) = f(b)$ for all $a, b \in G$. We must show that the map $\widehat{f} : G/\mathcal{H}^T_\Delta \longrightarrow H$ defined by $\widehat{f} \circ \pi = f$ preserves representation. Let $a, b, c \in G$ be so that $\pi(a) \in D_{G/\mathcal{H}^T_\Delta}(\pi(b), \pi(c))$. By item (b) there are elements $a' \in G, d_1, d_2 \in \Delta$ such that $a \sim_{\Delta,T} a'$ and $a' \in D(d_1b, d_2c)$. The congruence $a \sim_{\Delta,T} a'$ implies $f(a) = f(a')$ and, since f is a RS-morphism, $f(a) = f(a') \in D_H(f(d_1)f(b), f(d_2)f(c))$. Also, $d_j \in \Delta$ $(j = 1, 2)$ implies $d_j \sim_{\Delta,T} d_j^2$, and then $f(d_j) = f(d_j)^2$. Hence $f(a) \in D_H(f(d_1)^2f(b), f(d_2)^2f(c)) \subseteq D_H(f(b), f(c))$, showing that $\widehat{f}$ is a morphism of real semigroups. $\square$

Remarks II.3.6 (i) Saturated sets are also characterized as follows: for a RS, G, and $\mathcal{H} \subseteq X_G$, are equivalent:

(1) For all $g \in X_G$, $\bigcap_{h\in\mathcal{H}} P(h) \subseteq P(g)$ and $Z(g) \subseteq \bigcup_{h\in\mathcal{H}} Z(h)$ imply $g \in \mathcal{H}$.

(2) $\mathcal{H} = \mathcal{H}^T_\Delta$ for some multiplicative set $T \subseteq G$ containing 1 and some saturated subsemigroup Δ of G.

For (1) $\Rightarrow$ (2) take $T = G \setminus \bigcup_{h\in\mathcal{H}} Z(h)$ and $\Delta = \bigcap_{h\in\mathcal{H}} P(h)$. The implication (2) $\Rightarrow$ (1) is straightforward checking.

(ii) As can be expected, the closed subset $\mathcal{H}_{\sim_{\Delta,T}}$ associated to the congruence $\sim_{\Delta,T}$, defined in Proposition **??**, is identical with $\mathcal{H}^T_\Delta$.

Since clearly $\mathcal{H}^T_\Delta \subseteq \mathcal{H}_{\sim_{\Delta,T}}$, only the reverse inclusion needs proof. Let $h \in \mathcal{H}_{\sim_{\Delta,T}}$. Then there exists $\sigma \in X_{G/\mathcal{H}^T_\Delta}$ such that $h = \sigma \circ \pi$. Let $x \in \Delta$. Then $x \sim_{\Delta,T} x^2$, whence $\pi(x) = \pi(x^2)$, implying $h(x) \geq 0$. On the other hand, if $t \in T$, then $t^2 \sim_{\Delta,T} 1$, and $h(t^2) = (\sigma \circ \pi)(t^2) = 1$, which means $h(t) \neq 0$. Therefore $\Delta \subseteq P(h)$ and $Z(h) \cap T = \emptyset$, i.e., $h \in \mathcal{H}^T_\Delta$. □

Remarks II.3.7 (Convexity) (i) Let G be a real semigroup. Then every saturated set $\mathcal{H}^T_\Delta \subseteq X_G$ is convex under the specialization partial order $\rightsquigarrow$. Indeed, let $h, g \in \mathcal{H}^T_\Delta$ and let $k \in X_G$ be such that $h \rightsquigarrow k \rightsquigarrow g$. From $\Delta \subseteq P(h)$ and $P(h) \subseteq P(k)$ follows $\Delta \subseteq P(k)$. On the other hand, $k \rightsquigarrow g$ implies $Z(k) \subseteq Z(g)$. Since $Z(g) \cap T = \emptyset$, we get $Z(k) \cap T = \emptyset$, proving that $k \in \mathcal{H}^T_\Delta$.

(ii) Note, however, that the requirement of convexity under specialization on a procontructible subset $\mathcal{H}$ of X_G (G a RS) alone is not enough to guarantee that the quotient structure $G/\mathcal{H}$, defined as in **??**, is a real semigroup. Example I.3.6 is a counterexample: the (finite) set of characters of a reduced special group therein defined is proconstructible and convex under specialization (because specialization is just equality in that case), but the induced quotient structure does not verify axiom [RS3a].

(iii) Suppose G is a RS-fan (see Chapter VI) having three prime ideals I, J, K, with $I \subset J \subset K$ and let $\mathcal{H} = \{h \in X_G \,|\, Z(h) = I \text{ or } Z(h) = K\}$. Let $h_0, h_1, h \in X_G$ be such that $Z(h) = J$, $Z(h_0) = I$, $Z(h_1) = K$ and $h_0 \rightsquigarrow h_1$ (cf. Lemma VI.6.7). It is easy to check that $h_0 \rightsquigarrow h^2 h_0 \rightsquigarrow h_1$. On the other hand, both h_0, h_1 are in $\mathcal{H}$ and, since $Z(h^2 h_0) = J$, we get $h^2 h_0 \notin \mathcal{H}$. Therefore $\mathcal{H}$ is not convex under $\rightsquigarrow$ and, by (i), is not saturated either. However, since $\mathcal{H}$ is proconstructible and 3-closed (i.e., stable under product of any three of its members), it follows from Proposition VI.11.1 that $\equiv_\mathcal{H}$ is a congruence of RSs. So,

There are congruences of real semigroups other than those induced by saturated sets.

Further instances occur in the case of spectral RSs, studied in Chapter V: in this case congruences are induced by arbitrary proconstructible subsets of the character space, X_G, with the spectral topology (Theorem V.8.2 and Corollary V.8.4). It suffices to consider a suitable spectral RS, G, such that X_G contains three points $h_1 \underset{\neq}{\rightsquigarrow} h_2 \underset{\neq}{\rightsquigarrow} h_3$, and the proconstructible subset $\{h_1, h_3\}$. □

Next we consider some congruences of RSs induced by particular types of saturated subsets which frequently occur in practice.

C. Saturated subsemigroups. Let G be a RS and let $\Delta \subseteq G$ be a saturated subsemigroup of G. Let us consider the following set of characters:

$$\mathcal{H}_\Delta = \{h \in X_G \,|\, \Delta \subseteq P(h)\}.$$

Sets of this form are called *subspaces* in the terminology of [M]. The equivalence relation associated to $\mathcal{H}_\Delta$ will be denoted by $\sim_\Delta$ instead of $\sim_{\mathcal{H}_\Delta}$. Let $T_\Delta = \{x \in G \,|\, x^2 \sim_\Delta 1\}$. Clearly T_Δ is a multiplicative set containing 1, and $0 \notin T_\Delta$. An immediate consequence of Corollary I.4.11 is that for $a \in G$, $a \in \Delta \Leftrightarrow a \sim_\Delta a^2$, and then $a \in \Delta \cap -\Delta \Leftrightarrow a \sim_\Delta 0$. From this we get $(\Delta \cap -\Delta) \cap T_\Delta = \emptyset$. Moreover, suppose that $a \in G$, $x \in T_\Delta$ and $ax^2 \in \Delta$. If $h \in \mathcal{H}_\Delta$ then $h(ax^2) \geq 0$. Since $h(x^2) = 1$ we obtain $h(a) \geq 0$. Since h is arbitrary we conclude that $a \sim_\Delta a^2$, and then $a \in \Delta$. Therefore Δ and T_Δ satisfy the hypotheses of Theorem II.3.5.
Straightforward verification proves that $\mathcal{H}_\Delta = \mathcal{H}_\Delta^{T_\Delta}$. Hence $\mathcal{H}_\Delta$ is a saturated set and the corresponding equivalence relation $\sim_\Delta$ is a congruence of RSs. To ease notation we shall denote by G/Δ —instead of $G/\mathcal{H}_\Delta$— the quotient by $\sim_\Delta$.

The argument of II.3.7 (i) shows that the sets of characters $\mathcal{H}_\Delta$ defining subspaces are (*upwards*) *closed* under specialization: $h \in \mathcal{H}_\Delta$, $g \in X_G$ and $h \rightsquigarrow g$ imply $g \in \mathcal{H}_\Delta$. Since $\mathcal{H}_\Delta$ is a proconstructible subset of X_G, this additional property is equivalent to $\mathcal{H}_\Delta$ being closed for the *spectral* topology of X_G (cf. [DST], Corollary 6.1.6).

Theorem II.3.8 *Let G be a RS and let Δ be a saturated subsemigroup of G. Let $a, b, c \in G$. Then:*

(a) *$a \sim_\Delta b$ if and only if $ab \in \Delta$ and there are $d_1, d_2, d_3, d_4 \in \Delta$ such that $a^2 \in D_G(-d_1, d_2b^2)$ and $b^2 \in D_G(-d_3, d_4a^2)$.*

(b) *The following are equivalent:*

  (i) $\pi(a) \in D_{G/\Delta}(\pi(b), \pi(c))$.

  (ii) *There are $p, q, r \in G$ such that $p \sim_\Delta q \sim_\Delta r \sim_\Delta a^2$ and $ap \in D_G(bq, cr)$.*

  (iii) *There are $a' \in G$ and $d_1, d_2 \in \Delta$ such that $a \sim_\Delta a'$ and $a' \in D_G(d_1b, d_2c)$.*

(c) *The following are equivalent:*

  (i) $\pi(a) \in D^t_{G/\Delta}(\pi(b), \pi(c))$.

  (ii) *There are $x, y, z \in G$ such that $x \sim_\Delta a^2$, $y \sim_\Delta b^2$, $z \sim_\Delta c^2$ and $ax \in D^t_G(by, cz)$.*

(d) *The equivalence relation $\sim_\Delta$ determines a RS-congruence.*

Proof. First we show:

Claim. If $x \in G$, then $x \in T_\Delta$ iff there are $d_1, d_2 \in \Delta$ such that $x^2d_1 \in D^t_G(1, d_2)$.

Proof of Claim. ($\Leftarrow$) Assume the statement on the right-hand side and let $h \in \mathcal{H}_\Delta$. If $h(x) = 0$, then $h(x^2d_1) = 0$ and hence $h(d_2) = -1$ (I.2.5), which is impossible because $d_2 \in \Delta$. Then $h(x) \neq 0$, and hence $h(x^2) = 1$ for all $h \in \mathcal{H}_\Delta$, which means $x \in T_\Delta$.

($\Rightarrow$) Conversely, suppose $x \in T_\Delta$. Let $\Delta[-x^2]$ be the saturated subsemigroup generated by $\Delta \cup \{-x^2\}$. If $-1 \notin \Delta[-x^2]$, by Corollary I.4.11 we find a character $h \in X_G$ such that $\Delta[-x^2] \subseteq P(h)$. In particular $h \in \mathcal{H}_\Delta$, and since $h(-x^2) \geq 0$ we get $h(x^2) = 0$, contradicting $x \in T_\Delta$. Therefore $-1 \in \Delta[-x^2]$, and hence $-1 \in D(d_2, -d_1x^2)$ for some $d_1, d_2 \in \Delta$, which, by [RS6], implies $-1 \in D^t(d_2, -d_1x^2)$. It follows that $d_1x^2 \in D^t(1, d_2)$, proving the claim.

(a) ($\Leftarrow$) Assume $a, b \in G$ satisfy the conditions on the right-hand side of (i), and let $h \in \mathcal{H}_\Delta$. Since $ab \in \Delta$ we have $h(ab) \geq 0$. Clearly, $h(ab) = 1$ implies $h(a) = h(b) (\neq 0)$. Assuming $h(a) = 0$, we have $h(b^2) \in D_{\mathbf{3}}(-h(d_3), 0)$. Since $-h(d_3) \leq 0$ and $h(b^2) \geq 0$, we get $h(b^2) = 0$, and $h(b) = 0$. Thus, $h(a) = 0$ implies $h(b) = 0$. Similarly, the relation $a^2 \in D(-d_1, d_2 b^2)$ gives $h(b) = 0 \Rightarrow h(a) = 0$. Therefore, $h(a) = 0$ if and only if $h(b) = 0$, implying $h(a) = h(b)$ for all $h \in \mathcal{H}_\Delta$, i.e., $a \sim_\Delta b$.

($\Rightarrow$) Conversely, assume $a \sim_\Delta b$ for $a, b \in G$. If $ab \notin \Delta$, Corollary I.4.11 gives an $h \in \mathcal{H}_\Delta$ such that $h(ab) = -1$, contradiction. Therefore $ab \in \Delta$. By Theorem II.3.5 (a) there are elements $x \in T_\Delta$, $d_1, d_2 \in \Delta$ such that

$$(1)\quad a^2x^2 \in D^t(-d_1, a^2b^2x^2) \qquad \text{and} \qquad (2)\quad b^2x^2 \in D^t(-d_2, a^2b^2x^2).$$

By the Claim, $x^2 d_3 \in D^t(1, d_4)$ for some $d_3, d_4 \in \Delta$. From (1) we have $d_1 \in D^t(-a^2x^2, a^2b^2x^2)$ and scaling by d_3 we obtain $d_1 d_3 \in D^t(-a^2x^2d_3, a^2b^2x^2d_3) \subseteq D^t(-a^2, -a^2d_4, a^2b^2, a^2b^2d_4)$. It follows that $a^2 \in D(-d_1d_3, -a^2d_4, a^2b^2, a^2b^2d_4)$, and this implies $a^2 \in D(x, y)$ for some $y \in D(-d_1d_3, -a^2d_4)$ and some $z \in D(b^2a^2, b^2a^2d_4)$. Since d_1d_3, $a^2d_4 \in \Delta$ we have $-y \in \Delta$. We also have $z \in \Delta$ and $z = b^2z$ (I.2.3 (4)). Setting $d = -y \in \Delta$ we obtain $a^2 \in D(-d, b^2z)$. In a similar way it is shown that $b^2 \in D(-e, a^2z')$ for some $e, z' \in \Delta$.

Items (b), (c) and (d) follow from the corresponding statements in Theorem II.3.5. ☐

D. Transversally saturated subsemigroups. An interesting instance of quotients modulo saturated subsets treated in paragraph B, are the quotients of a RS modulo transversally saturated subsemigroups.

Notation II.3.9 Let G be a RS. A transversally saturated subsemigroup Γ of G (abbreviated **tss**), cf. I.4.1, is called **non-trivial** if $\mathcal{H}_\Gamma = \{h \in X_G \mid \Gamma \subseteq h^{-1}[1]\} \neq \emptyset$. ☐

Remarks. (i) It can be proved that the following conditions are equivalent for a tss Γ:

(1) $0 \notin \Gamma$;

(2) $\Gamma \cap -\mathrm{Id}(G) = \emptyset$;

(3) Γ is non-trivial (i.e., $\mathcal{H}_\Gamma \neq \emptyset$).

The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (2). If there is $a \in G$ so that $-a^2 \in \Gamma$, since Γ is multiplicative we have $(-a^2)^2 = a^2 \in \Gamma$. By Proposition I.2.3 (11), $0 \in D^t_G(a^2, -a^2) \subseteq \Gamma$, contradicting (1).

The proof of (2) $\Rightarrow$ (3) is non-trivial, and will be omitted.

(ii) Recall (I.4.2) that a subsemigroup of a RS is saturated if and only if it is tranversally saturated and contains all idempotents.

(iii) Natural examples of tss are:

— The set $a^\downarrow$ of predecessors of an element a of a RS, G in the representation partial order (I.6.5 (5)); more generally,

— The set $D^\downarrow$ of predecessors of elements of a (right) directed set $D \subseteq G$ under the representation partial order. By (i), these examples are non-trivial tss's if and only if $0 \not\leq a$ (resp., $0 \notin D^\downarrow$).

— The set $D^t_G(\varphi)$ of elements represented by a Pfister form over G (IV.5.8 (1); cf. also I.4.4 (2) ff.). This example is non-trivial unless the form φ is hyperbolic (i.e., $-1 \in D^t_G(\varphi)$). ☐

Next we show:

Proposition II.3.10 *Let G be a RS. The character set $\mathcal{H}_\Gamma$ of a non-trivial tss Γ of G is a saturated set; in fact, $\mathcal{H}^\Gamma_\Delta$ for a suitable saturated subsemigroup Δ.*

Proof. Observe that, with the terminology of paragraph B, the subsemigroup T is the given tss Γ itself. Set $\Delta :=$ the saturated subsemigroup of G generated by Γ. By Proposition I.4.6 (2), $\Delta = \bigcup\{D_G(\varphi) \mid \varphi$ a form over $\Gamma\}$. We show that $\mathcal{H}_\Gamma = \mathcal{H}^\Gamma_\Delta$.

To prove the inclusion $\subseteq$, let $h \in \mathcal{H}_\Gamma$, i.e., $\Gamma \subseteq h^{-1}[1]$. If $x \in D_G(\varphi)$ for some form φ over Γ, then $h(x) \in D_{\mathbf{3}}(h * \varphi) = D_{\mathbf{3}}(\langle 1, \ldots, 1\rangle) \subseteq \{0, 1\}$, showing $\Delta \subseteq P(h)$. On the other hand, since $\Gamma \subseteq h^{-1}[1]$, we have $Z(h) \cap \Gamma = \emptyset$. Hence, $h \in \mathcal{H}^\Gamma_\Delta$.

Conversely, let $h \in X_G$ be such that $\Delta \subseteq P(h)$ and $Z(h) \cap T = \emptyset$. Then, $\Gamma \subseteq \Delta \subseteq P(h)$ and $Z(h) \cap \Gamma = \emptyset$, showing $h \lceil \Gamma = 1$, as asserted. $\square$

The foregoing characterization of the set $\mathcal{H}_\Gamma$, Γ a non-trivial tss, together with Theorem II.3.5, yield:

Theorem II.3.11 *Let Γ be a non-trivial tranversally saturated subsemigroup of a RS, G. Let $\pi = \pi_\Gamma : G \longrightarrow G/\mathcal{H}_\Gamma$ denote the quotient map. For $a, b, c \in G$ we have:*

(a) *$a \sim_\Gamma b$ if and only if there are elements $t \in \Gamma$, $x, y \in G$, and forms φ, ψ, θ over Γ such that*

$$ab \in D^t_G(a^2b^2\varphi)\,,\ a^2b^2t^2 \in D^t_G(x^2\psi \oplus \langle a^2t^2\rangle) \text{ and } a^2b^2t^2 \in D^t_G(y^2\theta \oplus \langle b^2t^2\rangle).$$

(b) *$\pi(a) \in D_{G/\mathcal{H}_\Gamma}(\pi(b), \pi(c))$ if and only if there are $a' \in G$ and forms φ_1, φ_2 over Γ such that*

$$a' \sim_\Gamma a \text{ and } a' \in D_G(b\varphi_1 \oplus c\varphi_2)\,.$$

(c) *The equivalence relation $\sim_\Gamma$ is a RS-congruence of G.*

Proof. Let Δ be as in the preceding Proposition, so that $\mathcal{H}_\Gamma = \mathcal{H}^\Gamma_\Delta$; we apply Theorem II.3.5 to this situation.

(a) ($\Leftarrow$) Assume the right-hand side of (a) holds, and let $h \in \mathcal{H}_\Gamma$. Suppose first $h(a) = 0$. Since $h * \rho = \dim(\rho) \cdot \langle 1\rangle$ for any form ρ over Γ, $h(t)^2 = 1$, and $h(y)^2 \in \{0, 1\}$, taking images under h in the representation $a^2b^2t^2 \in D^t_G(y^2\theta \oplus \langle b^2t^2\rangle)$ yields $h(a^2b^2t^2) = 0 \in D_{\mathbf{3}}(h(y)^2 \dim(\theta) \cdot \langle 1\rangle \oplus \langle h(b^2)\rangle)$. This forces $h(b) = 0$. In fact, if $h(b)^2 = 1$, the right-hand side of the last representation is $\{1\}$ for both the values 0 and 1 of $h(y)^2$, contradiction. Likewise, the representation $a^2b^2t^2 \in D^t_G(x^2\psi \oplus \langle a^2t^2\rangle)$ yields $h(b) = 0 \Rightarrow h(a) = 0$.

If $h(a) \neq 0$, then $h(b) \neq 0$, and then $h(a^2b^2) \neq 0$. Taking images under h in $ab \in D^t_G(a^2b^2\varphi)$ yields $h(a)h(b) \in D_{\mathbf{3}}(h(a^2b^2)\dim(\varphi) \cdot \langle 1\rangle) = \{1\}$. Hence, $h(a) = h(b)$.

($\Rightarrow$) Assume $a \sim_\Gamma b$. From II.3.5 (a) we have $ab \in \Delta$, i.e., $ab \in D_G(\varphi)$ for some form φ over Γ; hence $ab \in D^t_G(a^2b^2\varphi)$ (I.2.8 (3)).

Next, the second assertion in II.3.5 (a) gives elements $t \in \Gamma$ and $d_1, d_2 \in \Delta$ such that $a^2t^2 \in D^t_G(-d_1, a^2b^2t^2)$ and $b^2t^2 \in D^t_G(-d_2, a^2b^2t^2)$. By Proposition II.3.10, $d_i \in \Delta$ implies $d_i \in D_G(\varphi_i)$ for forms φ_i over Γ, and hence $d_i \in D^t_G(d_i^2\psi_i)$ for $i = 1, 2$. The first of these representations yields $a^2b^2t^2 \in D^t_G(d_1, a^2t^2) \subseteq D^t_G(d_1^2\psi_1 \oplus \langle a^2t^2\rangle)$, as required (set $x := d_1$, $\psi := \psi_1$). The representation $a^2b^2t^2 \in D^t_G(y^2\theta \oplus \langle b^2t^2\rangle)$ is proved by the same argument from $b^2t^2 \in D^t_G(-d_2, a^2b^2t^2)$, setting $y := d_2$, $\theta := \psi_2$.

(b) ($\Leftarrow$) This implication is straightforward: $a' \sim_\Gamma a$ means $\pi(a') = \pi(a)$. Since π is a RS-homomorphism such that $\pi(g) = 1$ for $g \in \Gamma$, we have $\pi * \varphi_i = \dim(\varphi_i) \cdot \langle 1\rangle$ $(i = 1, 2)$; hence

$a' \in D_G(b\varphi_1 \oplus c\varphi_2)$ yields

$$\pi(a) = \pi(a') \in D_{G/\mathcal{H}_\Gamma}(\pi(b) \cdot (\pi * \varphi_1) \oplus \pi(c) \cdot (\pi * \varphi_2)) = D_{G/\mathcal{H}_\Gamma}(\pi(b), \pi(c))\,.$$

($\Rightarrow$) The lifting of representation in the real semigroup $G/\mathcal{H}^\Gamma_\Delta = G/\mathcal{H}_\Gamma$ given by II.3.5 (b.iii), together with Proposition II.3.10 yield:

$$\begin{aligned}\pi(a) \in D_{G/\mathcal{H}_\Gamma}(\pi(b), \pi(c)) \Leftrightarrow\ & \text{There are } a' \in G \text{ and } d_1, d_2 \in \Delta \text{ such that } a' \sim_{\Delta,\Gamma} a \text{ and} \\ & a' \in D_G(d_1 b, d_2 c) \\ \Rightarrow\ & \text{There are } a' \in G \text{ and forms } \varphi_1, \varphi_1 \text{ over } \Gamma \text{ such that } d_i \in D_G(\varphi_i) \\ & (i = 1, 2) \text{ and } a' \in D_G(d_1 b, d_2 c) \subseteq D_G(b\varphi_1 \oplus c\varphi_2)\,.\end{aligned}$$

(c) is a particular instance of Theorem II.3.5 (d). □

E. Quotients modulo saturated ideals. Before turning to the matter of the title, we prove a lemma that will be repeatedly used in this and the next paragraph.

Lemma II.3.12 *Let G be a RS, let I be a saturated prime ideal of G and let Δ_I denote the saturated subsemigroup generated by I. Then,*

(i) For $x \in G$, $x \in \Delta_I \Leftrightarrow \exists i \in I\,(x \in D_G(1, i))$.

(ii) The ideal I is radical (i.e., $x^2 \in I \Rightarrow x \in I$).

(iii) $\Delta_I \cap -\Delta_I = I$.

Proof. (i) The implication ($\Leftarrow$) is clear.

($\Rightarrow$) Let $x \in \Delta_I$. By I.4.6 (3) there are $n \in \mathbb{N}$ and a form φ over I such that $x \in D_G(n\langle 1\rangle \oplus \varphi)$. Then, there is $i' \in G$ such that $i' \in D_G(\varphi)$ and $x \in D_G(n\langle 1\rangle \oplus \langle i'\rangle)$. Since I is saturated, $i' \in I$. By [RS6], $x \in D^t_G(n\langle x^2\rangle \oplus \langle i\rangle)$, with $i = i'x^2 \in I$. Hence, there is $y \in D^t_G(n\langle x^2\rangle)$ such that $x \in D^t_G(y, i)$. The first of these representations entails $y = x^2$ (cf. Corollary IV.5.3 (ii)). Using [RS4] we have $x \in D^t_G(x^2, i) \subseteq D_G(x^2, i) \subseteq D_G(1, i)$, as asserted.

(ii) Assume there is $x \in G$ such that $x^2 \in I$ but $x \notin I$. We first note:

(*) $x \notin \Delta_I$.

By item (i), if $x \in \Delta_I$, then $x \in D_G(1, i))$ for some $i \in I$. By [RS6], $x \in D^t_G(x^2, i'))$ with $i' = ix^2 \in I$. Since $x^2 \in I$ and I is saturated, $x \in I$, contradiction.

Now we apply Corollary I.4.11 with $\Delta = \Delta_I$ and $T = \{1\}$; (*) shows $xT \cap \Delta_I = \emptyset$, and hence there is $h \in X_G$ so that $\Delta_I \subseteq P(h)$ and $h(x) = -1$. Since $x^2 \in I \subseteq \Delta_I \cap -\Delta_I \subseteq P(h) \cap -P(h) = Z(h)$ we have $h(x^2) = 0$, contradicting $h(x) = -1$.

(iii) Let $x \in \Delta_I \cap -\Delta_I$. By (i) there are $i, j \in I$ such that $x \in D_G(1, i), -x \in D_G(1, j)$. By I.2.8 (9) and saturatedness of I, there is $k \in I$ so that $-x^2 \in D(1, k)$. Invoking [RS6] we get $-x^2 \in D^t_G(x^2, k')$ with $k' \in I$, and (by I.2.3 (0),(6)) $-k' \in D^t(x^2, x^2)$, whence $-k' = x^2$, which, by (ii), yields $x \in I$. □

In this paragraph we consider the following setup: G is a real semigroup, I is a saturated ideal of G *not necessarily prime*, and $\mathcal{H}^I = \{h \in X_G \mid I \subseteq Z(h)\}$. The equivalence relation on G defined by $\mathcal{H}^I$ as in **??** (†) will be denoted by $\approx_I$. As above, Δ_I stands for the saturated subsemigroup generated by I. We first note:

Fact II.3.13 $\mathcal{H}^I = \mathcal{H}_{\Delta_I}$. *Hence, $\approx_I$ is a RS-congruence on G and, endowed with the ternary relation defined by $\mathcal{H}^I$ (cf. **??** (††)), $G/\mathcal{H}^I$ is a real semigroup.*

Proof. The stated equality just means, for $h \in X_G$,

$$I \subseteq Z(h) \ \Leftrightarrow\ \Delta_I \subseteq P(h).$$

The implication ($\Leftarrow$) is clear from $I \subseteq \Delta_I \cap -\Delta_I \subseteq P(h) \cap -P(h) = Z(h)$. Conversely, let $x \in \Delta_I$. By II.3.12 (i) there is $i \in I$ such that $x \in D_G(1,i)$. So, $h(x) \in D_{\mathbf{3}}(h(1),h(i)) = D_{\mathbf{3}}(1,0) = \{1,0\}$, i.e., $h(x) \geq 0$. □

Thus, Theorem II.3.8 can be applied with $\Delta = \Delta_I$, yielding the following intrinsic characterizations of the congruence $\approx_I$ and of both representation relations in $G/\mathcal{H}^I$:

Theorem II.3.14 *Let G be a RS and let I be a saturated ideal of G. Then, for $a,b,c \in G$ and with $\pi : G \longrightarrow G/\mathcal{H}^I$ canonical:*

(a) $a \approx_I b \Leftrightarrow \exists i,j,k \in I$ *such that* $a^2 \in D^t_G(a^2b^2,i)$, $b^2 \in D^t_G(a^2b^2,j)$ *and* $ab \in D^t_G(a^2b^2,k)$.

(b) *The following are equivalent:*

  (i) $\pi(a) \in D_{G/\mathcal{H}^I}(\pi(b),\pi(c))$.

  (ii) *There are* $p,q,r \in G$ *such that* $p \approx_I q \approx_I r \approx_I a^2$ *and* $ap \in D_G(bq,cr)$.

  (iii) *There are* $a' \in G$ *and* $k \in I$ *such that* $a \approx_I a'$ *and* $a' \in D_G(b,c,k)$.

(c) *The following are equivalent:*

  (i) $\pi(a) \in D^t_{G/\mathcal{H}^I}(\pi(b),\pi(c))$.

  (ii) *There are* $x,y,z \in G$ *such that* $x \approx_I a^2$, $y \approx_I b^2$, $z \approx_I c^2$ *and* $ax \in D_G(by,cz)$.

(d) *The relation $\approx_I$ defines a RS-congruence of G.*

Proof. The equivalence of (i) and (ii) in item (b) and (c) follows at once from the corresponding equivalences in Theorem II.3.8 applied with $\Delta = \Delta_I$; note that II.3.13 shows that the equivalence relation $\approx_I$ is identical with $\sim_{\Delta_I}$.

(a) From II.3.8 (a) we get, for $a,b \in G$:

(*) $\quad a \approx_I b \Leftrightarrow ab \in \Delta_I$ and there are $d_1,\ldots,d_4 \in \Delta_I$ such that $a^2 \in D_G(-d_1,d_2b^2)$ and $b^2 \in D_G(-d_3,d_4a^2)$.

By Lemma II.3.12 (i), for $\ell = 1,\ldots,4$ there are elements $m_\ell \in I$ such that $d_\ell \in D_G(1,m_\ell)$. The first representation in (*) then yields $a^2 \in D_G(-1,-m_1,b^2,m_2b^2)$; by saturatedness of I there is $m' \in I$ such that $a^2 \in D_G(-1,b^2,m')$. By [RS6] this implies $a^2 \in D^t_G(-a^2,a^2b^2,m)$, with $m = m'a^2 \in I$, and hence $m \in D^t_G(a^2,a^2,-a^2b^2)$; using I.2.8 (5) and I.2.3 (6) we get $m \in D^t_G(a^2,-a^2b^2)$, and hence $a^2 \in D^t_G(a^2b^2,i)$ with $i = -m \in I$.

In the same manner one proves $b^2 \in D^t_G(a^2b^2,j)$ for some $j \in I$.

Finally, invoking Lemma II.3.12 (i) again, $ab \in \Delta_I$ implies $ab \in D_G(1,i')$ for some $i' \in I$, whence $ab \in D_G(a^2b^2,k)$, with $k = i'a^2b^2 \in I$. This proves the implication ($\Rightarrow$) in (a).

For the converse, we must show $h(a) = h(b)$ for all $h \in X_G$ such that $I \subseteq Z(h)$. Suppose first $h(a) = 0$; since $j \in I$, $h(j) = 0$, and from $b^2 \in D^t_G(a^2b^2,j)$ comes $h(b^2) \in D^t_{\mathbf{3}}(h(a^2b^2),h(j)) =$

$D^t_{\mathbf{3}}(0,0) = \{0\}$, whence $h(b) = 0$. Conversely, the representation $a^2 \in D^t_G(a^2b^2, i)$ yields $h(b) = 0 \Rightarrow h(a) = 0$. Hence, $h(b) = 0 \Leftrightarrow h(a) = 0$.

Suppose $h(a), h(b) \neq 0$; then, $h(a^2b^2) = 1$, and the last transversal representation in (a) gives $h(ab) \in D^t_{\mathbf{3}}(h(a^2b^2), h(k)) = D^t_{\mathbf{3}}(1,0) = \{1\}$, whence $h(a) = h(b)$.

(b) As remarked above it only remains proving the equivalence of (i) and (iii). The corresponding equivalence in II.3.8 (b), with $\Delta = \Delta_I$ shows that (i) is equivalent to

There are $a' \in G$ and $d_1, d_2 \in \Delta_I$ such that $a \approx_I a'$ and $a' \in D_G(d_1 b, d_2 c)$.

By Lemma II.3.12 (i), $d_\ell \in D_G(1, k_\ell)$ with $k_\ell \in I$ ($\ell = 1,2$), and we get $a' \in D_G(b, k_1 b, c, k_2 c)$. Then, there is $k \in D_G(k_1 b, k_2 c) \subseteq I$ such that $a' \in D_G(b, c, k)$, proving (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). assume (iii) holds; $a \approx_I a'$ implies $\pi(a) = \pi(a')$. Since π preserves representation, $\pi(k) = 0$, and $G/\mathcal{H}^I$ is a RS, from $a' \in D_G(b,c,k)$ comes

$$\pi(a) = \pi(a') \in D_{G/\mathcal{H}^I}(\pi(b), \pi(c), \pi(k)) = D_{G/\mathcal{H}^I}(\pi(b), \pi(c), 0) = D_{G/\mathcal{H}^I}(\pi(b), \pi(c)),$$

which proves (i).

Items (c) and (d) are particular cases of the corresponding statements in Theorem II.3.8. $\square$

F. Residue spaces at saturated prime ideals. We shall now consider quotients of a RS, G, modulo saturated *prime* ideals, I, determined, however, by families of characters *different* from those considered in the preceding paragraph. Namely, we shall consider quotients of G modulo sets of characters of type $\{h \in X_G \mid Z(h) = I\}$, hereafter denoted by $\mathcal{H}_I$; the congruence induced by $\mathcal{H}_I$ will be denoted by $\sim_I$, and the corresponding quotient set by G/I.[2]

We note first that the sets of this form are saturated: it is routine to check that with $T_I = G \setminus I$ (a multiplicative set of G containing 1 and not 0) and Δ_I = the saturated subsemigroup of G generated by I, we have, with notation as in paragraph B, $\mathcal{H}^{T_I}_{\Delta_I} = \{h \in X_G \mid Z(h) = I\}$.

Let $\Gamma_I = \{x \in G \mid x \sim_I 1\}$. This set is a tss, i.e. is multiplicatively closed, contains 1 but not 0, and is closed under D^t, as easily verified using $D^t_{\mathbf{3}}(1,1) = \{1\}$. Our main result in this paragraph is:

Theorem II.3.15 *Let G be a RS and I be a saturated prime ideal of G. For $a, b, c \in G$, we have:*

(a) $a \sim_I b$ *if and only if there exist $y \notin I$ and $i \in I$ such that $i \in D^t_G(ay, -by)$.*

Moreover, $a \sim_I 0$ if and only if $a \in I$.

(b) *The following are equivalent:*

(i) $\pi(a) \in D_{G/I}(\pi(b), \pi(c))$.

(ii) *There are $x \notin I$ and $i \in I$ so that $ax^2 \in D_G(i, b, c)$.*

(iii) *There is $a' \in G$ such that $a' \sim_I a$ and $a' \in D_G(b,c)$.*

(c) *The following are equivalent:*

(i) $\pi(a) \in D^t_{G/I}(\pi(b), \pi(c))$.

[2] Quotients of this type have been considered by Marshall in the dual category of abstract real spectra; cf. [M], p. 102 and Cor. 6.6.9. Here we employ his terminology.

(ii) *Either* $a \in I$ *and* $c \sim_I -b$, *or* $a \notin I$ *and there are* $x \notin I$ *and* $b', c' \in G$ *so that* $b' \sim_I b$, $c' \sim_I c$ *and* $ax^2 \in D^t_G(b', c')$.

(iii) *There are* $a' \in G$ *and* $d \in \Gamma_I$ *so that* $a \sim_I a'$ *and* $a' \in D^t_G(db, dc)$.

(d) *The relation* $\sim_I$ *is a RS-congruence of* G, *and the set* $(G/I)\backslash\{\pi(0)\}$ *is a reduced special group under the representation* $D_{G/I}$.

Proof. (a) The implication ($\Leftarrow$) is routine checking.

($\Rightarrow$) Let $a \sim_I b$. By Theorem II.3.5 (a) there are $x \in T_I$ and $d_1, d_2 \in \Delta_I$ such that $a^2x^2 \in D^t(-d_1, a^2b^2x^2)$, $b^2x^2 \in D^t(-d_2, a^2b^2x^2)$ and $ab \in \Delta_I$. From $d_\ell \in \Delta_I$ ($\ell = 1, 2$) we get $d_\ell \in D(1, i_\ell)$ for some $i_\ell \in I$ (II.3.12 (i)), and then $d_\ell \in D^t(d_\ell^2, j_\ell)$ with $j_\ell = i_\ell d_\ell^2 \in I$. Therefore,

(1) $a^2x^2 \in D^t(-d_1^2, -j_1, a^2b^2x^2)$ and (2) $b^2x^2 \in D^t(-d_2^2, -j_2, a^2b^2x^2)$.

Using [RS4], (1) gives $a^2x^2 \in D(-1, -j_1, a^2b^2x^2)$, and then $a^2x^2 \in D^t(-a^2x^2, j_1', a^2b^2x^2)$ with $j_1' = -j_1a^2x^2 \in I$. This representation implies

(3) $a^2x^2 \in D^t(j_1', a^2b^2x^2)$.

[If $y \in D^t(-y, b, c)$, then there is $e \in D^t(b, c)$ such that $y \in D^t(-y, e)$ (cf. I.2.7); then, $-e \in D^t(-y, -y)$ which, by I.2.3 (6), implies $e = y$.]

Likewise, from (2) we obtain $b^2x^2 \in D^t(j_2', a^2b^2x^2)$ for some $j_2' \in I$. Hence, $a^2b^2x^2 \in D^t(-j_1', a^2x^2)$, and then $b^2x^2 \in D^t(j_2', -j_1', a^2x^2)$, which implies

(4) $b^2x^2 \in D^t(k, a^2x^2)$, with $k \in D^t(j_2', -j_1') \subseteq I$.

From $ab \in \Delta_I$, we get $ab \in D(1, i)$ for some $i \in I$ (II.3.12 (i)), and hence $ab \in D^t(a^2b^2, i')$ with $i' = a^2b^2i \in I$. Scaling (4) by b gives $bx^2 \in D^t(kb, a^2x^2b) = D^t(kb, ax^2(ab)) \subseteq D^t(kb, ab^2x^2, i'ax^2)$, and hence

(5) $bx^2 \in D^t(k', ab^2x^2)$ with $k' \in D^t(kb, i'ax^2) \subseteq I$.

If $b \in I$, item (3) implies $a^2x^2 \in I$, and (since $x \notin I$), $a \in I$. Taking $c \in D^t(a, -b)$ arbitrarily and $y = 1 \in T_I$ yields $c \in I$ and $c \in D^t(ay, -by)$, as required. If $b \notin I$, then $z = bx \notin I$, and (5) yields $bz^2 = bx^2 \in D^t(k', az^2)$, whence $-k' \in D^t(az^2, -bz^2)$ with $-k' \in I$ and $y = z^2 \notin I$, as required.

The second assertion in (a) is clear.

Proof of (b). The implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) are routine checking (for the first use $D(\langle 0\rangle \oplus \varphi) = D(\varphi)$).

(i) $\Rightarrow$ (ii). Assume $\pi(a) \in D_{G/I}(\pi(b), \pi(c))$. By Theorem II.3.5 (b), there are elements $a' \in G$, $d_1, d_2 \in \Delta_I$ such that $a \sim_I a'$ and $a' \in D(d_1b, d_2c)$. By (a), there are $x \notin I$, $i \in I$ so that $ax^2 \in D^t(i, a'x^2) \subseteq D(i, d_1x^2b, d_2x^2c) \subseteq D(i, d_1b, d_2c)$. Since $d_1, d_2 \in \Delta_I$ and I is saturated, from item (*) before the statement we easily get $ax^2 \in D(j, b, c)$ for some $j \in I$.

(ii) $\Rightarrow$ (iii). By (ii), $ax^2 \in D_G(i, b, c)$ for some $x \notin I$, $i \in I$. Passing to transversal representation ([RS6]) we get $ax^2 \in D^t_G(i', a^2x^2b, a^2x^2c)$, with $i' = ia^2x^2 \in I$. Then, there is $a' \in G$ such that $ax^2 \in D^t_G(i', a')$ and $a' \in D^t_G(a^2x^2b, a^2x^2c)$. Using [RS4] yields $a' \in D_G(b, c)$. Now we prove:

(*) $a \sim_I a'$.

Let $h \in X_G$ be such that $Z(h) = I$. If $h(a) = 0$, we get

$$h(a') \in D^t_{\mathbf{3}}(h(a)^2h(x)^2h(b), h(a)^2h(x)^2h(c)) = D^t_{\mathbf{3}}(0,0) = \{0\},$$

i.e., $h(a') = 0$. If $h(a) \neq 0$, then $h(a)^2 = 1$. Since $x \notin I$, $i \in I$, we have $h(x) \neq 0$, whence $h(x)^2 = 1$, and $h(i') = 0$. From $ax^2 \in D^t_G(i', a')$ follows $a' \in D^t_G(-i', ax^2)$, whence $h(a') \in D^t_{\mathbf{3}}(h(-i'), h(a)h(x)^2) = D^t_{\mathbf{3}}(0, h(a)) = \{h(a)\}$, i.e., $h(a') = h(a)$.

Proof of (c). (ii) $\Rightarrow$ (i). If $a \in I$ and $c \sim_I -b$, then $\pi(a) = \pi(0)$ and $\pi(c) = -\pi(b)$. Since in any RS, $0 \in D^t(y, -y)$ (I.2.3 (11)), we get $\pi(a) \in D^t_{G/I}(\pi(b), \pi(c))$.

If $a \notin I$ and the condition stated in (ii) holds, then $\pi(x) \neq 0$, and $\pi(x^2) = 1$. It follows that $\pi(a) = \pi(ax^2)$. Since $\pi(b') = \pi(b)$ and $\pi(c') = \pi(c)$, the required conclusion follows.

(iii) $\Rightarrow$ (i). Routine checking using that $h(d) = 1$ and $h(a) = h(a')$ for any $d \in \Gamma_I$ and any $h \in X_G$ such that $Z(h) = I$.

(i) $\Rightarrow$ (ii). Assume $\pi(a) \in D^t_{G/I}(\pi(b), \pi(c))$. If $a \in I$, then $\pi(a) = \pi(0)$, and I.2.3 (11) yields $\pi(c) = -\pi(b)$, i.e., $c \sim_I -b$.

Let $a \notin I$. Since, in particular, $\pi(a) \in D_{G/I}(\pi(b), \pi(c))$, by (b.(ii)) there are $x \in G \setminus I$ and $i \in I$ so that $ax^2 \in D_G(i, b, c)$. By [RS6], $ax^2 \in D^t_G(j, ba^2x^2, ca^2x^2)$ with $j = ia^2x^2 \in I$. Let $b' \in D^t_G(j, ba^2x^2)$ be such that $ax^2 \in D^t_G(b', ca^2x^2)$ (I.2.7). Setting $c' = ca^2x^2$ we have $c' \sim_I c$, as $h(x), h(a) \neq 0$ for all $h \in \mathcal{H}_I$. From $b' \in D^t_G(j, ba^2x^2)$ comes $b' \sim_I b$, as $h(j) = 0$ and $h(a^2x^2) = 1$ imply $h(b') = h(b)$ for all $h \in \mathcal{H}_I$ (cf. I.2.5).

(i) $\Rightarrow$ (iii). Assume (i); then (ii) holds as well. We consider two cases:

Case 1. $a \in I$.

From (ii) we have $c \sim_I -b$. By item (a) there are $y \notin I$ and $i \in I$ such that $i \in D^t_G(by, cy)$, whence $i' = yi \in D^t_G(by^2, cy^2)$. From $y \notin I$ we get $y^2 \in \Gamma_I$. Set $d := y^2$ and $a' := yi$. Since both $a, a' \in I$, we have $a \sim_I a'$.

Case 2. $a \notin I$.

This assumption gives $a^2 \in \Gamma_I$. From assumption (i) and item (b.iii) there is $a' \in G$ such that $a' \sim_I a$ and $a' \in D_G(b, c)$. By [RS6], $a' \in D^t_G(a'^2b, a'^2c)$. From $a' \sim_I a$ comes $a'^2 \sim_I a^2 \sim_I 1$, i.e., $a'^2 \in \Gamma_I$. Set $d := a'^2$.

(d) The first assertion is a particular instance of Theorem II.3.5 (d), and the second is an immediate consequence of (a) and the fact that $D_{G/I}$ verifies [RS3]. $\square$

As a corollary to Theorem II.3.15 we obtain the following algebraic version of Bröcker's weak local-global principle for forms of dimension 2.

Corollary II.3.16 *Let G be a RS and let $\mathrm{Spec}_{\mathrm{sat}}(G)$ be the family of all saturated prime ideals of G. For each $I \in \mathrm{Spec}_{\mathrm{sat}}(G)$, let $\pi_I : G \to G/I$ be the canonical projection. Then the map $\mu : G \to \prod_{I \in \mathrm{Spec}_{\mathrm{sat}}(G)} G/I$ defined by $\mu(g)_I = \pi_I(g)$ is a (necessarily injective) morphism of RSs satisfying the following condition for $a, b, c \in G$:*

$a \in D(b, c)$ if and only if $\pi_I(a) \in D_{G/I}(\pi_I(b), \pi_I(c))$ for all $I \in \mathrm{Spec}_{\mathrm{sat}}(G)$ (equivalently, $\mu(a) \in D_{\prod_{I \in \mathrm{Spec}_{\mathrm{sat}}(G)}}(\mu(b), \mu(c))$).

Proof. To ease notation we write $\widehat{G} = \prod_{I \in \mathrm{Spec_{sat}}(G)} G/I$. Clearly, μ is a morphism of RSs, where the semigroup operation, the constants $0, -1, 1$, and representation are coordinatewise defined in $\widehat{G}$. Hence $a \in D(b,c)$ implies $\mu(a) \in D_{\widehat{G}}(\mu(b), \mu(c))$. Conversely, if this relation holds and $a \notin D(b,c)$, we can find a character $h \in X_G$ such that $h(a) \notin D_{\mathbf{3}}(h(b), h(c))$. Let $I = Z(h)$. Hence $\pi_I(a) \notin D_{G/I}(\pi_I(b), \pi_I(c))$, contradiction. To see that μ is injective, assume $\mu(a) = \mu(b)$. Then, with signs interpreted in the obvious way, we have $\pm\mu(a) \in D_{\widehat{G}}(\mu(1), \pm\mu(b))$ and $\pm\mu(b) \in D_{\widehat{G}}(\mu(1), \pm\mu(a))$. Therefore, $\pm a \in D_G(1, \pm b)$ and $\pm b \in D_G(1, \pm a)$. In terms of the representation partial order $\leq_G$ (see I.6.2), we have $a \leq_G b$ and $b \leq_G a$. So $a = b$ (I.6.4 (a)).□

INCLUDE HERE REMARK ON MODEL-THEORETIC QUOTIENTS ?

II.4 RS-congruences and rings

To avoid repetitions we shall begin by proving a lemma providing fairly general sufficient conditions under which RS-quotients of two real semigroups of the form G_A, G_B (A, B semi-real rings) are isomorphic. Applying systematically this result we will then be able to obtain, in the case of RSs associated to rings, "concrete" realizations of the various types of quotients considered in section II.3. We begin by describing the setup for our result.

Preliminaries and Notation II.4.1 (1) Let A, B be semi-real rings and let $f : A \longrightarrow B$ be a ring homomorphism. $\mathrm{Sper}\, f : \mathrm{Sper}\,(B) \longrightarrow \mathrm{Sper}\,(A)$ denotes the real spectral morphism dual to f, given by $(\mathrm{Sper}\, f)(\beta) = f^{-1}[\beta]$, for $\beta \in \mathrm{Sper}\,(B)$. We remark in passing that $\mathrm{Sper}\, f$ is a continuous —in fact spectral— map; cf. [DST], § 24.4 for this and other properties (however, this fact is not used in the sequel).

The following equivalence will be frequently used below:

Fact II.4.2 *For $\alpha \in \mathrm{Sper}\,(A), \beta \in \mathrm{Sper}\,(B)$ and with f as above, the following are equivalent:*

(i) $\alpha = (\mathrm{Sper}\, f)(\beta)$;

(ii) For all $a \in A$, $\overline{a}(\alpha) = \overline{f(a)}(\beta)$.

Proof. (i) ⇒ (ii). Let $a \in A$. Using $\alpha = f^{-1}[\beta]$ the equality in (ii) is checked by cases according to the values of $\overline{a}(\alpha)$. It is clear that $\mathrm{supp}(\alpha) = f^{-1}[\mathrm{supp}(\beta)]$ and $\alpha \setminus (-\alpha) = f^{-1}[\beta \setminus (-\beta)]$. Then we have, e.g.:

$$\overline{a}(\alpha) = 1 \;\Leftrightarrow\; a \in \alpha \setminus (-\alpha) \;\Leftrightarrow\; f(a) \in \beta \setminus (-\beta) \;\Leftrightarrow\; \overline{f(a)}(\beta) = 1,$$

and similarly for the values $0, -1$.

(ii) ⇒ (i). For $a \in A$, $a \in \alpha \Leftrightarrow \overline{a}(\alpha) \geq 0 \Leftrightarrow \overline{f(a)}(\beta) \geq 0 \Leftrightarrow f(a) \in \beta$, i.e., $\alpha = f^{-1}[\beta] = (\mathrm{Sper}\, f)(\beta)$. □

The map $\overline{f} : G_A \longrightarrow G_B$ induced by f is defined by $\overline{f}(\overline{a}) := \overline{f(a)}$, for $a \in A$.

(2) We assume given sets $Y \subseteq \mathrm{Sper}\,(A)$, $X \subseteq \mathrm{Sper}\,(B)$. We denote by $\equiv_X$ (resp., $\equiv_Y$) the equivalence relation on G_B (resp., G_A) defined by X (resp., Y) as in clause $(\dagger)_{\mathcal{H}}$ of **??** (a), i.e., for $y, z \in B$,

$$\overline{y} \equiv_X \overline{z} \;:\Leftrightarrow\; \forall \beta \in X\, (\overline{y}(\beta) = \overline{z}(\beta)) \;\Leftrightarrow\; \overline{y} \restriction X = \overline{z} \restriction X.$$

Let D_X (resp., D_Y) denote the ternary relation on G_B (resp., G_A) defined as in clause $(\dagger\dagger)_{\mathcal{H}}$ of **??** (a), i.e., for $x, y, z \in B$, and with $\pi_X : G_B \longrightarrow G_B/\!\equiv_X$ canonical,

$$\pi_X(\overline{x}) \in D_X(\pi_X(\overline{y}), \pi_X(\overline{z})) \;:\Leftrightarrow\; \forall \beta \in X\, [\overline{x}(\beta) \in D^t_{\mathbf{3}}(\overline{y}(\beta), \overline{z}(\beta))]\,,$$

and similarly for D_Y.

(3) We shall make the following blanket assumptions:

Assumption I. $\equiv_X$ and $\equiv_Y$ define RS-congruences on G_B and G_A, respectively.

Assumption II. $Y = (\mathrm{Sper}\, f)[X]$.

Assumption III. For all $a \in A$ there is $b \in B$ such that $\overline{f(a)} \equiv_X \overline{b}$.

Remarks. (a) Assumption III amounts to saying that the map $\pi_X \circ \overline{f} : G_A \longrightarrow G_B/{\equiv_X}$ is surjective. It obviously holds if $\overline{f}$ is surjective —in particular, regardless of X, if f is surjective—, but below we will find examples where it holds even when $X = \mathrm{Sper}\,(B)$ and f is not surjective. All in all, Assumption III is a rather mild requirement.

(b) The map $\mathrm{Sper}\, f$ in Assumption II is not required to be injective but, in most examples below it turns out to be a homeomorphism between X and Y. $\square$

II.4.3 General Lemma *With notation as in II.4.1 and under Assumptions I – III, the RS-quotients $G_A/{\equiv_Y}$ and $G_B/{\equiv_X}$ are isomorphic.*

Proof. Assumption I guarantees that the quotient maps π_X, π_Y are RS-morphisms. Note they are surjective.

Claim 1. For $a, b \in A$, $\overline{a} \equiv_Y \overline{b} \Rightarrow (\pi_X \circ \overline{f})(\overline{a}) = (\pi_X \circ \overline{f})(\overline{b})$.

Proof of Claim 1. The implication of the statement amounts to $\overline{a} \equiv_Y \overline{b} \Rightarrow \overline{f(a)} \equiv_X \overline{f(b)}$. By Assumption II, $\beta \in X$ implies $\alpha = (\mathrm{Sper}\, f)(\beta) \in Y$. The hypothesis $\overline{a} \equiv_Y \overline{b}$ yields $\overline{a}(\alpha) = \overline{b}(\alpha)$, and II.4.2 (ii) entails $\overline{f(a)}(\beta) = \overline{f(b)}(\beta)$; since $\beta \in X$ is arbitrary, $\overline{f(a)} \equiv_X \overline{f(b)}$.

By the factoring condition II.2.1 (iii), $\pi_X \circ \overline{f}$ induces a RS-homomorphism $\widehat{\pi_X \circ \overline{f}} : G_A/{\equiv_Y} \longrightarrow G_B/{\equiv_X}$ such that $(\widehat{\pi_X \circ \overline{f}}) \circ \pi_Y = \pi_X \circ \overline{f}$.

Claim 2. The map $\widehat{\pi_X \circ \overline{f}}$ is injective.

Proof of Claim 2. We must show, for all $a, b \in A$,

$$(\widehat{\pi_X \circ \overline{f}})(\pi_Y(\overline{a})) = (\widehat{\pi_X \circ \overline{f}})(\pi_Y(\overline{b})) \;\Rightarrow\; \pi_Y(\overline{a}) = \pi_Y(\overline{b})\,;$$

this is equivalent to the converse of the implication in Claim 1, namely,

$$(\dagger) \qquad \overline{f(a)} \equiv_X \overline{f(b)} \;\Rightarrow\; \overline{a} \equiv_Y \overline{b}\,.$$

Let $\alpha \in Y$. By Assumption II there is $\beta \in X$ such that $\alpha = (\mathrm{Sper}\, f)(\beta)$. The antecedent of (†) gives $\overline{f(a)}(\beta) = \overline{f(b)}(\beta)$; from II.4.2 (ii) we conclude $\overline{a}(\alpha) = \overline{b}(\alpha)$, as required.

Finally we prove:

Claim 3. The map $\widehat{\pi_X \circ \overline{f}}$ reflects representation.

Proof of Claim . We must prove, for $a, b, c \in A$,

$$(\widehat{\pi_X \circ \overline{f}})(\pi_Y(\overline{a})) \in D_X((\widehat{\pi_X \circ \overline{f}})(\pi_Y(\overline{b})), (\widehat{\pi_X \circ \overline{f}})(\pi_Y(\overline{c})) \;\Rightarrow\; \pi_Y(\overline{a}) \in D_Y(\pi_Y(\overline{b}), \pi_Y(\overline{c}))\,.$$

Equivalently,

$$(\dagger\dagger) \qquad \pi_X(\overline{f(a)}) \in D_X(\pi_X(\overline{f(b)}), \pi_X(\overline{f(c)}) \;\Rightarrow\; \pi_Y(\overline{a}) \in D_Y(\pi_Y(\overline{b}), \pi_Y(\overline{c}))\,.$$

By definition of the relation D_Y we have

$$\pi_Y(\overline{a}) \in D_Y(\pi_Y(\overline{b}), \pi_Y(\overline{c})) \;\Leftrightarrow\; \forall\, \alpha \in Y\, [\, \overline{a}(\alpha) \in D^t_{\mathbf{3}}(\overline{b}(\alpha), \overline{c}(\alpha))\,],$$

and similarly for D_X.

To prove (††), let $\alpha \in Y$; Assumption II guarantees that there is a $\beta \in X$ such that $\alpha = (\mathrm{Sper}\, f)(\beta)$. From the antecedent of (††) comes $\overline{f(a)}(\beta) \in D^t_{\mathbf{3}}(\overline{f(b)}(\beta), \overline{f(c)}(\beta))$; by II.4.2 (ii) we get $\overline{a}(\alpha) \in D^t_{\mathbf{3}}(\overline{b}(\alpha), \overline{c}(\alpha))$. Since this holds for all $\alpha \in Y$, the conclusion in (††) follows, proving Claim 3.

Claims $1-3$ together show that $\widehat{\pi_X \circ \overline{f}}$ is the required RS-isomorphism between $G_A/{\equiv_Y}$ and $G_B/{\equiv_X}$. □

Remark. The role of Assumption I in the General Lemma is to ensure that the quotient $\mathcal{L}_{\mathrm{RS}}$- structures $(G_A/{\equiv_Y}, D_Y)\,, (G_B/{\equiv_X}, D_X)$ are real semigroups and verify the factoring condition II.2.1 (iii). In its absence, these quotient structures *may not* be RSs. However, since the quotient maps π_X and π_Y *are* $\mathcal{L}_{\mathrm{RS}}$*- morphisms* (verification of this, left to the reader, uses Assumption II), the proof of II.4.3 shows that, *even in the absence of Assumption I*, $(G_A/{\equiv_Y}, D_Y)$ and $(G_B/{\equiv_X}, D_X)$ are isomorphic as $\mathcal{L}_{\mathrm{RS}}$- structures. □

The sequel of this section is devoted to obtain explicit representations of the various quotient constructions considered in § II.3; the preceding General Lemma will be the main tool to get them.

A. Localizations.

Definition II.4.4 A subsemigroup T of a RS is called **proper** if $0 \notin T$. Likewise, a multiplicative subset S of a ring is called **proper** if $1 \in S$ and $0 \notin S$. □

Theorem II.4.5 *Let A be a semi-real ring and let H be a RS. The following are equivalent:*

(1) *H is a localization of G_A, i.e., $H = G_A/{\sim_T}$ for some proper subsemigroup T of G_A.*

(2) *There is a proper multiplicative subset S of A such that $H \simeq G_{S^{-1}A}$.*

Remark. $S^{-1}A$ denotes the ring of fractions of A by S. The requirement in (2) that $G_{S^{-1}A}$ be a RS forces the ring $S^{-1}A$ to be semi-real (Corollary I.4.13).

Preliminaries and Notation II.4.6 We fix a semi-real ring A and a proper multiplicative subset S of A.

(1) The following conditions are equivalent:

- (i) The ring $S^{-1}A$ is semi-real;
- (ii) $0 \notin \{\overline{s} \mid s \in S\}$ $(\subseteq G_A)$;
- (iii) $-S^2 \cap \sum A^2 = \emptyset$.

Proof. (ii) $\Leftrightarrow$ (iii). By Thm. 5.4.2 (1), pp. 93–94 of [M], for $s \in S$, condition $\overline{s} = 0$ is equivalent to $-s^{2k} \in \sum A^2$ for some integer $k \geq 0$. The case $k = 0$ is excluded since A is semi-real. Since S is multiplicative, $-s^{2k} \in \sum A^2$ with $k \geq 1$ is equivalent to $-S^2 \cap \sum A^2 \neq \emptyset$.

(i) $\Rightarrow$ (iii). If (iii) fails, $-s^2 = \sum a_i^2$ for some $s \in S$ and $a_i \in A$. Then, in $S^{-1}A$ we have $-1 = \frac{\sum a_i^2}{s^2} \in \sum (S^{-1}A)^2$.

(iii) $\Rightarrow$ (i). Assume $-1 = \sum (\frac{a_i}{s_i})^2$ with $a_i \in A\,, s_i \in S$. Chasing denominators and setting $s := \prod_i s_i \in S$ we have $-1 = \frac{\sum b_i^2}{s^2}$ with $b_i \in A$. Then, there is $s' \in S$ such that $s'(s^2 + \sum b_i^2) = 0$.

Scaling by s' and setting $t := s's \in S$, we get $-t^2 \in \sum A^2$, i.e., $-S^2 \cap \sum A^2 \neq \emptyset$. $\square$

(2) Let $\iota_S : A \longrightarrow S^{-1}A$ denote the canonical homomorphism $a \mapsto \frac{a}{1}$. Then,

(i) $\operatorname{Im}(\operatorname{Sper}\iota_S) = \{\alpha \in \operatorname{Sper}(A) \mid S \cap \operatorname{supp}(\alpha) = \emptyset\}$.

Further, for $\alpha \in \operatorname{Sper}(A)$ and $\beta \in \operatorname{Sper}(S^{-1}A)$,

(ii) $\alpha = (\operatorname{Sper}\iota_S)(\beta)\ (= \iota_S^{-1}[\beta]) \Leftrightarrow \beta = \{\frac{a}{s^2} \mid a \in \alpha \text{ and } s \in S\} = S^{-2}\alpha$.

In particular,

(iii) $\operatorname{Sper}\iota_S$ is injective.

(iv) $\operatorname{Sper}\iota_S$ is a homeomorphism of $\operatorname{Sper}(S^{-1}A)$ onto $\operatorname{Im}(\operatorname{Sper}\iota_S)$. $\square$

For a proof of these results we refer the reader to [DST], Prop. 23.4.19, or [DM6], Appendix C, § 3.

(3) Recall (I.5.5) that for a semi-real ring R and $\gamma \in \operatorname{Sper}(R)$, $h_\gamma := \operatorname{sgn}_\gamma \circ \pi_\gamma$ is the RS-character of G_R induced by γ, and that $Z(h_\gamma) = \{\overline{x} \mid x \in \operatorname{supp}(\gamma)\}$, $P(h_\gamma) = \{\overline{x} \mid x \in \gamma\}$.

(4) For $a \in A$, $s \in S$, in $G_{S^{-1}A}$ we have:

(i) $\overline{\iota_S}(\overline{s^2}) = 1$; (ii) $\overline{\left(\frac{a}{s}\right)} = \overline{\iota_S}(\overline{as})$.

In particular,

(iii) $\overline{\iota_S}$ is surjective.

Proof. (i) For $x \in a$ we have $\frac{x^2}{1} = (\frac{x}{1})^2 \in \beta$; hence $\overline{\iota_S}(\overline{x}^2)(\beta) = \overline{\left(\frac{x^2}{1}\right)}(\beta) \geq 0$ for all $\beta \in \operatorname{Sper}(S^{-1}A)$. Since in the ring $S^{-1}A$ for $s \in S$ we have $(\frac{s^2}{1})(\frac{1}{s})^2 = 1$, we obtain $\overline{\iota_S}(\overline{s^2}) \cdot \overline{\iota_S}\left(\overline{\left(\frac{1}{s}\right)}^2\right) = 1$, and hence $\overline{\iota_S}(\overline{s^2}) = 1$.

(ii) In $S^{-1}A$: $\frac{as}{1} = \frac{a}{s} \cdot (\frac{s}{1})^2$. By (i), $\overline{\iota_S}(\overline{as}) = \overline{\left(\frac{a}{s}\right)} \cdot \overline{\iota_S}(\overline{s^2}) = \overline{\left(\frac{a}{s}\right)}$. $\square$

The crux of the proof of Theorem II.4.5 is contained in the following

Proposition II.4.7 *Let A be a semi-real ring, S a proper multiplicative subset of A and T a proper subsemigroup of G_A. If $T = \{\overline{s} \mid s \in S\}$, then $G_{S^{-1}A}$ is isomorphic to $G_A/{\sim_T}$. In particular, $G_{S^{-1}A}$ is a real semigroup and a RS-quotient of G_A.*

Proof. This will follow from the General Lemma II.4.3 applied with $B = S^{-1}A$, $f = \iota_S$ (= the canonical homomorphism $a \mapsto \frac{a}{1}$), $X = \operatorname{Sper}(B)$ and $Y = \{\alpha \in \operatorname{Sper}(A) \mid S \cap \operatorname{supp}(\alpha) = \emptyset\}$. So, we have to check the validity of Assumptions I – III in II.4.3 for this choice of parameters.

<u>Assumption I.</u> With $X = \operatorname{Sper}(B)$ the equivalence relation $\equiv_X$ is just equality in G_B, and so it certainly defines a RS-quotient, namely G_B itself.

As to the equivalence relation $\equiv_Y$, we prove next that (under the identification $\alpha \leftrightarrow h_\alpha$ of I.5.5), $Y = U(T^2) = \{h \in X_{G_A} \mid h \lceil T^2 = 1\}$; hence $\equiv_Y$ equals $\sim_T$, already proved to be a RS-congruence in II.3.2 (3,4).

Claim. $U(T^2) = \{h_\alpha \mid \alpha \in Y\}$.

<u>Proof of Claim.</u> We must show $\alpha \in Y \Leftrightarrow h_\alpha \lceil T^2 = 1$ for $\alpha \in \operatorname{Sper}(A)$. By II.4.6 (2.(i)) this is equivalent to

(†) $$S \cap \operatorname{supp}(\alpha) = \emptyset \;\Leftrightarrow\; Z(h_\alpha) \cap T = \emptyset.$$

Since $Z(h_\alpha) = \{\overline{a} \mid a \in \operatorname{supp}(\alpha)\}$ (II.4.6 (3)) and (by assumption) $T = \{\overline{s} \mid s \in S\}$, we have, for $a \in A$:

$$\overline{a} \in Z(h_\alpha) \cap T \;\Leftrightarrow\; a \in \operatorname{supp}(\alpha) \;\text{ and }\; \overline{a} = \overline{s} \;\text{ for some } s \in S,$$

proving implication ($\Rightarrow$) in (†).

For the converse we invoke Corollary 5.4.3 in [M], p. 94, which shows:

$$\overline{a} = \overline{s} \;\Leftrightarrow\; \exists\, k \geq 0\; \exists\, x, y \in \textstyle\sum A^2 \text{ such that } xas = (a^2 + s^2)^k + y\,.$$

If $a \in \operatorname{supp}(\alpha)$, then $(a^2+s^2)^k + y \in \operatorname{supp}(\alpha)$ and (by the binomial formula), $s^{2k} + y \in \operatorname{supp}(\alpha)$. Since $\operatorname{supp}(\alpha)$ is a real prime ideal, we conclude $s \in S \cap \operatorname{supp}(\alpha)$, which proves ($\Leftarrow$).

Assumption II. As indicated in II.4.6 (2), $\operatorname{Sper} \iota_S$ is a bijection of $X = \operatorname{Sper}(B)$ onto $Y = \operatorname{Im}(\operatorname{Sper} \iota_S)$.

Assumption III. Since $\overline{\iota_S}$ is surjective, see II.4.6 (4.iii), this assumption holds. □

Proof of Theorem II.4.5. (1) $\Rightarrow$ (2). Given a proper subsemigroup T of G_A, set

$$S := A \setminus \bigcup \{\operatorname{supp}(\alpha) \mid \alpha \in \operatorname{Sper}(A) \text{ and } h_\alpha \lceil T^2 = 1\}\,.$$

Clearly, S is a proper multiplicative subset of A. Item (2) follows from Proposition II.4.7 upon proving

— $T = \{\overline{s} \mid s \in S\}$.

For the inclusion $\subseteq$, let $t \in T$. Since $t \in G_A$, $t = \overline{x}$ for some $x \in A$. Suppose $x \notin S$; then, there is $\alpha \in \operatorname{Sper}(A)$ such that $h_\alpha \lceil T^2 = 1$ and $x \in \operatorname{supp}(\alpha)$. Since $Z(h_\alpha) = \{\overline{a} \mid a \in \operatorname{supp}(\alpha)\}$, we have $h_\alpha(\overline{x}) = 0$. On the other hand, $t = \overline{x} \in T$ implies $h_\alpha(\overline{x}) \neq 0$, contradiction.

The reverse inclusion requires a finer touch, employing Lemma I.4.10. Assume, towards a contradiction, that there is $s_0 \in S$ such that $\overline{s_0} \notin T$. Firstly, we construct the saturated subsemigroup Δ of G_A generated by the subsemigroup $\{\overline{s}^2 \mid s \in S\}$, see Proposition I.4.6 (2). Note that $s \in S \Rightarrow -s \in S$. Then, $\overline{s}^2 \in \Delta \cap -\Delta$ for all $s \in S$. We apply Lemma I.4.10 with $I = \{0\}$ and the given subsemigroup T of G_A; with notation therein, we must check:

Claim. $I[\Delta] \cap T = \emptyset$.

Proof of Claim. Suppose $t \in I[\Delta]$ for some $t \in T$. By the definition of $I[\Delta]$, there is $d \in \Delta$ such that $-t^2 \in D_{G_A}(0, d)$; hence, $-t^2 \in D^t_{G_A}(0, t^2 d)$, and $0 \in D^t_{G_A}(t^2, t^2 d)$. By I.2.3 (11), $-t^2 = t^2 d$, which implies $-1 \sim_T d$, i.e., $\pi(-1) = -1 = \pi(d)$ (in $G_A/\!\sim_T$).

On the other hand, since $d \in \Delta$, there is a form φ with coefficients in $\{\overline{s}^2 \mid s \in S\}$ such that $d \in D_{G_A}(\varphi)$ (I.4.6 (2)), i.e., $d \in D_{G_A}(\langle \overline{s_1}^2, \ldots, \overline{s_n}^2 \rangle)$ with $s_i \in S$. Since π is a RS-morphism, $-1 = \pi(d) \in D_{G_A/\sim_T}(\langle \pi(\overline{s_1}^2), \ldots, \pi(\overline{s_n}^2) \rangle)$, which yields $-1 \in D^t_{G_A/\sim_T}(\langle \pi(\overline{s_1}^2), \ldots, \pi(\overline{s_n}^2) \rangle)$.

Since $X_{G_A/\sim_T} = U(T^2)$ (II.3.3 (i)), for every $h \in U(T^2)$ we have $-1 \in D^t_{\mathbf{3}}(\langle h(\pi(\overline{s_1}^2)), \ldots, h(\pi(\overline{s_n}^2)) \rangle)$. As $h(\pi(\overline{x}^2)) \geq 0$, the right-hand side of this transversal representation is $\{1\}$ (I.2.5), contradiction. This proves the Claim.

Lemma I.4.10 gives a character $h \in X_{G_A}$ such that $Z(h) \supseteq \Delta \cap -\Delta$ and $h \lceil T^2 = 1$. From $s_0 \in S$ follows $\overline{s_0}^2 \in \Delta \cap -\Delta$, whence $h(\overline{s_0}) = 0$. Since $h = h_\alpha$ for some $\alpha \in \operatorname{Sper}(A)$, we have $s_0 \in \operatorname{supp}(\alpha)$ and $h_\alpha \lceil T^2 = 1$, contradicting $s_0 \in S$, and proving (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Given a proper multiplicative subset S of A such that $G_{S^{-1}A} \models$RS (i.e., the ring $S^{-1}A$ is semi-real), set $T := \{\overline{s} \mid s \in S\}$. Item (1) in II.4.6 shows that $0 \notin T$, i.e., T is a proper subsemigroup of G_A; (1) follows, then, from Proposition II.4.7. $\square$

Warning. In the category of rings, localizations are not quotients: the canonical map ι_S : $A \longrightarrow S^{-1}A$ is not epimorphic (i.e., surjective). However, the preceding results show that in the category of real semigroups, $G_{S^{-1}A}$ is a (RS)-quotient of G_A. $\square$

B. Quotients modulo saturated sets.

Proposition II.4.8 *Let A be a semi-real ring and let $\mathcal{H} \subseteq X_{G_A}$ be a non-empty set of characters of G_A. The following are equivalent:*

(1) *$\mathcal{H}$ is a saturated set, i.e., there are a saturated subsemigroup $\Delta \subseteq G_A$ and a proper subsemigroup $P \subseteq G_A$ such that $\mathcal{H} = \mathcal{H}^P_\Delta$ (cf. § II.3 (B)).*

(2) *There is a preorder T of A and a proper multiplicative set $S \subseteq A$ such that, with notation as in I.5.5, $\mathcal{H} = \{h_\alpha \mid \alpha \in \mathrm{Sper}\,(A), T \subseteq \alpha$ and $S \cap \mathrm{supp}(\alpha) = \emptyset\}$.*

Proof. We first remark:

(a) Δ is a (proper) saturated subsemigroup of G_A iff $\{a \mid \overline{a} \in \Delta\}$ is a (proper) preorder of A;

(b) P is a (proper) subsemigroup of G_A iff $\{a \mid \overline{a} \in P\} \subseteq A$ is a (proper) multiplicative set.

The proof of (a) and (b) is straightforward. [For (a) note that $-1 \notin \{a \mid \overline{a} \in \Delta\}$; otherwise $\overline{-1} \in \Delta$ and, by I.2.3 (9), $G_A = D_{G_A}(1, -1) \subseteq \Delta$, i.e., Δ is improper.]

(1) $\Rightarrow$ (2). Recall (I.5.5) that every $h \in X_{G_A}$ is of the form h_α for a unique $\alpha \in \mathrm{Sper}\,(A)$. Set $T := \{a \mid \overline{a} \in \Delta\}$ and $S := \{a \mid \overline{a} \in P\}$.

Assuming $h \in \mathcal{H}^P_\Delta$, i.e., $\Delta \subseteq P(h_\alpha)$ and $Z(h_\alpha) \cap P = \emptyset$, it is easily checked that:

(i) $T \subseteq \alpha$, and (ii) $S \cap \mathrm{supp}(\alpha) = \emptyset$.

For (i): Let $a \in T$, i.e., $\overline{a} \in \Delta$; then, $h_\alpha(\overline{a}) = \mathrm{sgn}_\alpha(\pi_\alpha(a)) \geq 0$, i.e., $\overline{a}(\alpha) \geq 0$; by the definition of $\overline{a}$ this means $a \in \alpha$.

For (ii): If $a \in S \cap \mathrm{supp}(\alpha)$, then $\overline{a} \in P$ and $h_\alpha(\overline{a}) = 0$, whence, $\overline{a} \in Z(h_\alpha) \cap P \neq \emptyset$.

We conclude that $\mathcal{H}^P_\Delta \subseteq \{h_\alpha \mid \alpha \in \mathrm{Sper}\,(A), T \subseteq \alpha$ and $S \cap \mathrm{supp}(\alpha) = \emptyset\}$.

For the reverse inclusion, let h_α be in the right-hand side, and prove $\Delta \subseteq P(h_\alpha)$ and $Z(h_\alpha) \cap P = \emptyset$:

— The first condition follows from $T \subseteq \alpha$: Let $\overline{a} \in \Delta$; then, $a \in T$; by assumption, $a \in \alpha$; this gives $h_\alpha(\overline{a}) \geq 0$, i.e., $\overline{a} \in P(h_\alpha)$.

— If there is $\overline{a} \in Z(h_\alpha) \cap P$, then $a \in \mathrm{supp}(\alpha)$ and $a \in S$, whence $S \cap \mathrm{supp}(\alpha) \neq \emptyset$, contradiction.

(2) $\Rightarrow$ (1). Given $T, S \subseteq A$ as in (2), set $\Delta := \{\overline{a} \mid a \in T\}$ and $P := \{\overline{a} \mid a \in S\}$. Assuming $\{\alpha \in \mathrm{Sper}\,(A) \mid T \subseteq \alpha$ and $S \cap \mathrm{supp}(\alpha) = \emptyset\} \neq \emptyset$, we must show:

(*i′*) Δ is a (proper) saturated subsemigroup of G_A;

(*ii′*) P is a (proper) subsemigroup of G_A;

(*iii′*) $\mathcal{H}^P_\Delta = \{h_\alpha \mid \alpha \in \mathrm{Sper}\,(A)\,, T \subseteq \alpha$ and $S \cap \mathrm{supp}(\alpha) = \emptyset\}$.

Proof of (i′). Clearly, Δ is closed under product and contains $\overline{0}, \overline{1}$ (as T does).

— To prove saturatedness, let $\overline{x} \in D_{G_A}(\overline{a}, \overline{b})$, with $a, b \in T$ and $x \in A$. By [M], Prop. 5.5.1 (5) applied with G_A, there are $t_0, t_1, t_2 \in \sum A^2$ such that $t_0 x = t_1 a + t_2 b$ and $\overline{t_0 x} = \overline{x}$. Since $\sum A^2 \subseteq T$, we get $t_0 x \in T$, whence $\overline{x} = \overline{t_0 x} \in \Delta$.

— To show that Δ is proper, suppose $\overline{-1} \in \Delta$; then, $\overline{-1} = \overline{a}$ for some $a \in T$, and hence $\overline{a}(\alpha) = -1$ for all $\alpha \in \mathrm{Sper}\,(A)$. By assumption there is $\alpha_0 \in \mathrm{Sper}\,(A)$ so that $T \subseteq \alpha_0$ and $S \cap \mathrm{supp}(\alpha_0) = \emptyset$. This yields $a \in \alpha_0$, i.e., $\overline{a}(\alpha_0) \geq 0$, contradiction.

Proof of (ii′). Obviously, P is a subsemigroup of G_A. Suppose $\overline{0} \in P$, i.e., $\overline{0} = \overline{a}$ for some $a \in S$, that is, $a \in \mathrm{supp}(\alpha)$ for all $\alpha \in \mathrm{Sper}\,(A)$. With $\alpha = \alpha_0$ as above, we get $a \in S \cap \mathrm{supp}(\alpha_0)$, contradiction.

Proof of (iii′). The equality to be proved amounts to showing, for $\alpha \in \mathrm{Sper}\,(A)$,

$$\Delta \subseteq P(h_\alpha) \Leftrightarrow T \subseteq \alpha \quad \text{and} \quad Z(h_\alpha) \cap P = \emptyset \Leftrightarrow S \cap \mathrm{supp}(\alpha) = \emptyset.$$

The first of these equivalences is straightforward checking, while the second is proved exactly as the equivalence (†) in the proof of II.4.7. □

Preliminaries and Notation II.4.9 Let S be a multiplicative subset and T a preorder, of A. Then:

(1) $S^{-2}T$ is a preorder of $S^{-1}A$.

(2) $S^{-2}T$ is proper $\Leftrightarrow -S^2 \cap T = \emptyset \Leftrightarrow S \cap T \cap -T = \emptyset$.

(3) If $S^{-2}T$ is a proper preorder of $S^{-1}A$, then $(\mathrm{Sper}\,\iota_S) \lceil \mathrm{Sper}\,(S^{-1}A, S^{-2}T)$ is a homeomorphism of $\mathrm{Sper}\,(S^{-1}A, S^{-2}T)$ onto $\{\alpha \in \mathrm{Sper}\,(A) \,|\, T \subseteq \alpha \text{ and } S \cap \mathrm{supp}(\alpha) = \emptyset\}$.

Proof. (1) is straightforward and left to the reader.

(2) $\frac{-1}{1} \in S^{-2}T \Leftrightarrow$ There are $t \in T$ and $s \in S$ such that $\frac{-1}{1} = \frac{t}{s^2} \Leftrightarrow$ There is $s' \in S$ such that $s'(s^2 + t) = 0$.

Multiplying the last equality by s', we get $-(s's)^2 = s'^2 t \in -S^2 \cap T$. Conversely, if $-s^2 \in T$ for some $s \in S$, we get $\frac{-1}{1} = \frac{-s^2}{s^2} \in S^{-2}T$.

The second equivalence is routine.

(3) We already know (II.4.6 (2.iv)) that $\mathrm{Sper}\,\iota_S$ is a homeomorphism of $\mathrm{Sper}\,(S^{-1}A)$ onto $\mathrm{Im}(\mathrm{Sper}\,\iota_S) = \{\alpha \in \mathrm{Sper}\,(A) \,|\, S \cap \mathrm{supp}(\alpha) = \emptyset\}$. It suffices to show, for $\alpha = \mathrm{Sper}\,\iota_S(\beta)$,

$$S^{-2}T \subseteq \beta \;\Leftrightarrow\; T \subseteq \alpha.$$

($\Rightarrow$) Let $t \in T$; then $\frac{t}{1} \in S^{-2}T$, whence $\frac{t}{1} \in \beta$, which means $t \in \iota_S^{-1}[\beta] = \alpha$.

($\Leftarrow$) Let $\frac{t}{s^2} \in S^{-2}T$ $(t \in T, s \in S)$. By assumption $t \in \alpha$, which (by II.4.6 (2.ii)) implies $\frac{t}{s^2} \in S^{-2}\alpha = \beta$. □

Our next result shows that quotients modulo saturated sets of the real semigroup associated to a ring are of the form $G_{R,Q}$ for a suitable preorder Q of a ring R. This is obtained, again, by application of the General Lemma II.4.3.

Theorem II.4.10 *Let A be a ring, $T \subseteq A$ a proper preorder, $S \subseteq A$ a proper multiplicative set such that $-S^2 \cap T = \emptyset$, $\Delta \subseteq G_A$ a saturated subsemigroup, and $P \subseteq G_A$ a proper subsemigroup. Assume $P = \{\overline{a} \,|\, a \in S\}$ and $\Delta = \{\overline{a} \,|\, a \in T\}$. Then, $G_{S^{-1}A,\, S^{-2}T}$ is isomorphic to $G_A/\mathcal{H}^P_\Delta$.*

Proof. In this case the General Lemma II.4.3 will be applied with $B = S^{-1}A$, $f = \iota_S$ (the canonical homomorphism $a \mapsto \frac{a}{1}$), $X = \mathrm{Sper}\,(B, S^{-2}T) = \{\beta \in \mathrm{Sper}\,(B) \,|\, S^{-2}T \subseteq \beta\}$, and $Y = \{\alpha \in \mathrm{Sper}\,(A) \,|\, T \subseteq \alpha \text{ and } S \cap \mathrm{supp}(\alpha) = \emptyset\}$. Again, we will have to check the validity of Assumptions I – III in II.4.3 for this choice of parameters.

To ease notation we set $Q := S^{-2}T$.

Assumption I. The equivalence relation $\equiv_X$ is

$$\overline{y} \equiv_X \overline{z} \;\Leftrightarrow\; \overline{y_Q} = \overline{z_Q} \qquad\qquad (y, z \in B).$$

So, $G_B/{\equiv_X}$ is $G_{B,Q}$; this has been proved to be RS-quotient of G_B in Example II.2.2.

Proposition II.4.8 shows that, with $P = \{\overline{s} \,|\, s \in S\}$ and $\Delta = \{\overline{t} \,|\, t \in T\}$ we have $\mathcal{H} = \mathcal{H}^P_\Delta = \{h_\alpha \,|\, \alpha \in Y\}$. It follows that $\equiv_Y$ is identical to $\sim_{\Delta,P}$ and this was proved to be a RS-congruence in Theorem II.3.5 (d).

Assumption II. We must show that $(\mathrm{Sper}\,\iota_S)[X] = Y$. Fix $\alpha \in \mathrm{Sper}\,(A)$, $\beta \in \mathrm{Sper}\,(B)$ with $\alpha = (\mathrm{Sper}\,\iota_S)(\beta)$. By II.4.6 (2.ii), $\beta = S^{-2}\alpha$. This implies:

(a) $S^{-2}T \subseteq \beta \;\Leftrightarrow\; T \subseteq \alpha$.

Indeed, if $t \in T$, then $\iota_S(t) = \frac{t}{1} \in S^{-2}T \subseteq \beta$, whence $t \in \iota_S^{-1}[\beta] = \alpha$. Conversely, if $t \in T, s \in S$, then $t \in \alpha$, and $\frac{t}{s^2} \in S^{-2}\alpha = \beta$.

Since the elements of S are invertible in B, we also have:

(b) $S \cap \mathrm{supp}(\alpha) = \emptyset$.

If $s \in S$, by II.4.6 (4.i), $\overline{\iota_S}(\overline{s^2}) = 1$, whence $\overline{\iota_S(s)}(\beta) = \overline{\iota_S}(\overline{s})(\beta) \neq 0$; by II.4.2, $\overline{s}(\alpha) \neq 0$, which gives $s \notin \mathrm{supp}(\alpha)$.

Item (a) gives: $\alpha \in Y \Rightarrow T \subseteq \alpha \Rightarrow S^{-2}T \subseteq \beta \Rightarrow \beta \in X$. Conversely, if $\beta \in X$, from (a) comes $T \subseteq \alpha$, which, together with item (b), yields $\alpha \in Y$, as required to prove Assumption II.

Assumption III. As in the case of localizations, Assumption III holds because $\overline{\iota_S}$ is surjective (II.4.6 (4.iii)). □

C. Quotients modulo saturated subsemigroups (subspaces). Let now be given a (proper) saturated subsemigroup Δ of G_A, A a ring (necessarily semi-real). We already know (II.3.C) that the set of characters

$$\mathcal{H}_\Delta = \{h \in X_{G_A} \,|\, \Delta \subseteq P(h)\} = \{\alpha \in \mathrm{Sper}\,(A) \,|\, \Delta \subseteq P(h_\alpha)\}$$

defines a RS-congruence, $\sim_\Delta$, of G_A, i.e., $G_A/\mathcal{H}_\Delta (= G_A/\Delta)$ is a RS-quotient of G_A. (Note that Δ proper implies $\mathcal{H}_\Delta \neq \emptyset$: use Corollary I.4.11 with $T = \{1\}$ and $a = -1$.)

Let $T := \{a \in A \,|\, \overline{a} \in \Delta\}$; this is a proper preorder of A. Hence, the real semigroup $G_{A,T}$ is another RS-quotient of G_A, given by the RS-congruence

$$\overline{a} \equiv_T \overline{b} \;\Leftrightarrow\; \overline{a_T} = \overline{b_T} \quad (a, b \in A),$$

(cf. Example II.2.2). We get:

Proposition II.4.11 *With notation as above,* $G_{A,T} \simeq G_A/\Delta$.

Proof. This is a particular case of Theorem II.4.10, with $S = \{1\}$ —hence $P = \{\overline{1}\}$—, and Δ the given saturated subsemigroup of G_A. Then, $G_{S^{-1}A,\,S^{-2}T} = G_{A,T}$ is isomorphic to $G_A/\mathcal{H}^P_\Delta$.

Since $\mathcal{H}^P_\Delta = \mathcal{H}_\Delta$, then $G_A/\mathcal{H}^P_\Delta = G_A/\mathcal{H}_\Delta = G_A/\Delta$, and the result follows. $\square$

D. Quotients modulo transversally saturated subsemigroups. From Proposition II.3.10 and Theorem II.4.10 we obtain:

Proposition II.4.12 *Let A be a ring, let Γ be a non-trivial tranversally saturated subsemigroup of G_A with $\mathcal{H}_\Gamma = \{h \in X_G \,|\, \Gamma \subseteq h^{-1}[1]\}$, and let Δ be the saturated subsemigroup of G_A generated by Γ. Let $S := \{a \in A \,|\, \overline{a} \in \Gamma\}$ and $T := \{a \in A \,|\, \overline{a} \in \Delta\}$. Then, S is a proper multiplicative subset of A, T is a proper preorder of A, and the RS-quotient $G_A/\mathcal{H}_\Gamma$ is isomorphic to $G_{S^{-1}A,\, S^{-2}T}$.*

Proof. The assertions about S, T are straightforward. Further, we have $\Gamma = \{\overline{a} \in A \,|\, a \in S\}$ and $\Delta = \{\overline{a} \in A \,|\, a \in T\}$.

Note that $-S^2 \cap T = \emptyset$ (equivalently, $S \cap T \cap -T = \emptyset$). Otherwise, let $s \in S$ be such that $\pm\, s \in T$; then, $\overline{s} \in \Gamma$ and $\pm\, \overline{s} \in \Delta$. Picking $h \in \mathcal{H}_\Gamma$, we have $h(\overline{s}) = 1$ and $h(-\overline{s}) \geq 0$, i.e., $h(\overline{s}) \leq 0$, contradiction.

Since $\mathcal{H}_\Gamma = \mathcal{H}^\Gamma_\Delta$ (II.3.10), we conclude from Theorem II.4.10 that $G_A/\mathcal{H}^\Gamma_\Delta = G_A/\mathcal{H}_\Gamma$ is isomorphic to $G_{S^{-1}A,\, S^{-2}T}$. $\square$

Remark. One may wonder whether the preorder T has an explicit characterization in terms of S (i.e., of Γ). We have:

Fact. *For $a \in A$, $a \in T \Leftrightarrow$ There is $t_0 \in \sum A^2$ such that $\overline{t_0 a} = \overline{a}$ and $t_0 a \in \sum A^2 S$.*

Proof. Since Δ is the saturated subsemigroup generated by Γ,

$$\overline{a} \in \Delta \Leftrightarrow \text{There is a form } \varphi \text{ over } \Gamma \text{ such that } \overline{a} \in D_{G_A}(\varphi).$$

(Cf. I.4.6 (2).) Say $\varphi = \langle \overline{g_1}, \ldots, \overline{g_n} \rangle$, with $\overline{g_i} \in \Gamma$. By [M], Prop. 5.5.1(5), $\overline{a} \in D_{G_A}(\langle \overline{g_1}, \ldots, \overline{g_n} \rangle)$ holds if and only if there are $t_0, \ldots, t_n \in \sum A^2$ such that $\overline{t_0 a} = \overline{a}$ and $t_0 a = \sum_{i=1}^n t_i g_i \in \sum A^2 S$.

Conversely, from $t_0 a \in \sum A^2 S$ and $\overline{t_0 a} = \overline{a}$, say $t_0 a = \sum_{i=1}^n a_i^2 g_i$ with $g_i \in S$ (i.e., $\overline{g_i} \in \Gamma$), we get $\overline{a} = \overline{t_0 a} \in D_{G_A}(\langle \overline{g_1}, \ldots, \overline{g_n} \rangle) \subseteq \Delta$. $\square$

Note that $\overline{t_0 a} = \overline{a}$ means $\forall \alpha \in \mathrm{Sper}\,(A) [\, t_0 \in \mathrm{supp}(\alpha) \Rightarrow a \in \mathrm{supp}(\alpha)\,]$.

E. Quotients modulo saturated ideals. We shall now analyze the type of quotients considered in II.3.E, in the case of real semigroups associated to a ring.

Let us assume that A is a ring and I is a real ideal of A (see II.1.15); under this assumption the quotient ring A/I is semi-real, and so is A. Hence, endowed with the representation relation defined in [M], Prop. 5.5.1 (5), both G_A and $G_{A/I}$ are real semigroups. The set $\overline{I} = \{\overline{a} \,|\, a \in I\}$ (see II.1.A) is a saturated ideal of G_A which, as remarked in II.1.7, is proper. The equivalence relation $\approx_{\overline{I}}$ introduced in II.3.E is a RS-congruence (II.3.13) and hence $G_A/{\approx_{\overline{I}}}$ is a RS-quotient of G_A. Under the identification $\alpha \leftrightarrow h_\alpha$, see I.5.5, the family of characters defining $\approx_{\overline{I}}$ is $\mathcal{H}^{\overline{I}} = \{h_\alpha \,|\, \alpha \in \mathrm{Sper}\,(A) \text{ and } I \subseteq \mathrm{supp}(\alpha)\}$.

We prove:

Proposition II.4.13 *Let A be a ring and let I be a real ideal of A. Then, the real semigroup $G_A/{\approx_{\overline{I}}}$ is isomorphic to $G_{A/I}$.*

Proof. We will apply the General Lemma II.4.3 with $B = A/I$, $f = \pi : A \longrightarrow B$ the canonical quotient map, $X = \mathrm{Sper}\,(B)$ and $Y = \{\alpha \in \mathrm{Sper}\,(A) \,|\, I \subseteq \mathrm{supp}(\alpha)\}$. Assumptions I – III of the Lemma are verified, as follows:

Assumption I. The relation $\equiv_X$ is identity on G_B; then it certainly gives a RS-congruence.

As remarked before the statement, $\mathcal{H}^{\overline{I}} = \{h_\alpha \mid \alpha \in Y\}$; hence, the equivalence relations $\equiv_Y$ and $\approx_{\overline{I}}$ on G_A are identical and, after II.3.13, the latter defines a RS-congruence.

Assumption II. Since $X = \mathrm{Sper}\,(B)$, $(\mathrm{Sper}\,\pi)[X] = \mathrm{Im}(\mathrm{Sper}\,\pi)$. We check that $\mathrm{Im}(\mathrm{Sper}\,\pi) = Y$.

Let $\alpha = (\mathrm{Sper}\,\pi)(\beta)$, $\beta \in \mathrm{Sper}\,(B)$, and let $i \in I$; then, $\pi(i) = 0 \in \mathrm{supp}(\beta)$, i.e., $i \in \pi^{-1}[\mathrm{supp}(\beta)] = \mathrm{supp}(\alpha)$, which proves the inclusion $\subseteq$.

Conversely, given $\alpha \in Y$, i.e., $I \subseteq \mathrm{supp}(\alpha)$, let $\beta := \pi[\alpha]$. The fact that π *is surjective* entails that $\beta \in \mathrm{Sper}\,(B)$ and $\alpha = \pi^{-1}[\beta] = (\mathrm{Sper}\,\pi)(\beta)$ as required. The only non-routine point in proving $\beta \in \mathrm{Sper}\,(B)$ is to show that, for $x, y \in B$, $xy \in \beta \Rightarrow x \in \beta$ or $-y \in \beta$. Let $a, b \in A$ and $c \in \alpha$ be such that $x = \pi(a), y = \pi(b)$ and $xy = \pi(c)$; then $ab - c \in I \subseteq \mathrm{supp}(\alpha)$, whence $ab \in \alpha$. Since $\alpha \in \mathrm{Sper}\,(A)$ it follows $a \in \alpha$ or $-b \in \alpha$, whence $x \in \beta$ or $-y \in \beta$.

Assumption III. Since f is surjective, so is $\overline{f}$, and this assumption holds automatically. □

An analog of the preceding Proposition holds, as well, for the real semigroups $G_{A,T}$, T a preorder of A; namely,

Proposition II.4.14 *Let A be a ring, T be a (proper) preorder of A, and I be a T-compatible ideal of A (cf. Definition II.1.2 (1)). Then the set $T/I = \{t/I \mid t \in T\}$ is a proper preorder of A/I, and the equivalence relation on $G_{A,T}$ given by*

$$\overline{a_T} \equiv_{T,I} \overline{b_T} \;:\Leftrightarrow\; \overline{(a/I)_{T/I}} = \overline{(b/I)_{T/I}} \quad (a, b \in A),$$

determines a RS-congruence under which the real semigroups $G_{A,T}/\equiv_{T,I}$ and $G_{A/I\,,T/I}$ are isomorphic. □

The proof is similar to that of Proposition II.4.13 (though notationally more involved); it is left as an exercise to the reader. [Remark that the set of characters of $G_{A,T}$ defining the equivalence relation $\equiv_{T,I}$ is $\{\alpha \in \mathrm{Sper}\,(A) \mid I \subseteq \mathrm{supp}(\alpha) \text{ and } T/I \subseteq \alpha/I\}$.]

F. Residue spaces at saturated prime ideals. Next we give a representation result for residue spaces of RSs of type G_A, A a ring (necessarily semi-real), at a saturated prime ideal I. Quotients of this form, for arbitrary RSs, were studied in II.3.F.

The residue space at I is defined by an equivalence relation $\sim_I$ determined, in turn, by the set of characters $\mathcal{H}_I = \{h \in X_{G_A} \mid Z(h) = I\}$. Under the identification $\alpha \leftrightarrow h_\alpha$ of I.5.5, $\mathcal{H}_I$ corresponds to the set $\{\alpha \in \mathrm{Sper}\,(A) \mid \mathrm{supp}(\alpha) = \widehat{I}\}$, where $\widehat{I} = \{a \in A \mid \overline{a} \in I\}$, a *prime* ideal of A (see II.1.A (2) and II.1.1 (iv)).

Setting $S = A \setminus \widehat{I}$ the set above can be written as

$$\{\alpha \in \mathrm{Sper}\,(A) \mid \widehat{I} \subseteq \mathrm{supp}(\alpha)\} \cap \{\alpha \in \mathrm{Sper}\,(A) \mid S \cap \mathrm{supp}(\alpha) = \emptyset\},$$

which can be thought of as a combination of a localization (at S) with a quotient at the saturated ideal $\widehat{I}$ (as in paragraph C). The ring of fractions of A by S is the (ring-theoretic) localization at the prime ideal $\widehat{I}$ —usually denoted by $A_{\widehat{I}}$—, a local ring with maximal ideal $M = \widehat{I} \cdot A_{\widehat{I}}$. With this notation in hand, our representation result reads as follows:

Theorem II.4.15 *Let A be a ring (necessarily semi-real) and let I be a saturated prime ideal* *of G_A. With notation as above, the residue space G_A/I is isomorphic to the real semigroup G_B, where B is the field $A_{\widehat{I}}/M$. Furthermore:*

(1) *The field B is formally real.*

(2) *G_B is $G_{\text{red}}(B) \cup \{0\}$, the reduced special group of the field B with an added zero.*

(3) *The character space X_{G_B} is homeomorphic to the space of orders of B, hence a Boolean space.*

Remarks. For undefined notions concerning reduced special groups (RSG), see [DM1]. For RSs obtained by adding a zero to a RSG, see Examples I.1.2 (d) and I.2.2 (3). Orders of a field F are naturally identified with group characters of the associated RSG, $G_{\text{red}}(F)$, with values in $\{\pm 1\}$. $\square$

Proof. The result will be obtained by applying the General Lemma II.4.3 with

— $B = A_{\widehat{I}}/M$;

— $f = \pi \circ \iota : A \longrightarrow B$ given by composition of the morphisms $\iota : A \longrightarrow A_{\widehat{I}}$ ($a \mapsto \frac{a}{1}$) and the canonical quotient morphism $\pi : A_{\widehat{I}} \longrightarrow B$;

— $X = \text{Sper}\,(B)$ and $Y = \{\alpha \in \text{Sper}\,(A) \,|\, \text{supp}(\alpha) = \widehat{I}\}$.

Verification of Assumptions I – III in II.4.3 goes as follows:

Assumption I. Since $X = \text{Sper}\,(B)$ the equivalence relation $\equiv_X$ is equality in G_B, and defines G_B as a RS-quotient of itself.

Since $\mathcal{H}_I = \{h_\alpha \,|\, \alpha \in Y\}$, the equivalence relation $\equiv_Y$ is identical with $\sim_I$; the latter was proved to be a RS-congruence in Theorem II.3.15 (d).

Assumption II. We show: $\text{Im}(\text{Sper}\, f) = Y$.

($\subseteq$). Let $\beta \in \text{Sper}\,(B) = X$ and $\alpha \in (\text{Sper}\, f)(\beta)$. To prove $\alpha \in Y$, i.e., $\text{supp}(\alpha) = \widehat{I}$, set $\gamma := (\text{Sper}\,\pi)(\beta) \in \text{Sper}\,(A_{\widehat{I}})$; then, $\text{supp}(\gamma) = \pi^{-1}[\text{supp}(\beta)]$. Since B is a field and $\text{supp}(\beta)$ is a proper ideal, $\text{supp}(\beta) = \{0\}$, and hence $\text{supp}(\gamma) = \pi^{-1}[\{0\}] = M$. Since $f = \pi \circ \iota$, it follows that $\text{supp}(\alpha) = \iota^{-1}[\text{supp}(\gamma)] = \iota^{-1}[M]$. From $M = \widehat{I} \cdot A_{\widehat{I}}$, we get $\widehat{I} = \text{supp}(\alpha)$, as required.

($\supseteq$). Given $\alpha \in Y$, i.e., $\text{supp}(\alpha) = \widehat{I}$, we must find $\beta \in \text{Sper}\,(B)$ such that $\alpha \in (\text{Sper}\, f)(\beta)$. By II.4.6 (2.i),

$$\text{Im}(\text{Sper}\,\iota) = \{\alpha \in \text{Sper}\,(A) \,|\, (A \setminus \widehat{I}) \cap \text{supp}(\alpha) = \emptyset\} = \{\alpha \in \text{Sper}\,(A) \,|\, \text{supp}(\alpha) \subseteq \widehat{I}\}.$$

Then, there is $\gamma \in \text{Sper}\,(A_{\widehat{I}})$ such that $\alpha \in (\text{Sper}\,\iota)(\gamma)$. Since π is surjective, the argument used in the proof of Assumption II in II.4.13 shows that $\beta := \pi[\gamma] \in \text{Sper}\,(B)$ and $\gamma = \pi^{-1}[\beta] = (\text{Sper}\,\pi)(\beta)$. It follows that $\alpha = (\text{Sper}\,\iota)((\text{Sper}\,\pi)(\beta)) = (\text{Sper}\, f)(\beta)$, as asserted.

Assumption III. The map $\overline{\iota} : G_A \longrightarrow G_{A_{\widehat{I}}}$ was shown to be surjective in II.4.6 (4.iii). The homomorphism π is surjective, and hence so is $\overline{\pi}$. It follows that $\overline{f} = \overline{\pi} \circ \overline{\iota}$ is surjective , which entails the validity of Assumption III.

Concerning the remaining assertions, we prove

(1) Suppose, towards a contradiction, that $-1 \in \sum B^2$, i.e., $-1 = \sum_i (\frac{a_i}{s_i})^2$ with $a_i \in \widehat{I}$ and $s_i \notin \widehat{I}$. Chasing denominators we can write $-1 = \frac{1}{s^2} \sum a_i^2$, with $s = \prod_i s_i \notin \widehat{I}$ ($\widehat{I}$ prime). Then, in A we have $s^2 + \sum a_i^2 = 0 \in \widehat{I}$. Since I is saturated, the ideal $\widehat{I}$ is $\sum A^2$-radical (Theorem

II.1.12); then, $s \in \widehat{I}$, contradiction.

Assertions (2) and (3) are standard facts from the theory of reduced special groups, and hence we omit their proofs. $\square$

II.5 Extensions of reduced special groups by 3-semigroups

The construction of an *extension* of a special group by a group of exponent 2 has been extensively treated in the literature; see, e.g., [DM1], [M] (complete refs). Attempts to extend this construction to real semigroups —extending them by, e.g., ternary semigroups— has so far proved to be unsuccessful; a summary of the obstructions to such a construction is given in sections $7-9$ of the unpublished notes [**?**].

The aim of this section is to study a natural notion of extension of reduced special groups (or quasi-RSG's) by 3-semigroups satisfying an additional requirement, and prove that this construction yields a real semigroup.

Definition and Notation II.5.1 (a) Recall from I.1.1 that a **3-semigroup** is a commutative semigroup, Δ, with unit satisfying the identity $x^3 = x$ for all elements $x \in \Delta$. The only constant is 1. We shall denote by $\chi(\Delta)$ the set of all semigroup homomorphisms of a 3-semigroup Δ into **3**. Note that Δ may or may not have an absorbent element 0 and it does not have a *distinguished* element -1. Thus, the morphisms of $\chi(\Delta)$ are only required to preserve product and send 1 to 1. In particular, the constant map $\mathbb{1}$ sending all of Δ to 1 is in $\chi(\Delta)$.

(b) All 3-semigroups considered in this section are required to satisfy the additional condition

[Z] For all $a, b \in \Delta$, $a^2b^2 = a^2$ or $a^2b^2 = b^2$.

Theorem I.1.13 shows that this condition is equivalent to the requirement that the family of zero-sets of elements $x \in \Delta$, $Z(x) = \{h \in \chi(\Delta) \mid h(x) = 0\}$, is totally ordered under inclusion; cf. the proofs of I.6.5 and VI.1.2. Note that condition [Z] implies that Δ does not have zero-divisors: for $a, b \in \Delta$, $ab = 0 \Rightarrow a = 0$ or $b = 0$. This observation is implicitly used in the sequel. $\square$

The next definition gives the notion of extension that we shall work with in this section:

Definition and Notation II.5.2 Let $G = G^*$ be a reduced special group (RSG) or a quasi-RSG, i.e., a RSG, G^*, with an added zero, as in I.2.2 (3). Let Δ be a 3-semigroup; we set $\Delta^o = \Delta$ if Δ has an absorbent element, 0, or $\Delta^o = \Delta \cup \{0\}$, with $\delta \cdot 0 = 0 \cdot \delta = 0$ for $\delta \in \Delta^o$, otherwise. We consider the following equivalence relation in $G^* \times \Delta^o$:

$$(g_1, 0) \sim (g_2, 0), \text{ for all } g_1, g_2 \in G^*,$$
$$(g_1, \delta_1) \sim (g_2, \delta_2) \;\Leftrightarrow\; g_1 = g_2 \text{ and } \delta_1 = \delta_2, \text{ if } \delta_1 \text{ or } \delta_2 \neq 0.$$

Straightforward checking shows that the equivalence $\sim$ is compatible with the coordinatewise product in $G^* \times \Delta^o$; thus, the quotient set $(G^* \times \Delta^o)/_\sim$, henceforth denoted by $G[\Delta]$, carries a natural product operation, denoted by $\cdot$. We set: $\widehat{0} := (1,0)/_\sim$, $\widehat{1} := (1,1)/_\sim$, $\widehat{-1} := (-1,1)/_\sim$.

Remark. To ease notation we shall assume that the given 3-semigroup Δ has an absorbent element 0. $\square$

Fact II.5.3 *With notation as in II.5.2, if the 3-semigroup Δ satisfies condition [Z] in II.5.1 (b), then $\langle G[\Delta], \cdot, \widehat{1}, \widehat{-1}, \widehat{0} \rangle$ is a ternary semigroup. Further, the idempotents of $G[\Delta]$ are* $\mathrm{Id}(G[\Delta])$ $= \{(1, \delta^2)/_\sim \mid \delta \in \Delta\}$. *Hence, $G[\Delta]$ also satisfies condition [Z] in II.5.1 (b).*

Proof. Verification of axioms [TS1] − [TS4] of ternary semigroups, see I.1.1, is straightforward and left to the reader; we only check (the contrapositive of) axiom [TS5]. Let $x \in G[\Delta], x \neq 0$; then $x = (g, \delta)/_\sim$ with $g \in G^*$ and $\delta \in \Delta, \delta \neq 0$. If $x = -x$, we get $(g, \delta) \sim (-g, \delta)$, whence $g = -g$, impossible since G^* is a RSG.

The assertion about idempotents is clear: if $x = (g, \delta)/_\sim \in G[\Delta]$ with $x^2 = x$, then $(g^2, \delta^2) \sim (g, \delta)$. If $\delta = 0$, then $(g, \delta) \sim (1, \delta^2)$. If $\delta \neq 0$, then $g^2 = g$, which entails $g = 1$ as G^* is a RSG, and $\delta = \delta^2$. Since Δ is assumed to satisfy condition [Z] in II.5.1 (b), this characterization of the idempotents implies that the zero-sets of elements of $G[\Delta]$ are also totally ordered under inclusion (note that $Z(x) = Z(x^2)$ in any TS). □

Next, we characterize the ternary semigroup characters of $G[\Delta]$ in terms of those of the group G^* of exponent 2 and of the 3-semigroup Δ. We denote by $\mathcal{X}(G^*)$ the set of all group homomorphisms (characters) $f : G^* \longrightarrow \mathbb{Z}_2 = \{1, -1\}$, and by $X_{G[\Delta]}$ the set of TS-characters of $G[\Delta]$.

Proposition II.5.4 *Let G^* be a RSG and let Δ be a 3-semigroup satisfying condition $[Z]$ in II.5.1 (b). The following are equivalent for any map $h : G[\Delta] \longrightarrow \mathbf{3}$:*

(i) $h \in X_{G[\Delta]}$ *(i.e., h is a TS-character of $G[\Delta]$).*

(ii) *There are unique characters $h_G \in \mathcal{X}(G^*)$ and $h_\Delta \in \mathcal{X}(\Delta)$ such that $h(g, \delta) = h_G(g) \cdot h_\Delta(\delta)$, for all $g \in G^*, \delta \in \Delta$.*

Hence, the action $(h, \alpha) \mapsto h \cdot \alpha$, where $h \in \mathcal{X}(G^*)\,, \alpha \in \mathcal{X}(\Delta)$ and $(h, \alpha)((g, \delta)/_\sim) = h(g)\alpha(\delta)$ for $g \in G^*\,, \delta \in \Delta$, identifies the set $X_{G[\Delta]}$ of TS-characters of $G[\Delta]$ with $\mathcal{X}(G^*) \times \mathcal{X}(\Delta)$.

In particular, the set $X_{G[\Delta]}$ of TS-characters of $G[\Delta]$) separates points.

Proof. (i) ⇒ (ii). Given $h \in X_{G[\Delta]}$, we define maps $h_G : G^* \longrightarrow \{1, -1\}$ and $h_\Delta : \Delta \longrightarrow \mathbf{3}$ as follows:

— $h_G(g) = h((g, 1)/_\sim)$, for $g \in G^*$, and

— $h_\Delta(\delta) = h((1, \delta)/_\sim)$, for $\delta \in \Delta$.

Since $(g, 1)^2 \sim (1, 1)$ it follows that $h_G(g) \neq 0$ for all $g \in G^*$, which proves that h_G is well defined; it is clear that this map is a character of groups of exponent 2. Likewise, it is obvious that $h_\Delta \in \mathcal{X}(\Delta)$. For $(g, \delta) \in G^* \times \Delta$ we have $(g, \delta) \sim (g, 1) \cdot (1, \delta)$, whence $h((g, \delta)/_\sim) = h((g, 1)/_\sim \cdot (1, \delta)/_\sim) = h((g, 1)/_\sim) \cdot h((1, \delta)/_\sim) = h_G(g) \cdot h_\Delta(\delta)$ as claimed. It is clear that the characters $h_G\,,\, h_\Delta$ verifying (ii) are unique. The implication (ii) ⇒ (i) is trivial.

Separation of points in $G[\Delta]$) is quite clear: given $(g_i, \delta_i) \in G^* \times \Delta$ $(i = 1, 2)$ so that $(g_1, \delta_1) \not\sim (g_2, \delta_2)$, either $g_1 \neq g_2$ and there is $h \in X_{G[\Delta]}$ such that $h(g_1) \neq h(g_2)$, and therefore $h(g_1)\mathbb{1}(\delta_1) \neq h(g_2)\mathbb{1}(\delta_2)$, or $g_1 = g_2\,, \delta_1 \neq \delta_2$; by I.1.13 there is $\alpha \in \mathcal{X}(\Delta)$ so that $\alpha(\delta_1) \neq \alpha(\delta_2)$, and therefore $h(g_1)\alpha(\delta_1) \neq h(g_2)\alpha(\delta_2)$ for any $h \in X_{G[\Delta]}$. □

Definition and Notation II.5.5 Given a RSG G^* and a 3-semigroup Δ satisfying condition $[Z]$ in II.5.1 (b), we define a ternary relation in $G[\Delta]$ by the following stipulation: given $p = (g_3, \delta_3)/_\sim\,,\, q = (g_1, \delta_1)/_\sim$ and $r = (g_2, \delta_2)/_\sim$ in $G[\Delta]$, we set

$p \in D^t_{G[\Delta]}(q, r)$ iff For all $h \in X_{G^*}$ and $\alpha \in \mathcal{X}(\Delta)$, $h(g_3)\alpha(\delta_3) \in D^t_{\mathbf{3}}(h(g_1)\alpha(\delta_1), h(g_2)\alpha(\delta_2))$. □

The following theorem gives a closed formula for the relation $D^t_{G[\Delta]}$ just defined.

Theorem II.5.6 *Given a RSG G^* and a 3-semigroup Δ satisfying condition $[Z]$ in II.5.1 (b), the following holds for all $(g_i, \delta_i) \in G^* \times \Delta$, $i = 1, 2$:*

$$D^t_{G[\Delta]}((g_1,\delta_1)/_\sim,(g_2,\delta_2)/_\sim)=\begin{cases}\{(g_1,\delta_1)/_\sim\} & \textit{if}\ \ Z(\delta_1)\subsetneqq Z(\delta_2)\\ \{(g_2,\delta_2)/_\sim\} & \textit{if}\ \ Z(\delta_2)\subsetneqq Z(\delta_1)\\ \{(g_1,\delta_1)/_\sim,(g_2,\delta_2)/_\sim\} & \textit{if}\ \ Z(\delta_1)=Z(\delta_2)\text{ and }\delta_1\neq\delta_2\\ (G^*\times\delta_i^2\cdot\Delta)/_\sim & \textit{if}\ \ \delta_1=\delta_2\text{ and }\ g_1=-g_2\\ \{(g,\delta_i)/_\sim\,|\,g\in D_{G^*}(g_1,g_2)\} & \textit{if}\ \ \delta_1=\delta_2\ \textit{and}\ g_1\neq -g_2,\end{cases}$$

where $Z(\delta)=\{\alpha\in\chi(\Delta)\,|\,\alpha(\delta)=0\}$, *for* $\delta\in\Delta$.

Remark II.5.7 Invoking the identification of $X_{G[\Delta]}$ with $\chi(G^*)\times\chi(\Delta)$ in II.5.4, and observing that for $a_i=(g_1,\delta_1)/_\sim$ $(i=1,2)$ we have $Z(a_i)=\chi(G^*)\times Z(\delta_i)$ and $Z(a_1)\subseteq Z(a_2)\Leftrightarrow Z(\delta_1)\subseteq Z(\delta_2)$, Theorem II.5.6 can be rephrased as follows:

$$D^t_{G[\Delta]}(a_1,a_2)=\begin{cases}\{a_1\} & \text{if}\ \ Z(a_1)\subsetneqq Z(a_2)\\ \{a_2\} & \text{if}\ \ Z(a_2)\subsetneqq Z(a_1)\\ \{a_1,a_2\} & \text{if}\ \ Z(a_1)=Z(a_2)\ \textit{and}\ \delta_1\neq\delta_2\\ a_i^2\,G[\Delta] & \text{if}\ \ a_1=-a_2\\ \{(g,\delta_i)/_\sim\,|\,g\in D_{G^*}(g_1,g_2)\} & \text{if}\ \ \delta_1=\delta_2\text{ and }\ g_1\neq -g_2.\end{cases}$$

This formulation makes it clear the relationship between Theorem II.5.6 and the characterization of transversal representation in RS-fans given in Theorem VI.2.1. See also Corollary II.5.9 below. □

Proof. Note first that

(†) $(g,\delta)/_\sim\in D^t_{G[\Delta]}((g_1,\delta_1)/_\sim,(g_2,\delta_2)/_\sim)\ \Rightarrow\ g\in D_{G^*}(g_1,g_2)$.

By taking an arbitrary RSG-character $h\in X_{G^*}$, the assumption and II.5.5 entail $h(g)\cdot\mathbb{1}(\delta)=h(g)\in D^t_{\mathbf{3}}(h(g_1)\cdot\mathbb{1}(\delta_1),h(g_2)\cdot\mathbb{1}(\delta_2))=D^t_{\mathbf{3}}(h(g_1),h(g_2))$ for all $h\in X_{G^*}$, which, by the separation theorem for RSG's (add ref) yields $g\in D_{G^*}(g_1,g_2)$.

Direct inspection of the definition of $D^t_{G[\Delta]}$ in II.5.5 shows:

(††) $Z(\delta_2)\subseteq Z(\delta_1)\Rightarrow(g_2,\delta_2)/_\sim\in D^t_{G[\Delta]}((g_1,\delta_1)/_\sim,(g_2,\delta_2)/_\sim)$ (and, symmetrically, exchanging δ_1 and δ_2).

Next, we prove the various assertions in Theorem II.5.6.

(1) $Z(\delta_1)\subsetneqq Z(\delta_2)\ \Rightarrow\ D^t_{G[\Delta]}((g_1,\delta_1)/_\sim,(g_2,\delta_2)/_\sim)=\{(g_1,\delta_1)/_\sim\}$.

Proof of (1). Assume $Z(\delta_1)\subsetneqq Z(\delta_2)$, and fix $\alpha\in\chi(\Delta)$ with $\alpha(\delta_2)=0$ and $\alpha(\delta_1)\neq 0$. Let $(g,\delta)/_\sim\in D^t_{G[\Delta]}((g_1,\delta_1)/_\sim,(g_2,\delta_2)/_\sim)$.

First we show that $Z(\delta)=Z(\delta_1)$. Let $\beta\in\chi(\Delta)$ be such that $\beta(\delta_1)=0$; then, $\beta(\delta_2)=0$. Pick $h\in X_{G^*}$ arbitrarily. Then, $h(g)\beta(\delta)\in D^t_{\mathbf{3}}(h(g_1)\beta(\delta_1),h(g_2)\beta(\delta_2))=D^t_{\mathbf{3}}(0,0)=\{0\}$. Since $h(g)\neq 0$, we get $\beta(\delta)=0$, and then $Z(\delta_1)\subseteq Z(\delta)$. Conversely, let $\beta\in\chi(\Delta)$ be such that $\beta(\delta)=0$, and let $\gamma:=\alpha\cdot\beta$. Clearly, $\gamma\in\chi(\Delta)$ and, since $\alpha(\delta_2)=0=\gamma(\delta_2)$, we get $h(g)\gamma(\delta)=0\in D^t_{\mathbf{3}}(h(g_1)\gamma(\delta_1),h(g_2)\gamma(\delta_2))=D^t_{\mathbf{3}}(h(g_1)\gamma(\delta_1),0)$, which implies $h(g_1)\gamma(\delta_1)=h(g_1)\alpha(\delta_1)\beta(\delta_1)=0$. Since $h(g_1)\alpha(\delta_1)\neq 0$, we conclude $\beta(\delta_1)=0$, and $Z(\delta)=Z(\delta_1)$.

Next we prove $g=g_1$. Otherwise, there is a RSG-character $h\in X_{G^*}$ such that $h(g)\neq h(g_1)$; let $\beta=\alpha^2$. Since $\alpha(\delta_1)\neq 0$ and $Z(\delta)=Z(\delta_1)$ we get $\beta(\delta)=\beta(\delta_1)=1$, and $h(g)\beta(\delta)=h(g)\in D^t_{\mathbf{3}}(h(g_1)\beta(\delta_1),h(g_2)\beta(\delta_2))=D^t_{\mathbf{3}}(h(g_1),0)$, whence $h(g)=h(g_1)$, contrary to the choice of h.

Finally, we get $\delta = \delta_1$. Otherwise, there is $\beta \in \mathcal{X}(\Delta)$ such that $\beta(\delta) \neq \beta(\delta_1)$. Pick $h \in X_{G^*}$ arbitrarily, and let $\gamma = \beta \cdot \alpha^2$. We obtain:

$$\begin{aligned} h(g)\gamma(\delta) = h(g)\beta(\delta) \in D^t_{\mathbf{3}}(h(g_1)\gamma(\delta_1), h(g_2)\gamma(\delta_2)) = D^t_{\mathbf{3}}(h(g_1)\gamma(\delta_1), 0) &= D^t_{\mathbf{3}}(h(g_1)\beta(\delta_1), 0) \\ &= \{h(g_1)\beta(\delta_1)\}, \end{aligned}$$

whence $h(g)\beta(\delta) = h(g_1)\beta(\delta_1) = h(g)\beta(\delta_1)$, which implies $\beta(\delta) = \beta(\delta_1)$, contradiction. This proves (1).

(2)The proof of the second item in the statement (obtained by interchanging δ_1 and δ_2) is similar to (1).

(3) $Z(\delta_1) = Z(\delta_2)$ and $\delta_1 \neq \delta_2 \Rightarrow D^t_{G[\Delta]}((g_1,\delta_1)/_\sim, (g_2,\delta_2)/_\sim) = \{(g_1,\delta_1)/_\sim, (g_2,\delta_2)/_\sim\}$.

Proof of (3). Suppose $Z(\delta_1) = Z(\delta_2)$ and $\delta_1 \neq \delta_2$ and let $(g,\delta)/_\sim \in D^t_{G[\Delta]}((g_1,\delta_1)/_\sim, (g_2,\delta_2)/_\sim)$. First we show that $Z(\delta) = Z(\delta_1) = Z(\delta_2)$. The argument in the first paragraph of the proof of (1) proves that $Z(\delta_1) = Z(\delta_2) \subseteq Z(\delta)$. Suppose the reverse inclusion fails, i.e., $Z(\delta) \not\subseteq Z(\delta_1) = Z(\delta_2)$, and let $\beta \in \mathcal{X}(\Delta)$ be such that $\beta(\delta) = 0$ and $\beta(\delta_i) \neq 0$, for $i = 1, 2$. Fix $\alpha \in \mathcal{X}(\Delta)$ so that $\alpha(\delta_1) \neq \alpha(\delta_2)$. Let $\gamma := \alpha \cdot \beta \in \mathcal{X}(\Delta)$, and pick $h \in X_{G^*}$ arbitrarily. Then, $h(g)\gamma(\delta) = 0 \in D^t_{\mathbf{3}}(h(g_1)\gamma(\delta_1), h(g_2)\gamma(\delta_2))$; this implies $h(g_1)\gamma(\delta_1) = -h(g_2)\gamma(\delta_2)$. Let $\zeta := \alpha \cdot \beta^2 \in \mathcal{X}(\Delta)$; then $\zeta(\delta) = 0$. On the other hand, $Z(\delta_1) = Z(\delta_2)$ and $\alpha(\delta_1) \neq \alpha(\delta_2)$ imply $\alpha(\delta_i) \neq 0$, for $i = 1, 2$. A similar argument using $\zeta(\delta) = 0$ and $\beta^2(\delta_1) = \beta^2(\delta_2) = 1$ shows that $h(g_1)\beta(\delta_1) = -h(g_2)\beta(\delta_2) \neq 0$. Cancelling out these terms in the equality $h(g_1)\gamma(\delta_1) = h(g_1)\alpha(\delta_1)\beta(\delta_1) = -h(g_2)\gamma(\delta_2) = -h(g_2)\alpha(\delta_2)\beta(\delta_2)$ we get $\alpha(\delta_1) = \alpha(\delta_2)$, contrary to the choice of α, and proving $Z(\delta) = Z(\delta_1) = Z(\delta_2)$.

Note that $\alpha(\delta_1) \neq 0$, $\alpha(\delta_2) \neq 0$ and $\alpha(\delta_1) \neq \alpha(\delta_2)$ imply $\alpha(\delta_1) = -\alpha(\delta_2)$. From the equality of zero-sets proved above follows $\alpha(\delta) \neq 0$, which implies $\alpha(\delta) = \alpha(\delta_1)$ or $\alpha(\delta) = \alpha(\delta_2)$. We analyze these two cases separately:

(3.a) $\alpha(\delta) = \alpha(\delta_1)$.

In this case we show that $g = g_1$ and $\delta = \delta_1$. Assuming $g \neq g_1$, there is $h \in X_{G^*}$ so that $h(g) \neq h(g_1)$. Invoking (†), we have $g \in D_{G^*}(g_1, g_2)$, and hence $h(g) = h(g_2)$. Thus, we have $h(g)\alpha(\delta) \in D^t_{\mathbf{3}}(h(g_1)\alpha(\delta_1), h(g_2)\alpha(\delta_2))$. Since $h(g)\alpha(\delta) \neq 0$ we get $h(g)\alpha(\delta) = h(g_1)\alpha(\delta_1) = h(g_1)\alpha(\delta)$ or $h(g)\alpha(\delta) = h(g_2)\alpha(\delta_2) = h(g)\alpha(\delta_2)$. The first alternative is impossible as $\alpha(\delta) \neq 0$ and $h(g) \neq h(g_1)$. Then, $h(g)\alpha(\delta) = h(g)\alpha(\delta_2)$, and hence $\alpha(\delta) = \alpha(\delta_2) = \alpha(\delta_1)$, contrary to the choice of α. This proves $g = g_1$.

As for the equality $\delta = \delta_1$, if this fails, there is $\beta \in \mathcal{X}(\Delta)$ so that $\beta(\delta) \neq \beta(\delta_1)$. The equality of zero-sets proved above implies that $\beta(\delta) \neq 0$ and $\beta(\delta_1) \neq 0$. Picking $h \in X_{G^*}$ arbitrarily, we have $h(g)\beta(\delta) \neq h(g)\beta(\delta_1) = h(g_1)\beta(\delta_1)$. Since $(g,\delta)/_\sim \in D^t_{G[\Delta]}((g_1,\delta_1)/_\sim, (g_2,\delta_2)/_\sim)$ and $h(g)\beta(\delta) \neq 0$, we obtain $h(g)\beta(\delta) = h(g_2)\beta(\delta_2)$. From the equality $Z(\delta) = Z(\delta_1) = Z(\delta_2)$ we get $\beta(\delta_2) \neq 0$, and hence $\beta(\delta) = \beta(\delta_2)$ or $\beta(\delta) = -\beta(\delta_2)$. The first of these equalities entails $h(g) = h(g_2)$ for all $h \in X_{G^*}$, whence $g = g_2$. Likewise, the second equality implies $g = -g_2$. Next we prove that either of these equalities leads to a contradiction, which proves $g = g_1$ and $\delta = \delta_1$ in case (3.a).

— Assuming $g = g_2$ (and hence $\beta(\delta) = \beta(\delta_2)$), we have $\alpha \cdot \beta \in \mathcal{X}(\Delta)$; pick $h \in X_{G^*}$ arbitrarily. From $g = g_1 = g_2$, $(g,\delta)/_\sim \in D^t_{G[\Delta]}((g_1,\delta_1)/_\sim, (g_2,\delta_2)/_\sim)$ and $h(g)\alpha(\delta)\beta(\delta) \neq 0$ we get

$(*)$ $h(g)\alpha(\delta)\beta(\delta) = h(g_i)\alpha(\delta_i)\beta(\delta_i)$, for $i = 1$ or $i = 2$.

If $(*)$ holds for $i = 1$, from $h(g) = h(g_1)$ and $\alpha(\delta) = \alpha(\delta_1)$ we get $\beta(\delta) = \beta(\delta_1)$, contradiction. If $(*)$ holds for $i = 2$, from $h(g) = h(g_2)$ and $\beta(\delta) = \beta(\delta_2)$ we get $\alpha(\delta) = \alpha(\delta_2)$, contrary to $\alpha(\delta) = \alpha(\delta_1) \neq \alpha(\delta_2)$.

— If $g = -g_2$ (and $\beta(\delta) = -\beta(\delta_2)$), as in the previous case the equality (*) holds for any $h \in X_{G^*}$. If $(*)$ holds for $i = 1$, as above we get $\beta(\delta) = \beta(\delta_1)$, contradiction. If $(*)$ holds for $i = 2$, $h(g) = -h(g_2)$ and $\beta(\delta) = -\beta(\delta_2)$ entail $\alpha(\delta) = \alpha(\delta_2)$, again a contradiction.

(3.b) $\alpha(\delta) = \alpha(\delta_2)$.
The same proof as in case (3.a), interchanging δ_1 and δ_2, shows that $g = g_2$ and $\delta = \delta_2$.

(4) $\delta_1 = \delta_2$ and $g_1 = -g_2 \;\Rightarrow\; D^t_{G[\Delta]}((g_1, \delta_1)/_\sim, (g_2, \delta_2)/_\sim) = (G^* \times \delta_i^2 \cdot \Delta)/\sim$.
Proof of (4). Set $\delta := \delta_1 = \delta_2$. An easy computation using II.5.5 shows that $(g, \delta')/_\sim \in D^t_{G[\Delta]}((g_1, \delta)/_\sim, (g_2, \delta)/_\sim)$ entails $Z(\delta) \subseteq Z(\delta')$ which, in turn, is equivalent to $\delta' = \delta^2\,\delta'$. Hence $(g, \delta') \in G^* \times \delta^2 \cdot \Delta$. The reverse inclusion follows from $(g, \delta^2 \cdot d)/_\sim \in D^t_{G[\Delta]}((g_1, \delta)/_\sim, (-g_1, \delta)/_\sim)$, for $d \in \Delta$ and $g \in G^*$. In fact, for all $h \in X_{G^*}$ and $\alpha \in \mathcal{X}(\Delta)$ we have
$(*)\quad h(g)\alpha(\delta^2)\alpha(d) \in D^t_{\mathbf{3}}(h(g_1)\alpha(\delta), -h(g_1)\alpha(\delta))$,
as this representation is equivalent to $0 \in D^t_{\mathbf{3}}(0, 0) = \{0\}$ if $\alpha(\delta) = 0$, and to $h(g)\alpha(d) \in D^t_{\mathbf{3}}(1, -1) = \mathbf{3}$ if $\alpha(\delta) \neq 0$.

(5) $\delta_1 = \delta_2$ and $g_1 \neq -g_2 \;\Rightarrow\; D^t_{G[\Delta]}((g_1, \delta_1)/_\sim, (g_2, \delta_2)/_\sim) = \{(g, \delta_i)/\!\sim \mid g \in D_{G^*}(g_1, g_2)\}$.
Proof of (5). Set $\delta := \delta_1 = \delta_2$. The implication
$$g \in D_{G^*}(g_1, g_2) \;\Rightarrow\; (g, \delta)/_\sim \in D^t_{G[\Delta]}((g_1, \delta)/_\sim, (g_2, \delta)/_\sim)$$
is immediate. Conversely, we already know that $(g, \delta')/_\sim \in D^t_{G[\Delta]}((g_1, \delta)/_\sim, (g_2, \delta)/_\sim)$ implies $g \in D_{G^*}(g_1, g_2)$ (cf. (†)) and $\delta' = \delta^2\,\delta'$ (cf. proof of (4)). Since $g_1 \neq -g_2$, there is $h \in X_{G^*}$ so that $h(g_1) = h(g_2)$, and this gives $h(g) = h(g_1) = h(g_2)$. Pick $\alpha \in \mathcal{X}(\Delta)$ arbitrarily. We have
$$h(g)\alpha(\delta') \in D^t_{\mathbf{3}}(h(g)\alpha(\delta), h(g)\alpha(\delta)) = \{h(g)\alpha(\delta)\},$$
whence $\alpha(\delta') = \alpha(\delta)$. Since α is arbitrary, from I.1.13 we get $\delta' = \delta$, as claimed. $\square$

We shall now prove:

Theorem II.5.8 *Given a RSG G^* and a 3-semigroup Δ satisfying condition $[Z]$ in II.5.1(b), the structure $\langle G[\Delta], \cdot, D^t_{G[\Delta]}, \widehat{1}, \widehat{-1}, \widehat{0}\rangle$ is a real semigroup.*

Proof. By II.5.4 the set $X_{G[\Delta]}$ of TS-characters of $G[\Delta]$ separates points. Therefore, by I.3.3, it suffices to prove that $G[\Delta]$ satisfies axiom [RS3]. To ease notation in this proof we write D^t for $D^t_{G[\Delta]}$.

Let $p, a, b, c, d \in G[\Delta]$ be elements such that $p \in D^t(a, b)$ and $b \in D^t(c, d)$; we must find an x so that
$(+)\quad x \in D^t(a, c)$ and $p \in D^t(x, d)$.

Claim 1. If at least one of the parameters p, a, b, c, d is 0, then there is x so that (+) holds.

Proof of Claim 1. We check it by cases.
— $p = 0$. In this case, $a = -b$ and $x = -d$ satisfies (+), since $0 \in D^t(d, -d)$ and $-a = b \in D^t(c, d)$ entails $-d \in D^t(a, c)$.
— $a = 0$. Then, $p = b$, and $x = c$ satisfies (+) since $c \in D^t(0, c)$ and $b \in D^t(c, d)$ by assumption.
— $b = 0$. Here we have $p = a$ and $c = -d$. Then, any $x \in D^t(a, c)$ (axiom [RS3b]) satisfies (+), since $p = a \in D^t(x, d) \Leftrightarrow -x \in D^t(-a, d) \Leftrightarrow x \in D^t(a, -d) = D^t(a, c)$.
— $c = 0$. Then, $b = d$, and $x = a$ satisfies (+), as $a \in D^t(a, 0)$ and $p \in D^t(a, d)$.
— $d = 0$. Then, $b = c$, and $x = p$ satisfies (+), as $p \in D^t(p, 0)$ and $p \in D^t(a, c)$.

We shall argue according to the inclusion between the zero-sets of the parameters p, a, b, c, d (these are totally ordered under inclusion, see II.5.3).

Case 1. $Z(a) \subsetneq Z(b)$. By II.5.6, $p = a$. Condition (+) is then equivalent to

(++) $\quad x \in D^t(a, c) \cap D^t(a, -d)$.

If $c = -d$, any $x \in D^t(a, c)$ satifies (++). Assume $c \neq -d$; we show that in this case $Z(a) \subseteq Z(c) \cap Z(d)$, and then $x = a$ verifies (++). If $Z(d) \subsetneq Z(a)$, then $Z(d) \subsetneq Z(-b)$, and from II.5.6 we get $D^t(-b, d) = \{d\}$. But $b \in D^t(c, d)$ implies $-c \in D^t(-b, d)$, whence $-c = d$, contradiction. Hence, $Z(a) \subseteq Z(d)$. Next, if $Z(c) \subsetneq Z(a)$, we get $Z(c) \subsetneq Z(d)$, whence $D^t(c, d) = \{c\}$, and therefore $b = c$, contradicting Case 1 assumption: $Z(c) \subsetneq Z(a) \subsetneq Z(b)$.

Case 2. $Z(b) \subsetneq Z(a)$. By II.5.6, $p = b$, whence $p \in D^t(c, d)$. If $Z(c) \subseteq Z(a)$, then $c \in D^t(a, c)$, showing that $x = c$ satisfies (+). If $Z(a) \subsetneq Z(c)$, we have $Z(b) \subsetneq Z(a) \subsetneq Z(c)$. By II.5.6, this inclusion implies $D^t(-b, c) = \{-b\}$. On the other hand, $b \in D^t(c, d)$ yields $-d \in D^t(-b, c)$, whence $p = b = d$, and then $x = a$ satisfies (+), as $Z(a) \subsetneq Z(c)$.

Case 3. $Z(b) = Z(a)$. If $D^t(a, b) = \{a, b\}$, then $p = a$ or $p = b$. In either of these alternatives an element x satisfying (+) can be proved to exist by arguments similar to those in cases 1 or 2 above; details are left to the reader. Let us suppose $D^t(a, b) \neq \{a, b\}$. We consider two subcases:

(3.a) $b = -a$. By the clause before last in II.5.6 we have $p = b^2 p$; also $-a = b \in D^t(c, d)$, gives $-d \in D^t(c, a)$. If $Z(d) \subseteq Z(b)$, this and $p = b^2 p \in D^t(-d, d)$ show that (+) is verified with $x = -d$. If $Z(b) \subsetneq Z(d)$, then $b \in D^t(c, d)$ entails $-c \in D^t(-b, d) = \{-b\}$, whence $c = b = -a$. Now, if $Z(d) \subseteq Z(p)$, then $p \in D^t(-d, d)$, and hence $x = -d$ satisfies (+). If $Z(p) \subsetneq Z(d)$, then $p \in D^t(p, d) = \{p\}$, and $p = b^2 p = a^2 p$ implies $p \in D^t(a, c) = D^t(a, b)$; this proves that $x = p$ satisfies (+).

(3.b) $b \neq -a$. We now argue according to the inclusions of the zero-sets of c and d.

(i) $Z(c) \subsetneq Z(d)$. By II.5.6 we have $D^t(c, d) = \{c\}$, hence $b = c$. By hypothesis, $p \in D^t(a, b) = D^t(a, c)$. Since $Z(p) = Z(a) = Z(b) = Z(c) \subsetneq Z(d)$, we have $p \in D^t(p, d)$, showing that $x = p$ satisfies (+).

(ii) $Z(d) \subsetneq Z(c)$. By II.5.6 we have $b \in D^t(c, d) = \{d\}$. Since $Z(a) = Z(b) = Z(d) \subsetneq Z(c)$ implies $a \in D^t(a, c)$, and $p \in D^t(a, b) = D^t(a, d)$, it follows that $x = a$ satisfies (+).

(iii) $Z(c) = Z(d)$. If $D^t(c, d) = \{c, d\}$, then $b = c$ or $b = d$. The same argument used in (i), resp. (ii), above shows that in the first case $x = p$ satisfies (+), and in the second $x = a$ satisfies (+). If $c = -d$, then $x = c$ satisfies (+). In fact, $b \in D^t(c, d) = D^t(c, -c)$ implies $Z(c) \subseteq Z(b) = Z(a)$; the first, third and fourth items in II.5.6 show that this inclusion implies $c \in D^t(a, c)$. On the other hand, $p \in D^t(a, b)$ and $Z(b) = Z(a)$ yield $Z(a) = Z(p)$ and hence $Z(c) \subseteq Z(p)$. This, in turn is equivalent to $p \in D^t(c, -c)$ (see end of I.2.3), i.e., $p \in D^t(c, d)$.

For the remaining case we assume:

(iv) $Z(c) = Z(d)$, $D^t(c, d) \neq \{c, d\}$ and $c \neq -d$. With notation as in the last clause of II.5.6 let $g, g_1, \ldots, g_4 \in G^*$ and $\delta \in \Delta$ be such that $p = (g, \delta)/_\sim$, $a = (g_1, \delta)/_\sim$, $b = (g_2, \delta)/_\sim$, $c = (g_3, \delta)/_\sim$, $d = (g_4, \delta)/_\sim$, with $g \in D^t_{G^*}(g_1, g_2)$ and $g_2 \in D^t_{G^*}(g_3, g_4)$. Since G^*, being a RSG, satisfies axiom [RS3], there is $x \in D^t_{G^*}(g_1, g_3)$ so that $g \in D^t_{G^*}(x, g_4)$. Clearly, $p \in D^t((x, \delta)/_\sim, (g_4, \delta)/_\sim)$ and $(x, \delta)/_\sim \in D^t((g_1, \delta)/_\sim, (g_3 \delta)/_\sim)$, which shows that $(x, \delta)/_\sim$ satisfies

(+), ending the proof of Theorem II.5.8. □

Remark. Ordinary representation in $G[\Delta]$ is given by:

$$p \in D_{G[\Delta]}(q, r) \;\Leftrightarrow\; p \in D^t_{G[\Delta]}(p^2 q, p^2 r),$$

for $p, q, r \in G[\Delta]$; cf. Theorem **??** (2). □

An interesting corollary of Theorem II.5.8 is:

Corollary II.5.9 *Let G be a reduced special group and let Δ be a 3-semigroup satisfying condition [Z] in II.5.1 (b). Then G is a RSG-fan (i.e., a fan in the category of reduced special groups) if and only if $G[\Delta]$ is a RS-fan (i.e., a fan in the category of real semigroups).* [3]

Proof. Assume first that G is a (RSG-)fan and let $a_1, a_2 \in G[\Delta]$ with, say $a_i = (g_i, \delta_i)/_\sim$, $g_i \in G$, $\delta_i \in \Delta$ $(i = 1, 2)$. Then, we have $D_G(g_1, g_2) = \{g_1, g_2\}$ if $g_1 \neq -g_2$ and $D_G(g_1, -g_1) = G$. Recalling (II.5.7) that inclusion of the zero-sets of a_1 and a_2 correspond to that of the zero-sets of δ_1 and δ_2, the last item in the description of $D^t_{G[\Delta]}$ therein reduces to $D^t_{G[\Delta]}(a_1, a_2) = \{a_1, a_2\}$ whenever $Z(a_1) = Z(a_2)$ and $a_1 \neq -a_2$; thus, we have:

$$D^t_{G[\Delta]}(a_1, a_2) = \begin{cases} \{a_1\} & \text{if } \; Z(a_1) \subsetneqq Z(a_2) \\ \{a_2\} & \text{if } \; Z(a_2) \subsetneqq Z(a_1) \\ \{a_1, a_2\} & \text{if } \; Z(a_1) = Z(a_2) \text{ and } a_1 \neq -a_2 \\ a_i^2 \cdot G[\Delta] & \text{if } \; a_1 = -a_2, \end{cases}$$

which is exactly the description of transversal representation in RS-fans given by Theorem VI.2.1 (see also **??**).

Conversely, suppose that $G[\Delta]$ is a RS-fan, and let $g_1, g_2 \in G$ be so that $g_1 \neq g_2$. Pick $\delta \in \Delta$ arbitrarily and set $a_i := (g_i, \delta)/_\sim$ $(i = 1, 2)$. Since $Z(a_1) = Z(a_2)$ and $a_1 \neq -a_2$, the third and the last items in II.5.7 yield $D^t_{G[\Delta]}(a_1, a_2) = \{a_1, a_2\} = \{g_1, g_2\} \times \{\delta\} = D_G(g_1, g_2) \times \{\delta\}$, whence $D_G(g_1, g_2) = \{g_1, g_2\}$, proving that G is a RSG-fan. □

The following is a simple example of the type of extension of RSG's presented above.

Example II.5.10 Let $A := \mathbb{R}[\![X]\!]$ be the ring of formal power series in one variable over the reals (or any other real closed field). The real semigroup G_A associated to A is the extension of the RSG, $G(\mathbb{R}) = \mathbb{Z}_2$, by the 3-semigroup $\Delta = \{1, X, X^2\}$ (where $X = X^3$). Clearly, Δ verifies condition [Z] of II.5.1 (b). □

[3] For the definition of a RS-fan, see **??**, and for the definition of a RSG-fan, see [DM1], Example 1.7, pp. 8-9.

Chapter III

Sheaf Representation and Projective Limits

New section added Nov. 2011; results from Dec. 2009.

A classical theme in commutative algebra and algebraic geometry is the representation of algebraic structures —frequently rings— by means of continuous global sections of sheaves of other algebraic structures —usually with better properties— over topological spaces.

Archetypal of results of this kind is Grothendieck's representation of any ring (commutative, unitary) by continuous sections of a sheaf of local rings over its prime (Zariski) spectrum. Hofmann [Ho] contains a survey of results of this type (up to the early 1970's).

III.1 Sheaf representation of real semigroups

In this section we shall prove a result of the above mentioned type for real semigroups; namely:

Theorem III.1.1 *Any real semigroup, G, is isomorphic to the RS of (continuous) global sections of a sheaf over the spectral space* $\mathrm{Spec}_{\mathrm{sat}}(G)$ *of saturated prime ideals of G, whose stalk at each $P \in \mathrm{Spec}_{\mathrm{sat}}(G)$ has a quasi reduced special group as a quotient.*

This result drastically refines Corollary II.3.16 by characterizing the image of the embedding μ therein in terms continuous sections of the spectral topology on the index set $\mathrm{Spec}_{\mathrm{sat}}(G)$. (A similar remark applies to many representation results by continuous sections as well.)

The proof will require a number of preliminaries.

III.1.2 Reminder. (a) Recall from I.2.2 (3) that a quasi reduced special group (QRSG) is a RSG with an added zero (absorbent element), and representation and transversal representation defined therein. We shall employ the notation $G^* = G \cup \{0\}$, where $G \models \mathrm{RSG}$.

(b) We have $X_{G^*} = X_G$. To be precise, each $\sigma \in X_G$ extends uniquely to a RS-character $\sigma^* \in X_{G^*}$ by setting $\sigma^*(0) = 0$ and $\sigma^* \lceil G = \sigma$. The map $\sigma \mapsto \sigma^*$ is a homeomorphism.

(c) (I.6.19) For a real semigroup, H, the set $\mathrm{Spec}_{\mathrm{sat}}(H)$ is endowed with the (Zariski) topology, having the family

$$D_H(a) = D(a) = \{P \in \mathrm{Spec}_{\mathrm{sat}}(H) \,|\, a \notin P\} \qquad (a \in G),$$

as a subbasis of quasi-compact opens. Since $D(a) \cap D(b) = D(ab)$ $(a, b \in G)$, this family *is a basis* for the topology, that we call $\mathcal{D}(H)$ (or just $\mathcal{D}$, if H is clear from context). Note that $D(a) = \mathrm{Spec}_{\mathrm{sat}}(H)$ for any $a \in H^{\times}$, and $D(0) = \emptyset$. The specialization order in $\mathrm{Spec}_{\mathrm{sat}}(H)$ is inclusion: for $P, Q \in \mathrm{Spec}_{\mathrm{sat}}(H)$, $P \rightsquigarrow Q \Leftrightarrow P \subseteq Q$.

(d) If $P \in \mathrm{Spec}_{\mathrm{sat}}(H)$, the real semigroup H/P is a quasi-RSG, see Theorem II.3.15 (d). Note also that

$$X_{H/P} = \{\sigma \in X_H \mid \sigma^{-1}[0] = P\},$$

a proconstructible subset of X_H; cf. **??**. Characterizations of $D_{H/P}$ and $D^t_{H/P}$ in terms of D_H and D^t_H are given in items (b) and (c) of Theorem II.3.15.

(e) Adequate references on presheaves and sheaves are [MacL] and [Mit]. These references (as well as most others on this subject) deal with (pre-)sheaves of *algebraic* structures (no relations in the language other than equality). The theory for general first-order structures is not significantly different, though care has to be exerted on a number of points; cf. [Mir], Chs. 16, 17. (OJO; check!) □

A. Construction of a presheaf basis of real semigroups. Given a RS, G, we shall now define a presheaf basis of $\mathcal{L}_{\mathrm{RS}}$-structures over the basis $\mathcal{D}\,(= \mathcal{D}(G))$ of the (Zariski) topology on $\mathrm{Spec}_{\mathrm{sat}}(G)$, introduced above (III.1.2 (c)).

(a) For $a \in G$ we denote by $G_{(a)}$ the localization of G at the multiplicative set $\{1, a, a^2\}$, see II.3.A; the RS-congruence determined by this set (cf. II.3.1) will be denoted by $\sim_a$, and the canonical quotient map $G \longrightarrow G_{(a)}$ by π_a. Thus, for $x, y, z \in G$, we have

$$(\dagger) \qquad \pi_a(x) = \pi_a(y) \;\Leftrightarrow\; x \sim_a y :\Leftrightarrow\; xa = ya \;\Leftrightarrow\; xa^2 = ya^2,$$

with representation in $G_{(a)}$ given by,

$$(\dagger\dagger) \qquad \pi_a(x) \in D_{G_{(a)}}(\pi_a(y), \pi_a(z)) \;:\Leftrightarrow\; xa \in D_G(ya, za).$$

[Note that $a^2 \sim_a 1$, since $a^2 \cdot a = a = 1 \cdot a$.] By Proposition II.3.2, $G_{(a)}$ is a real semigroup.

The following observations will frequently be used in the sequel:

Fact III.1.3 *Let G be a RS and $a \in G$. The map $\gamma \mapsto \gamma \circ \pi_a$ $(\gamma \in X_{G_{(a)}})$ is a homeomorphism of $X_{G_{(a)}}$ onto $\{\sigma \in X_G \mid \sigma(a) \neq 0\}$.*

Proof. The composition $\gamma \circ \pi_a : G \longrightarrow \mathbf{3}$ $(\gamma \in X_{G_{(a)}})$ is obviously a RS-homomorphism. From $a^2 \sim_a 1$ follows $\pi_a(a^2) = 1$, whence $(\gamma \circ \pi_a)(a^2) = 1$, and $(\gamma \circ \pi_a)(a) \neq 0$.

Conversely, given $\sigma \in X_G$ such that $\sigma(a) \neq 0$, define $\gamma : G_{(a)} \longrightarrow \mathbf{3}$ by the functional equation $\gamma \circ \pi_a = \sigma$. This map is well defined: for $x, y \in G$, $\pi_a(x) = \pi_a(y) \Rightarrow \sigma(x) = \sigma(y)$. Indeed, by $(\dagger)$ above, the assumption gives $xa = ya$; since $\sigma(a) \neq 0$, taking images under σ yields $\sigma(x) = \sigma(y)$. The equivalence $(\dagger\dagger)$ and $\sigma \in X_G$ readily imply $\gamma \in X_{G_{(a)}}$. We leave it to the reader to check that $\gamma \mapsto \gamma \circ \pi_a$ is a homeomorphism. □

Remark. Since $\sigma(a) \neq 0 \Leftrightarrow \sigma^{-1}[0] \in D(a)$, with a slight abuse of notation Fact III.1.3 can be restated as:

$$X_{G_{(a)}} = X_G \cap D(a)\,. \qquad \square$$

Fact III.1.4 *Let G be a RS. For $a, b \in G$, we have:*

(*i*) $D(a) \subseteq D(b) \;\Leftrightarrow\; a = ab^2 \;\Leftrightarrow\; b^2 \sim_a 1$.

(*ii*) $D(a) = D(b) \;\Leftrightarrow\; a^2 = b^2$.

Proof. (i) Since every $P \in \mathrm{Spec}_{\mathrm{sat}}(G)$ is $P = \sigma^{-1}[0]$ for some $\sigma \in X_G$ (I.4.9), then $D(a) = X_G \setminus Z(a)$, where $Z(a) = \{\sigma \in X_G \,|\, \sigma(a) = 0\}$. Thus, $D(a) \subseteq D(b) \Leftrightarrow Z(b) \subseteq Z(a)$, and the equivalence of (ii) and (iii) in Proposition I.6.5 (1) yields $D(a) \subseteq D(b) \Leftrightarrow a = ab^2$; this identity is clearly equivalent to $b^2 \sim_a 1$.

(ii) follows at once from (i). □

(b) Every inclusion $D(a) \subseteq D(b)$ $(a, b \in G)$ induces a map $\varphi_{ba} : G_{(b)} \longrightarrow G_{(a)}$ given by:

$$\varphi_{ba}(x) = \pi_a(x) \quad (x \in G).$$

Fact III.1.5 *For $a, b \in G \models RSG$ and $D(a) \subseteq D(b)$, φ_{ba} is a well defined RS-homomorphism.*

Proof. (i) φ_{ba} is well defined.

According to (†) above, we must prove, for $x, y \in G$: $\quad xb = yb \Rightarrow xa = ya$. Scale the antecedent by ab and use III.1.4 (a) above.

(ii) φ_{ba} is a RS-homomorphism.

Clearly, φ_{ba} preserves product and sends the constants $0, 1, -1$ of $G_{(b)}$ onto the corresponding constants of $G_{(a)}$. To show that φ_{ba} preserves representation, in view of (††) above we must show, for $g, g_1, g_2 \in G$,

$$bg \in D_G(bg_1, bg_2) \;\Rightarrow\; ag \in D_G(ag_1, ag_2).$$

Multiply the antecedent by ab and use III.1.4 (a) above. □

Thus, we have shown:

Fact III.1.6 *Let $G \models RS$. The assignment $\mathcal{G}(G)\,(= \mathcal{G})$*

$$D(a) \mapsto G_{(a)}, \qquad D(a) \subseteq D(b) \mapsto \varphi_{ba} \qquad (a, b \in G)$$

defines a contravariant functor from $\mathcal{D}$ into the category **RS** *of real semigroups.* □

In the next two Propositions we show that $\mathcal{G}$ is a sheaf.

Proposition III.1.7 *Let G be a real semigroup. Then, $\mathcal{G}$ verifies the* extensionality axiom*: For every atomic $\mathcal{L}_{\mathrm{RS}}$-formula $\theta(v_1, v_2, v_3)$ and elements a, a_i $(i \in I), g_1, g_2, g_3 \in G$, if $D(a) = \bigcup_{i \in I} D(a_i)$, then*

$$(*) \qquad \forall\, i \in I \;\; G_{(a_i)} \models \theta[\varphi_{aa_i}(\pi_a(g_1)), \varphi_{aa_i}(\pi_a(g_2)), \varphi_{aa_i}(\pi_a(g_3))] \;\Rightarrow$$
$$\Rightarrow\; G_{(a)} \models \theta[\pi_a(g_1)), \pi_a(g_2)), \pi_a(g_3))].$$

Proof. To illustrate the argument we do the proof for $\theta(v_1, v_2, v_3) : v_1 \in D(v_2, v_3)$, leaving to the reader the (even simpler) verification for other atomic $\mathcal{L}_{\mathrm{RS}}$-formulas.

In this case, the antecedent of (*) is equivalent to

$$\forall\, i \in I \; \forall\, \gamma \in X_{G_{(a_i)}} [(\gamma \circ \pi_{a_i})(g_1) \in D_{\mathbf{3}}((\gamma \circ \pi_{a_i})(g_2), (\gamma \circ \pi_{a_i})(g_3))].$$

In turn, by Fact III.1.3 this is equivalent to:

(**) $\qquad \forall\, i \in I\ \forall\, \sigma \in X_G\, (\sigma(a_i) \neq 0 \ \Rightarrow\ \sigma(g_1) \in D_{\mathbf{3}}(\sigma(g_2), \sigma(g_3)).$

Invoking the same Fact again, the conclusion to be proved amounts to:

$$\forall\, \sigma \in X_G\, (\sigma(a) \neq 0 \ \Rightarrow\ \sigma(g_1) \in D_{\mathbf{3}}(\sigma(g_2), \sigma(g_3)).$$

Fix $\sigma \in X_G$ such that $\sigma(a) \neq 0$. With $P := \sigma^{-1}[0] \in \mathrm{Spec}_{\mathrm{sat}}(G)$, we have $P \in D(a)$. Since, by assumption, the $D(a_i)$'s cover $D(a)$, there is $i_0 \in I$ so that $P \in D(a_{i_0})$, i.e., $\sigma(a_{i_0}) \neq 0$. Applying (**) with $i = i_0$ we conclude $\sigma(g_1) \in D_{\mathbf{3}}(\sigma(g_2), \sigma(g_3))$, as required. □

Proposition III.1.8 *Let G be a real semigroup. Then, $\mathcal{G}$ verifies the* gluing axiom*: Assume $D(a) = \bigcup_{i\in I} D(a_i)$, with $a, a_i \in G$ $(i \in I)$. Let $\{s_i \,|\, i \in I\}$ be a family of sections with* $\mathrm{dom}(s_i) = D(a_i)$ *$(i \in I)$, and pairwise compatible, i.e.,*

$$\textit{For all } i, j \in I,\ \ s_i \restriction (D(a_i) \cap D(a_j)) = s_j \restriction (D(a_i) \cap D(a_j))\,.$$

Then, there is a section s such that $\mathrm{dom}(s) = D(a)$ *and $s \restriction D(a_i) = s_i$ for all $i \in I$.*

Proof. Since $D(a)$ is quasi-compact, we may assume I finite, say $I = \{1, \ldots, n\}$. Each s_i is of the form $\pi_{a_i}(t_i)$, with $t_i \in G$, and the restriction maps are (by definition of $\mathcal{G}$) the functions φ_{bc}, with $D(c) \subseteq D(b)$ defined in III.1 (b). Since $D(a_i) \cap D(a_j) = D(a_i a_j)$, the compatibility condition of the statement becomes

(+) $\qquad$ For all $i, j \in \{1, \ldots, n\}$, $\ \pi_{a_i a_j}(t_i) = \pi_{a_i a_j}(t_j)\,.$

By induction it suffices to do the proof for $n = 2$.

Using axiom [RS3b] (I.2.4), pick an element t such that

(++) $\quad t \in D^t_G(t_1 a_1^2, t_2 a_2^2).$

We must show that, for $i = 1, 2$, $\ \pi_{a_i}(t) = \pi_{a_i}(t_i)$, i.e., by III.1 (a.(†)), $t a_i = t_i a_i$. We prove this equality using characters in case $i = 1$, the case $i = 2$ being similar.

Let $\sigma \in X_G$. The equality $\sigma(t a_1) = \sigma(t_1 a_1)$ being clear whenever $\sigma(a_1) = 0$, we assume $\sigma(a_1) \neq 0$, i.e., $\sigma(a_1^2) = 1$.

— If $\sigma(a_2) \neq 0$, then $\sigma(a_1 a_2) \neq 0$ and, by (+), $t_1 a_1 a_2 = t_2 a_1 a_2$. Taking images under σ and cancelling out $\sigma(a_1 a_2)$ yields $\sigma(t_1) = \sigma(t_2)$. Taking images under σ in (++) gives

$$\sigma(t) \in D^t_{\mathbf{3}}(\sigma(t_1)\sigma(a_1^2), \sigma(t_2)\sigma(a_2^2)) = D^t_{\mathbf{3}}(\sigma(t_1), \sigma(t_1)) = \{\sigma(t_1)\},$$

i.e., $\sigma(t) = \sigma(t_1)$.

— If $\sigma(a_2) = 0$, (++) yields

$$\sigma(t) \in D^t_{\mathbf{3}}(\sigma(t_1)\sigma(a_1^2), \sigma(t_2)\sigma(a_2^2)) = D^t_{\mathbf{3}}(\sigma(t_1), 0) = \{\sigma(t_1)\},$$

i.e., $\sigma(t) = \sigma(t_1)$.

Thus, in either case, $\sigma(t) = \sigma(t_1)$, which obviously yields $\sigma(t a_1) = \sigma(t_1 a_1)$, as claimed. □

B. The stalks of the sheaf $\mathcal{G}$. Our next order of business is to compute explicitly the stalk of the sheaf $\mathcal{G}\,(= \mathcal{G}(G))$ at each point $P \in \mathrm{Spec}_{\mathrm{sat}}(G)$, that we denote by $G(P)$.

III.1.9 Reminder. Recall that, by definition, $G(P)$ is the inductive limit of an (any) inductive system $\langle\, G_{(a_i)}, \varphi_{a_i a_j} \,|\, i \leq j \text{ in } I\,\rangle$, where $\{D(a_i) \,|\, i \in I\}$ $(a_i \in G)$ is *any* neighborhood basis of P in $\mathrm{Spec}_{\mathrm{sat}}(G)$, i.e., a family of neighborhoods of P such that

$$\forall x \in G\,(P \in D(x) \;\Rightarrow\; \exists i \in I\,(D(a_i) \subseteq D(x)).$$

Explicitly, $G(P) = (\coprod_{i\in I} G_{(a_i)})/{\equiv}$ (disjoint union), where $\equiv$ is the equivalence relation on $\coprod_{i\in I} G_{(a_i)}$ induced by the family of morphims $\{\varphi_{a_i a_j} \,|\, i \leq j \text{ in } I\}$ (that we will call φ_{ij} to ease notation): for $x \in G_{(a_i)}, y \in G_{(a_j)}$ $(i, j \in I)$,

$$x \equiv y \;\Leftrightarrow\; \exists\, k \in I \text{ such that } k \geq i, j \text{ and } \varphi_{ik}(x) = \varphi_{jk}(y)\,.$$

[Since the index set I, ordered under $i \leq j \;:\Leftrightarrow D(a_j) \subseteq D(a_i)$, is directed, the reader can easily check that $\equiv$ is independent of the neighborhood basis of P.]

For completeness we also recall that, setting $\pi_i := \pi_{a_i}$ for $i \in I$, the $\mathcal{L}_{\mathrm{RS}}$-structure of $G(P)$ is given by:

— The denotation of a constant $c \in \{1, 0, -1\}$ in $G(P)$ is $\pi_i(c)/{\equiv}$ (any $i \in I$).

— Product in $G(P)$ is: for $x \in G_{(a_i)}, y \in G_{(a_j)}$,

$$(x/{\equiv}) \cdot (y/{\equiv}) \;:=\; (\varphi_{ik}(x) \cdot \varphi_{jk}(y))/{\equiv}, \quad \text{any } k \geq i, j.$$

— Representation in $G(P)$ is defined as follows: for $x \in G_{(a_i)}, y_1 \in G_{(a_{j_1})}, y_2 \in G_{(a_{j_2})}$,

$$x/{\equiv} \in D_{G(P)}(y_1/{\equiv}, y_2/{\equiv}) \;\;:\Leftrightarrow\; \varphi_{ik}(x) \in D_{G_{(a_k)}}(\varphi_{j_1 k}(y_1), \varphi_{j_2 k}(y_2)),$$

for any $k \in I$ such that $k \geq i, j_1, j_2$.

Checking well-definedness of these notions is routine and, hence, omitted. Since the axioms for real semigroups are universal-existential ($\forall\exists$) sentences in the language $\mathcal{L}_{\mathrm{RS}}$ —and hence preserved under inductive limits—, $G(P)$ is a real semigroup. $\square$

The real semigroup $G(P)$ can be recast as a quotient of G modulo a certain equivalence relation $\approx_P$ that we define below. With notation as above, if $x, y \in \coprod_{i\in I} G_{(a_i)}$, then $x = \pi_i(g)$, $y = \pi_j(h)$ for some $g, h \in G$, and unique indices $i, j \in I$; we have:

$$\begin{aligned} x \equiv y \;\; &\Leftrightarrow\; \exists\, k \geq i, j \text{ such that } \varphi_{ik}(\pi_i(g)) = \varphi_{jk}(\pi_j(h)) \\ &\Leftrightarrow\; \exists\, k \geq i, j \text{ such that } \pi_k(g) = \pi_k(h) \;\; [\text{definition of } \varphi_{ik}] \\ &\Leftrightarrow\; \exists\, k \geq i, j \text{ such that } g \sim_{a_k} h \;\Leftrightarrow\; \exists\, k \geq i, j \text{ such that } g a_k = h a_k \\ &\Leftrightarrow\; \exists\, k \in I \text{ such that } g a_k = h a_k. \end{aligned}$$

[The last equivalence comes from III.1.4 (ii) and the fact that I is right-directed under $\leq$.]

So, we can see $\equiv$ as the equivalence relation $\approx_P$ on G defined by:

$$g \approx_P h \;\;:\Leftrightarrow\; \exists\, k \in I \text{ such that } g a_k = h a_k,$$

that is,

$$\approx_P \;=\; \bigcup_{i\in I} \sim_{a_i}.$$

It is easily checked that $\approx_P$ is an equivalence relation compatible with the product of G, and that the quotient $G/{\approx_P}$ is a ternary semigroup with constants $c/{\approx_P}$, where $c \in \{1, 0, -1\} \subseteq G$.

Proposition III.1.10 *For $G \models RS$ and $P \in \mathrm{Spec}_{\mathrm{sat}}(G)$, we have*

$$G(P) \cong \langle G/{\approx_P}, \cdot\,, 1/{\approx_P}, 0/{\approx_P}, -1/{\approx_P}, D_{G/\approx_P} \rangle,$$

where, for $g, g_1, g_2 \in G$,

$$g/\approx_P \in D_{G/\approx_P}(g_1/\approx_P, g_2/\approx_P) \quad :\Leftrightarrow\ \exists\, i \in I\, (ga_i \in D_G(g_1 a_i\,, g_2 a_i)).$$

Proof. The definition of the inductive limit structure gives RS-morphisms $\psi_i : G_{(a_i)} \longrightarrow G(P)$ $(i \in I)$ such that, for all $i \leq j$ in I, the diagram

$$\begin{tikzcd} G_{(a_i)} \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\psi_i"'] & & G_{(a_j)} \arrow[dl, "\psi_j"] \\ & G(P) & \end{tikzcd}$$

commutes; ψ_i is given by: $\psi_i(x) := x/\equiv$, for $x \in G_{(a_i)}$.

We define a map $\theta_P : G/\approx_P \longrightarrow G(P)$ by

$$\theta_P(g/\approx_P) = \pi_i(g)/\equiv \quad (= \psi_i(\pi_i(g))),$$

where $g \in G$ and $i \in I$ arbitrary. We show that θ_P is an isomorphism of $\mathcal{L}_{\mathrm{RS}}$-structures.

(1) θ_P is well-defined.

We have to show two things:

(i) $\theta_P(g/\approx_P)$ does not depend on the index $i \in I$ chosen in the preceding definition, i.e., for $g \in G$ and $i, j \in I$, $\pi_i(g) \equiv \pi_j(g)$.

Since I is directed, there is $k \geq i, j$; clearly we have:

$$\varphi_{ik}(\pi_i(g)) = \pi_k(g) = \varphi_{jk}(\pi_j(g)),$$

whence $\pi_i(g) \equiv \pi_j(g)$.

(ii) $\theta_P(g/\approx_P)$ does not depend on the representative modulo $\approx_P$, i.e., for $g, h \in G, i \in I$,

$$g \approx_P h \ \Rightarrow\ \pi_i(g) \equiv \pi_i(h).$$

By definition, $g \approx_P h \ \Leftrightarrow\ \exists\, j \in I\, (ga_j = ha_j)$. Pick $k \geq i, j$; thus, $D(a_k) \subseteq D(a_i), D(a_j)$ and, by Fact III.1.4 (i), $a_k = a_k a_j^2$. Scaling $ga_j = ha_j$ by $a_k a_j$ we get $ga_k a_j^2 = ga_k = ha_k = ha_k a_j^2$. Hence, $\pi_k(g) = \pi_k(h)$, which entails $\varphi_{ik}(\pi_k(g)) = \varphi_{ik}(\pi_k(h))$, i.e., $\pi_i(g) \equiv \pi_i(h)$.

(2) θ_P is injective.

We must prove that $\pi_i(g) \equiv \pi_i(h) \ \Rightarrow\ g \approx_P h$, for $i \in I$, $g, h \in G$. The assumption means there is $k \geq i$ such that $\varphi_{ik}(\pi_i(g)) = \varphi_{ik}(\pi_i(h))$, i.e., $\pi_k(g) \equiv \pi_k(h)$, i.e., $g \sim_{a_k} h$. Since $\sim_{a_k} \subseteq \approx_P$, we conclude that $g \approx_P h$.

(3) θ_P is surjective.

Every $x \in \coprod_{i \in I} G_{(a_i)}$ is of the form $x = \pi_i(g)$ for some $g \in G$ and a unique $i \in I$. Then, $\theta_P(g/\approx_P) = \pi_i(g)/\equiv = x/\equiv$.

Obviously,

(4) θ_P sends constants onto the corresponding constants.

(5) θ_P preserves product.

Let $g, h \in G$. Since $\pi_i : G \longrightarrow G_{(a_i)}$ preserves product and $\equiv$ is compatible with product in $G(P)$, altogether we have,

$$\begin{aligned}\theta_P(g/{\approx_P} \cdot h/{\approx_P}) &= \theta_P((gh)/{\approx_P}) = \pi_i(gh)/{\equiv} = (\pi_i(g)\pi_i(h))/{\equiv} \\ &= \pi_i(g)/{\equiv} \cdot \pi_i(h)/{\equiv} = \theta_P(g/{\approx_P})\, \theta_P(g/{\approx_P})\,.\end{aligned}$$

(6) θ_P preserves and reflects representation, i.e., for $g, g_1, g_2 \in G$,

(*) $\qquad g/{\approx_P} \in D_{G/\approx_P}(g_1/{\approx_P}, g_1/{\approx_P}) \;\Leftrightarrow\; \theta_P(g/{\approx_P}) \in D_{G(P)}(\theta_P(g_1/{\approx_P}), \theta_P(g_2/{\approx_P}))\,.$

By definition, the left-hand side of (*) means $\exists i \in I\,(ga_i \in D_G(g_1a_i, g_2a_i))$, which, by (††) in III.1 (a), is equivalent to

$$\exists i \in I\,(\pi_i(g) \in D_{G_{(a_i)}}(\pi_i(g_1), \pi_i(g_2)).$$

In turn, by the definition of θ_P, the right-hand side of (*) is:

$$\pi_j(g)/{\equiv} \in D_{G(P)}(\pi_j(g_1)/{\equiv}, \pi_j(g_2)/{\equiv}) \qquad (\text{any } j \in I),$$

i.e.,

$$\psi_j(\pi_j(g)) \in D_{G(P)}(\psi_j(\pi_j(g_1))\,, \psi_j(\pi_j(g_2)))\,.$$

So, the proof of (*) boils down to showing the equivalence

(**) $\exists i \in I\,[\psi_i(\pi_i(g)) \in D_{G(P)}(\psi_i(\pi_i(g_1))\,, \psi_i(\pi_i(g_2)))] \Leftrightarrow \exists i \in I\,[\pi_i(g) \in D_{G_{(a_i)}}(\pi_i(g_1), \pi_i(g_2))]$.

Proof of (**). The implication ($\Leftarrow$) is clear beacuse $\psi_i G_{(a_i)} \longrightarrow G(P)$ is a RS-morphism.

($\Rightarrow$) Since $G(P)$ is the inductive limit of $\langle\, G_{(a_i)}, \varphi_{a_ia_j} \mid i \leq j \text{ in } I\,\rangle$, representation in $G(P)$ is given by: for $x, y, z \in \coprod_{i\in I} G_{(a_i)}$ and with i_x denoting the unique index $i \in I$ such that $x \in G_{(a_i)}$ (similarly for y, z),

$$\psi_{i_x}(x) \in D_{G(P)}(\psi_{i_y}(y), \psi_{i_z}(z)) \;\Leftrightarrow\; \exists\, k \geq i_x, i_y, i_z\,[\,\varphi_{i_xk}(x) \in D_{G_{(a_i)}}(\varphi_{i_yk}(y), \varphi_{i_zk}(z))].$$

Applying this equivalence with $x = \pi_i(g), y = \pi_i(g_1), z = \pi_i(g_2)$ (and $i_x = i_y = i_z = i$), we conclude

$$\exists\, k \geq i\,[\,\varphi_{ik}(\pi_i(g)) \in D_{G_{(a_i)}}(\varphi_{ik}(\pi_i(g_1)), \varphi_{ik}(\pi_i(g_2))],$$

whence, $\exists k \in I\,(\pi_k(g) \in D_{G_{(a_k)}}(\pi_k(g_1), \pi_k(g_2)))$, as required. □

Our next result gives the desired description of the stalks of the sheaf $\mathcal{G}$ as real semigroups whose canonical quotients are quasi reduced special groups, i.e., RSG's with an added zero (cf. I.2.2 (3)).

Proposition III.1.11 *Let G be a RS, and $P \in \mathrm{Spec}_{\mathrm{sat}}(G)$. There is a saturated prime ideal $\widehat{P}$ of the stalk $G(P)$ of $\mathcal{G}$ at P such that the quotient $G(P)/\widehat{P}$ is isomorphic to the residue G/P of G at P (cf. II.3.15). In particular, $G(P)/\widehat{P}$ is a quasi reduced special group.*

Remark. To simplify the proof we shall identify the stalk $G(P)$ with the structure $G/{\approx_P}$ by means of the isomorphism θ_P constructed in the proof of Proposition III.1.10. To avoid risk of confusion, the elements of $G/{\approx_P}$ will be denoted, as above, by $g/{\approx_P}$, while those of G/P will be denoted by g/P ($g \in G$). □

Before starting the proof we make a couple of observations needed therein.

Fact III.1.12 *Let G be a RS, and $P \in \mathrm{Spec}_{\mathrm{sat}}(G)$. For $g, h, a \in G$ we have:*

(1) $g \approx_P h \;\Rightarrow\; g/P = h/P$.

(2) $g^2 \approx_P 1 \;\Leftrightarrow\; g^2/P = 1 \;\; (\Leftrightarrow\; g \notin P)$.

(3) $a \notin P \;\Rightarrow\; g \approx_P ga^2$.

Proof. As before, we fix a neighborhood basis $\{D(a_i) \mid i \in I\}$ of P in $\mathrm{Spec}_{\mathrm{sat}}(G)$.

(1) By definition, $g \approx_P h \Leftrightarrow \exists i \in I\,(ga_i = ha_i)$, cf. paragraph following III.1.9. Then, $ga_i = -(-ha_i)$ and, by I.2.3 (11), $0 \in D^t_G(ga_i, -ha_i)$. Since $a_i \notin P$, Theorem II.3.15 (a) yields $g/P = h/P$.

(2) By (1), only the implication ($\Leftarrow$) needs proof. Assume $g^2/P = 1$; then, $g/P \neq 0$, whence $g \notin P$, and $P \in D(g)$. By assumption there is $i \in I$ so that $P \in D(a_i) \subseteq D(g)$, which, by Fact III.1.4 (i), yields $g^2 a_i = a_i = 1 \cdot a_i$, whence $g^2 \approx_P 1$. The last assertion is clear.

(3) follows at once from (2): $a \notin P \Rightarrow a^2 \approx_P 1 \Rightarrow ga^2 \approx_P g \cdot 1 = g$. □

Next, we observe that transversal representation in $G\approx_P$ has a characterization in terms of that of G similar to that of ordinary representation.

Fact III.1.13 *With notation as in Proposition III.1.10, for $g, g_1, g_2 \in G$ we have,*

(†) $$g/\!\approx_P \,\in D^t_{G/\approx_P}(g_1/\!\approx_P, g_2/\!\approx_P) \;\Leftrightarrow\; \exists\, i \in I\,(ga_i \in D^t_G(g_1 a_i\,, g_2 a_i)).$$

Proof. ($\Rightarrow$) Using the definition of D^t in terms of D (cf. [t-rep], § I.2), the left-hand side of (†) is:

$$g/\!\approx_P \,\in D_{G/\approx_P}(g_1/\!\approx_P, g_2/\!\approx_P) \;\wedge - g_1/\!\approx_P \,\in D^t_{G/\approx_P}(-g/\!\approx_P, g_2/\!\approx_P) \,\wedge$$
$$\wedge - g_2/\!\approx_P \,\in D^t_{G/\approx_P}(g_1/\!\approx_P, -g/\!\approx_P)\,.$$

By the definition of $D_{G/\approx_P}$ (III.1.10) there are $i, j, k \in I$ such that

(*) $$ga_i \in D_G(g_1 a_i\,, g_2 a_i) \wedge - g_1 a_j \in D_G(-ga_j\,, g_2 a_j) \wedge - g_2 a_k \in D_G(g_1 a_k\,, -ga_k)\,.$$

Since I is directed, there is $\ell \in I$ such that $\ell \geq i, j, k$, i.e., $D(a_\ell \subseteq D(a_i), D(a_j), D(a_k)$; whence, by Fact III.1.4 (i),

(**) $\;a_\ell = a_\ell a_i^2 = a_\ell a_j^2 = a_\ell a_k^2\,.$

Suitably scaling (*), using (**) and [t-rep], § I.2, we get $ga_\ell \in D^t_G(g_1 a_\ell\,, g_2 a_\ell)\,.$

($\Leftarrow$) Using [t-rep] on the right-hand side of (†), we get $i \in I$ such that

$$ga_i \in D_G(g_1 a_i\,, g_2 a_i) \wedge - g_1 a_i \in D_G(-ga_i\,, g_2 a_i) \wedge - g_2 a_i \in D_G(g_1 a_i\,, -ga_i)\,.$$

The definition of $D_{G/\approx_P}$ (III.1.10) and [t-rep] again, yield $g/\!\approx_P \,\in D^t_{G/\approx_P}(g_1/\!\approx_P, g_2/\!\approx_P)$, as required. □

Proof of Proposition III.1.11. We define a map $\tau_P : G/\!\approx_P \longrightarrow G/P$ by

$$\tau_P(g/\!\approx_P) = g/P \qquad (g \in G)\,.$$

Item (1) in Fact III.1.12 shows

(1) τ_P is well-defined,

and, by definition,

(2) τ_P is surjective.

(3) τ_P is a RS-homomorphism.

The only non-trivial point to check is that τ_P preserves representation. Assume $g/{\approx_P} \in D_{G/\approx_P}(g_1/{\approx_P}, g_2/{\approx_P})$. Then (III.1.10), $ga_i \in D_G(g_1a_i\,, g_2a_i)$ for some $i \in I$, and hence

$$ga_i^2 \in D_G(g_1a_i^2\,, g_2a_i^2) \subseteq D_G(0, g_1a_i^2\,, g_2a_i^2)\,.$$

With $\pi_P : G \longrightarrow G/P$ canonical (i.e., $\pi_P(g) = g/P$), Theorem II.3.15 (b) and axiom [RS4] give

$$\pi_P(g) \in D_{G/P}(\pi_P(g_1)\pi_P(a_i^2)\,, \pi_P(g_2)\pi_P(a_i^2)) \subseteq D_{G/P}(\pi_P(g_1), \pi_P(g_2)),$$

as required.

Straightforward verification shows:

(4) $\widehat{P} = P/{\approx_P} := \{g/{\approx_P} \mid g \in P\}$ is a saturated prime ideal of $G/{\approx_P}$.

From Theorem II.3.15 (d) we get,

(5) The quotient $G/{\approx_P}/\widehat{P}$ is a quasi-RSG.

Let us write $\widehat{\pi_P}$ for the canonical quotient map $G/{\approx_P} \longrightarrow (G/{\approx_P})/\widehat{P}$. We claim:

(6) The quotients $G/{\approx_P}/\widehat{P}$ and G/P are isomorphic via the map induced by τ_P :

$$\widehat{\tau_P}(\widehat{\pi_P}(g/{\approx_P})) := \tau_P(g/{\approx_P}) = g/P \qquad (g \in G).$$

(6.i) $\widehat{\tau_P}$ is well-defined and injective.

Altogether, these assertions reduce to the equivalence:

$$\text{(I)} \qquad \widehat{\pi_P}(g/{\approx_P}) = \widehat{\pi_P}(h/{\approx_P}) \;\Leftrightarrow\; g/P = h/P \qquad (g, h \in G).$$

By II.3.15 (a) applied with $G/{\approx_P}$ and $\widehat{P}$, the left-hand side reads

$$\text{(II)} \qquad \exists z \notin \widehat{P}\, \exists i \in \widehat{P} \text{ such that } i \in D^t_{G/\approx_P}(g/{\approx_P} \cdot z, -h/{\approx_P} \cdot z).$$

Since z, i are of the form $z = x/{\approx_P}\,, i = p/{\approx_P}$ with $x, p \in G$ (and $z \notin \widehat{P} \Leftrightarrow x \notin P$, (II) is in turn equivalent to

$$\exists x \notin P\, \exists p \in P \text{ such that } p/{\approx_P} \in D^t_{G/\approx_P}(gx/{\approx_P}, -hx/{\approx_P}),$$

which, by Fact III.1.13 translates as:

$$\text{(III)} \qquad \exists i \in I\, \exists x \notin P\, \exists p \in P\, (pa_i \in D^t_G(gxa_i\,, -hxa_i))\,.$$

Since $xa_i \notin P$ and $pa_i \in P$, (III) and II.3.15 (a) entail $g/P = h/P$, proving the implication ($\Rightarrow$) in (I).

Conversely, by II.3.15 (a) applied with G and P, the equality $g/P = h/P$ translates as

$$\exists y \notin P\, \exists j \in P\, (j \in D^t_G(gy\,, -hy))\,.$$

Scaling this transversal representation by any a_i and invoking Fact III.1.13, we get (III), and, by the equivalence of this with (II), $\widehat{\pi_P}(g/{\approx_P}) = \widehat{\pi_P}(h/{\approx_P})$, proving the implication ($\Leftarrow$) in (I).

Since $\widehat{\pi_P}$ and τ_P are surjective RS-homomorphisms, we get

(6.ii) $\widehat{\tau_P}$ is a surjective RS-homomorphism.

Finally, we prove:

(6.iii) $\widehat{\tau_P}$ reflects representation.

This amounts to,

(IV) $\qquad g/P \in D_{G/P}(g_1/P, g_2/P) \;\Rightarrow\; \widehat{\pi_P}(g/{\approx_P}) \in D_{(G/\approx_P)/\widehat{P}}(\widehat{\pi_P}(g_1/{\approx_P}), \widehat{\pi_P}(g_2/{\approx_P}))$.

Applying II.3.15 (a) with G and P, the antecedent of (IV) is equivalent to

$$\exists x \notin P\, \exists i \in P \text{ such that } gx^2 \in D_G(i, g_1, g_2)\,;$$

scaling this representation by a_k^2 (any $k \in I$) and setting $z = xa_k/{\approx_P} \notin \widehat{P}$, $j = ia_k^2/{\approx_P} \in \widehat{P}$, Proposition III.1.10 and Fact III.1.12 (3) give $g/{\approx_P} \cdot z^2 \in D_{G/\approx_P}(j\,, g_1a_k^2/{\approx_P}\,, g_2a_k^2/{\approx_P}) = D_{G/\approx_P}(j\,, g_1/{\approx_P}\,, g_2/{\approx_P})$. By II.3.15 (a) applied with $G/{\approx_P}$ and $\widehat{P}$, this representation entails the consequent in (IV), as asserted. $\square$

Summarizing, the foregoing results amount, altogether, to the

Proof of Theorem III.1.1. Given a real semigroup G, the presheaf $\mathcal{G} = \mathcal{G}(G)$ over the basis $\mathcal{D}$ of $\mathrm{Spec}_{\mathrm{sat}}(G)$ constructed in paragraph A is a sheaf of real semigroups (Propositions III.1.7 and III.1.8). The localization $G_{(a)}$ is the RS of sections of $\mathcal{G}$ over $D(a) \in \mathcal{D}$ ($a \in G$); hence $G = G_{(1)}$ is the RS of sections over $D(1) = \mathrm{Spec}_{\mathrm{sat}}(G)$, i.e., the RS of global sections of $\mathcal{G}$. The stalk of $\mathcal{G}$ at $P \in \mathrm{Spec}_{\mathrm{sat}}(G)$ is the real semigroup $G(P)$ (III.1.9), canonically isomorphic to $G/{\approx_P}$ (Proposition III.1.10). The quotient $G(P)/\widehat{P}$ is a quasi reduced special group, canonically isomorphic to G/P (Proposition III.1.11). $\square$

C. Behaviour of the stalks under specialization. In this paragraph we study the behaviour of the stalks of the sheaf $\mathcal{G}$ under specialization in the base space $\mathrm{Spec}_{\mathrm{sat}}(G)$. We shall prove:

Proposition III.1.14 *Let $G \models RS$ and let $P, Q \in \mathrm{Spec}_{\mathrm{sat}}(G)$ be such that $P \rightsquigarrow Q$ (Q specializes P). Then, the stalk $G(P)$ is a homomorphic image of $G(Q)$.*

Remarks and Notation III.1.15 Recall that $P \rightsquigarrow Q$ means $Q \in \overline{\{P\}}$ (closure in $\mathrm{Spec}_{\mathrm{sat}}(G)$); in the present case this boils down to $P \subseteq Q$.

In the sequel we fix a neighborhood basis $\{D(b_j) \mid j \in J\}$ of Q. Since $Q \in \overline{\{P\}}$, we have $P \in D(b_j)$ for all $j \in J$. However, $\{D(b_j) \mid j \in J\}$ is not, in general, a neighborhood basis of P, i.e., there may be neighborhoods $D(x)$ of P such that for no $j \in J$, $D(b_j) \subseteq D(x)$. Given a neighborhood basis $\{D(a_i) \mid i \in I\}$ of P, for all $j \in J$ there is $i \in I$ such that $D(a_i) \subseteq D(b_j)$. Hence, there is a function $f : J \longrightarrow I$ so that for all $j \in J$ and all $i \in I$, $i \geq f(j) \Rightarrow D(a_i) \subseteq D(b_j)$. Hereafter we fix such a function f.

(1) Note that, if $D(a) \subseteq D(b)$ ($a, b \in G$), the equivalence relation $\sim_a$ (cf. item (a) in paragraph A) is coarser than $\sim_b$: for $g, h \in G$, using III.1.4 (i), we have

$$g \sim_b h \;\Leftrightarrow\; gb = hb \;\Leftrightarrow\; gab^2 = hab^2 \;\Leftrightarrow\; ga = ha \;\Leftrightarrow\; g \sim_a h.$$

(2) For $g, h \in G$, $g \approx_Q h \;\Rightarrow\; g \approx_P h$.

In fact, from

$$\approx_P = \bigcup_{i \in I} \sim_{a_i}, \quad \approx_Q = \bigcup_{j \in J} \sim_{b_j} \quad \text{and} \quad \sim_{b_j} \subseteq \sim_{a_{f(j)}},$$

we get

$$\approx_Q = \bigcup_{j\in J} \sim_{b_j} \subseteq \bigcup_{j\in J} \sim_{a_{f(j)}} \subseteq \bigcup_{i\in I} \sim_{a_i} = \approx_P . \qquad \square$$

Proof of Proposition III.1.14. As before, we work with $G/{\approx_Q}$ and $G/{\approx_P}$ instead of $G(Q)$ and $G(P)$, respectively. The required surjective RS-homomorphism is the map $\iota_{QP} : G/{\approx_Q} \longrightarrow G/{\approx_P}$ given by $\iota_{QP}(g/{\approx_Q}) := g/{\approx_P}$; III.1.15 (2) shows that it is well-defined; it clearly is surjective.

We prove that ι_{QP} preserves representation. Assume $g/{\approx_Q} \in D_{G/\approx_Q}(g_1/{\approx_Q}, g_2/{\approx_Q})$ $(g, g_1, g_2 \in G)$. By III.1.10 there is $j \in J$ such that $gb_j \in D_G(g_1 b_j, g_2 b_j)$. Pick $i \in I$ so that $P \in D(a_i) \subseteq D(b_j)$ (III.1.15). Then, $a_i = a_i b_j^2$ (III.1.4 (i)). Scaling the last representation by $a_i b_j$ we get $ga_i \in D_G(g_1 a_i, g_2 a_i)$, which, again by III.1.10, yields $g/{\approx_P} \in D_{G/\approx_P}(g_1/{\approx_P}, g_2/{\approx_P})$. $\square$

IMPORTANT NOTE (February 2018). Previous section III.2 “Projective limits of real semi-groups”, from Feb. 2014, omitted. Results there particular cases of products, as noted by Chico in 2016-17. If needed, original is in file RS-fev-16-rev, pp. 115-118.

III.2 Transversally 2-regular morphisms.

In this paragraph we consider a class of morphisms for structures of signature $\mathcal{L}_{\mathrm{RS}}$ —the language of real semigroups—, which turns out to be of interest in the study of quotients of RSs.

Definition III.2.1 (a) Let G, H be structures of language $\mathcal{L}_{RS} = \{1, 0, -1, \cdot, D\}$. An $\mathcal{L}_{\mathrm{RS}}$-morphism $f : G \longrightarrow H$ is called **transversally 2-regular** if and only if for all $a, b, c, d \in G$,

$$\begin{aligned} D^t_H(f(a), f(b)) \cap D^t_H(f(c), f(d)) \neq \emptyset \;\Rightarrow\; & \text{There are } a', b', c', d' \in G \text{ so that } f(a) = f(a'), \ldots, \\ & f(d) = f(d') \text{ and } D^t_G(a', b') \cap D^t_G(c', d') \neq \emptyset. \end{aligned}$$

(b) If G is a RS and $\mathcal{H} \subseteq X_G$, we say that $\mathcal{H}$ is **transversally 2-regular** if the quotient map $\pi : G \longrightarrow G/\mathcal{H}$ is transversally 2-regular, with representation in $G/\mathcal{H}$ defined as in II.2.12, $(\dagger\dagger)_{\mathcal{H}}$.

As we shall see below, in a number of natural quotients of RSs,

(1) The set of characters determining them is transversally 2-regular, and this is relatively easy to check; and

(2) Transversal 2-regularity of a set of characters of a RS guarantees that the quotient it determines is a RS and, further, that the induced equivalence relation —see $(\dagger)_{\mathcal{H}}$ in II.2.12— is a RS-congruence in the sense of Definition II.2.1.

Remark. Quotients of reduced special groups by saturated subgroups with a property resembling transversal 2-regularity have been considered in [DMM], Prop. 2.13, p. 37. For the notion of a regular map of RSGs, see [DM1], Def. 2.22 , p. 43. $\square$

Proposition III.2.2 *Let G be a RS and let $\mathcal{H} \subseteq X_G$ be a transversally 2-regular set of characters. Then:*

(1) *The quotient $G/\mathcal{H}$ is a RS.*

(2) *The equivalence relation on G induced by $\mathcal{H}$ is a RS-congruence.*

Proof. (1) By Theorem II.2.16 and Proposition I.2.10 it suffices to check that $G/\mathcal{H}$ verifies axiom [RS3′]. Let $a, b, c, d \in G$ be so that $D^t_{G/\mathcal{H}}(\pi(a), \pi(b)) \cap D^t_{G/\mathcal{H}}(\pi(c), \pi(d)) \neq \emptyset$. By the regularity assumption there are $a', b', c', d' \in G$ such that $D^t_G(a', b') \cap D^t_G(c', d') \neq \emptyset$ and $\pi(a) = \pi(a'), \ldots, \pi(d) = \pi(d')$ i.e., $a \equiv_{\mathcal{H}} a', \ldots, d \equiv_{\mathcal{H}} d'$. Since G verifies [RS3′], we have $D^t_G(a', -c') \cap D^t_G(-b', d') \neq \emptyset$, and since π preserves D^t and $\pi(a) = \pi(a'), \ldots,$ we get $D^t_{G/\mathcal{H}}(\pi(a), -\pi(c)) \cap D^t_{G/\mathcal{H}}(-\pi(b), \pi(d)) \neq \emptyset$, as required.

(2) We must check the factoring condition (iii) of Definition II.2.1. Let H be a RS and let $f : G \longrightarrow H$ be a RS-morphism such that

(*) $\qquad a \equiv_{\mathcal{H}} b \;\Rightarrow\; f(a) = f(b) \quad$ for all $a, b \in G$.

We must show that the unique (and, by (*), well defined) map $\widehat{f} : G/\mathcal{H} \longrightarrow H$ given by the functional equation $\widehat{f} \circ \pi = f$ is a RS-morphism, i.e., for $a, b, c \in G$,

$$\pi(a) \in D_{G/\mathcal{H}}(\pi(b), \pi(c)) \;\Rightarrow\; f(a) \in D_H(f(b), f(c)).$$

By (1) we already know that $G/\mathcal{H}$ is a RS. From $\pi(a) \in D_{G/\mathcal{H}}(\pi(b), \pi(c))$ we get $\pi(a) \in D^t_{G/\mathcal{H}}(\pi(b)\pi(a)^2, \pi(c)\pi(a)^2)$ (by axiom [RS6], in $G/\mathcal{H}$). By Proposition I.2.3 (6) we have $\pi(a) \in D^t_{G/\mathcal{H}}(\pi(a), \pi(a))$. Transversal 2-regularity of π entails the existence of $a', b', c', x \in G$ such that $a' \equiv_{\mathcal{H}} a$, $b' \equiv_{\mathcal{H}} ba^2$, $c' \equiv_{\mathcal{H}} ca^2$ and $x \in D^t_G(b', c') \cap D^t_G(a', a')$. Invoking I.2.3 (6) again, $x \in D^t_G(a', a')$ implies $x = a'$, whence $x \equiv_{\mathcal{H}} a$. Assumption (*) yields, then, $f(x) = f(a)$; likewise, $f(b') = f(ba^2)$ and $f(c') = f(ca^2)$. Since f is a RS-morphism, $x \in D^t_G(b', c')$ implies $f(x) = f(a) \in D^t_H(f(b'), f(c')) = D^t_H(f(b)f(a)^2, f(c)f(a)^2)$, whence $f(a) \in D_H(f(b)f(a)^2, f(c)f(a)^2)$; by [RS4] (in H), $f(a) \in D_H(f(b), f(c))$, as required. $\square$

In some cases where transversal representation in the quotient $G/\mathcal{H}$ has an explicit lifting to G —for example in the cases localizations (Proposition II.3.2 (5)) and of quotients by saturated prime ideals (Theorem II.3.15 (c.ii))— it is easily proved that the corresponding quotient maps are transversally 2-regular.

Example III.2.3 Localizations at multiplicative sets are transversally 2-regular.

Proof. Using item (5) of Proposition II.3.2, if $\pi(x) \in D^t_{G/\mathcal{H}^T}(\pi(a), \pi(b)) \cap D^t_{G/\mathcal{H}^T}(\pi(c), \pi(d))$, there are $t, t' \in T$ so that $xt^2 \in D^t_G(at^2, bt^2)$ and $xt'^2 \in D^t_G(ct'^2, dt'^2)$. Scaling these representations by t'^2 and t^2 respectively, and setting $z = tt' \in T$, we get $xz^2 \in D^t_G(az^2, bz^2) \cap D^t_G(cz^2, dz^2)$. Observing that $y \sim_T yt^2$, i.e., $\pi(y) = \pi(yt^2)$, for all $y \in G$ and $t \in T$ (because $h(t^2) = 1$ for all $h \in \mathcal{H}^T$), proves our contention. $\square$

Example III.2.4 Residue spaces at saturated prime ideals are transversally 2-regular.

Proof. With the setting of Theorem II.3.15, assume $\pi(x) \in D^t_{G/I}(\pi(a), \pi(b)) \cap D^t_{G/I}(\pi(c), \pi(d))$. If $x \in I$, the first case of II.3.15 (c.ii) gives $a \sim_I -b$ and $c \sim_I -d$. Since $x \sim_I 0$ (II.3.15 (a)) and $0 \in D^t_G(b, -b) \cap D^t_G(d, -d)$ (I.2.3 (11)), the result holds.

If $x \notin I$, the second alternative in II.3.15 (c.ii) shows that there are $y, z \notin I$ and $a' \sim_I a$, $b' \sim_I b$, $c' \sim_I c$, $d' \sim_I d$ so that $xy^2 \in D^t_G(a', b')$, $xz^2 \in D^t_G(c', d')$. Scaling these representations by z^2 and y^2, respectively, we get $xy^2z^2 \in D^t_G(a'z^2, b'z^2) \cap D^t_G(c'y^2, d'y^2)$. Since $vw^2 \sim_I v$ for all $v \in G$ and $w \in G \setminus I$, the result follows. $\square$

Remark. In Chapters IV and VI we shall prove transversal 2-regularity for arbitrary quotients of Post algebras (seen as RSs, Theorem IV.4.11), and for all quotients of fans (Theorem VI.11.3).

However, the example that follows shows that the property fails, in general, for quotients by saturated subsemigroups (§ II.3 (C)). □

Example III.2.5 A quotient that is not transversally 2-regular.

The example is a quotient of the form $G_A \longrightarrow G_{A,T}$, with $\langle A, T \rangle$ a p-ring (cf. II.2.2). Recall (II.4.C) that $G_{A,T}$ is a quotient of G_A by the saturated subsemigroup $\Delta = \{\overline{a} \in G_A \mid a \in T\}$. For the choice of the p-ring $\langle A, T \rangle$, recall from [DM6], Lemma 8.29, p. 111, that, given a topological space X and a closed subset $K \subseteq X$, the set $P_K = \{f \in C(X) \mid f\lceil K \geq 0\}$ is a proper preorder of the ring $C(X)$ of real-valued continuous functions on X. We take $A = C(X)$ and $T = P_K$, with $K \subsetneq X$, and show:

Proposition III.2.6 *With notation as above, the quotient map $G_{C(X)} \longrightarrow G_{C(X),P_K}$ <u>is not</u> transversally 2-regular, for any compact Hausdorff space X and any closed subset $K \subsetneq X$.*

For ready reference we recall:

Remarks and Notation III.2.7 (a) For a ring $A, a \in A$ and $\alpha \in \mathrm{Sper}\,(A)$, we write $\overline{a}(\alpha) = \mathrm{sgn}_{\leq_\alpha}(\pi_\alpha(a))$, with $\pi_\alpha : A \longrightarrow A/\mathrm{supp}(\alpha)$ canonical, and $\leq_\alpha$ the total order in $A/\mathrm{supp}(\alpha)$ determined by α.

(b) If $\alpha, \beta \in \mathrm{Sper}\,(A)$ and $\alpha \rightsquigarrow \beta$ (i.e., $\alpha \subseteq \beta$), the map $\pi_{\alpha,\beta} : A/\mathrm{supp}(\alpha) \longrightarrow A/\mathrm{supp}(\beta)$ given by $a/\mathrm{supp}(\alpha) \mapsto a/\mathrm{supp}(\beta)$ $(a \in A)$ is a homomorphism of ordered rings of $(A/\mathrm{supp}(\alpha), \leq_\alpha)$ onto $(A/\mathrm{supp}(\beta), \leq_\beta)$.

(c) Given a compact Hausdorff space X and $\alpha \in \mathrm{Sper}\,(C(X))$ there is a <u>unique</u> point $x \in X$ such that $\alpha \rightsquigarrow \alpha_x$, where $\alpha_x = \{f \in C(X) \mid f(x) \geq 0\}$. [The existence of x requires compactness; cf. [GJ], 4.6, 4.8, 4.9(a), pp. 56-57]. Further, $\pi_{\alpha_x} = f(x)$ for $f \in C(X)$. □

Lemma III.2.8 *For any topological space X and $f, g \in C(X)$ we have:*

$$\overline{f} = \overline{g} \;\Rightarrow\; \forall x \in X\,(\mathrm{sgn}(f(x)) = \mathrm{sgn}(g(x))).$$

Proof. By [M], Cor. 5.4.3, p. 94 (with $T = C(X)^2$),

(†) $\quad \overline{f} = \overline{g} \;\Leftrightarrow\;$ There are $s, t \in C(X)$ and $k \geq 0$ such that $s^2 fg = (f^2 + g^2)^k + t^2$.

Suppose the conclusion fails, and first assume $f(y)g(y) < 0$ for some $y \in X$. If $s(y) \neq 0$, then $s^2(y) > 0$, and $(s^2 fg)(y) < 0$. But, clearly, $((f^2 + g^2)^k + t^2)(y) \geq 0$, contradicting (†). Hence, $s(y) = 0$, whence $((f^2 + g^2)^k + t^2)(y) = 0$, which clearly entails $f(y) = g(y) = t(y) = 0$ if $k \geq 1$, and $(1 + t^2)(y) = 0$ if $k = 0$, a contradiction in either case.

Next, suppose, e.g., $f(y) = 0$ and $g(y) \neq 0$ for some $y \in X$. Then, both sides of the equality in the right-hand side of (†) vanish at y, whence $(g^{2k} + t^2)(y) = 0$, which entails $g(y) = 0$ if $k \geq 1$, and $(1 + t^2)(y) = 0$ if $k = 0$, again a contradiction. □

Fact III.2.9 *Let X be a topological space, $f, a, b \in C(X)$ and $y \in X$. Assume:*
(i) $\overline{f} \in D^t_{C(X)}(\overline{a}, \overline{b})$;
(ii) $\mathrm{sgn}(a(y)) = \mathrm{sgn}(b(y))$.
Then, $\mathrm{sgn}(f(y)) = \mathrm{sgn}(a + b)(y))$.

Proof. By [M], § 5.5, p. 96, assumption (i) is equivalent to the existence of $a', b' \in C(X)$ so that $\overline{a'} = \overline{a}, \overline{b'} = \overline{b}$ and $\overline{f} = \overline{a' + b'}$. From Lemma III.2.8 we get, for all $x \in X$:

— $\operatorname{sgn}(a(x)) = \operatorname{sgn}(a'(x))$.

— $\operatorname{sgn}(b(x)) = \operatorname{sgn}(b'(x))$.

— $\operatorname{sgn}(f(x)) = \operatorname{sgn}(a' + b')(x))$.

An easy calculation using assumption (i) and these sign equalities yields $\operatorname{sgn}((a' + b')(y)) = \operatorname{sgn}((a + b)(y))$, and the conclusion follows. $\square$

Corollary III.2.10 *Given a completely regular space X and a proper closed subset K, there are functions $a, c \in C(X)$ such that $a \lceil K = c \lceil K = 1$ and $D^t_{C(X)}(\overline{a}, \overline{a}) = D^t_{C(X)}(\overline{c}, \overline{c}) = \emptyset$.*

Proof. Fix $y \in X \setminus K$. By complete regularity there are functions $a, c \in C(X)$ so that $a \lceil K = c \lceil K = 1, a(y) < 0$ and $c(y) > 0$. If $\overline{f} \in D^t_{C(X)}(\overline{a}, \overline{a}) \cap D^t_{C(X)}(\overline{c}, \overline{c})$, Fact III.2.9 (with $b = a$) gives $\operatorname{sgn}(f(y)) = \operatorname{sgn}(2a)(y)) = \operatorname{sgn}(a(y)) = -1$ and $\operatorname{sgn}(f(y)) = \operatorname{sgn}(2c)(y)) = \operatorname{sgn}(c(y)) = 1$, contradiction. $\square$

In order to complete the proof of Proposition III.2.6, we show:

Proposition III.2.11 *Given a compact Hausdorff space X, a proper closed subset K, and functions $a, c \in C(X)$ such that $a \lceil K = c \lceil K > 0$, we have:*

$$D^t_{C(X), P_K}(\pi(\overline{a}), \pi(\overline{a})) \cap D^t_{C(X), P_K}(\pi(\overline{c}), \pi(\overline{c})) \neq \emptyset.$$

Fact III.2.12 *Let X and K be as in III.2.11. Given $\alpha \in \operatorname{Sper}(C(X), P_K)$, let x be the unique point in X such that $\alpha \rightsquigarrow \alpha_x$ (III.2.7 (c)). Then, $x \in K$.*

Proof. Otherwise (since X is completely regular) there is $g \in C(X)$ so that $g \lceil K = 1$ and $g(x) = -1$. Then, $g \in P_K \subseteq \alpha$, whence $g/\operatorname{supp}(\alpha) \geq_\alpha 0$; by III.2.7 (b), $\pi_{\alpha, \alpha_x}(g/\operatorname{supp}(\alpha)) = g/\operatorname{supp}(\alpha_x) = g(x) \geq 0$, contradiction. $\square$

Fact III.2.13 *Under the same assumptions as in Fact III.2.12,*

$$f \in C(X) \text{ and } f \lceil K \text{ imply } \overline{f_{P_K}} = (\overline{f} \lceil \operatorname{Sper}(C(X), P_K) = 1.$$

Proof. We must show that $\overline{f}(\alpha) = 1$, i.e., $f/\operatorname{supp}(\alpha) >_\alpha 0$ for all $\alpha \in \operatorname{Sper}(C(X), P_K)$. Let x be the unique point of X such that $\alpha \rightsquigarrow \alpha_x$. By Fact III.2.12, $x \in K$, and thus $f(x) > 0$. If the conclusion fails, i.e., $f/\operatorname{supp}(\alpha) \leq_\alpha 0$, then, $\pi_{\alpha, \alpha_x}(f/\operatorname{supp}(\alpha)) = f(x) \leq 0$, contradiction. $\square$

Proof of Proposition III.2.11. Choosing the functions a, c as in Corollary III.2.10, we have $a \lceil K = c \lceil K > 0$; by Fact III.2.13, $\overline{a_{P_K}} = \overline{(2a)_{P_K}} = 1 = \overline{c_{P_K}} = \overline{(2c)_{P_K}}$. Since $\overline{(x + y)_T} \in D^t_{A,T}(\overline{x_T}, \overline{y_T})$ in any p-ring $\langle A, T \rangle$, we get

$$\overline{a_{P_K}} = \overline{c_{P_K}} \in D^t_{C(X), P_K}(\overline{a_{P_K}}, \overline{a_{P_K}}) \cap D^t_{C(X), P_K}(\overline{c_{P_K}}, \overline{c_{P_K}}),$$

whence

$$D^t_{C(X), P_K}(\pi(\overline{a}), \pi(\overline{a})) \cap D^t_{C(X), P_K}(\pi(\overline{c}), \pi(\overline{c})) \neq \emptyset. \qquad \square$$

To be moved elsewhere. ↓

Definition III.2.14 Let G, H be RSs and let X_G, X_H be their character spaces. Let $Z \subseteq X_G$ and let $F : Z \longrightarrow Y$ be a map. We say that F **preserves 3-products** (in Z) iff for all $h_1, h_2, h_3 \in Z$,

$$h_1 h_2 h_3 \in Z \;\Rightarrow\; F(h_1 h_2 h_3) = F(h_1) F(h_2) F(h_3). \qquad \square$$

Proposition III.2.15 (Small representation theorem). *Let G be a RS. The following conditions are equivalent for a map $f : X_G \longrightarrow \mathbf{3}$:*

(1) *a) f is continuous in the constructible topology of X_G.*

b) f preserves 3-products in X_G.

(2) *f is represented by an element of G: there is $a \in G$ so that $f = \widehat{a}$.*

[Recall that $\widehat{a} : X_G \longrightarrow \mathbf{3}$ denotes "evaluation at a": for $h \in X_G$, $\widehat{a}(h) = h(a)$.]

Proof. (2) $\Rightarrow$ (1) is clear since the evaluation maps have properties (1.a) and (1.b).

(1) $\Rightarrow$ (2). We use the representation theorem [M], Cor. 8.3.6, p. 162. It suffices to check that the assumptions of this theorem as well as one of the equivalent conditions in its conclusion hold under our hypothesis (1). With our notation, the conditions to be checked are: for $x, y \in X_G$,

(†) $f(x) = 0$ and $Z(x) \subseteq Z(y)$ implies $f(y) = 0$.

(††) $f(x) \neq 0$ and $x^{-1}[0,1] \supseteq y^{-1}[0,1]$ implies $f(x) = f(y)$.

(†††) For any saturated prime ideal I of G, either
(i) $f\lceil\{u \in X_G \,|\, Z(u) = I\} = 0$, or
(ii) $\prod_{i=1}^{4} f(x_i) = 1$ for any 4-element AOS-fan $\{x_1, \ldots x_4\}$ in $\{u \in X_G \,|\, Z(u) = I\}$.

— Condition (†) follows at once from Lemma I.1.19 (2) (as $Z(x) \subseteq Z(y) \Rightarrow y = yx^2$).

— Condition (††) follows from Lemma I.1.18 (3),(5): $x^{-1}[0,1] \supseteq y^{-1}[0,1] \Rightarrow x = x^2y$. Since $f(x) \neq 0 \Rightarrow f(x^2) = 1$, assumption (1.b) implies $f(x) = f(x^2)f(y) = f(y)$.

— As for (†††), if (i) does not hold, (†) implies $f(u) \neq 0$ for all $u \in X_G$ such that $Z(u) = I$. Let $\{x_1, \ldots x_4\}$ be an AOS-fan in $\{u \in X_G \,|\, Z(u) = I\}$. Thus, $x_4 = x_1x_2x_3$ and $f(x_i) \neq 0$ for $i = 1, \ldots, 4$. Assumption (1.b) gives $f(x_4) = f(x_1)f(x_2)f(x_3) \neq 0$, i.e., $\prod_{i=1}^{4} f(x_i) = 1$. $\square$

Chapter IV

Post Algebras

Introduction

The subject matter of this chapter is a detailed study of the first significant class of real semigroups: the *Post algebras*.[1]

The Preface to this monograph gives a brief account of how these algebraic structures originating in non-classical propositional calculus emerged in the context of real semigroups. A more detailed account of this question appears in [DP2] (ADD PAGE NOS.). We shall only stress here that Post algebras are connected with the *constructible* topology of the character space of real semigroups, as suggested by the Representation Theorem IV.1.7.

Section IV.1 contains a brief summary —for use in the present and later chapters— of the main notions and basic results concerning Post algebras and Kleene algebras.

Next, we introduce (in § IV.2) a natural structure of type $\mathcal{L}_{\mathrm{RS}}$ in Post algebras and check (in IV.2.7) that it verifies the axioms for real semigroups. The same theorem gives a remarkable characterization of the resulting transversal representation relation in terms of the order and the lattice and "modal" operations of the Post algebra. We also show that for this natural RS structure, RS-morphisms coincide with Post-algebra morphisms (Proposition IV.2.11).

In § IV.3 we give two characterizations of those real semigroups that are Post algebras. The first, Theorem IV.3.2, is basically in terms of their character spaces, and implies immediately that the spectral topology of the character space of a Post algebra is Boolean. The second, Theorem IV.3.5, gives, in particular, explicit (first-order) definitions of the Post algebra operations in terms of the RS structure.

The aim of § IV.4 is to show that every RS, G, can be functorially embedded into (the real semigroup associated to) a certain, canonically determined, Post algebra P_G —the *Post hull* of G—, namely the set of all continuous maps from the character space X_G endowed with its (Boolean) constructible topology, into **3** with the discrete topology. This construction is an analog for RSs of the construction of the Boolean hull of a reduced special group, carried out in [DM1], Ch. 4 (Thm. 4.17). The analogy between these constructions extends to several other results; an important instance is Theorem IV.4.5 —an analog to Theorem 5.2 in [DM1]— giving several characterizations of *complete* embeddings of RSs, among others by the injectivity of the associated Post algebra morphisms; the proof, however, is far more delicate than that of [DM1], Thm. 5.2. In Theorem IV.4.8 we show that the Post hull construction commutes with

[1] All Post algebras considered in this monograph are of order 3, i.e., algebraic counterparts of the *three-valued* propositional calculus, Post version.

arbitrary quotients, and in Theorem IV.4.11 that quotients of Post algebras have the powerful 2-regular transversality property, introduced in § III.2.

In Section § IV.6 we prove some model-theoretic results concerning Post algebras in their guise as real semigroups. The main results are that these structures admit a first-order universal/positive-primitive axiomatisation in the language $\mathcal{L}_{\text{RS}}$, a result that yields preservation under several constructions. We also prove some results concerning pure embeddings of Post algebras.

The following section deals with the question of determining those rings whose associated real semigroup is a Post algebra. A complete answer is given in Proposition IV.7.1, which automatically yields natural examples, notably amongst von-Neumann regular rings. We also prove (Theorem IV.7.4) that every Post algebra is "realized" by a ring, i.e., isomorphic to the RS associated to some ring.

Finally, in § IV.5 the techniques previously developed are used to characterize representation and transversal representation by arbitrary quadratic forms in Post algebras in terms of order and the operations existing therein. Under certain conditions, this information "descends" from the Post hull of a RS to the given real semigroup yielding, for instance, information about the value set of Pfister forms (and multiples of them) over arbitrary real semigroups. This way we obtain (weak) versions of some classical properties of those value sets (see Corollaries IV.5.8 and IV.5.11) and other related results of interest.

IV.1 Łukasiewicz-Post algebras. Kleene algebras

The structures of the title are the algebraic counterparts of the n-valued propositional calculus developed independently by Łukasiewicz and Post in the early 1920's. They were introduced by Moisil and Rosenbloom in the early 40's. The monograph [BFGR] contains an exhaustive study of these algebraic structures; Chapters X and XI of [BD] give an account of the basic results. Here we shall only consider the case $n = 3$. We begin by summarizing some basic notions and results about Łukasiewicz and Post algebras of order 3.

Definition IV.1.1 A **three-valued Łukasiewicz algebra** is a structure $(L, \wedge, \vee, \neg, \nabla, \top)$ fulfilling the following requirements:

[L1] $(L, \wedge, \vee, \top)$ is a distributive lattice with last element $\top$.

[L2] The unary operation $\neg$ (*negation*) verifies the De Morgan laws:

(i) $\neg\neg\, x = x$.

(ii) $\neg\,(x \wedge y) = \neg\, x \vee \neg\, y$.

[L3] The unary operation ∇, called the *possibility operator*, verifies:

(i) $\neg\, x \vee \nabla x = \top$.

(ii) $x \wedge \neg\, x = \neg\, x \wedge \nabla x$.

(iii) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$.

A **Post algebra** of order 3 is a three-valued Łukasiewicz algebra with a *center*, that is, a distinguished element $\underline{\mathbf{c}}$ verifying $\neg\,\underline{\mathbf{c}} = \underline{\mathbf{c}}$. □

Henceforth we omit the words "three-valued" and "order 3".

Remarks IV.1.2 (a) It follows at once from the De Morgan laws that every Łukasiewicz algebra L satisfies the dual law $\neg(x \vee y) = \neg x \wedge \neg y$. Moreover, L is a bounded distributive lattice with last element $\top$ and first element $\bot = \neg\top$.

(b) In addition, every Łukasiewicz algebra satisfies the so-called **Kleene inequality**:

$$\text{For all } x, y \ (x \wedge \neg x \le y \vee \neg y).$$

This inequality implies that the center in a Post algebra is necessarily unique, and $x \wedge \neg x \le \underline{\mathbf{c}} \le y \vee \neg y$. □

A number of other, conceptually important operations are definable in terms of the operator ∇. We mention, for instance:

(i) The *necessity operator* Δ defined by $\Delta x = \neg\nabla\neg x$. The operators Δ and ∇ are also called *modal operators*.

(ii) The *Łukasiewicz implication*, a binary operation defined by

$$x \to y \ = \ ((\nabla\neg x) \vee y) \wedge ((\nabla y) \vee \neg x).$$

(iii) The *arithmetical* (or MV-algebra) operations

$$x \oplus y = \neg x \to y, \qquad\qquad x \odot y = \neg(\neg x \oplus \neg y).$$

To be sure, the presentation of Łukasiewicz algebras given above is one of several equivalent alternatives. For example, implication or truncated sum ($\oplus$) are frequently used as primitive notions, instead of ∇.

Examples IV.1.3 (a) Every Boolean algebra B with its usual negation becomes a Łukasiewicz algebra upon defining $\nabla x = x$. However, Boolean algebras are never Post algebras. Indeed, the existence of a center $\underline{\mathbf{c}}$ such that $\neg\underline{\mathbf{c}} = \underline{\mathbf{c}}$, and the laws $\underline{\mathbf{c}} \vee \neg\underline{\mathbf{c}} = \top$ and $\underline{\mathbf{c}} \wedge \neg\underline{\mathbf{c}} = \bot$ lead to a collapse: $\underline{\mathbf{c}} = \bot = \top$.

(b) The simplest Post algebra is the three-element chain $\mathbf{3} = \{\bot, \underline{\mathbf{c}}, \top\}$ with $\bot < \underline{\mathbf{c}} < \top$, $\neg\bot = \top$, $\neg\top = \bot$, $\neg\underline{\mathbf{c}} = \underline{\mathbf{c}}$ and the operator ∇ defined by $\nabla\bot = \bot$ and $\nabla\underline{\mathbf{c}} = \nabla\top = \top$.

(c) Further examples of Post algebras —in fact, all possible examples— are given in the Representation Theorem IV.1.7 below.

Note. It is customary to denote the elements $\bot, \underline{\mathbf{c}}, \top$ of a Post algebra by the symbols $0, \frac{1}{2}, 1$, respectively. We have changed this usual notation in order to prevent confusion with our previous notation for the distinguished elements of ternary semigroups and real semigroups, which in due course will also enter into the picture. □

Proposition IV.1.4 *Let L be a three-valued Łukasiewicz algebra. Then the modal operators ∇ and Δ satisfy the following conditions: for all $x, y \in L$,*

(a) $\Delta x \le x \le \nabla x$.

(b) $\Delta\bot = \nabla\bot = \bot$, $\Delta\top = \nabla\top = \top$, *and if L has a center, $\underline{\mathbf{c}}$, then* $\Delta\underline{\mathbf{c}} = \bot$ *and* $\nabla\underline{\mathbf{c}} = \top$.

(c) Δ *and* ∇ *are lattice homomorphisms, i.e.,* $\Delta(x \vee y) = \Delta x \vee \Delta y$ *and* $\Delta(x \wedge y) = \Delta x \wedge \Delta y$*; similar equalities hold for* ∇.

(d) $\Delta^2 x = \Delta x$ *and* $\nabla^2 x = \nabla x$.

(e) $\nabla\Delta x = \Delta x$ *and* $\Delta\nabla x = \nabla x$.

(*f*) $\nabla x \wedge \neg \nabla x = \bot$ *and* $\Delta x \vee \neg \Delta x = \bot$.

(*g*) $\nabla x = x$ *if and only if* $x \vee \neg\, x = \top$ *and* $\Delta x = x$ *if and only if* $x \wedge \neg\, x = \bot$.

(*h*) *If* $\Delta x = \Delta y$ *and* $\nabla x = \nabla y$ *then* $x = y$.

(*i*) *If* L *has a center,* $\underline{\mathbf{c}}$*, then* $x = (\underline{\mathbf{c}} \wedge \nabla x) \vee \Delta x = (\underline{\mathbf{c}} \vee \Delta x) \wedge \nabla x$. □

Remark IV.1.5 Elements of a Łukasiewicz algebra verifying $x \wedge \neg\, x = \bot$ (equivalently, $x \vee \neg\, x = \top$) are called **complemented** or **Boolean**. Under the induced operations the set of Boolean elements is a Boolean algebra, denoted $B(L)$. Note that $\nabla x, \Delta x \in B(L)$ for all $x \in L$ (IV.1.4 (f)). The modal operators Δ and ∇ are, in fact, characterized by: for $x \in L$,

$$\Delta x = \text{the largest Boolean element} \leq x,$$

$$\nabla x = \text{the smallest Boolean element} \geq x\,.$$

This is straightforward using IV.1.4 (g). Then, these operators are first-order definable in the language $\{\wedge\,, \vee\,, -\,, \bot\}$ (for the class of three-valued Łukasiewicz algebras). □

We shall now outline the theory of characters of Łukasiewicz and Post algebras, i.e. of homomorphisms of such structures with values in **3**. We employ the notions of a filter and a prime filter with their standard meaning in distributive lattices. In a Łukasiewicz algebra the operation

$$g(P) = \{a \in L \mid \neg\, a \notin P\}$$

defines an involution in the family of prime filters under the order of inclusion. Furthermore, the Kleene inequality (IV.1.2 (b)) implies that $P \subseteq g(P)$ or $g(P) \subseteq P$ for every prime filter P, and the existence of a center entails that $P \neq g(P)$ for every such P. We have:

Theorem IV.1.6 *Let* L *be a Łukasiewicz algebra. Then:*

(1) *Every prime filter of* L *is either minimal or maximal* (*under inclusion*).

(2) *If* L *is a Post algebra, no prime filter of* L *is simultaneously maximal and minimal. Hence, the order structure of the set of prime filters of* L *under inclusion is a disjoint sum of 2-element chains.*

(3) *A prime filter* P *of* L *is minimal if and only if it is implicative* (*that is,* $x \in P$ *and* $x \rightarrow y \in P$ *imply* $y \in P$).

(4) *If* L *is a Post algebra and* $h : L \longrightarrow \mathbf{3}$ *is a homomorphism of Post algebras, then* $h^{-1}[\{\top\}]$ *is a minimal prime filter and* $h^{-1}[\{\top, \underline{\mathbf{c}}\}]$ *is a maximal prime filter. Hence* $h^{-1}[\{\top, \underline{\mathbf{c}}\}] = g(h^{-1}[\{\top\}])$.

(5) *Conversely, every* minimal *prime filter* P *in a Post algebra induces a character by setting:*

$$h_P(x) = \begin{cases} \top & \text{if } x \in P \\ \underline{\mathbf{c}} & \text{if } x \in g(P) \backslash P \\ \bot & \text{if } x \notin g(P) \end{cases}$$

(6) *The correspondence* $P \longmapsto h_P$ *is a bijection.* □

A central result in the theory of Post algebras is an analog of Stone's representation theorem for Boolean algebras; we shall repeatedly use this result in the sequel. Let X_L denote the set of characters of a Post algebra L. The topology induced on X_L by $\mathbf{3}^L$ (product of the discrete topology in **3**) makes it a Boolean space.

Theorem IV.1.7 (Representation Theorem for Post algebras; [BD], Thm. X.4.5, p. 198).

(i) Let X be a Boolean space. Then, the set $\mathcal{C}(X) = \mathcal{C}(X, \mathbf{3})$ of continuous functions of X into $\mathbf{3}$ under pointwise operations is a Post algebra.

(ii) If L is a Post algebra, the evaluation map $ev : L \longrightarrow \mathcal{C}(X_L)$ defined by $ev(a)(h) = h(a)$ for $a \in L$, $h \in X_L$, is a Post-algebra isomorphism. Hence, every Post algebra is isomorphic to one of the form $\mathcal{C}(X)$, where X is a Boolean space. $\square$

Remark. It can be shown (same reference) that the Boolean space X_L in (ii) is the Stone space of the Boolean algebra $B(L)$ of Boolean elements of L. $\square$

It follows from this theorem that every finite Post algebra is of the form 3^n for some positive natural number n. It also follows that if L is a Post algebra and a, b are elements of L such that $a \not\leq b$, then there exists a character $h \in X_L$ such that $h(a) \not\leq h(b)$, i.e., $h(a) > h(b)$. This last remark is frequently used in the sequel.

It is well known in the theory of Post algebras that there exists a natural correspondence between the set $Con(P)$ of congruences —i.e., equivalence relations compatible with the Post-algebra operations— and the family $\mathcal{I}(P)$ of lattice ideals of P closed under the operator ∇. More precisely,

Proposition IV.1.8 (ADD REF.!) *Given a Post algebra P and $I \in \mathcal{I}(P)$, define an equivalence relation $\equiv_I$ in P, as follows: for $a, b \in P$,*

$$a \equiv_I b \quad \textit{if and only if} \quad \Delta a \mathbin{\underline{\triangle}} \Delta b \in I \ \textit{ and } \ \nabla a \mathbin{\underline{\triangle}} \nabla b \in I,$$

(where $\underline{\triangle}$ denotes symmetric difference). Then, the correspondence $I \longmapsto \equiv_I$ is a bijection between $\mathcal{I}(P)$ and $Con(P)$. $\square$

Closing this section we introduce the notion of a Kleene algebra and a basic construction of Kleene algebras. This material will be of use later in this text, especially in the study of spectral real semigroups (§ V.7).

Definition IV.1.9 A **Kleene algebra with a center** is a structure $(K, \wedge, \vee, \neg, \bot, \underline{\mathbf{c}})$ satisfying the following requirements:

[K1] $(K, \wedge, \vee)$ is a distributive lattice with first element $\bot$.

[K2] $\neg$ (negation) is a unary operation verifying, for all $x, y \in K$:

(a) $\neg\neg x = x$; (b) $\neg(x \wedge y) = \neg x \vee \neg y$; (c) (Kleene inequality) $x \wedge \neg x \leq y \vee \neg y$.

[K3] $\underline{\mathbf{c}}$ is a distinguished element (called the **center**) verifying $\underline{\mathbf{c}} = \neg\underline{\mathbf{c}}$. $\square$

Note that $\top := \neg\bot$ is the largest element of K, the center is unique, $x \wedge \neg x \leq \underline{\mathbf{c}} \leq y \vee \neg y$, and $\neg(x \vee y) = \neg x \wedge \neg y$ —the dual to [K2] (b)— holds.

Since we shall only consider Kleene algebras with a center, we omit the modifier.

IV.1.10 Construction. There is a classical construction of a Kleene algebra from any bounded distributive lattice $(L, \wedge, \vee, \bot, \top)$, due to Kalman [Kal]. Let $L^{\rm inv}$ denote the distributive lattice inverse to L, i.e., the lattice with the same underlying set, where order is reversed and the operations and constants $\wedge, \vee, \bot, \top$ become their duals: $\wedge^{\rm inv} := \vee$, $\bot^{\rm inv} := \top$, etc. Consider the set

$$K(L) := \{(x, y) \in L \times L^{\rm inv} \mid x \wedge y = \bot \text{ (in } L)\},$$

with operations $\wedge, \vee, \neg$ defined, in terms of those of L, as follows:

$(x,y) \vee (x',y') := (x \vee y, x' \wedge y')$, $\quad (x,y) \wedge (x',y') := (x \wedge y, x' \vee y')$ $\quad$ and $\quad \neg\,(x,y) := (y,x)$.

One checks without difficulty that, with these operations $K(L)$ is a Kleene algebra having $(\bot\,,\top)$ as its first element, $(\top\,,\bot)$ as a last element, and $(\bot\,,\bot)$ as its center. $\square$

The Kleene algebras of form $K(L)$ admit a purely algebraic characterization:

Proposition IV.1.11 *Let K be a Kleene algebra and let $\underline{\mathbf{c}}$ denote its center. The following are equivalent:*

(1) *There is a bounded distributive lattice such that $K \simeq K(L)$.*

(2) *K verifies the following condition:*

[dec][2] *For all $a, b \in K$ such that $a \wedge b = \underline{\mathbf{c}}$, there is $t \in K$ so that $t \vee \underline{\mathbf{c}} = a$ and $\neg\, t \vee \underline{\mathbf{c}} = b$.*

Proof. (1) $\Rightarrow$ (2). Let L be a bounded distributive lattice; we prove that $K(L)$ verifies condition [dec]. With notation as in IV.1.10, let $a = (x,y)$, $b = (x',y') \in K(L)$ be such that $(x,y) \wedge (x',y') = (\bot\,,\bot)$. Then, $x \wedge y = x' \wedge y' = x \wedge x' = \bot$ and $y \vee y' = \bot$, which implies $y = y' = \bot$. Setting $t := (x,x')$, we have $t \vee (\bot\,,\bot) = (x,\bot) = a$ and $\neg\, t \vee (\bot\,,\bot) = (x',x) \vee (\bot\,,\bot) = (x',\bot) = b$, as required.

(2) $\Rightarrow$ (1). Let K be a Kleene algebra verifying [dec] and let $L = \{x \in K \,|\, x \geq \underline{\mathbf{c}}\}$. Clearly, L is a bounded distributive lattice with smallest element $\underline{\mathbf{c}}$ and largest element $\top$. To show $K \simeq K(L)$, let $h : K \longrightarrow K(L)$ be the map

$$h(x) = (x \vee \underline{\mathbf{c}}\,, \neg\, x \vee \underline{\mathbf{c}}) \quad \text{for } x \in K\,.$$

h takes on values in $K(L)$ because $\underline{\mathbf{c}} \leq x \vee \underline{\mathbf{c}}, \neg\, x \vee \underline{\mathbf{c}}$ and $(x \vee \underline{\mathbf{c}}) \wedge (\neg\, x \vee \underline{\mathbf{c}}) = (x \wedge \neg\, x) \vee \underline{\mathbf{c}} = \underline{\mathbf{c}}$. Routine checking proves that h is a Kleene algebra homomorphism. Further,

— h is injective. Let $x, y \in K$ be such that $x \vee \underline{\mathbf{c}} = y \vee \underline{\mathbf{c}}$ and $\neg\, x \vee \underline{\mathbf{c}} = \neg\, y \vee \underline{\mathbf{c}}$; by the dual to [K2.b], $x \wedge \underline{\mathbf{c}} = y \wedge \underline{\mathbf{c}}$. But it is well known that, in a distributive lattice, for any z, $x \vee z = y \vee z$ and $x \wedge z = y \wedge z$ imply $x = y$.

— h is surjective. Let $(x,y) \in K(L)$; hence, $x \wedge y = \underline{\mathbf{c}}$. By assumption (2) there is $t \in K$ so that $t \vee \underline{\mathbf{c}} = x$ and $\neg\, t \vee \underline{\mathbf{c}} = y$, whence $h(t) = (x,y)$. $\square$

IV.2 The real semigroup structure of a Post algebra

In this section we describe how every Post algebra can be endowed with a ternary relation which makes it into a RS. We first define its ternary semigroup structure.

As in the case of Boolean algebras, **symmetric difference**, $\underline{\triangle}$, is defined in Post algebras as follows:

$$x \,\underline{\triangle}\, y = (x \wedge \neg\, y) \vee (y \wedge \neg\, x).$$

It is easy to check that $\underline{\triangle}$ satisfies the following conditions:

(i) $\underline{\triangle}$ is commutative; $\qquad$ (ii) $x \,\underline{\triangle}\, \bot = x$; $\qquad$ (iii) $x \,\underline{\triangle}\, \top = \neg\, x$.

Kleene's inequality IV.1.2 (b) also yields:

(iv) $x \,\underline{\triangle}\, \underline{\mathbf{c}} = \underline{\mathbf{c}}$; $\qquad$ (v) $x = \neg\, x$ implies $x = \underline{\mathbf{c}}$.

Below we prove:

[2] [dec] = "decomposition".

(vi) $\underline{\triangle}$ is associative; (vii) $x \,\underline{\triangle}\, x \,\underline{\triangle}\, x = x$.

These observations amount to:

Proposition IV.2.1 *Let P be a Post algebra. Then $(P, \underline{\triangle}, \bot, \underline{\mathbf{c}}, \top)$ is a ternary semigroup where $\bot$ is the unit, $\underline{\mathbf{c}}$ is the absorbent element, and $\top$ is the distinguished element -1 (cf. Definition IV.1.1).*

Proof. It only remains to prove:

(vi) $\underline{\triangle}$ is associative.

Let $x, y, z \in P$. Then, $x \,\underline{\triangle}\, (y \,\underline{\triangle}\, z) = x \,\underline{\triangle}\, ((y \wedge \neg z) \vee (z \wedge \neg y))$. The De Morgan laws and distributivity give:

$x \,\underline{\triangle}\, (y \,\underline{\triangle}\, z) = (x \wedge \neg y \wedge \neg z) \vee (x \wedge y \wedge z) \vee (x \wedge \neg y \wedge y) \vee (x \wedge z \wedge \neg z) \vee (y \wedge \neg x \wedge \neg z) \vee (z \wedge \neg x \wedge \neg y)$.

In the same way we get:

$(x \,\underline{\triangle}\, y) \,\underline{\triangle}\, z = (x \wedge \neg y \wedge \neg z) \vee (x \wedge y \wedge z) \vee (z \wedge \neg y \wedge y) \vee (z \wedge x \wedge \neg x) \vee (y \wedge \neg x \wedge \neg z) \vee (z \wedge \neg x \wedge \neg y)$.

Note that the terms $(x \wedge \neg y \wedge \neg z)$, $(x \wedge y \wedge z)$, $(y \wedge \neg x \wedge \neg z)$, $(z \wedge \neg x \wedge \neg y)$ appear in both formulas. On the other hand, Kleene's inequality IV.1.2 (b) implies $x \wedge y \wedge \neg y \leq (x \wedge z) \vee (x \wedge \neg z)$. Hence, $x \wedge y \wedge \neg y \leq (x \wedge z \wedge y \wedge \neg y) \vee (x \wedge \neg z \wedge y \wedge \neg y) \leq (x \wedge y \wedge z) \vee (x \wedge \neg z \wedge \neg y) \leq (x \,\underline{\triangle}\, y) \,\underline{\triangle}\, z$. A similar reasoning proves $x \wedge z \wedge \neg z \leq (x \,\underline{\triangle}\, y) \,\underline{\triangle}\, z$. These inequalities, together with the remark above, show $x \,\underline{\triangle}\, (y \,\underline{\triangle}\, z) \leq (x \,\underline{\triangle}\, y) \,\underline{\triangle}\, z$, and a similar argument proves the reverse inequality, yielding $(x \,\underline{\triangle}\, y) \,\underline{\triangle}\, z = x \,\underline{\triangle}\, (y \,\underline{\triangle}\, z)$, as asserted.

(vii) $x \,\underline{\triangle}\, x \,\underline{\triangle}\, x = x$.

Since $x \,\underline{\triangle}\, x = x \wedge \neg x$, we have $x \,\underline{\triangle}\, x \,\underline{\triangle}\, x = x \,\underline{\triangle}\, (x \wedge \neg x) = (x \wedge (x \vee \neg x)) \vee ((x \wedge \neg x) \wedge \neg x) = x \vee (x \wedge \neg x) = x$. $\square$

IV.2.2 Remark on notation. In view of this Proposition, for notational consistency we will identify the chain $\{\bot, \underline{\mathbf{c}}, \top\}$ with the chain $1 < 0 < -1$; i.e., $\bot$ will be identified with 1, $\underline{\mathbf{c}}$ with 0, and $\top$ with -1. The distinguished elements in general Post algebras continue to be denoted by $\bot$, $\underline{\mathbf{c}}$, and $\top$. $\square$

Note that in a Post algebra an element x is invertible for symmetric difference if and only if it is Boolean: $x \,\underline{\triangle}\, x = \bot \;\Leftrightarrow\; x \wedge \neg x = \bot$.

Definition IV.2.3 For a Post algebra P, define a ternary relation as follows: for $x, y, z \in P$,

$$x \in D_P(y, z) \;\Leftrightarrow\; y \wedge z \wedge \underline{\mathbf{c}} \leq x \leq y \vee z \vee \underline{\mathbf{c}}. \qquad \square$$

Remark. In the particular case that $x \in B(P)$ the formula above reduces to:

$$x \in D_P(y, z) \;\Leftrightarrow\; y \wedge z \leq x \leq y \vee z.$$

[<u>Proof</u>. For the non-trivial implication $(\Rightarrow)$, assuming $x \in D_P(y, z)$, i.e., $y \wedge z \wedge \underline{\mathbf{c}} \leq x \leq y \vee z \vee \underline{\mathbf{c}}$, applying to these inequalities the operators Δ, ∇, and using $\Delta x = \nabla x = x$, we get $\nabla y \wedge \nabla z \leq x \leq 1$ and $0 \leq x \leq \Delta y \vee \Delta z$; the conclusion follows, then, from $y \wedge z \leq \nabla y \wedge \nabla z$ and $\Delta y \vee \Delta z \leq y \vee z$.]

Thus, restricting the representation relation D_P to the Boolean algebra $B(P)$ we retrieve the formula obtained in [DM1], Cor. 7.13, p. 149, for binary representation in BA's. This shows:

Fact IV.2.4 *Let P be a Post algebra. The structure $(B(P), \underline{\triangle}, D_P, \bot, \top)$ is a Boolean algebra. In particular, it is a reduced special group.* $\square$

The next Proposition is crucial for the proof of many subsequent results. Note that, by Theorem I.5.4, items (ii) and (iii) of this result are available as soon as we know that a Post algebra is a RS (which we don't yet); the point is that we shall use the Proposition precisely to establish this fact, in Theorem IV.2.7 below. To prove IV.2.5 we shall then compute using characters, which in turn rely on the structure of the prime ideals in a Post algebra, set out in Theorem IV.1.6.

Proposition IV.2.5 *Let P be a Post algebra, and let $x, y, z \in P$. Then,*

(i) $x \leq y$ *if and only if for for each Post-algebra character h, $h(x) \leq h(y)$ (in* **3***).*

(ii) $x \in D_P(y, z)$ *if and only if for each Post-algebra character h, either $h(x) = 0$, or $h(x) = h(y)$, or $h(x) = h(z)$.*

(iii) $x \in D^t_P(y, z)$ *if and only if for each Post-algebra character h, either $h(x) = 0$ and $h(y) = \neg\, h(z)$, or $h(x) = h(y)$, or $h(x) = h(z)$.* □

Proof. (i) Immediate consequence of the Representation Theorem IV.1.7 (ii).

(ii) (⇒) Suppose $x \in D_P(y, z)$, and take $h \in X_P$. From $y \wedge z \wedge \underline{\mathbf{c}} \leq x \leq y \vee z \vee \underline{\mathbf{c}}$ (IV.2.3), follows $h(y) \wedge h(z) \wedge 0 \leq h(x) \leq h(y) \vee h(z) \vee 0$. Let $h(x) \neq 0$. If $h(x) = 1$, the inequality $h(y) \wedge h(z) \wedge 0 \leq h(x)$ yields $h(y) = 1$ or $h(z) = 1$, while $h(x) = -1$ and $h(x) \leq h(y) \vee h(z) \vee 0$ yield $h(y) = -1$ or $h(z) = -1$. In either case we conclude $h(x) = h(y)$ or $h(x) = h(z)$.

To prove the other implication assume $x \notin D_P(y, z)$. Hence either $x \not\leq y \vee z \vee \underline{\mathbf{c}}$ or $y \wedge z \wedge \underline{\mathbf{c}} \not\leq x$. In the first case, say, item (i) shows there is a character $h \in X_P$ such that $h(x) > h(y) \vee h(z) \vee 0$, i.e, $h(x) = -1$. The hypothesis gives, then, $h(y) = -1$ or $h(z) = -1$, a contradiction. A similar argument proves that $y \wedge z \wedge \underline{\mathbf{c}} \not\leq x$ leads to a contradiction, proving $x \in D_P(y, z)$ as required.

(iii) Recall that $x \in D^t_P(y, z)$ if and only if $x \in D_P(y, z)$, $\neg\, y \in D_P(\neg x, z)$, and $\neg\, z \in D_P(y, \neg\, x)$. Suppose first $x \in D^t_P(y, z)$. By (ii) it suffices to prove that $h(x) = 0$ implies $h(y) = \neg\, h(z)$. If $h(y) \neq 0$, then $h(\neg\, y) \neq 0$. Since $\neg\, y \in D_P(\neg x, z)$, from (ii) and the assumption $h(x) = 0$ we infer in this case that $\neg\, h(y) = h(\neg\, y) = h(z)$. In case $h(y) = 0$, the relation $\neg\, z \in D_P(y, \neg\, x)$ yields $h(y) = \neg\, h(z) = 0$. The converse is easily proved by repeatedly using item (ii). □

IV.2.6 Comment. (On 3-valued "truth-table" checking.) Proposition IV.2.5 makes the old and well-known method of "truth-table checking" available to prove the validity of certain "atomic" formulas —such as inequalities, representations, transversal representations and, of course, equalities— in all Post algebras, and for all values of the variables occuring in them, by the expedient of checking their validity in **3** (for all assignments of values therein to their variables). Complex combinations ("terms") of the primitives $\wedge$, $\vee$, $\underline{\triangle}$, $\neg$, ∇, Δ, $\perp$, $\underline{\mathbf{c}}$, $\top$, of the language for Post algebras may occur in these relations. Concrete examples occur, e.g., in Theorem IV.2.7 (i), Proposition IV.2.10, Lemma IV.5.10, and other places below. When the validity of a formula can be routinely established in this way (which not always is the case), we shall say that the formula is proved by "truth-table checking", and often omit details. □

Theorem IV.2.7 *Let P be a Post algebra. Then,*

(i) Transversal representation takes the following form: for $x, y, z \in P$,

$$x \in D^t_P(y, z) \;\Leftrightarrow\; (y \wedge \nabla z) \vee (z \wedge \nabla y) \leq x \leq (y \vee \Delta z) \wedge (z \vee \Delta y).$$

(*ii*) *The structure* $(P, \underline{\triangle}, \perp, \underline{\mathbf{c}}, \top, D_P)$ *is a real semigroup.*

(*iii*) *Its associated abstract real spectrum is the Boolean space* X_P.

Proof. Throughout this proof we omit the subscript P from the notation.

(i) ($\Rightarrow$). Assume $x \in D^t(y,z)$ and $s := (y \wedge \nabla z) \vee (z \wedge \nabla y) \not\leq x$. By IV.2.5 (i) there is a character $h \in X_P$ such that $h(s) > h(x)$. We argue according to the values of $h(x)$.

— $h(x) = 1$. In this case $x \in D^t(y,z)$ entails $h(y) = 1$ or $h(z) = 1$. In either case, $h(s) = 1$, a contradiction.

— $h(x) = 0$. It follows that $h(s) = -1$, and (from $x \in D^t(y,z)$) $h(y) = -h(z)$. But $h(s) = -1$ implies that one of $h(y \wedge \nabla z)$ or $h(z \wedge \nabla y)$ is -1. If the first is -1, we have $h(y) = h(\nabla z) = -1$, whence $h(z) = -h(y) = 1$, which implies $h(\nabla z) = 1$, contradiction. Same argument if the second term is -1. This contradiction shows that $s \leq x$, and a similar argument proves the inequality $x \leq (y \vee \Delta z) \wedge (z \vee \Delta y)$.

($\Leftarrow$). Conversely, the relation $x \in D^t(y,z)$ is proved by use of IV.2.5 (iii), arguing by cases according to the values of characters h at x.

— $h(x) = 1$. Then, the left-hand side of $(y \wedge \nabla z) \vee (z \wedge \nabla y) \leq x$ also takes value 1 at h, which implies $h(y \wedge \nabla z) = h(z \wedge \nabla y) = 1$; in turn, this entails that one of $h(y)$ or $h(z)$ equals 1.

— $h(x) = -1$. The same argument, using $x \leq (y \vee \Delta z) \wedge (z \vee \Delta y)$ instead, shows that one of $h(y)$ or $h(z)$ is -1.

— $h(x) = 0$. We now argue according to the values of $h(y)$, to show $h(y) = -h(z)$.

a) $h(y) = 0$. If $h(z) = 1$, then $h(y \vee \Delta z) = 0$ and $h(z \vee \Delta y) = 1$, whence $h((y \vee \Delta z) \wedge (z \vee \Delta y)) = 1$; the right-hand side inequality in (i) yields $h(x) = 1$, a contradiction. In a similar way, $h(z) = -1$ contradicts the left-hand side inequality in (i). This shows that $h(z) = 0$, whence $h(y) = -h(z)$.

b) $h(y) = 1$. The right-hand side inequality in (i) implies $h(x) = 0 \leq h(\Delta z) \wedge h(z) = \Delta h(z)$, which in turn yields $h(z) = -1$, whence $h(z) = -h(y)$.

c) $h(y) = -1$. The argument in (b), using instead the left-hand side inequality in (i), proves $h(z) = 1 = -h(y)$.

(ii) The verification of all axioms for RSs, except [RS3], is straightforward truth-table checking, using the characterizations of representation and transversal representation given in Proposition IV.2.5, and is left to the reader.

For [RS3], we prove weak associativity and $D^t(y,z) \neq \emptyset$ for all $y, z \in P$ which, together, (in presence of the other axioms) are equivalent to strong associativity; cf. I.2.4.

a) For all $y, z \in P$, $D^t(y,z) \neq \emptyset$.

By the characterization of $D^t(y,z)$ in item (i), it suffices to prove

$$(y \wedge \nabla z) \vee (z \wedge \nabla y) \leq (y \vee \Delta z) \wedge (z \vee \Delta y),$$

which, in turn, amounts to proving that each term of the disjunction in the left-hand side is $\leq$ than each term of the conjunction in the right-hand side. By symmetry it is enough to show:

$$(1)\ \ y \wedge \nabla z \ \leq \ y \vee \Delta z \qquad \text{and} \qquad (2)\ \ y \wedge \nabla z \ \leq \ z \vee \Delta y.$$

(1) is trivial. (2) is proved by truth-table checking, taking into account that elements of the

form ∇x, Δx are Boolean, and hence never take value 0 in **3**. To illustrate the argument we do this proof. Let $h \in X_P$. If $h(y \wedge \nabla z) = 1$ or $h(z \vee \Delta y) = -1$ there is nothing to prove.

— If $h(z \vee \Delta y) = 1$, then $h(z) = h(\Delta y) = 1$. Since $h(y \wedge \nabla z) \in \{0, -1\}$ and $h(\nabla z) \neq 0$, then $h(\nabla z) = -1$, which implies $h(z) \neq 1$, contradiction.

— If $h(z \vee \Delta y) = 0$, since $h(\Delta y) \neq 0$, then $h(z) = 0$. But $h(\Delta y) \neq -1$ (else $h(z \vee \Delta y) = -1$), whence $h(\Delta y) = 1$, which implies $h(y) \in \{0, 1\}$. But $h(y) = 1$ gives $h(y \wedge \nabla z) = 1$, excluded by assumption. Then, $h(y) = 0$, and hence $h(y \wedge \nabla z) = h(y) \wedge \nabla h(z) = 0 \wedge \nabla 0 = 0 = h(z \vee \Delta y)$.

b) Weak associativity. If $x \in D(y, z)$ and $z \in D(s, t)$, there is $u \in P$ so that $u \in D(y, s)$ and $x \in D(u, t)$.

The hypotheses amount to (Definition IV.2.3):

$$(*)\ \ y \wedge z \wedge \underline{\mathbf{c}} \leq x \leq y \vee z \vee \underline{\mathbf{c}} \qquad \text{and} \qquad (**)\ \ s \wedge t \wedge \underline{\mathbf{c}} \leq z \leq s \vee t \vee \underline{\mathbf{c}}\,.$$

We claim that the element $u = (\Delta x \wedge \neg \Delta t) \vee (y \wedge s)$ does the job. Then, we must prove:

$$(\dagger)\ \ y \wedge s \wedge \underline{\mathbf{c}} \leq u \leq y \vee s \vee \underline{\mathbf{c}} \qquad \text{and} \qquad (\dagger\dagger)\ \ u \wedge t \wedge \underline{\mathbf{c}} \leq x \leq u \vee t \vee \underline{\mathbf{c}}\,.$$

Proof of ($\dagger$). The left inequality is clear by the definition of u. For the other it is enough to show $\Delta x \wedge \neg \Delta t \leq y \vee s \vee \underline{\mathbf{c}}$. Since $\Delta \underline{\mathbf{c}} = \bot$, applying the operator Δ to (*) and (**) gives $\Delta x \leq \Delta y \vee \Delta s \vee \Delta t \leq y \vee s \vee \Delta t$. Hence, $\Delta x \wedge \neg \Delta t \leq (y \wedge \neg \Delta t) \vee (s \wedge \neg \Delta t) \vee (\Delta t \wedge \neg \Delta t) \leq y \vee s \leq y \vee s \vee \underline{\mathbf{c}}$, as required.

Proof of ($\dagger\dagger$). For the left inequality we have:

$$(\natural) \qquad \begin{aligned} u \wedge t \wedge \underline{\mathbf{c}} &= ((\Delta x \wedge \neg \Delta t) \vee (y \wedge s)) \wedge t \wedge \underline{\mathbf{c}} = (\Delta x \wedge \neg \Delta t \wedge t \wedge \underline{\mathbf{c}}) \vee (y \wedge s \wedge t \wedge \underline{\mathbf{c}}) \\ &\leq x \vee (y \wedge s \wedge t \wedge \underline{\mathbf{c}}). \end{aligned}$$

On the other hand, the left inequalities in (*) and (**) give $x \geq y \wedge z \wedge \underline{\mathbf{c}} \geq y \wedge s \wedge t \wedge \underline{\mathbf{c}}$, which prove that the last term in ($\natural$) is $\leq x$.

Finally, for the right inequality in ($\dagger\dagger$) we note that

$$u \vee t \vee \underline{\mathbf{c}} = (\Delta x \wedge \neg \Delta t) \vee (y \wedge s) \vee t \vee \underline{\mathbf{c}} = (\Delta x \vee (y \wedge s) \vee t \vee \underline{\mathbf{c}}) \wedge (\neg \Delta t \vee t \vee \underline{\mathbf{c}} \vee (y \wedge s));$$

since $\neg \Delta t \vee t = \top$ (axiom [L3,(i)], IV.1.1), we get $u \vee t \vee \underline{\mathbf{c}} = \Delta x \vee \underline{\mathbf{c}} \vee t \vee (y \wedge s)$. Now, Proposition IV.1.4 (i) yields $x = (\underline{\mathbf{c}} \vee \Delta x) \wedge \nabla x \leq \Delta x \vee \underline{\mathbf{c}}$, and hence $x \leq u \vee t \vee \underline{\mathbf{c}}$, completing the proof of ($\dagger\dagger$), and that Post algebras are real semigroups.

(iii) With notation as in the Duality Theorem I.5.1 the ARS dual to the real semigroup $P = (P, \underline{\triangle}, D_P, \ldots)$ is $(X_P, \overline{P})$, where $\overline{P} = \{\overline{a} \,|\, a \in P\}$, $\overline{a} = ev(a)$ denotes evaluation at a, X_P is the set of RS-characters of P, and representation in $\overline{P}$ is given by:

$$\begin{aligned} \overline{a} \in D_{X_P}(\overline{b}, \overline{d}) \ &\Leftrightarrow\ \forall h \in X_P\, (\overline{a}(h) = 0 \vee \overline{a}(h) = \overline{b}(h) \vee \overline{a}(h) = \overline{d}(h) \\ &\Leftrightarrow\ \forall h \in X_P\, (h(a) = 0 \vee h(a) = h(b) \vee h(a) = h(d). \end{aligned}$$

Via the obvious identification of P with $\overline{P}$, it suffices to prove that D_{X_P} coincides with the relation D_P defined in IV.2.3. But this is exactly the content of item (ii) in Proposition IV.2.5 (modulo the fact, easily verified by truth-table checking, that Post-algebra characters coincide with RS-characters for $(P, \underline{\triangle}, D_P, \ldots)$; cf. Proposition IV.2.11 below). $\square$

Notation IV.2.8 To keep matters straight, real semigroups of the type $(P, \underline{\triangle}, \bot, \underline{\mathbf{c}}, \top, D_P)$ where P is a Post algebra, will be called **RS-Post algebras**, abbreviated **RS-PA**.

Remark IV.2.9 [REVISE!!] The foregoing theorem says, in particular, that the dual spectra of Post algebras are zero-dimensional ARS's, see [M], §7.6; indeed, in the terminology of this monograph, they coincide (CAREFUL!) with the "zero-dimensional real-closed" ARS's mentioned in [M], Ex. (1), p. 181. The ∇ operator occurs, in a somewhat hidden form, in Lemmas 7.6.2 and 7.6.3 of [M], pp. 144–145. The operation ∇ is, in fact, definable (*within the class of* RS*s which are Post algebras*) in terms of the relations D_P and D^t_P; see Theorem IV.3.5 below. $\square$

The next result exhibits some elementary relations between Post-algebra operations and the representation relation defined in IV.2.3. It follows at once from Proposition IV.2.5.

Proposition IV.2.10 *Let P be a Post algebra and let $x, y, z \in P$. Then:*

(i) $x \leq y$ *if and only if* $x \in D_P(\bot, y)$ *and* $\neg y \in D_P(\bot, \neg x)$*.*

(ii) $x \wedge y \in D_P(x, y)$ *and* $x \vee y \in D_P(x, y)$*.*

(iii) $x \in D_P(\nabla x, \nabla x)$ *and* $x \in D_P(\Delta x, \Delta x)$*.*

Proof. (i) The validity of (i) in **3** is proved by routine truth-table checking, using Corollary I.2.5. Its validity in arbitrary Post algebras follows, then, from items (i) and (ii) of Proposition IV.2.5.

(ii) is obvious using Definition IV.2.3.

(iii) Proposition IV.1.4 (i) implies $\nabla x \wedge \underline{\mathbf{c}} \leq x \leq \Delta x \vee \underline{\mathbf{c}}$. By use of Definition IV.2.3, this, together with $\Delta x \leq x \leq \nabla x$, shows that (iii) holds. $\square$

Remark. Compare item (i) in IV.2.10 with Definition I.6.2. $\square$

The following result shows that in the category of Post algebras homomorphisms coincide with RS-morphisms.

Proposition IV.2.11 *Let P_1, P_2 be Post algebras and let $f : P_1 \longrightarrow P_2$ be a map. The following are equivalent:*

(i) f is a morphism of real semigroups.

(ii) f is a Post-algebra homomorphism.

Proof. It is clear from the definition of the representation relation in IV.2.3 that every Post-algebra homomorphism is a morphism of real semigroups, so we have (ii) $\Rightarrow$ (i). The opposite implication is trickier.

(i) $\Rightarrow$ (ii). Let f be a morphism of real semigroups. In particular, f is a morphism of ternary semigroups, and then symmetric difference as well as the constants $\bot$, $\underline{\mathbf{c}}$, $\top$ are preserved. Since $\neg x = \top \,\underline{\triangle}\, x$, negation is also preserved. Since f preserves representation, Proposition IV.2.10 (i) implies that f is order-preserving. The observation that the Boolean elements of a Post algebra are exactly the units for $\underline{\triangle}$, entails $x \in B(P_1) \Rightarrow f(x) \in B(P_2)$.

It remains to prove that f preserves ∇ and the lattice operations $\wedge$, $\vee$.

a) For $x \in P_1$, $f(\nabla x) = \nabla f(x)$.

Since $x \leq \nabla x$ and f is order-preserving, we get $f(x) \leq f(\nabla x)$, and then $\nabla f(x) \leq \nabla f(\nabla x)$. Since $\nabla x \in B(P_1)$ (IV.1.4 (f)), we get $f(\nabla x) \in B(P_2)$, and IV.1.4 (g) gives $\nabla f(\nabla x) = f(\nabla x)$. We conclude that $\nabla f(x) \leq f(\nabla x)$.

On the other hand, $x \in D_{P_1}(\nabla x, \nabla x)$ (IV.2.10 (iii)), implies $f(x) \in D_{P_2}(f(\nabla x), f(\nabla x))$, which in particular gives $f(\nabla x) \wedge \underline{\mathbf{c}} \leq f(x)$ (IV.2.3). It follows that $\nabla(f(\nabla x) \wedge \underline{\mathbf{c}}) \leq \nabla f(x)$; since

$$\nabla(f(\nabla x) \wedge \underline{\mathbf{c}}) = \nabla f(\nabla x) \wedge \nabla \underline{\mathbf{c}} = \nabla f(\nabla x) = f(\nabla x),$$

we get $f(\nabla x) \leq \nabla f(x)$, and hence the equality asserted in (a).

b) For $x, y \in P_1$, $f(x \vee y) = f(x) \vee f(y)$.

The idea is to show

$$(\dagger)\ \ \Delta f(x \vee y) = \Delta(f(x) \vee f(y)) \qquad \text{and} \qquad (\dagger\dagger)\ \ \nabla f(x \vee y) = \nabla(f(x) \vee f(y)).$$

By Proposition IV.1.4 (h), (†) and (††) together imply the equality asserted in (b).

Clearly, $x, y \leq x \vee y$ and the fact that f is order-preserving imply:

(*) $f(x) \vee f(y) \leq f(x \vee y)$.

From $x \vee y \in D_{P_1}(x, y)$ (IV.2.10 (ii)), we get $f(x \vee y) \in D_{P_2}(f(x), f(y))$, which in turn gives $f(x \vee y) \leq f(x) \vee f(y) \vee \underline{\mathbf{c}}$. Applying Δ, we obtain $\Delta(f(x \vee y)) \leq \Delta(f(x)) \vee \Delta(f(y)) = \Delta(f(x) \vee f(y))$, since $\Delta\underline{\mathbf{c}} = \perp$ and Δ is a lattice homomorphism. The reverse inequality follows by applying Δ to (*), and proves (†).

In order to prove (††), first recall that $\nabla x \vee \nabla y \in D_{P_1}(\nabla x, \nabla y)$ (IV.2.10 (iii)). Since f preserves representation, from (a) we get:

$$f(\nabla x \vee \nabla y) \in D_{P_2}(f(\nabla x), f(\nabla y)) = D_{P_2}(\nabla f(x), \nabla f(y)).$$

By Definition IV.2.3 this condition gives, in particular:

$$f(\nabla x \vee \nabla y) \leq \nabla f(x) \vee \nabla f(y) \vee \underline{\mathbf{c}}.$$

Now, using the easily checked fact that, if r, s are Boolean elements, $r \leq s \vee \underline{\mathbf{c}} \Rightarrow r \leq s$, we obtain,

(**) $f(\nabla x \vee \nabla y) \leq \nabla f(x) \vee \nabla f(y)$.

Using (*) with x, y replaced by $\nabla x, \nabla y$, and invoking (a), we get:

(***) $\nabla f(x) \vee \nabla f(y) \leq f(\nabla x \vee \nabla y)$.

(**) and (***) show $f(\nabla x \vee \nabla y) = \nabla f(x) \vee \nabla f(y) = \nabla(f(x) \vee f(y))$, proving (††), and thus completing the proof of Theorem IV.2.11. □

Remark. Proposition IV.2.11 shows that the categories **PA** of Post algebras and **RS-PA** of RS-Post algebras (IV.2.8) have the same morphisms, differing only by their languages, i.e., by the choice of the primitive notions. The language of Post algebras is $\{\underline{\triangle}, \perp, \underline{\mathbf{c}}, \top, \nabla\}$ while that of RS-Post algebras is $\mathcal{L}_{\mathrm{RS}}$, the language of real semigroups. □

IV.3 Characterizations of Post algebras as real semigroups

New section; added Nov. 2011. Gathers together former Thms. III.2.12 and III.4.2.

In this section we give two characterizations of those real semigroups which are Post algebras. The first, Theorem IV.3.2, is done in terms of the character space (and also of transversal representation). The second, Theorem IV.3.5, does it in terms of the (first-order) definability of the lattice and the modal operations of a Post algebra in the language $\mathcal{L}_{\mathrm{RS}}$ for real semigroups.

We begin with a preliminary result, needed in the proof of Theorem IV.3.2, giving an algebraic characterization of those RSs whose saturated prime ideals are pairwise incomparable for inclusion.

Proposition IV.3.1 *Let G be a RS. Then, the following are equivalent:*

(a) *For all saturated prime ideals I, J, of G, $I \subseteq J \Rightarrow I = J$.*

(b) *For all $x \in G$ there exists $y \in D^t_G(1, -x^2)$ such that $xy = 0$.*

Proof. (a) ⇒ (b). Let $x \in G$. By axiom [RS5], $\mathrm{Ann}(x) = \{a \in G \,|\, ax = 0\}$ is a saturated ideal and, clearly, $T_x = D^t(1, -x^2)$ is a multiplicative set. The set $\overline{T_x} = T_x \cup x^2 T_x$ is also multiplicative. Suppose $\mathrm{Ann}(x) \cap \overline{T_x} = \emptyset$, and pick a saturated ideal I containing $\mathrm{Ann}(x)$ and maximal for the property of being disjoint with $\overline{T_x}$; such an I is prime. Since $1 \in T_x$, we have $x^2 \in \overline{T_x}$, and hence $x^2 \notin I$. Then, the saturated ideal $I[x^2]$ generated by $I \cup \{x^2\}$ properly contains I. We claim that $I[x^2]$ is proper. Otherwise, $1 \in I[x^2]$, which means $1 \in D(i, x^2 p)$ for some $i \in I$ and $p \in G$, see Proposition I.4.6 (1′). By [RS8], $1 \in D(i^2, x^2 p^2) \subseteq D(i^2, x^2)$, and then $1 \in D^t(i^2, x^2)$. It follows that $-i^2 \in D^t(-1, x^2)$, and therefore $i^2 \in I \cap D^t(1, -x^2)$, a contradiction. Any saturated prime ideal J containing $I[x^2]$ will contain I properly, contradicting assumption (a). This shows that $\mathrm{Ann}(x) \cap \overline{T_x} \neq \emptyset$. If $\mathrm{Ann}(x) \cap T_x \neq \emptyset$, there exists $y \in D^t_G(1, -x^2)$ such that $xy = 0$. If $\mathrm{Ann}(x) \cap x^2 T_x \neq \emptyset$, there is $z \in T_x = D^t(1, -x^2)$ such that $(x^2 z)x = xz = 0$, as required. In both cases assertion (b) holds.

(b) ⇒ (a). Suppose there are saturated prime ideals I, J of G so that $I \subset J$, and let $x \in J \setminus I$. Invoking (b), let $y \in D^t(1, -x^2)$ be such that $xy = 0$. Since I is prime and $x \notin I$, we have $y \in I$, and then $y \in J$. We get $-1 \in D^t(-y, -x^2)$, whence $1 \in D^t(y, x^2)$ with $y, x^2 \in J$; since J is saturated, $1 \in J$, absurd. □

Theorem IV.3.2 *Let G be a RS. The following are equivalent:*

(1) *G is a Post algebra.*

(2) *If $h_1, h_2, h_3 \in X_G$ and $h_1 h_2 h_3 \in X_G$, then $h_1 = h_2 = h_3$.*

(3) *If Y_1, Y_2 are (non-empty) disjoint closed subsets of X_G, there are $a, b \in G$ such that $a \lceil Y_1 = b \lceil Y_1 = 1$, $a \lceil Y_2 = -1$, and $b \lceil Y_2 = 0$.*

(4) *i)* *For all $x \in G$ there exists $y \in D^t_G(1, -x^2)$ such that $xy = 0$ (i.e., G verifies condition IV.3.1 (b)).*

ii) *For each $a \in G$ there are elements $x \in D^t_G(a^2, -a)$ and $y \in D^t_G(a^2, a)$ such that $xy = 0$.*

Proof. To ease notation we write $X := X_G$.

(1) ⇒ (3). Assume G is a Post algebra, and let Y_1, Y_2 be sets as in (3). Since X is a Boolean space, disjoint closed sets can be separated by clopens; let U_1, U_2 be disjoint clopens of X, necessarily non-empty, such that $Y_i \subseteq U_i$, $i = 1, 2$. Let $a, b : X \longrightarrow \mathbf{3}$ be defined by:

$$a(x) = \begin{cases} 1 & \text{if } x \in U_1 \\ -1 & \text{if } x \notin U_1, \end{cases} \qquad\qquad b(x) = \begin{cases} 1 & \text{if } x \in U_1 \\ 0 & \text{if } x \notin U_1. \end{cases}$$

Obviously, these maps are continuous, i.e., $a, b \in \mathcal{C}(X, \mathbf{3}) = G$, (cf. IV.1.7 (ii)), and have the required properties.

(3) ⇒ (2). Assume $h_1, h_2, h_3 \in X$ and $h_1 h_2 h_3 \in X$. Set $h := h_1 h_2 h_3$, $Y_1 = \{h_1\}$, and $Y_2 = \{h_2, h_3, h\}$. If $Y_1 \cap Y_2 = \emptyset$, condition (3) gives an element $a \in G$ such that $h_1(a) =$

1, $h_2(a) = h_3(a) = -1$ and $h(a) = h_2(a)h_3(a) = -1$, obviously impossible. Hence, $Y_1 \cap Y_2 \neq \emptyset$, which just means $h_1 = h_2$, or $h_1 = h_3$, or $h_1 = h$.

— In case $h_1 = h_2$ we have $h_2^2 h_3 \in X$. If $h_2^2 h_3 \neq h_2$, statement (3) applied to the sets $\{h_2^2 h_3\}$ and $\{h_2\}$ yields an element $b \in G$ verifying $h_2^2(b)\, h_3(b) = 1$ and $h_2(b) = 0$, a contradiction. Hence, $h_2^2 h_3 = h_2$. A similar argument shows that $h_2^2 h_3 = h_3$, proving that all three characters are equal. The same reasoning yields the desired conclusion also in case $h_1 = h_3$.

— If $h_1 = h = h_1 h_2 h_3$ and $h_1 \neq h_2$, by (3) we get a $c \in G$ verifying $h_2(c) = 0$ and $h_1(c) = 1$, which is impossible. Therefore, $h_1 = h_2$, and in a similar way we get $h_1 = h_3$, showing, again that all three characters are equal.

(2) $\Rightarrow$ (1). It suffices to prove $G = \mathcal{C}(X, \mathbf{3})$; only the inclusion $\supseteq$ requires proof. It follows from (2) that any $f \in \mathcal{C}(X, \mathbf{3})$ trivially preserves 3-products in X (cf. III.2.14). The small representation theorem III.2.15 shows, then, that f is represented by an element of G.

(1) $\Rightarrow$ (4). (4.i). Assume G is a Post algebra, and let $x \in G$. By Theorem IV.2.7 (i), $z \in D^t(1, -x^2)$ iff $z \leq \Delta(x \vee \neg\, x)$. Let $y = \Delta(x \vee \neg\, x) \wedge \underline{\mathbf{c}}$; hence $y \in D^t(1, -x^2)$. The claim that follows proves (4.i).

Claim. $xy = x \,\underline{\triangle}\, y = \underline{\mathbf{c}}$.

<u>Proof of Claim</u>. By the definition of symmetric difference, using the definition of y, and distributing, we have:

$$\begin{aligned} x \,\underline{\triangle}\, y &= [x \wedge (\neg\,\Delta(x \vee \neg\, x) \vee \underline{\mathbf{c}})] \vee [\Delta(x \vee \neg\, x) \wedge \underline{\mathbf{c}} \wedge \neg\, x] \\ &= (x \wedge \nabla(x \wedge \neg\, x)) \vee (x \wedge \underline{\mathbf{c}}) \vee (\Delta x \wedge \underline{\mathbf{c}} \wedge \neg\, x) \vee (\Delta \neg\, x \wedge \underline{\mathbf{c}} \wedge \neg\, x) \\ &= (x \wedge \nabla \neg\, x) \vee (x \wedge \underline{\mathbf{c}}) \vee (\Delta \neg\, x \wedge \underline{\mathbf{c}}), \end{aligned}$$

because $\Delta x \wedge \underline{\mathbf{c}} \wedge \neg\, x \leq \Delta x \wedge \neg\, x = 1$. On the other hand, it is easy to verify (e.g., by truth-table checking) that $x \wedge \nabla \neg\, x \leq \underline{\mathbf{c}} \leq x \vee \Delta \neg\, x$. Hence,

$$(x \wedge \nabla \neg\, x) \vee (x \wedge \underline{\mathbf{c}}) \vee (\Delta \neg\, x \wedge \underline{\mathbf{c}}) = (x \wedge \underline{\mathbf{c}}) \vee (\Delta \neg\, x \wedge \mathbf{c}) = (x \vee \Delta \neg\, x) \wedge \underline{\mathbf{c}} = \underline{\mathbf{c}},$$

as required.

(4.ii) Let $a \in G$. By Theorem IV.2.7 (i), $z \in D^t(a^2, -a)$ iff

(*) $\quad (\nabla(a \wedge \neg\, a) \wedge \neg\, a) \vee (a \wedge \neg\, a \wedge \nabla \neg\, a) \leq z \leq ((a \wedge \neg\, a) \vee \Delta \neg\, a) \wedge (\Delta(a \wedge \neg\, a) \vee \neg\, a)$.

Now, observe that $\nabla(a \wedge \neg\, a) \wedge \neg\, a = \nabla a \wedge \neg\, a$ and $a \wedge \neg\, a \wedge \nabla \neg\, a = a \wedge \neg\, a$. Substituting these identities in (*), the left-hand side boils down to $\nabla a \wedge \neg\, a$. Likewise, since $\Delta(a \wedge \neg\, a) = \perp$ and $(a \wedge \neg\, a) \vee \Delta \neg\, a = \neg\, a$, the right-hand side of (*) equals $\neg\, a$. Hence, $z \in D^t(a^2, -a)$ iff $\neg\, a \wedge \nabla a \leq z \leq \neg\, a$.

A similar computation proves: $z \in D^t(a^2, a)$ iff $a \wedge \nabla \neg\, a \leq z \leq a$.

Set $x = (a \wedge \nabla \neg\, a) \vee (a \wedge \underline{\mathbf{c}})$ and $y = (\neg\, a \wedge \nabla a) \vee (\neg\, a \wedge \underline{\mathbf{c}})$. The conditions just proved show that $x \in D^t(a^2, a)$ and $y \in D^t(a^2, -a)$. Straightforward truth-table checking shows that $x \,\underline{\triangle}\, y = \underline{\mathbf{c}}$, as required.

(4) $\Rightarrow$ (2). Let $h_1, h_2, h_3 \in X$ be such that $h_1 h_2 h_3 \in X$. By Lemma II.2.11 (2), $Z(h_1 h_2 h_3) = Z(h_i)$ for some $i \in \{1, 2, 3\}$, whence $Z(h_j) \subseteq Z(h_i)$ for all $j \in \{1, 2, 3\}$. Since G is assumed to verify condition (4.i), Proposition IV.3.1 implies that these three zero-sets are equal.

Suppose, towards a contradiction, e.g., $h_1 \neq h_2$. Since $Z(h_1) = Z(h_2)$, there is an $a \in G$ so that $h_1(a) = 1$ and $h_2(a) = -1$. Using (4.ii) take $x \in D^t(a^2, -a)$ and $y \in D^t(a^2, a)$ such that $xy = \underline{\mathbf{c}}$. From, $h_1(a) = 1$ comes $h_1(y) \in D^t_{\mathbf{3}}(h_1(a^2), h_1(a)) = D^t_{\mathbf{3}}(1, 1) = \{1\}$; similarly,

$h_2(a) = -1$ implies $h_2(x) = 1$. But $xy = \underline{\mathbf{c}}$ implies $h_1(x) = 0$, i.e., $x \in Z(h_1) \setminus Z(h_2)$, a contradiction. This shows $h_1 = h_2$, and a similar argument gives $h_2 = h_3$, proving condition (2), and ending the proof of Theorem IV.3.2. $\square$

Corollary IV.3.3 *Let P be a RS-Post algebra. The (spectral) space X_P of RS-characters of P is Boolean.*

Proof. It is well-known (cf. [DST], ADD REF.) that a spectral space is Boolean if and only if its order of specialization is "flat" (i.e., $x \rightsquigarrow y \Rightarrow x = y$).

Let $g, h \in X_P$ be such that $g \rightsquigarrow h$. From Lemma I.1.18 (5) we get $h = h^2 g \in X_P$, and IV.3.2 (2) then gives $g = h$, as required. $\square$

Corollary IV.3.4 *Let P be a RS-Post algebra. For every saturated prime ideal I of P there is a* unique *RS-character $g \in X_P$ such that $I = g^{-1}[0]$.*

Proof. Corollary I.4.9 shows that there is at least one such g. Assume $g, h \in X_P$ are such that $Z(g) = Z(h)$. By Lemma I.1.19, $g^2 = h^2$. Hence $h = h^2 h = g^2 h \in X_P$, and IV.3.2 (2) gives $g = h$, as required. $\square$

Theorem IV.3.5 *Let G be a RS and let $G^\times$ denote the set of invertible elements of G. Then, G is (the RS of) a Post algebra if and only if G verifies:*

(1) *For all $a \in G$ there is $x \in G^\times$ (necessarily unique) such that $a \in D_G(x, x)$ and $D_G(1, -a) \cap G^\times = D_G(1, -x) \cap G^\times$.*

Write ∇a for such x and set $\Delta a = -\nabla(-a)$.

(2) *For all $a, b \in G$ there exists $y \in G$ (necessarily unique) such that:*

$$\nabla y \in D_G(\nabla a, \nabla b), \qquad \Delta y \in D_G(\Delta a, \Delta b),$$
$$D^t_G(1, \nabla y) = D^t_G(1, \nabla a) \cap D^t_G(1, \nabla b), \qquad \text{and}$$
$$D^t_G(1, \Delta y) = D^t_G(1, \Delta a) \cap D^t_G(1, \Delta b).$$

If these conditions hold, then $y = a \wedge b$ (and hence join is given by $a \vee b = -(-a \wedge -b)$).

Proof. ($\Rightarrow$). First we prove that if G ($= P$) is the RS associated to a Post algebra, as in Theorem IV.2.7 (ii), then, for $a, b \in P$,

(a) ∇a is the unique $x \in P$ verifying clause (1) of the statement, and

(b) $a \wedge b$ is the unique $y \in P$ verifying clause (2) of the statement.

<u>Proof of (a)</u>. The definition of representation in P (IV.2.3) gives:

(*) $\qquad q \in D_P(\bot, p) \;\Leftrightarrow\; q \leq p \vee \underline{\mathbf{c}}$.

This, together with $\neg\nabla a \leq \neg\, a \leq \neg\, a \vee \underline{\mathbf{c}}$ (cf. IV.1.4 (b)), yields $D_P(\bot, \neg\nabla a) \subseteq D_P(\bot, \neg\, a)$. Conversely, let $p \in D_P(\bot, \neg\, a) \cap B(P)$ (note that $P^\times = B(P)$, cf. ....). Then, (*) gives $p \leq \neg\, a \vee \underline{\mathbf{c}}$, whence, $a \wedge \underline{\mathbf{c}} \leq \neg\, p$. Since p is a Boolean element and ∇ preserves the lattice operations, $\nabla a \leq \neg\, p$, whence $p \leq \neg\nabla a$, implying $p \in D_P(\bot, \neg\nabla a)$.

As for uniqueness, if $x \in B(P)$ verifies clause (1) of the statement, since $\neg\, x \in D_P(\bot, \neg\, x)$, we have $\neg\, x \in D_P(\bot, \neg\, a)$, whence, by (*), $\neg\, x \leq \neg\, a \vee \underline{\mathbf{c}}$; from $\neg\, x \in B(P)$, we get $\neg\, x =$

$\Delta\neg x \leq \Delta\neg a = \neg\nabla a$, and hence $\nabla a \leq x$. Also $a \in D_P(x,x)$ and the definition of D_P entail $x \wedge \underline{\mathbf{c}} \leq a$, whence $\nabla x \wedge \nabla \underline{\mathbf{c}} = \nabla x = x \leq \nabla a$. We have proved $x = \nabla a$.

Proof of (b). Let $y = a \wedge b$. Since ∇, Δ preserve meet, Proposition IV.2.10 (ii) gives $\nabla y \in D_G(\nabla a, \nabla b)$ and $\Delta y \in D_G(\Delta a, \Delta b)$. Further, from Theorem IV.2.7 (i) we have:

(**) $\quad q \in D^t_P(\bot, p) \;\Leftrightarrow\; (\bot \wedge \nabla p) \vee (p \wedge \nabla\bot) \leq q \leq (\bot \vee \Delta p) \wedge (p \vee \Delta\bot) \;\Leftrightarrow\; q \leq \Delta p.$

Using items (c), (e) and (d) of Proposition IV.1.4, this equivalence yields at once:

$$D^t_P(1, \nabla y) = D^t_P(1, \nabla a) \cap D^t_P(1, \nabla b), \quad \text{and} \quad D^t_P(1, \Delta y) = D^t_P(1, \Delta a) \cap D^t_P(1, \Delta b).$$

As for uniqueness, let $z \in P$ be another element satisfying clause (2) in the statement. In particular, we get $D^t_P(1, \nabla y) = D^t_P(1, \nabla z)$ and $D^t_P(1, \Delta y) = D^t_P(1, \Delta z)$. By (**), this implies $\nabla y = \nabla z$ and $\Delta y = \Delta z$, and then Proposition IV.1.4 (h) yields $y = z$.

($\Leftarrow$). We omit the subscript G in D_G, D^t_G. We begin by proving that the clauses (1), (2) of the statement imply:

(c) Under the representation partial order (I.6.4), G is a lattice with first element 1, last element -1 and $a \wedge b =$ the unique element y verifying (2). We shall need:

Fact. *For $a \in G$, setting $\nabla a = x$, where $x \in G^\times$ is as in item* (1) *of the statement, $\Delta a = \neg\nabla\neg a$, and with $\leq$ denoting the representation partial order of G, we have:*

(i) $\Delta a \leq a \leq \nabla a$;

(ii) $\nabla a \leq \nabla b$ *and* $\Delta a \leq \Delta b$ *imply* $a \leq b$;

(iii) $\nabla a = \nabla b$ *and* $\Delta a = \Delta b$ *imply* $a = b$.

Proof of Fact. (i) It suffices to prove $a \leq \nabla a\,(= x)$ for all $a \in G$. In fact, this inequality applied to $-a$ and the fact that the operation "$-$" reverses the order $\leq$ (I.6.4 (a)) yield $\Delta a \leq a$.

By the definition of the representation partial order, we must show $a \in D(1, \nabla a)$ and $-\nabla a \in D(1, -a)$. Since $-x \in D(1, -x)$ ([RS1]) and $-x$ is invertible, (1) implies $-x = -\nabla a \in D(1, -a)$. We also have $x^2 = 1$; passing to transversal representation we get $-x = -\nabla a \in D^t(1, -a)$, whence $a \in D^t(1, x) = D^t(1, \nabla a) \subseteq D(1, \nabla a)$.

(ii) Assume $\nabla a \leq \nabla b$ and $\Delta a \leq \Delta b$. Scaling the representation $-a \in D(\nabla(-a), \nabla(-a))$ (see (1)) by -1 yields $a \in D(\Delta a, \Delta a)$. Since $\Delta a \in D(1, \Delta b)$ (assumption $\Delta a \leq \Delta b$), we get $a \in D(1, \Delta b, 1, \Delta b) = D(1, \Delta b)$; from $\Delta b \leq b$, we then conclude that $a \in D(1, b)$. On the other hand, $b \in D(\nabla b, \nabla b)$ ((1)) and $\nabla a \leq \nabla b$ imply $-b \in D(-\nabla b, -\nabla b) \subseteq D(1, -\nabla a, 1, -\nabla a) = D(1, -\nabla a) \subseteq D(1, -a)$. Thus, $a \leq b$, as required.

(iii) follows at once from (ii). $\Box$

Proof of (c). Since ∇y is invertible, $\nabla y \in D^t(1, \nabla y)$, and (2) yields $\nabla y \in D^t(1, \nabla a) \cap D^t(1, \nabla b)$, implying $\nabla y \leq \nabla a, \nabla b$. Likewise, we get $\Delta y \leq \Delta a, \Delta b$, and item (ii) of the Fact shows that $y \leq a, b$.

Let $z \in G$ be so that $z \leq a, b$. Since $-\nabla a \in D(1, -\nabla a)$, $-\nabla a \leq -a$, $-a \in D(1, -z)$ and $-z \in D(-\nabla z, -\nabla z)$ (see (1)), by transitivity we get $-\nabla a \in D(1, -\nabla z, -\nabla z) = D(1, -\nabla z)$. Since $-\nabla a$ is invertible, it follows that $-\nabla a \in D^t(1, -\nabla z)$, which yields $\nabla z \in D^t(1, \nabla a)$. A similar argument gives $\nabla z \in D^t(1, \nabla b)$. From (2) we conclude $\nabla z \in D^t(1, \nabla y)$. We also get $-\nabla y \in D^t(1, -\nabla z)$, and these transversal representations together yield $\nabla z \leq \nabla y$. A similar argument proves that $\Delta z \leq \Delta y$. By item (ii) of the Fact, $z \leq y$, as required to prove $y = a \wedge b$.

Remark. Item (2) of the statement implies that the set $G^{\times}$ of invertible elements of G also is a lattice under the representation partial order. In fact, if $a, b \in G^{\times}$, we have $\nabla a = \Delta a = a$ and $\nabla b = \Delta b = b$. Let $y = a \wedge b$. These identities and the last two equalities in (2) give $D^t(1, \nabla y) = D^t(1, \Delta y)$, hence $\nabla y \in D^t(1, \Delta y)$. This implies $\nabla y \leq \Delta y$, and, by (i) of the Fact above, $\nabla y = \Delta y = y$, i.e., $y \in G^{\times}$. The join of $G^{\times}$ is, of course, $a \vee b = -(-a \wedge -b)$.

Before proving that the lattice $(G, \wedge, \vee)$ is distributive, we check that the operator ∇ defined above verifies the axioms [L3] for a three-valued Łukasiewicz algebra (Definition IV.1.1). (The axioms [L2] are obvious, since $a \vee b$ is defined to be $-(-a \wedge -b)$.)

[L3 (i)]. Truth-table checking in **3** shows that $D^t(1, a) \cap D^t(1, -a) = \{1\}$ for all a. It follows from (2) that $D^t(1, \nabla(\nabla a \wedge -\nabla a)) = \{1\}$. Since $\nabla(\nabla a \wedge -\nabla a) \in D^t(1, \nabla(\nabla a \wedge -\nabla a))$ we conclude $\nabla(\nabla a \wedge -\nabla a) = 1$, and then $\nabla a \wedge -\nabla a = 1$. Since $a \leq \nabla a$, we get $a \wedge -\nabla a = 1 = \bot$, and hence $-a \vee \nabla a = \top$.

[L3 (iii)]. Let $a, b \in G$. Condition (1) implies at once that $p \in G^{\times} \Leftrightarrow \nabla p = p \Leftrightarrow \Delta p = p$. Set $y = a \wedge b$ and $z = \nabla(a \wedge b) = \nabla y$. Then, $\nabla z = \Delta z = \nabla y$, and by use of IV.1.4 (d), (e), assumption (2) shows that the element z verifies:

$\nabla z = \nabla y \in D(\nabla a, \nabla b) = D(\nabla\nabla a, \nabla\nabla b); \qquad \Delta z = \nabla y \in D(\nabla a, \nabla b) = D(\Delta\nabla a, \Delta\nabla b);$

$D^t(1, \nabla z) = D^t(1, \nabla y) = D^t(1, \nabla a) \cap D^t(1, \nabla b) = D^t(1, \nabla\nabla a) \cap D^t(1, \nabla\nabla b), \quad \text{and}$

$D^t(1, \Delta z) = D^t(1, \nabla y) = D^t(1, \nabla a) \cap D^t(1, \nabla b) = D^t(1, \Delta\nabla a) \cap D^t(1, \Delta\nabla b).$

Uniqueness of the element satisfying assumtion (2) implies, then: $\nabla(a \wedge b) = z = \nabla a \wedge \nabla b$, as required.

A similar argument shows that Δ also preserves meets.

[L3 (ii)]. Let $a \in G$ and set $x = a \wedge -a$ and $y = -a \wedge \nabla a$. Then, $\nabla x = \nabla a \wedge \nabla(-a)$ and $\nabla y = \nabla(-a) \wedge \nabla\nabla a = \nabla(-a) \wedge \nabla a$, i.e., $\nabla x = \nabla y$. We claim that $\Delta x = \Delta y = 1$. Indeed, since $x \leq y$ and Δ is order-preserving, it suffices to prove $\Delta y = 1$. But $\Delta y = 1 \Leftrightarrow \nabla(-y) = -1 \Leftrightarrow \nabla(a \vee -\nabla a) = -1 \Leftrightarrow \nabla a \vee -\nabla a = -1$. It suffices then to prove:

$$a \in G^{\times} \;\Rightarrow\; a \vee -a = -1 \;\; (\text{or, equivalently, } a \wedge - a = 1).$$

Since by definition $a \wedge - a$ is the unique $y \in G$ verifiying the conditions in assumption (2) (for $b = -a$), it suffices to check that 1 verifies those conditions. Since $a = \nabla a = \Delta a$ for $a \in G^{\times}$, this reduces to check the validity of

$$1 \in D(a, -a) \qquad \text{and} \qquad D^t(1, a) \cap D^t(1, -a) = \{1\}.$$

Both these assertions are obvious (the second was already remarked). Thus, we have proved $\nabla x = \nabla y$ and $\Delta x = \Delta y$; item (iii) of the Fact yields $x = y$, as desired.

To complete the proof of the Theorem we show:

(d) The lattice $(G, \vee, \wedge)$ is distributive.

It is a well-known result that a lattice is distributive if and only if for all a, b, z,

$$(\dagger) \qquad a \vee z = b \vee z \;\; \text{and} \;\; a \wedge z = b \wedge z \quad \text{imply} \quad a = b.$$

(see [BD], Remark following Thm. 9, Ch. II, p. 51.) As a first step we show that this holds for $a, b, z \in G^{\times}$ i.e., that the sublattice $(G^{\times}, \vee, \wedge)$ is distributive. We prove that $h(a) = h(b)$ for all $h \in X_G$. Since a, b, z are invertible, any RS-character takes only values ± 1 on them. Note also that, by the definition of the lattice operations given by clause (2) of the assumption, any RS-character preserves meets and joins. So, assuming $h(a) = 1$, the antecedent of (†) gives $h(a) \vee h(z) = h(z) = h(b) \vee h(z)$ whence $h(b) \leq h(z)$, and $h(a) \wedge h(z) = 1 = h(b) \wedge h(z) = h(b)$. This proves $h(a) = 1 \Rightarrow h(b) = 1$. Exchanging a and b, the same argument gives the reverse implication; hence $h(a) = h(b)$, as asserted.

In the general case, by item (iii) of the Fact it suffices to prove $\nabla a = \nabla b$ and $\Delta a = \Delta b$. Since ∇, Δ preserve the lattice operations (by [L3 (iii)]), the assumptions of (†) yield:

$$(\dagger\dagger) \quad \begin{cases} \nabla a \vee \nabla z = \nabla b \vee \nabla z, & \nabla a \wedge \nabla z = \nabla b \wedge \nabla z, \\ \Delta a \vee \Delta z = \Delta b \vee \Delta z, & \Delta a \wedge \Delta z = \Delta b \wedge \Delta z. \end{cases}$$

Taking meets with $-\nabla z$ in the first of these equalities, since distributivity holds for invertible elements of G, we obtain:

$$\nabla a \wedge - \nabla z = (\nabla a \vee \nabla z) \wedge - \nabla z = (\nabla b \vee \nabla z) \wedge - \nabla z = \nabla b \wedge - \nabla z.$$

Taking join of these terms with $\nabla a \wedge \nabla z$ and $\nabla b \wedge \nabla z$, respectively, and using again distributivity for elements of $G^{\times}$, we conclude that $\nabla a = \nabla b$. A similar argument using the second line of (††) shows, in turn, that $\Delta a = \Delta b$, as required.

By duality we infer that the RS structure of this Post algebra coincides with that of G. □

IV.4 The Post hull of a real semigroup

Our aim in this section is to show that every RS can be functorially embedded into (the real semigroup associated to) a certain, canonically determined, Post algebra. This is the analog in the context of real semigroups of the main result of [DM1], Ch. 4 (Thm. 4.17), the latter for reduced special groups and Boolean algebras, respectively. The structure of the argument is similar to that case. □

The **Post hull** P_G of a real semigroup G is the RS of the Post algebra $\mathcal{C}((X_G)_{\mathrm{con}})$ of continuous functions of the Boolean space $(X_G)_{\mathrm{con}}$ into $\mathbf{3}$, given by Theorem IV.1.7. Recall (I.1.17 (b)) that $(X_G)_{\mathrm{con}}$ denotes the set X_G of RS-characters of G *endowed with the* **constructible** (*also called* **patch**) *topology*, having as a *sub*-basis the (clopen) sets $[\![\, a = \delta \,]\!] := \{ g \in X_G \mid g(a) = \delta \}$, for arbitrary $a \in G$ and $\delta \in \mathbf{3} = \{1, 0, -1\}$. A useful observation, frequently used in the sequel, is that the sets

$$(*) \qquad \bigcap\nolimits_{i=1}^{n} [\![\, a_i = 1 \,]\!] \cap [\![\, b = 0 \,]\!], \quad \text{with } a_1, \ldots, a_n, b \in G,$$

are a basis of clopens for this topology; cf. [M], Note (1), p. 111. The canonical embedding $\varepsilon_G : G \longrightarrow P_G$ is the evaluation map: for $a \in G$ and $h \in X_G$,

$$\varepsilon_G(a)(h) = h(a).$$

We start with the following basic:

Proposition IV.4.1 *If G is a RS, then $\varepsilon_G : G \longrightarrow P_G$ has the following properties:*

(i) ε_G is well-defined, i.e., $\varepsilon_G(a) \in P_G$ for all $a \in G$.

(ii) $\varepsilon_G(1)$, $\varepsilon_G(0)$ and $\varepsilon_G(-1)$ are the constant maps with values $\perp$, $\underline{\mathbf{c}}$ and $\top$, respectively.

(iii) ε_G is a RS-morphism satisfying the following condition: for $a, b, d \in G$,

$$a \in D_G(b, d) \quad \textit{if and only if} \quad \varepsilon_G(a) \in D_{P_G}(\varepsilon_G(b), \varepsilon_G(d)).$$

In particular, ε_G is injective.

(iv) P_G is generated by $\mathrm{Im}(\varepsilon_G)$ *as a Post algebra (i.e., using the operators ∇, Δ, in addition to the lattice operations).*

Proof. (i) With notation as above we clearly have $\varepsilon_G(a)^{-1}[\delta] = [\![\, a = \delta \,]\!]$ $(a \in G;\, \delta \in \mathbf{3})$, showing that $\varepsilon_G(a)$ is a continuous map.

(ii) is obvious, while (iii) follows from Theorem I.5.4 (1) and the fact that representation is pointwise defined in the Post algebra $\mathcal{C}(X, \mathbf{3})$.

(iv) Let $g \in P_G$. Using the formula $g = (\underline{\mathbf{c}} \wedge \nabla g) \vee \Delta g$ in Proposition IV.1.4(i), since $\underline{\mathbf{c}} = \varepsilon_G(0) \in \mathrm{Im}(\varepsilon_G)$, the problem reduces to show that the Boolean elements Δg, ∇g of P_G are in the Post subalgebra generated by $\mathrm{Im}(\varepsilon_G)$. Thus, it suffices to prove:

Every Boolean element of P_G is in the Post subalgebra generated by $\mathrm{Im}(\varepsilon_G)$.

Observe that a function $g \in \mathcal{C}(X_G, \mathbf{3})$ is a Boolean element of P_G iff g only takes on values ± 1. Thus, the sets $U = g^{-1}[1]$, $X \setminus U = g^{-1}[-1]$ partition X_G into two clopens. Since the sets of form (*) above are a basis for the topology of X_G, we have,

$$U = \bigcup\nolimits_{j=1}^{m} \big(\bigcap\nolimits_{i=1}^{n} [\![\, a_{ij} = 1 \,]\!] \cap [\![\, b_j = 0 \,]\!]\big),$$

with $a_{ij}, b_j \in G$. We search a Post-algebraic combination f of the functions $\varepsilon_G(a_{ij}), \varepsilon_G(b_j)$ $(j = 1, \ldots, m;\, i = 1, \ldots, n)$ with the property that, for all $h \in X_G$,

$$(\dagger) \qquad h \in U \;\Rightarrow\; f(h) = 1 \quad \text{and} \quad h \in X \setminus U \;\Rightarrow\; f(h) = -1.$$

By induction on n, m, it is sufficient to prove:

a) If for $\ell = 1, 2$ we have maps f_ℓ in the Post subalgebra of P_G generated by $\mathrm{Im}(\varepsilon_G)$ verifying $(\dagger)$ for $U = U_\ell$, then $f = f_1 \wedge f_2$ (resp., $f = f_1 \vee f_2$) verifies $(\dagger)$ for $U = U_1 \cup U_2$ (resp., $U = U_1 \cap U_2$), and

b) If $U = [\![\, a = 1 \,]\!]$ or $U = [\![\, b = 0 \,]\!]$, there is a map f in the Post subalgebra of P_G generated by $\mathrm{Im}(\varepsilon_G)$ verifying $(\dagger)$.

Statement (a) is clear, for if $h \in U_1 \cup U_2$, one of $f_i(h)$ is 1, whence $f(h) = 1$, while if $h \notin U_1 \cup U_2$, then both $f_i(h)$ equal -1, hence $f(h) = -1$; similarly for $U_1 \cap U_2$.

b) If $U = [\![\, a = 1 \,]\!]$, it suffices to take $f = \nabla \varepsilon_G(a)$, since

$$h(a) = 1 \Rightarrow (\nabla \varepsilon_G(a))(h) = \nabla(h(a)) = 1, \quad \text{and}$$

$$h(a) \neq 1 \Rightarrow h(a) \in \{0, -1\} \Rightarrow (\nabla \varepsilon_G(a))(h) = \nabla(h(a)) = -1.$$

If $U = [\![\, b = 0 \,]\!]$, then $f = -\nabla \varepsilon_G(b^2)$ works, for

$$h(b) = 0 \Rightarrow (-\nabla \varepsilon_G(b^2))(h) = -\nabla(h(b^2)) = -\nabla 0 = 1, \quad \text{and}$$

$$h(b) \neq 0 \Rightarrow h(b) \in \{1, -1\} \Rightarrow h(b)^2 = 1 \Rightarrow (-\nabla \varepsilon_G(b^2))(h) = -\nabla(h(b)^2) = -1. \qquad \square$$

Let **PA** denote the category of Post algebras and Post-algebra homomorphisms. The correspondence $G \mapsto P_G$ can be extended to morphisms by composition: given a RS-homomorphism $f : G \longrightarrow H$, Theorem I.5.1 yields a continuous map $f^* : X_H \longrightarrow X_G$ defined by: $f^*(\sigma) = \sigma \circ f$ for $\sigma \in X_H$. Next we define $P(f) : P_G \longrightarrow P_H$ by $P(f)(\gamma) = \gamma \circ f^*$ for $\gamma \in P_G$. This correspondence is functorial and has the very important property given in item (ii) of the following:

Theorem IV.4.2 *Let G, H be real semigroups. Then*

(i) The correspondence $G \mapsto P_G$ and $f \mapsto P(f)$ is a covariant functor from the category **RS** *to the category* **P**.

(ii) If $f : G \longrightarrow H$ is a RS-morphism, then $\varepsilon_H \circ f = P(f) \circ \varepsilon_G$, i.e., the diagram (D) below left is commutative:

$$
\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "\varepsilon_G"'] \arrow[dr, phantom, "(D)"] & H \arrow[d, "\varepsilon_H"] \\
P_G \arrow[r, "P(f)"'] & P_H
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
G \arrow[r, "\varepsilon_G"] \arrow[d, "f"'] & P_G \arrow[dl, "\varepsilon_L^{-1} \circ P(f)"] \\
L &
\end{tikzcd}
$$

Moreover, $P(f)$ is the unique Post-algebra homomorphism which makes the square above commutative.

(iii) The pair (P_G, ε_G) is a hull for G in the category **P***; that is, given a (RS-)Post algebra L, any RS-morphism $f : G \longrightarrow L$ factors through ε_G, i.e. the triangle above right is commutative.*

(iv) The Post hull of a Post algebra is canonically isomorphic to the given algebra.

In particular,

(v) Up to canonical isomorphism, the Post hull functor is idempotent.

Remark. Item (iii) says that the Post hull functor in (i) is left adjoint to the forgetful functor from **P** to **RS**.

Proof. (i) It is clear from the definition of $P(f)$ that $P(id_G) = id_{P_G}$ and $P(f \circ g) = P(f) \circ P(g)$, which proves (i).

(ii) For $x \in G$, $P(f)(\varepsilon_G(x)) = \varepsilon_G(x) \circ f^*$. Let $h \in X_H$; then, $(\varepsilon_G(x) \circ f^*)(h) = \varepsilon_G(x)(h \circ f) = h(f(x)) = \varepsilon_H(f(x))(h)$; this proves $P(f)(\varepsilon_G(x)) = \varepsilon_H(f(x))$, as asserted. Uniqueness: assume $F : P_G \longrightarrow P_H$ is a Post algebra homomorphism that makes the square (D) commute; then, $F \circ \varepsilon_G = \varepsilon_H \circ f = P(f) \circ \varepsilon_G$. This just means that F and $P(f)$ coincide on $\mathrm{Im}(\varepsilon_G)$; since this set generates $P(f)$ (IV.4.1(iv)), both maps are equal.

(iii) With $H = L$, the commutative square (D) above gives $\varepsilon_L \circ f = P(f) \circ \varepsilon_G$. By definition $\varepsilon_L : L \longrightarrow \mathcal{C}(X_L) = P_L$ is the evaluation map which, by (ii) of the Representation Theorem IV.1.7, is a Post algebra isomorphism.

(iv) When f is the embedding $\varepsilon_G : G \longrightarrow P_G$ we obtain a Post-algebra homomorphism $P(\varepsilon_G) : P_G \longrightarrow P_{P_G}$. The Representation Theorem IV.1.7 applied with $L = P_G$ shows that $P_G \simeq \mathcal{C}(X_{P_G}) \simeq P_{P_G}$; further, the uniqueness statement in (ii) implies that the isomorphism defined in IV.1.7 is identical to $P(\varepsilon_G)$. $\square$

A conceptually important, and rather powerful by-product of the Post-hull construction is Theorem IV.4.5, an analog of Theorem 5.2 of [DM1] (p. 75 ff.) for RS's. The following notion, used in this theorem, is an adaptation of Definition 5.1 of [DM1] to the context of RSs, formulated in terms of the notion of Witt-equivalence used in our context (cf. I.2.7 (c)).

Definition IV.4.3 A RS-homomorphism $f : G \longrightarrow H$ between RS's, G, H, is called a **complete embedding** if for every pair of forms φ, ψ, over G (possibly of different dimensions),

$$\varphi \cong_G \psi \;\Leftrightarrow\; f * \varphi \cong_H f * \psi. \qquad \square$$

Here follow some basic properties of complete embeddings of RSs.

Fact IV.4.4 (*a*) *The implication* ($\Rightarrow$) *in IV.4.3 holds automatically.*

(*b*) *Complete embeddings preserve and reflect binary representation: for* $a, b, c \in G$,

$$a \in D_G(b, c) \;\Leftrightarrow\; f(a) \in D_H(f(b), f(c)).$$

A similar result holds for transversal representation.

(*c*) *Complete embeddings are* (*order*) *monomorphisms for the representation partial order. In particular, they are injective.*

Proof. (a) Since f is a RS-homomorphism, $h \circ f \in X_G$, for $h \in X_H$; then, $\varphi \cong_G \psi$ implies $\mathrm{sgn}_h(f * \varphi) = \mathrm{sgn}_{h\circ f}(\varphi) = \mathrm{sgn}_{h\circ f}(\psi) = \mathrm{sgn}_h(f * \psi)$ for all $h \in X_H$, i.e., $f * \varphi \cong_H f * \psi$.

(b) Only the implication ($\Leftarrow$) needs proof. A straightforward computation invoking the separation theorem for RSs (I.5.4) shows:

$$a \in D_G(b, c) \;\Leftrightarrow\; \langle a^2 b, a^2 c \rangle \cong_G \langle a, abc \rangle,$$

(cf. [M], § 6.2, p. 105). The required implication is easily derived from this characterization of representation and our definition of $\cong$. For transversal representation, just use its definition in terms of (ordinary) binary representation, cf. [t-rep] in Section I.2.

(c) Since $a \le b \;\Leftrightarrow\; a \in D(1, b) \wedge -b \in D(1, -a)$, item (b) obviously implies $a \le_G b \;\Leftrightarrow\; f(a) \le_H f(b)$. Injectivity follows from the antisymmetry of the representation partial order (I.6.4 (a)). $\square$

Theorem IV.4.5 *Let* $f : G \longrightarrow H$ *be a RS-morphism. Let* $f^* : X_H \longrightarrow X_G$ *be the map dual to* f ($f^*(\sigma) = \sigma \circ f$ *for* $\sigma \in X_H$), *and let* $P(f) : P_G \longrightarrow P_H$ *denote the Post-algebra homomorphism associated to* f. *The following are equivalent:*

(1) f^* *is surjective.*

(2) $\mathrm{Im}(f^*)$ *is dense in* X_G.

(3) $P(f)$ *is injective.*

(4) $P(f)$ *is a Post-algebra isomorphism from* P_G *onto the Post subalgebra of* P_H *generated by* $\mathrm{Im}(\varepsilon_H \circ f)$.

(5) *For every Pfister form* φ *over* G *and every* $a \in G$, $f(a) \in D_H(f * \varphi) \;\Rightarrow\; a \in D_G(\varphi)$.

(6) f *is a complete embedding.*

Remark. The proof of IV.4.5 follows a pattern similar to that of Theorem 5.2 in [DM1], except for the implications (5) $\Rightarrow$ (2) and (6) $\Rightarrow$ (2), which require a finer argument.

Proof. (1) $\Rightarrow$ (2) is obvious, while (2) $\Rightarrow$ (1) comes from the fact that any dense compact set in a Hausdorff space X must be equal to X. The equivalence of (3) and (4) is also clear.

(1) $\Rightarrow$ (3). Since "injective = monic" in the category of Post algebras ([BD], Thm. X.3.2, pp. 193–194), it suffices to show that $P(f)$ is monic (i.e., right-cancellable). Let g, h be homomorphisms from a Post algebra P into P_G such that $P(f) \circ g = P(f) \circ h$. For $p \in P$

write $g(p)=\gamma_1, h(p)=\gamma_2 \in P_G=\mathcal{C}(X_G,\mathbf{3})$, so that $P(f)(\gamma_1)=P(f)(\gamma_2)$. By the definition of $P(f)$ we have $\gamma_1\circ f^*=\gamma_2\circ f^*$; since f^* is surjective, $\gamma_1=\gamma_2$. Thus, $g(p)=h(p)$ for all $p\in P$, i.e., $g=h$.

(3) $\Rightarrow$ (1). Since "surjective = epic" in the category of Boolean spaces (ADD REF.), it suffices to prove f^* epic (i.e., left-cancellable). To this end, recall that any continuous map $\rho : X_G \longrightarrow X$ into a Boolean space X induces a dual map $\overline{\rho} : \mathcal{C}(X,\mathbf{3}) \longrightarrow \mathcal{C}(X_G,\mathbf{3})$ by left composition: $\overline{\rho}(g)=g\circ\rho$ for $g\in\mathcal{C}(X,\mathbf{3})$. We leave to the reader the straightforward checking that $\overline{\rho}$ is a Post algebra homomorphism from $P=\mathcal{C}(X,\mathbf{3})$ into $P_G=\mathcal{C}(X_G,\mathbf{3})$. In particular, we get a Post algebra homomorphism $\overline{\rho\circ f^*}: P\longrightarrow P_H$, and, for $g\in P$,

$$(P(f)\circ\overline{\rho})(g)=P(f)(g\circ\rho)=g\circ\rho\circ f^*=(\overline{\rho\circ f^*})(g),$$

i.e., $P(f)\circ\overline{\rho}=\overline{\rho\circ f^*}$.

Assume now that $\rho_1,\rho_2 : X_G\longrightarrow X$ are continuous maps such that $\rho_1\circ f^*=\rho_2\circ f^*$; then, $\overline{\rho_1\circ f^*}=\overline{\rho_2\circ f^*}$, and by the above, $P(f)\circ\overline{\rho_1}=P(f)\circ\overline{\rho_2}$. Since $P(f)$ is injective, we conclude that $\overline{\rho_1}=\overline{\rho_2}$. To finish the proof it only remains to prove $\rho_1=\rho_2$. Otherwise, $\rho_1(h)\neq\rho_2(h)$ for some $h\in X_G$. Taking disjoint clopen neighborhoods U_i of $\rho_i(h)$ in X $(i=1,2)$, and setting, say, $g\lceil U_1=1$, $g\lceil U_2=-1$, $g\lceil X\setminus(U_1\cup U_2)=0$, gives a function $g\in\mathcal{C}(X,\mathbf{3})=P$ so that $g(\rho_1(h))\neq g(\rho_2(h))$, i.e., $\overline{\rho_1}(g)\neq\overline{\rho_2}(g)$, a contradiction.

(1) $\Rightarrow$ (5). Let φ be a Pfister form with entries in G, $a\in G$, and assume $f(a)\in D_H(f*\varphi)$. Corollary I.5.7 tells us:

(i) This assumption is equivalent to $\forall h\in X_H\,(h(f(a))\in D_{\mathbf{3}}((h\circ f)*\varphi))$, and

(ii) The conclusion $a\in D_G(\varphi)$ is equivalent to $\forall g\in X_G\,(g(a)\in D_{\mathbf{3}}(g*\varphi))$.

Since f^* is surjective, for every $g\in X_G$ there is $h\in X_H$ such that $g=f^*(h)=h\circ f$. Then, (ii) follows at once from (i), proving (5).

(1) $\Rightarrow$ (6). Let $\varphi=\langle a_1,\ldots,a_n\rangle$, $\psi=\langle b_1,\ldots,b_m\rangle$ be arbitrary forms over G, and assume $f*\varphi\cong_H f*\psi$, i.e.,

(+) $\qquad \sum_{i=1}^n h(f(a_i))=\sum_{j=1}^m h(f(b_j))$ for all $h\in X_H$.

Given $g\in X_G$ pick $h\in X_H$ so that $g=f^*(h)=h\circ f$. Then, (+) yields at once $\sum_{i=1}^n g(a_i)=\sum_{j=1}^m g(b_j)$ for all $g\in X_G$, i.e., $\varphi\cong_G\psi$.

The proofs of (5) $\Rightarrow$ (2) and (6) $\Rightarrow$ (2) rest on:

Fact. *Let G be a RS, and $a_1,\ldots,a_n,b\in G$. Let $V=\bigcap_{i=1}^n [\![a_i=1]\!] \cap [\![b=0]\!] \subseteq X_G$. Let φ and ψ respectively denote the Pfister forms $\langle\langle a_1,\ldots,a_n,-b^2\rangle\rangle$ and $2^{n-1}\langle\langle -1,b^2\rangle\rangle$, if $n\geq 1$, and $\psi=\langle 1,-b^2\rangle$, $\varphi=2\cdot\psi=\psi\oplus\psi$, if $n=0$. Set $d=\prod_{i=1}^n a_i^2$, if $n\geq 1$, and $d=1$, if $n=0$. The following hold:*

i) If $V=\emptyset$, then $-d\in D_G(\varphi)$ and $d\varphi\cong_G\psi$.

ii) If $V\neq\emptyset$, then $-d\notin D_G(\varphi)$ and $d\varphi\not\cong_G\psi$.

Note. In case $V=\bigcap_{i=1}^n [\![a_i=1]\!]$ $(n\geq 1)$, just omit the entries $-b^2$ in φ and b^2 in ψ.

Proof of Fact. By Corollary I.5.7, condition $-d\in D_G(\varphi)$ is equivalent to

(†) $\qquad \forall g\in X_G\,[\,g(a_i)\in\{0,1\}$ for $i=1,\ldots,n$, and $g(-b^2)=0\Rightarrow g(-d)=0]$.

Any $g \in V$ verifies $g(a_i) = 1$, $g(b) = 0$, whence $g(d) = \prod_{i=1}^{n} g(a_i)^2 = 1$, for $n \geq 1$ (and, obviously, also for $n = 0$). Thus, (†) fails at every $g \in V$. Hence, $V \neq \emptyset \Rightarrow -d \notin D_G(\varphi)$.

As for the second assertion in (ii), first note that if $n \geq 1$, then $\mathrm{sgn}_g(\langle\langle -1, x \rangle\rangle) = \mathrm{sgn}_g(\langle 1, -1, x, -x \rangle) = 0$, whence $\mathrm{sgn}_g(\psi) = 0$, for all $g \in X_G$. Next, observe that if $g \in V$, then $\mathrm{sgn}_g(d\varphi) = g(d)\cdot(1-g(b^2))\cdot\prod_{i=1}^{n}(1+g(a_i)) = 2^n$. Hence, $\mathrm{sgn}_g(d\varphi) \neq \mathrm{sgn}_g(\psi)$ whenever $n \geq 1$. If $n = 0$, we have $\mathrm{sgn}_g(\psi) = 1$ and $\mathrm{sgn}_g(\varphi) = 2$.

(i) If $V = \emptyset$, then for any $g \in X_G$ either $g(b) \neq 0$ (i.e., $g(b^2) = 1$) or $g(a_i) \neq 1$ for some $i \in \{1, \ldots, n\}$. If $g(b) \neq 0$ or $g(a_i) = -1$ for some i, (†) holds because its antecedent fails. If $g(b) = 0$ and $g(a_i) = 0$ for some i, then $g(-d) = 0$, and (†) holds. Hence, (†) holds for every $g \in X_G$, which entails $-d \in D_G(\varphi)$.

To prove the last assertion in (i), it suffices to show that $\mathrm{sgn}_g(d\varphi) = 0$ for all $g \in X_G$. Since $g \notin V$, if $g(b) \neq 0$, the factor $1 - g(b)^2$ is 0, and $\mathrm{sgn}_g(d\varphi) = 0$. Likewise, if $g(a_i) = 0$ for some i, the factor $g(d)$ vanishes, and if $g(a_i) = -1$ for some i, then $1 + g(a_i) = 0$; in either case $\mathrm{sgn}_g(d\varphi) = 0$, as required.

(5) $\Rightarrow$ (2). Assume $\mathrm{Im}(f^*)$ is not dense in X_G. Then, there is a non-empty clopen set U of the form $\bigcap_{i=1}^{n} [\![a_i = 1]\!] \cap [\![b = 0]\!]$ $(a_1, \ldots, a_n, b \in G)$, such that $U \cap \mathrm{Im}(f^*) = \emptyset$, i.e., $f^{*-1}[U] = \emptyset$. Statement (ii) of the Fact with $V = U$ yields $-d \notin D_G(\varphi)$, while item (i) with $V = f^{*-1}[U] = \bigcap_{i=1}^{n} [\![f(a_i) = 1]\!] \cap [\![f(b) = 0]\!] \subseteq X_H$ (and $f * \varphi$) gives $f(-d) \in D_H(f * \varphi)$. This contradicts assumption (5).

(6) $\Rightarrow$ (2). Similar to the preceding proof: assuming (2) fails, the last assertion in item (ii) of the Fact with $V = U$ yields $d\varphi \not\cong_G \psi$, while that of (i) applied to $V = f^{*-1}[U]$ gives $f * (d\varphi) \cong_G f * \psi$, contradicting (6). This ends the proof of Theorem IV.4.5. □

Remark IV.4.6 Let (5′) stand for the generalization of (5) obtained by replacing "multiple of Pfister form" instead of just "Pfister form". Then (5′) is still equivalent to the remaining assertions in Theorem IV.4.5. In fact, (5) $\Rightarrow$ (5′) upon observing:

(i) A map f verifying (5) must be injective (this is a particular case of (5) $\Rightarrow$ (3), but may be checked directly, as in Fact IV.4.4(c));

(ii) For $a, b \in G$ and a Pfister form φ over G,

$$a \in D_G(b\varphi) \;\Leftrightarrow\; a = b^2 a \;\text{ and }\; ab \in D_G(\varphi).$$

[Proof. The implication ($\Rightarrow$) follows from Proposition I.2.8(3) and (4), respectively, since $a \in D(b\varphi) \Rightarrow ab \in D(b^2\varphi) \subseteq D(\varphi)$. Conversely, $ab \in D(\varphi)$ implies $a = ab^2 \in D(b\varphi)$.] □

Theorem IV.4.5 together with IV.4.2 (iv), IV.4.6, and IV.4.1 (iv), yields:

Corollary IV.4.7 *Let G be a RS, and let $\varepsilon_G : G \longrightarrow P_G$ be its Post-hull embedding. Then:*

(1) *The dual map $\varepsilon_G^* : X_{P_G} \longrightarrow (X_G)_{\mathrm{con}}$ is a homeomorphism.* Notation corrected Nov. 2011

(2) *Every RS-character $h \in X_G$ extends uniquely to a Post algebra character $\hat{h} : P_G \longrightarrow \mathbf{3}$, i.e., $h = \hat{h} \circ \varepsilon_G$.*

(3) *ε_G is a complete embedding. It preserves and reflects representation by arbitrary binary forms and by (multiples of) Pfister forms: if φ is such a form with entries in G, and $a \in G$,*

$$a \in D_G(\varphi) \;\Leftrightarrow\; \varepsilon_G(a) \in D_{P_G}(\varepsilon_G * \varphi).$$

A similar equivalence holds for transversal representation.

Proof. (1) By definition, $\varepsilon_G^*(\gamma) = \gamma \circ \varepsilon_G$ for $\gamma \in P_G$; hence, ε_G^* is continuous. Since X_{P_G} and $(X_G)_{\rm con}$ are compact Hausdorff spaces (cf. Add ref. for the first!), it suffices to show that ε_G^* is bijective.

(i) ε_G^* is surjective.

By (3) $\Rightarrow$ (1) in Theorem IV.4.5 it suffices to show that $P(\varepsilon_G)$ is injective. This follows from item (ii) of the Representation Theorem IV.1.7: as noted in the proof of IV.4.2 (iv), $P(\varepsilon_G)$ is the evaluation isomorphism between P_G and P_{P_G}.

(ii) ε_G^* is injective.

This is a straightforward consequence of IV.4.1(iv). Let $\gamma_1, \gamma_2 \in X_{P_G}$ be so that $\varepsilon_G^*(\gamma_1) = \varepsilon_G^*(\gamma_2)$, i.e., $\gamma_1 \circ \varepsilon_G = \gamma_2 \circ \varepsilon_G$; then, $\gamma_1 \lceil \mathrm{Im}(\varepsilon_G) = \gamma_2 \lceil \mathrm{Im}(\varepsilon_G)$. Since $\mathrm{Im}(\varepsilon_G)$ generates P_G and the γ_i's are Post algebra homomorphisms (IV.2.11), we conclude $\gamma_1 = \gamma_2$.

(2) Just set $\widehat{h} = (\varepsilon_G^*)^{-1}(h)$ for $h \in X_G$; then, $h = \varepsilon_G^*(\widehat{h}) = \widehat{h} \circ \varepsilon_G$. Uniqueness: if for a fixed $h \in X_G$ a Post algebra character $H : P_G \longrightarrow \mathbf{3}$ verifies $h = H \circ \varepsilon_G$, then $H \lceil \mathrm{Im}(\varepsilon_G) = \widehat{h} \lceil \mathrm{Im}(\varepsilon_G) = h$, and IV.4.1(iv) entails $\widehat{h} = H$. $\square$

(3) For the first two assertions, the non-trivial implication ($\Leftarrow$) follows at once from item (1) above and the equivalence of (1), (5) and (6) in Theorem IV.4.5 (and Remark IV.4.6). The statement for transversal representation is derived from that for ordinary representation by use of Theorem I.2.8(10), which gives an explicit expression for D^t in terms of D. $\square$

COMMENT HERE ON DEFINITION OF COMPLETE EMBEDDING AND REPRESENTATION BY NON-PFISTER FORMS IN THE CASE OF RSGs.

IV.4.8 The Post hull of a quotient.

Given a congruence $\equiv$ on a RS, G, (see § II.2) we examine here the structure of the Post hull $P_{G/\equiv}$ of the quotient real semigroup $G/\!\equiv$; we prove, in fact, that formation of the Post hull "commutes" with the quotient operation. Recall from Proposition IV.1.8 that quotients of Post algebras are obtained modulo lattice ideals closed under ∇.

Theorem IV.4.9 *Let G be a RS and let $\equiv$ be a congruence of G. Let $\mathcal{H} = \mathcal{H}_\equiv$ be the proconstructible subset of X_G associated to $\equiv$ (cf. Proposition ??). Let $I = I_{\mathcal{H}_\equiv} = \{f \in P_G \,|\, f(h) = \perp$ for all $h \in \mathcal{H}\}$. Then, I is a lattice ideal closed under ∇, and the Post hull $P_{G/\equiv}$ is isomorphic to the quotient P_G/I. Furthermore, there is a* unique *Post-algebra morphism $\mu : P_G \longrightarrow P_{G/\equiv}$ making the following diagram commute:*

$$\begin{tikzcd}
G \arrow[r, "\varepsilon_G"] \arrow[d, "\pi"'] & P_G \arrow[ddl, dotted, "\mu"] \arrow[dd, "\pi_I"] \\
G/_\equiv \arrow[d, "\varepsilon_{G/\equiv}"'] & \\
P_{G/\equiv} & P_G/I \arrow[l, "\overline{\mu}"]
\end{tikzcd}$$

Proof. The stated property of I is obvious, since the Post-algebra operations in P_G are pointwise defined, and $\nabla\bot = \bot$ (IV.1.4 (b)). Proposition **??** (ii) shows that the Boolean spaces $X_{G/\equiv}$ and $\mathcal{H}$ are homeomorphic via the correspondence that assigns to a character $\sigma \in X_{G/\equiv}$ the composition $\sigma \circ \pi$, where $\pi : G \longrightarrow G/_{\equiv}$ is the canonical projection. Hence the Post hull of the quotient $G/_{\equiv}$ is isomorphic to the Post algebra $\mathcal{C}(\mathcal{H}, \mathbf{3})$. Let $\mu : P_G \longrightarrow \mathcal{C}(\mathcal{H}, \mathbf{3})$ be the map defined by $\mu(f) = f\lceil\mathcal{H}$ ($=$ restriction of f to $\mathcal{H}$). Clearly, μ is a morphism of Post algebras whose kernel is I. In order to complete the proof we must show that μ is surjective. Let $f : \mathcal{H} \longrightarrow \mathbf{3}$ be a continuous function, and let $U_\bot = f^{-1}[\{\bot\}]$, and $U_\top = f^{-1}[\{\top\}]$; these are disjoint clopens of $\mathcal{H}$. Hence, there are clopens U', U'' of X_G such that $U_\bot = U' \cap \mathcal{H}$ and $U_\top = U'' \cap \mathcal{H}$. Replacing, if necessary, U' and U'' by $U' \setminus U''$ and $U'' \setminus U'$ respectively, we may assume U', U'' disjoint. Then, the map $\hat{f} : X_G \longrightarrow \mathbf{3}$ defined by:

$$\hat{f}(x) = \begin{cases} \bot & \text{if } x \in U' \\ c & \text{if } x \notin U' \cup U'' \\ \top & \text{if } x \in U'', \end{cases}$$

is a continuous function extending f, and therefore $\mu(\hat{f}) = f$. The required isomorphism $\overline{\mu} : P_G/I \longrightarrow P_{G/\equiv}$ is canonically induced by μ. □

IV.4.10 Transversal 2-regularity of quotients of Post algebras.

Theorem IV.4.11 *Let P be a Post algebra and let I be a lattice ideal of P closed under ∇. Then the quotient map $\pi : P \longrightarrow P/I$ is transversally 2-regular.*

Proof. We have to show, for $\alpha, \beta, \gamma, \delta \in P/I$, that $D^t_{P/I}(\alpha, \beta) \cap D^t_{P/I}(\gamma, \delta) \neq \emptyset$ implies the existence of liftings $a, b, c, d \in P$ of $\alpha, \beta, \gamma, \delta$, respectively, so that $D^t_P(a, b) \cap D^t_P(c, d) \neq \emptyset$.

Recalling the characterization of transversal representation in a Post algebra A given by Theorem IV.2.7 (i), condition $D^t_A(x, y) \cap D^t_A(z, w) \neq \emptyset$ is equivalent to the existence of a $r \in A$ such that $(x \wedge \nabla y) \vee (y \wedge \nabla x) \leq r \leq (x \vee \Delta y) \wedge (y \vee \Delta x)$ and $(z \wedge \nabla w) \vee (w \wedge \nabla z) \leq r \leq (z \vee \Delta w) \wedge (w \vee \Delta x)$. This, in turn, is obviously equivalent to

(†) $\quad (x \wedge \nabla y) \vee (y \wedge \nabla x) \vee (z \wedge \nabla w) \vee (w \wedge \nabla z) \leq (x \vee \Delta y) \wedge (y \vee \Delta x) \wedge (z \vee \Delta w) \wedge (w \vee \Delta x).$

[For the rest of this proof A will stand either for P or for P/I, and the variables x, y, z, w either for $a, b, c, d \in P$ or for $\alpha, \beta, \gamma, \delta \in P/I$, respectively.]

By assumption, the inequality (†) is valid in P/I for $\alpha, \beta, \gamma, \delta$, and our problem boils down to finding liftings a, b, c, d of $\alpha, \beta, \gamma, \delta$, so that (†) is still valid in P.

From Proposition **??** (ii) we know that the character space $X_{P/I}$ of P/I is (homeomorphic to) a closed subset C of the character space X_P of P (the topology in these spaces being, necessarily, the constructible topology). Hence (cf. Theorem IV.1.7 (ii)), $\alpha, \beta, \gamma, \delta$ can be viewed as continuous functions from C (with the topology induced by that of X_P) into $\mathbf{3}$. Likewise, a, b, c, d are continuous functions $X_P \longrightarrow \mathbf{3}$. The requirement that $a/I = \alpha$, etc., simply means that the function a extends α from C to X_P, and hence we must extend the functions $\alpha, \beta, \gamma, \delta$ to continuous functions $a, b, c, d : X_P \longrightarrow \mathbf{3}$ in such a way that the inequality (†) is preserved. The result follows, then, from the well-known

Lemma IV.4.12 *Let X be a Boolean space and let C be a closed subset. Let $U_1, \ldots, U_n$ be clopens of X such that $U_1 \cap C, \ldots, U_n \cap C$ form a partition of C into non-empty sets (obviously clopen in C). Then, there is a partition $\{V_1, \ldots, V_n\}$ of X into (necessarily non-empty) clopens so that $V_i \cap C = U_i \cap C$, for $i = 1, \ldots, n$.* [Sketch of proof below.]

Continuing the proof of Theorem IV.4.11, for every quadruple $i, j, k, \ell \in \{-1, 0, 1\}$ let

$$(\dagger\dagger) \qquad C_{i,j,k,\ell} = [\![\, \alpha = i \,]\!] \cap [\![\, \beta = j \,]\!] \cap [\![\, \gamma = k \,]\!] \cap [\![\, \delta = \ell \,]\!],$$

where $[\![\, \chi = m \,]\!] = \{x \in C \,|\, \chi(x) = m\}$, for $\chi \in \mathcal{C}(C, \mathbf{3})$ and $m \in \{-1, 0, 1\}$, see I.1.17. By continuity, the sets $C_{i,j,k,\ell}$ are clopen in C; let $\{C_m \,|\, 1 \leq m \leq N\}$ be an indexing without repetitions of the non-empty sets among the $C_{i,j,k,\ell}$. Then $C_m = U_m \cap C$ for some clopens U_m of X $(1 \leq m \leq N)$. Let $\{V_1, \ldots, V_N\}$ be a partition of X into clopens so that $V_m \cap C = U_m \cap C$, for $m = 1, \ldots, N$, as given by Lemma IV.4.12. Now we define $a, b, c, d \in \mathcal{C}(X_P, \mathbf{3})$ by their values on the sets V_m, as follows: if $C_m = C_{i,j,k,\ell}$ for $i, j, k, \ell \in \{-1, 0, 1\}$ (necessarily unique), we set:

$$(\dagger\dagger\dagger) \qquad a \lceil V_m = i\,, \;\; b \lceil V_m = j\,, \;\; c \lceil V_m = k\,, \;\; d \lceil V_m = \ell\,.$$

To check the validity of the inequality (†) for the functions a, b, c, d at each point $x \in X$, let $m \in \{1, \ldots, N\}$ be the unique index such that $x \in V_m$, and let $y \in V_m \cap C = C_m$. Since a, b, c, d have constant sign over each V_m, (††) and (†††) guarantee that $a(x) = \alpha(y)$, $b(x) = \beta(y)$, $c(x) = \gamma(y)$, $d(x) = \delta(y)$, and the stated inequality follows, then, from the fact that (†) holds for the functions $\alpha, \beta, \gamma, \delta$ at y. □

Remark IV.4.13 The method of the foregoing proof, using Lemma IV.4.12, gives also the following

Proposition. *Let P be a Post algebra, and let I be a lattice ideal of P closed under ∇. Let $\tau_1, \ldots, \tau_n, \sigma_1, \ldots, \sigma_n$ be terms of the language $\{\neg, \vee, \wedge, \nabla, \bot, \underline{\mathbf{c}}\}$ for Post algebras in the free variables $v_1, \ldots, v_k$ (i.e., well-formed propositional expressions of the indicated language, built from the variables $v_1, \ldots, v_k$). Assume that the system of inequalities*

$$(*) \qquad \tau_i(v_1, \ldots, v_k) \leq \sigma_i(v_1, \ldots, v_k) \qquad (i = 1, \ldots, n)$$

has a solution $\alpha_1, \ldots, \alpha_k$ in P/I. Then, there are liftings $a_1, \ldots, a_k \in P$ of $\alpha_1, \ldots, \alpha_k$ (i.e., $a_j/I = \alpha_j$ for $j = 1, \ldots, k$), which are a solution of the system $(*)$ in P. □

Note that one may include equalities $\tau_i = \sigma_i$ in the system $(*)$, as $\tau_i = \sigma_i \Leftrightarrow \tau_i \leq \sigma_i$ and $\sigma_i \leq \tau_i$.

Sketch of proof of Lemma IV.4.12. Induction on $n \geq 1$. For $n = 1$ just take $V_1 = X$. For the induction step $(n - 1 \to n)$ consider the non-empty clopens $U_i \setminus U_n$ $(1 \leq i \leq n - 1)$ of X. With $C' = C \setminus U_n$, the sets $\{(U_i \setminus U_n) \cap C' \,|\, 1 \leq i \leq n - 1\}$ obviously partition the closed set C'. By induction hypothesis there is a partition $\{V'_1, \ldots, V'_{n-1}\}$ of X into clopens so that $V'_i \cap C' = (U_i \setminus U_n) \cap C'$, $1 \leq i \leq n - 1$. Set $V_i = V'_i \setminus U_n$ for $1 \leq i \leq n - 1$, $V_n = U_n$, and check that these sets have the required property. □

IV.5 Value sets of quadratic forms

The Post-algebraic techniques developed in previous sections of this chapter are applied here to study the properties of value sets of quadratic forms over real semigroups. We start with a general result (Theorem IV.5.1) characterizing representation and transversal representation of arbitrary quadratic forms over Post algebras in terms of the operators Δ, ∇, and the order and lattice operations of the algebra. By considering the Post hull $\varepsilon_G : G \longrightarrow P_G$ of a given RS, G, this characterization may be used to obtain information on the structure of the value sets of higher-dimensional forms over G itself. The possibility of transferring information about quadratic forms φ with entries in G from P_G back to G depends, however, on the validity of the implication

$$\varepsilon_G(x) \in D_{P_G}(\varepsilon_G * \varphi) \;\Rightarrow\; x \in D_G(\varphi),$$

which does not hold, in general, for arbitrary φ's (the reverse implication is valid without restrictions on φ). However, the weak local-global principle (Corollary IV.4.7 (3)) shows that this implication holds for arbitrary binary forms and for Pfister forms and their multiples, yielding interesting consequences in these cases (IV.5.6, IV.5.8 and IV.5.11).

In the sequel we shall repeatedly use the definition of representation and the characterization of transversal representation in a Post algebra given in Definition IV.2.3 and Theorem IV.2.7 (i), respectively. Recall that $\underline{\mathbf{c}}$ denotes the center of a Post algebra.

Our point of departure is:

Theorem IV.5.1 *Let P be a Post algebra, and let $x, b_1, \ldots, b_n \in P$ $(n \geq 1)$.*

(A) *The following are equivalent:*

(1) $x \in D_P(\langle b_1, \ldots, b_n \rangle)$,

(2) $b_1 \wedge \ldots \wedge b_n \wedge \underline{\mathbf{c}} \leq x \leq b_1 \vee \ldots \vee b_n \vee \underline{\mathbf{c}}$.

(B) *The following are equivalent:*

(1) $x \in D^t_P(\langle b_1, \ldots, b_n \rangle)$,

(2) *a)* $\nabla b_1 \wedge \ldots \wedge \nabla b_n \leq \nabla x \leq (\nabla b_1 \wedge \ldots \wedge \nabla b_n) \vee (\Delta b_1 \vee \ldots \vee \Delta b_n)$, *and*

b) $(\nabla b_1 \wedge \ldots \wedge \nabla b_n) \wedge (\Delta b_1 \vee \ldots \vee \Delta b_n) \leq \Delta x \leq \Delta b_1 \vee \ldots \vee \Delta b_n$.

Proof. (A) Induction on n for both implications. Recalling that $D_G(\langle b \rangle) = \{y^2 b \mid y \in G\}$ (I.2.7 (a)), the case $n = 1$ is dealt with by:

Fact. $b_1 \wedge \underline{\mathbf{c}} \leq x \leq b_1 \vee \underline{\mathbf{c}} \;\Leftrightarrow\; x = x^2 b_1$.

Proof of Fact. ($\Leftarrow$) Since product in P is symmetric difference, we have $x = x^2 b_1 = (x^2 \wedge \neg\, b_1) \vee (b_1 \wedge \neg\, x^2)$, and $x^2 = x \wedge \neg\, x$; hence, $\neg\, x^2 = x \vee \neg\, x$. Kleene's inequality IV.1.2 (b) implies $x^2 \leq \underline{\mathbf{c}} \leq \neg\, x^2$. Therefore,

$$x = (x^2 \wedge \neg\, b_1) \vee (b_1 \wedge \neg\, x^2) \;\geq\; b_1 \wedge \neg\, x^2 \;\geq\; b_1 \wedge \underline{\mathbf{c}}, \text{ and}$$

$$x \;\leq\; (\underline{\mathbf{c}} \wedge \neg\, b_1) \vee (b_1 \wedge \neg\, x^2) \;\leq\; (\underline{\mathbf{c}} \wedge \neg\, b_1) \vee b_1 \;\leq\; b_1 \vee \underline{\mathbf{c}}.$$

($\Rightarrow$) We prove the required identity by truth-table checking. Let $h : P \longrightarrow \mathbf{3}$ be a character. Recall that $h(\underline{\mathbf{c}}) = 0$. We argue by cases according to the values of $h(b_1)$.

— $h(b_1) = 1$. By assumption, $h(b_1) \wedge 0 = 1\,(= \bot) \leq h(x) \leq h(b_1) \vee 0 = 0$. Hence, $h(x) \in \{0, 1\}$; this yields $h(x) = h(x^2)$, implying $h(x) = h(x^2)\, h(b_1)$.

— $h(b_1) = 0$. By assumption, $h(b_1) \wedge 0 = 0 \leq h(x) \leq h(b_1) \vee 0 = 0$ i.e., $h(x) = 0$, which clearly yields $h(x) = h(x^2)\, h(b_1) = 0$.

— $h(b_1) = -1$. By assumption, $h(b_1) \wedge 0 = 0 \leq h(x) \leq h(b_1) \vee 0 = -1\,(= \top)$. Thus, $h(x) \in \{0, -1\}$, which gives $h(x) = -h(x^2) = h(x^2)\, h(b_1)$. □

The case $n = 2$ holds by the definition of D_P (cf. IV.2.3).

Induction step $n - 1 \to n$, $n \geq 3$.

(A.1) $\Rightarrow$ (A.2). Assume $x \in D_P(\langle b_1, \ldots, b_n \rangle)$. By the inductive definition of D in RSs (cf. I.2.7 (a)), there is $y \in D_P(\langle b_2, \ldots, b_n \rangle)$ so that $x \in D_P(\langle b_1, y \rangle)$. By induction hypothesis and the case $n = 2$, we have

$$b_2 \wedge \ldots \wedge b_n \wedge \underline{\mathbf{c}} \leq y \leq b_2 \vee \ldots \vee b_n \vee \underline{\mathbf{c}}, \quad \text{and} \quad b_1 \wedge y \wedge \underline{\mathbf{c}} \leq x \leq b_1 \vee y \vee \underline{\mathbf{c}}.$$

These inequalities clearly imply condition (A.2) for n.

(A.2) $\Rightarrow$ (A.1). Assume $b_1 \wedge \ldots \wedge b_n \wedge \underline{\mathbf{c}} \leq x \leq b_1 \vee \ldots \vee b_n \vee \underline{\mathbf{c}}$. Set $y = [((b_2 \vee \ldots \vee b_n \vee \underline{\mathbf{c}}) \wedge x) \vee (b_2 \wedge \ldots \wedge b_n \wedge \underline{\mathbf{c}})] \in P$. By induction hypothesis it suffices to show

$$b_2 \wedge \ldots \wedge b_n \wedge \underline{\mathbf{c}} \leq y \leq b_2 \vee \ldots \vee b_n \vee \underline{\mathbf{c}}, \quad \text{and} \quad b_1 \wedge y \wedge \underline{\mathbf{c}} \leq x \leq b_1 \vee y \vee \underline{\mathbf{c}};$$

this gives $y \in D_P(\langle b_2, \ldots, b_n \rangle)$ and $x \in D_P(\langle b_1, y \rangle)$, which, by the inductive definition of D_P, yields (A.1).

Clearly, $y \geq b_2 \wedge \ldots \wedge b_n \wedge \underline{\mathbf{c}}$. From $(b_2 \vee \ldots \vee b_n \vee \underline{\mathbf{c}}) \wedge x \leq b_2 \vee \ldots \vee b_n \vee \underline{\mathbf{c}}$ it follows that $y \leq b_2 \vee \ldots \vee b_n \vee \underline{\mathbf{c}}$. As for the other inequalities, we have:

$$b_1 \wedge y \wedge \underline{\mathbf{c}} = (b_1 \wedge \ldots \wedge b_n \wedge \underline{\mathbf{c}}) \vee (b_1 \wedge \underline{\mathbf{c}} \wedge x \wedge (b_2 \vee \ldots \vee b_n \vee \underline{\mathbf{c}})).$$

Since $b_1 \wedge \ldots \wedge b_n \wedge \underline{\mathbf{c}} \leq x$, both terms of the disjunction are $\leq x$. Also,

$$b_1 \vee y \vee \underline{\mathbf{c}} = [(b_1 \vee \ldots \vee b_n \vee \underline{\mathbf{c}}) \wedge (b_1 \vee \underline{\mathbf{c}} \vee x)] \vee (b_2 \wedge \ldots \wedge b_n \wedge \underline{\mathbf{c}}).$$

By the assumption $x \leq b_1 \vee \ldots \vee b_n \vee \underline{\mathbf{c}}$, both terms of the first disjunct are $\geq x$; hence $b_1 \vee y \vee \underline{\mathbf{c}} \geq x$, as required.

(B) First we treat the cases $n = 1, 2$, and then proceed by induction on n.

$n = 1$) Condition $x \in D^t(b_1)$ just means $x = b_1$. Condition (B.2) boils down to:

$$\nabla b_1 \leq \nabla x \leq \nabla b_1 \vee \Delta b_1 = \nabla b_1 \quad \text{and} \quad \Delta b_1 = \nabla b_1 \wedge \Delta b_1 \leq \Delta x \leq \Delta b_1,$$

i.e., $\nabla x = \nabla b_1$ and $\Delta x = \Delta b_1$. By IV.1.4 (h), this is equivalent to $x = b_1$.

$n = 2$) By Theorem IV.2.7 (i), condition (B.1) is equivalent to

$$(1') \qquad (b_1 \wedge \nabla b_2) \vee (b_2 \wedge \nabla b_1) \leq x \leq (b_1 \vee \Delta b_2) \wedge (b_2 \vee \Delta b_1).$$

We show this is equivalent to condition (B.2). For the implication $(1') \Rightarrow$ (B.2) it suffices to apply the operations ∇ and Δ to $(1')$ —taking into account that these are lattice homomorphisms, and using IV.1.4(d),(e)—, and then perform trivial Boolean transformations.

For the reverse implication, (B.2) $\Rightarrow (1')$, we use the identities

$$(\dagger) \qquad z = (\underline{\mathbf{c}} \wedge \nabla z) \vee \Delta z = (\underline{\mathbf{c}} \vee \Delta z) \wedge \nabla z,$$

(cf. IV.1.4(i)) to retrieve the inequalities $(1')$ from those in (B.2). We illustrate the argument by proving the left-hand side inequality in $(1')$:

$$\begin{aligned} x &= (\underline{\mathbf{c}} \vee \Delta x) \wedge \nabla x \geq [\underline{\mathbf{c}} \vee (\nabla b_1 \wedge \nabla b_2 \wedge (\Delta b_1 \vee \Delta b_2))] \wedge \nabla b_1 \wedge \nabla b_2 = \\ &= (\underline{\mathbf{c}} \vee \nabla b_1) \wedge (\underline{\mathbf{c}} \vee \nabla b_2) \wedge (\underline{\mathbf{c}} \vee \Delta b_1 \vee \Delta b_2) \wedge \nabla b_1 \wedge \nabla b_2 = \\ &= (\underline{\mathbf{c}} \vee \Delta b_1 \vee \Delta b_2) \wedge \nabla b_1 \wedge \nabla b_2 = [((\underline{\mathbf{c}} \vee \Delta b_1) \wedge \nabla b_1) \wedge \nabla b_2] \vee [((\underline{\mathbf{c}} \vee \Delta b_2) \wedge \nabla b_2) \wedge \nabla b_1] = \\ &= (b_1 \wedge \nabla b_2) \vee (b_2 \wedge \nabla b_1), \end{aligned}$$

where the last equality is obtained using the second identity in $(\dagger)$. A similar argument proves the right-hand side inequality in $(1')$, using the first identity in $(\dagger)$.

Induction step $n - 1 \to n$, $n \geq 3$.

(B.1) $\Rightarrow$ (B.2). Assume $x \in D^t_P(\langle b_1, \ldots, b_n \rangle)$. By the inductive definition of D^t (I.2.7 (a)), there is $y \in D^t_P(\langle b_2, \ldots, b_n \rangle)$ so that $x \in D^t_P(\langle b_1, y \rangle)$. From the induction hypothesis and the case $n = 2$, we get, respectively

$$\nabla b_2 \wedge \ldots \wedge \nabla b_n \leq \nabla y \leq (\nabla b_2 \wedge \ldots \wedge \nabla b_n) \vee \Delta b_2 \vee \ldots \vee \Delta b_n \,,$$

$$\nabla b_1 \wedge \nabla y \leq \nabla x \leq (\nabla b_1 \wedge \nabla y) \vee \Delta b_1 \vee \Delta y \,,$$

and

$$\nabla b_2 \wedge \ldots \wedge \nabla b_n \wedge (\Delta b_2 \vee \ldots \vee \Delta b_n) \leq \Delta y \leq \Delta b_2 \vee \ldots \vee \Delta b_n \,,$$

$$\nabla b_1 \wedge \nabla y \wedge (\Delta b_1 \vee \Delta y) \leq \Delta x \leq \Delta b_1 \vee \Delta y \,.$$

Putting these inequalities together easily yields the desired conclusion.

(B.2) $\Rightarrow$ (B.1). Assume the inequalities (2.a) and (2.b) of the statement. In order to check that $x \in D^t_P(\langle b_1, \ldots, b_n \rangle)$ via the inductive definition of D^t, we set:

$$(\dagger\dagger) \qquad t = (\nabla x^2 \wedge \neg \nabla b_1) \vee (x \wedge \neg \Delta b_1) \vee [(\bigwedge_{i=2}^{n} \nabla b_i) \wedge (\bigvee_{i=2}^{n} b_i)].$$

First we prove:

Claim. $\nabla x^2 \wedge \neg \nabla b_1 \leq \nabla x \wedge \neg \nabla b_1 \leq \bigvee_{i=2}^{n} \Delta b_i$.

Proof of Claim. The left inequality follows from $\nabla x^2 = \nabla (x \wedge \neg x) \leq \nabla x$. For the other, from the right-hand side inequality in asumption (2.a) we get:

$$\begin{aligned} \nabla x \wedge \neg \nabla b_1 &\leq \neg \nabla b_1 \wedge [(\bigwedge_{i=1}^{n} \nabla b_i) \vee (\bigvee_{i=1}^{n} \Delta b_i)] = \\ &= [(\neg \nabla b_1 \wedge \nabla b_1) \wedge \bigwedge_{i=2}^{n} \nabla b_i] \vee (\neg \nabla b_1 \wedge \Delta b_1) \vee \bigvee_{i=2}^{n} (\neg \nabla b_1 \wedge \Delta b_i) = \\ &= \bigvee_{i=2}^{n} (\neg \nabla b_1 \wedge \Delta b_i) \leq \bigvee_{i=2}^{n} \Delta b_i \,, \end{aligned}$$

(here we use that $\neg \nabla b_1 \wedge \Delta b_1 \leq \neg \nabla b_1 \wedge \nabla b_1 = \perp$, as $\Delta b_1 \leq \nabla b_1$, and $\Delta b_1, \nabla b_1$ are Boolean elements).

Next, we explicitly compute ∇t and Δt. Applying the operator ∇ on both sides of $(\dagger\dagger)$ and recalling that ∇ is a lattice homomorphism which is the identity on Boolean elements (IV.1.4), that $\nabla z, \Delta z$ and their negations are Boolean, and using $\nabla x^2 \leq \nabla x$ and $\neg \nabla b_1 \leq \neg \Delta b_1$, we obtain:

$$(\mathrm{I}) \quad \nabla t = (\nabla x^2 \wedge \neg \nabla b_1) \vee (\nabla x \wedge \neg \Delta b_1) \vee [(\bigwedge_{i=2}^{n} \nabla b_i) \wedge (\bigvee_{i=2}^{n} \nabla b_i)] = (\nabla x \wedge \neg \Delta b_1) \vee \bigwedge_{i=2}^{n} \nabla b_i;$$

thus, $\nabla t \geq \bigwedge_{i=2}^{n} \nabla b_i$. On the other hand, as in the proof of the Claim, we get:

$$\begin{aligned} \nabla x \wedge \neg \Delta b_1 &\leq (\neg \Delta b_1 \wedge \nabla b_1 \wedge \bigwedge_{i=2}^{n} \nabla b_i) \vee (\neg \Delta b_1 \wedge \Delta b_1) \vee \bigvee_{i=2}^{n} (\neg \Delta b_1 \wedge \Delta b_i) \\ &\leq (\bigwedge_{i=2}^{n} \nabla b_i) \vee (\bigvee_{i=2}^{n} \Delta b_i) \,, \end{aligned}$$

which, together with (I), clearly implies $\nabla t \leq (\bigwedge_{i=2}^{n} \nabla b_i) \vee (\bigvee_{i=2}^{n} \Delta b_i)$.

Next, applying Δ on the equality $(\dagger\dagger)$ defining t, we obtain:

$$(\mathrm{II}) \quad \Delta t = (\nabla x^2 \wedge \neg \nabla b_1) \vee (\Delta x \wedge \neg \Delta b_1) \vee [(\bigwedge_{i=2}^{n} \nabla b_i) \wedge (\bigvee_{i=2}^{n} \Delta b_i)].$$

Hence, $\Delta t \geq (\bigwedge_{i=2}^{n} \nabla b_i) \wedge (\bigvee_{i=2}^{n} \Delta b_i)$. For the inequality $\Delta t \leq \bigvee_{i=2}^{n} \Delta b_i$, observe that the last disjunct in (II) and, by the Claim, also the first disjunct, are $\leq \bigvee_{i=2}^{n} \Delta b_i$; the inequality $\Delta x \leq \bigvee_{i=1}^{n} \Delta b_i$ (assumption (2.b)) obviously implies that the middle term in (II) is $\leq \bigvee_{i=2}^{n} \Delta b_i$ as well.

By induction hypothesis, the foregoing bounds on Δt and ∇t imply $t \in D^t_P(\langle b_2, \ldots, b_n \rangle)$. To complete the proof we must show that $x \in D^t_P(b_1, t)$. The simplest way of doing this is by truth-table checking the definition of transversal representation, i.e., showing that, for all characters $h : P \longrightarrow \mathbf{3}$,

$$h(x) \in \{\bot, \top\} \;\Rightarrow\; h(x) = h(b_1) \text{ or } h(x) = h(t), \quad \text{and} \quad h(x) = 0 \;\Rightarrow\; h(b_1) = -h(t).$$

— Let $h(x) = \bot$. Then, $h(\nabla x^2) = \bot$, and (††) yields $h(t) = (\bigwedge_{i=2}^n \nabla h(b_i)) \wedge (\bigvee_{i=2}^n h(b_i))$.

Now, we argue according to the values of $h(b_1)$. If $h(b_1) = \bot$, then $h(x) = h(b_1)$. If $h(b_1) \neq \bot$, then $\nabla h(b_1) = \top$. Since $\bigwedge_{i=1}^n \nabla b_i \leq \nabla x$ (2.a), and $h(\nabla x) = \bot$, we get $h(t) = \bigwedge_{i=2}^n \nabla h(b_i) = \bot$, and then $h(t) = \bot = h(x)$.

— If $h(x) = \top$, since $h(\nabla x^2) = \bot$, (††) yields:

$$h(t) = \neg\, \Delta h(b_1) \vee [(\bigwedge_{i=2}^n \nabla h(b_i)) \wedge (\bigvee_{i=2}^n h(b_i))].$$

Arguing again according to the values of $h(b_1)$, we have: if $h(b_1) = \top$, then $h(x) = h(b_1)$; otherwise, $\Delta h(b_1) = \bot$, whence $h(t) = \top = h(x)$.

— Finally, if $h(x) = 0$, then $h(\nabla x^2) = \top$, and

$$h(t) = \neg \nabla h(b_1) \vee (0 \wedge \neg\, \Delta h(b_1)) \vee [(\bigwedge_{i=2}^n \nabla h(b_i)) \wedge (\bigvee_{i=2}^n h(b_i))].$$

If $h(b_1) = \bot$, then $\neg \nabla h(b_1) = \top$, and $h(t) = \top = -h(b_1)$. If $h(b_1) = \top$, then $\neg \nabla h(b_1) = \neg\, \Delta h(b_1) = \bot$, and $h(t) = (\bigwedge_{i=2}^n \nabla h(b_i)) \wedge (\bigvee_{i=2}^n h(b_i))$. Since $(\bigwedge_{i=1}^n \nabla b_i) \wedge (\bigvee_{i=1}^n \Delta b_i) \leq \Delta x$ (cf. (2.b)), and $h(\Delta x) = \bot$, we get $\bigwedge_{i=2}^n \nabla h(b_i) = \bot$, and hence $h(t) = \bot = -h(b_1)$.

If $h(b_1) = 0$, then $\neg \nabla h(b_1) = \bot$, $\neg\, \Delta h(b_1) = \top$, and

$$h(t) = 0 \vee [(\bigwedge_{i=2}^n \nabla h(b_i)) \wedge (\bigvee_{i=2}^n h(b_i))].$$

We prove that the last term of this disjunction is ≤ 0. Otherwise it equals $\top$, which implies $\bigwedge_{i=2}^n \nabla h(b_i) = \bigvee_{i=2}^n h(b_i) = \top$, and hence also $\bigvee_{i=2}^n \Delta h(b_i) = \top$. Since $\nabla h(b_1) = \top$, we get $\top = (\bigwedge_{i=1}^n \nabla h(b_i)) \wedge (\bigvee_{i=1}^n \Delta h(b_i)) \leq \Delta h(x) = \bot$, a contradiction. It follows that $h(t) = 0 = -h(b_1)$, completing the proof of Theorem IV.5.1. □

Theorem IV.5.1 shows a remarkable symmetry of the intervals involved in the formulas for D_P and D^t_P. If a RS, G, is identified with its image via ε_G inside its Post hull P_G, it shows, in particular, that for every form φ over G,

— $D_G(\varphi)$ is included in an interval of the lattice P_G;

— $D^t_G(\varphi)$ is included in the intersection of the inverse image of two intervals of P_G by the (monotone) operators ∇ and Δ.

In either case the endpoints of the relevant intervals are Post-algebraic functions of the entries of the given form φ.

Theorem IV.5.1 yields precise information concerning the value sets of forms, also over arbitrary real semigroups. First we state some general corollaries and deal later on with the case of Pfister forms.

Proposition IV.5.2 *Let P be a Post algebra, let $x, b_1, \ldots, b_n \in P$ and let $\varphi := \langle b_1, \ldots, b_n \rangle$. Then:*

(1) *$D_P(\varphi)$ is a sublattice of P containing $\underline{\mathbf{c}}$.*

(2) *$D^t_P(\varphi)$ is a sublattice of P closed under Δ and ∇. It may not contain either $0, 1$ or -1. However,*

(3) *If any two of $0, 1, -1$ are transversally represented by φ, then the third is also transversally represented, and $D^t_P(\varphi) = P$.* [3]

In case 1 (or -1) is represented or transversally represented by φ, the conditions in Theorem IV.5.1 take a simpler form and give further information; namely:

(4) *i) If $1 \in D_P(\varphi)$, then $x \in D_P(\varphi) \Leftrightarrow x \leq \underline{\mathbf{c}} \wedge \bigvee_{i=1}^n b_i$.*

ii) If $1 \in D^t_P(\varphi)$, then $x \in D^t_P(\varphi) \Leftrightarrow \nabla x \leq \bigvee_{i=1}^n \Delta b_i$.

Hence,

iii) If $1 \in D_P(\varphi)$ (resp., $1 \in D^t_P(\varphi)$), then $D_P(\varphi)$ (resp., $D^t_P(\varphi)$) is a lattice ideal; in particular, $D_P(\varphi)$ is closed under Δ.

(5) *i) If $-1 \in D_P(\varphi)$, then $x \in D_P(\varphi) \Leftrightarrow \underline{\mathbf{c}} \wedge \bigwedge_{i=1}^n b_i \leq x$.*

Hence,

ii) If $-1 \in D^t_P(\varphi)$, then $x \in D^t_P(\varphi) \Leftrightarrow \bigwedge_{i=1}^n \nabla b_i \leq \Delta x$.

iii) If $-1 \in D_P(\varphi)$ (resp., $-1 \in D^t_P(\varphi)$), then $D_P(\varphi)$ (resp., $D^t_P(\varphi)$) is a lattice filter; in particular, $D_P(\varphi)$ is closed under ∇.

Proof. (1) Straightforward verification shows that conditions (A.2) and (B.2 (a,b)) in Theorem IV.5.1 are preserved under $\wedge$ and $\vee$. The case of transversal representation uses the fact (Proposition IV.1.4 (c)) that Δ and ∇ are lattice endomorphisms of P.

(2) Items (d) and (e) in IV.1.4 imply that all the inequalities in IV.5.1 (B.2) are preserved under Δ and ∇.

For the remaining assertions we shall need:

Claim. (I) $-1 \in D^t_P(\varphi)) \Leftrightarrow \bigvee_{i=1}^n \Delta b_i = -1$.

(II) $1 \in D^t_P(\varphi)) \Leftrightarrow \bigwedge_{i=1}^n \nabla b_i = 1$.

(III) $0 \in D^t_P(\varphi)) \Leftrightarrow (\bigwedge_{i=1}^n \nabla b_i) \wedge (\bigvee_{i=1}^n \Delta b_i) = 1$ and $(\bigwedge_{i=1}^n \nabla b_i) \vee (\bigvee_{i=1}^n \Delta b_i) = -1$.

<u>Proof of Claim.</u> (I) By IV.5.1 (B.2 (b)), $-1 \in D^t_P(\varphi))$ implies $\Delta(-1) = -1 \leq \bigvee_{i=1}^n \Delta b_i = -1$. Conversely, if $\bigvee_{i=1}^n \Delta b_i = -1$, both inequalities in the right-hand side of (B.2 (a,b)) hold for $x = -1$; since $\Delta(-1) = \nabla(-1) = -1$, the left hand side ones trivially hold.

(II) Proof similar to that in (I).

(III) The implication ($\Rightarrow$) comes from items (B.2 (a,b)) in IV.5.1, as $\Delta 0 = 1$ and $\nabla 0 = -1$.

($\Leftarrow$) The inequalities in the right-hand side of (III) entail that the right-hand side inequality in (B.2 (a)) and the left one in (B.2 (b)) hold with $x = 0$. The remaining inequalities in (B.2) come from $\Delta 0 = 1$ and $\nabla 0 = -1$.

<u>Proof of (3).</u> If $1, -1 \in D^t_P(\varphi))$, items (I) and (II) give $\bigvee_{i=1}^n \Delta b_i = -1$ and $\bigwedge_{i=1}^n \nabla b_i = 1$,

[3] I.e., the form φ is *universal*. Note that the statement is about <u>transversal</u> representation.

implying that the right-hand side of (III) holds, whence $0 \in D^t_P(\varphi)$).

If $0, -1 \in D^t_P(\varphi)$), then (I) and the first inequality in (III) yield $\bigwedge_{i=1}^n \nabla b_i = 1$, i.e., $1 \in D^t_P(\varphi)$.

Similar argument in the case $0, 1 \in D^t_P(\varphi)$).

Concerning universality, the equivalences (I) and (II) entail that all ineqiualities in (B.2 (a)) and (B.2 (b)) hold for any $x \in P$, i.e., $x \in D^t_P(\varphi)$.

The proof of (5) is similar, using item (I) in the Claim. $\square$

Preceding Prop. added February 2014.

As an illustration we give a Post-algebraic proof of the following result, (already) used many times in this text.

Corollary IV.5.3 *Let G be a RS. Then:*

(i) ([M], Prop. 6.1.5). *For all $a, b \in G$ there is a unique x such that $D^t_G(a^2, b^2) = \{x\}$ and $x = x^2$. The same holds for $D^t_G(\langle a_1^2, \ldots, a_n^2 \rangle)$ $(a_i \in G)$.*

(ii) For all $a, b, d \in G$ there is a unique y such that $D^t_G(a^2d, b^2d) = \{y\}$ and $y = y^2d$. The same holds for $D^t_G(\langle a_1^2 d, \ldots, a_n^2 d \rangle)$ $(a_i \in G)$.

Proof. We only prove (ii) for binary forms. If $y \in D^t_G(a^2d, b^2d)$, Proposition I.2.3(4) gives $y = d^2y = y^2d$.

For uniqueness we show that $y \in D^t_G(a^2d, b^2d)$ implies $\Delta y = \Delta a^2 d \vee \Delta b^2 d$ and $\nabla y = \nabla a^2 d \vee \nabla b^2 d$, which then follows from Proposition IV.1.4 (h). The asserted identities are consequences of:

Fact IV.5.4 *For $x, y \in P$ (P a Post algebra),*

(i) $\Delta x^2 y = \Delta y \wedge \neg \nabla x^2$; *(ii)* $\nabla x^2 y = \nabla y \vee \nabla x^2$.

Proof. Since in a Post algebra product (as RS) is symmetric difference, we have $x^2y = (x^2 \wedge \neg\, y) \vee (y \wedge \neg\, x^2)$. Hence,

$$\Delta x^2 y = (\Delta x^2 \wedge \Delta \neg\, y) \vee (\Delta y \wedge \Delta \neg\, x^2) = \Delta y \wedge \neg \nabla x^2,$$

as $\Delta x^2 = \bot$ (cf. IV.1.2(b)), and $\Delta \neg\, x^2 = \neg \nabla x^2$. Also,

$$\nabla x^2 y = (\nabla x^2 \wedge \nabla \neg\, y) \vee (\nabla y \wedge \nabla \neg\, x^2) = (\nabla x^2 \wedge \neg\, \Delta y) \vee \nabla y = \nabla x^2 \vee \nabla y,$$

since $\nabla \neg\, x^2 = \neg \Delta x^2 = \top$ and $\neg\, \Delta y \vee \nabla y \geq \neg\, \Delta y \vee \Delta y = \top$. $\square$

To finish the proof of Corollary IV.5.3, the Fact implies $\Delta a^2 d = \Delta d \wedge \neg \nabla a^2 \leq \Delta d \leq \nabla d \leq \nabla d \vee \nabla b^2 = \nabla b^2 d$; hence, $\Delta a^2 d \vee \Delta b^2 d \leq \nabla b^2 d \wedge \nabla a^2 d$. By (2.a) of Theorem IV.5.1, we have:

$$\nabla a^2 d \wedge \nabla b^2 d \leq \nabla y \leq (\nabla a^2 d \wedge \nabla b^2 d) \vee \Delta a^2 d \vee \Delta b^2 d = \nabla a^2 d \wedge \nabla b^2 d,$$

i.e., $\nabla y = \nabla a^2 d \wedge \nabla b^2 d$. Item (2.b) of the same theorem gives:

$$(\nabla a^2 d \wedge \nabla b^2 d) \wedge (\Delta a^2 d \vee \Delta b^2 d) = \Delta a^2 d \vee \Delta b^2 d \leq \Delta y \leq \Delta a^2 d \vee \Delta b^2 d,$$

i.e., $\Delta y = \Delta a^2 d \vee \Delta b^2 d$, as required. $\square$

Omit next two corollaries?.

It also follows from Theorem IV.5.1 that the value set and the transversal value set of an arbitrary form with entries in a Post algebra contain the same Boolean elements (recall an element x is Boolean if $\nabla x = \Delta x = x$).

Corollary IV.5.5 *Let P be a Post algebra, $B(P)$ be the set of its Boolean elements, and let $x, b_1, \ldots, b_n \in P$. Then,*

$$x \in B(P) \cap D_P(\langle b_1, \ldots, b_n \rangle) \;\Rightarrow\; x \in D^t_P(\langle b_1, \ldots, b_n \rangle).$$

Proof. The assumption and IV.5.1(1) give

$$b_1 \wedge \ldots \wedge b_n \wedge \underline{\mathbf{c}} \le x \le b_1 \vee \ldots \vee b_n \vee \underline{\mathbf{c}}.$$

Applying the operators ∇ and Δ on these inequalities, and recalling that they are lattice homomorphisms such that $\nabla\underline{\mathbf{c}} = \top$ and $\Delta\underline{\mathbf{c}} = \bot$, gives

$$\bigwedge\nolimits_{i=1}^{n} \nabla b_i \le \nabla x \le \top \qquad \text{and} \qquad \bot \le \Delta x \le \bigvee\nolimits_{i=1}^{n} \Delta b_i,$$

whence (using $\nabla x = \Delta x = x$),

$$\bigwedge\nolimits_{i=1}^{n} \nabla b_i \le \nabla x = \Delta x \le \bigvee\nolimits_{i=1}^{n} \Delta b_i \le (\bigwedge\nolimits_{i=1}^{n} \nabla b_i) \vee (\bigvee\nolimits_{i=1}^{n} \Delta b_i), \qquad \text{and}$$

$$(\bigwedge\nolimits_{i=1}^{n} \nabla b_i) \wedge (\bigvee\nolimits_{i=1}^{n} \Delta b_i) \le \bigwedge\nolimits_{i=1}^{n} \nabla b_i \le \nabla x = \Delta x \le \bigvee\nolimits_{i=1}^{n} \Delta b_i.$$

By Theorem IV.5.1(2) this establishes the desired conclusion. $\square$

For arbitrary RSs this yields:

Corollary IV.5.6 *Let G be a RS and let x be an invertible element of G. If φ is either a binary form or the multiple of a Pfister form with (arbitrary) entries in G, we have*

$$x \in D_G(\varphi) \;\Rightarrow\; x \in D^t_G(\varphi).$$

Proof. This follows from the previous corollary and the fact that both representation and transversal representation are reflected from P_G down to G for forms of the stated type (Corollary IV.4.7 (3)), upon observing that x is invertible in G (i.e., $x^2 = 1$) if and only if $\varepsilon_G(x)$ is a Boolean element of P_G; the reader can easily verify this assertion. $\square$

Now we turn to the characterization of representation and transversal representation by Pfister forms in arbitrary real semigroups.

Theorem IV.5.7 *Let G be a RS, let $x, a_1, \ldots, a_n \in G$, and let $\varphi = \langle\langle a_1, \ldots, a_n \rangle\rangle = \bigotimes_{i=1}^{n} \langle 1, a_i \rangle$. Then:*

(1) $x \in D_G(\varphi) \;\Leftrightarrow\; \varepsilon_G(x) \le \bigvee\nolimits_{i=1}^{n} \varepsilon_G(a_i) \vee \underline{\mathbf{c}}$ *(in P_G)*.

(2) $x \in D^t_G(\varphi) \;\Leftrightarrow\; \varepsilon_G(x) \le \bigvee\nolimits_{i=1}^{n} \Delta(\varepsilon_G(a_i))$ *(in P_G)*.

Proof. By Corollary IV.4.7(3) we may assume without loss of generality that G is a Post algebra —which we call P—, and omit ε_G in the right-hand side of the statement.

Let C denote the set of entries of φ, i.e., C consists of 1 and all products $\prod_{i=1}^{n} a_i^{\eta_i}$, with $\eta_i \in \{0, 1\}$, and $y^0 = 1$, $y^1 = y$. By Theorem IV.5.1 we have:

(1′) $x \in D_G(\varphi) \;\Leftrightarrow\; \underline{\mathbf{c}} \wedge \bigwedge\nolimits_{b \in C} b \le x \le \underline{\mathbf{c}} \vee \bigvee\nolimits_{b \in C} b$, and

(2′) $x \in D^t_G(\varphi) \;\Leftrightarrow\; \bigwedge\nolimits_{b \in C} \nabla b \le \nabla x \le (\bigwedge\nolimits_{b \in C} \nabla b) \vee (\bigvee\nolimits_{b \in C} \Delta b)$, and
$(\bigwedge\nolimits_{b \in C} \nabla b) \wedge (\bigvee\nolimits_{b \in C} \Delta b) \le \Delta x \le \bigvee\nolimits_{b \in C} \Delta b$.

Now observe:

— $1 \in C$ implies $\bigwedge_{b\in C} b = \bigwedge_{b\in C} \nabla b = 1 = \bot$;

— $x \,\underline{\triangle}\, y \leq x \vee y$ implies $\prod_{i=1}^{n} a_i^{\eta_i} \leq \bigvee_{i=1}^{n} a_i^{\eta_i} \leq \bigvee_{i=1}^{n} a_i$ and similarly with a_i replaced with Δa_i.

Hence, the right-hand sides of $(1')$ and $(2')$ boil down to

$(*)\quad \bot \leq x \leq \underline{\mathbf{c}} \vee \bigvee_{i=1}^{n} a_i \qquad$ and $\qquad \bot \leq \Delta x, \nabla x \leq \bigvee_{i=1}^{n} \Delta a_i,$

respectively. The first of these inequalities proves (1). As for (2) we observe that the last inequality in $(*)$ is equivalent to

$(**)\quad x \leq \bigvee_{i=1}^{n} \Delta a_i.$

In fact, $(*) \Rightarrow (**)$ since $x \leq \nabla x$. Conversely, applying successively the operators Δ and ∇ to $(**)$ and using Proposition IV.1.4(c)-(e), we get the last inequality in $(*)$. $\square$

Next Prop. added Feb. 2014 (improvement from previous versions).

Proposition IV.5.8 *Let G be a RS, and let $\varphi = \langle\langle a_1, \ldots, a_n\rangle\rangle$ be a Pfister form over G. Then,*

(1) $D_G(\varphi)$ and $D^t_G(\varphi)$ are subsemigroups, downwards closed under the representation partial order of G. Further,

(2) $0 \in D^t_G(\varphi) \ \Rightarrow\ D^t_G(\varphi) = G$.

(3) If $-1 \in D_{P_G}(\langle a_1, \ldots, a_n\rangle)$, then the Pfister form φ is universal in G.

Further, we get an alternative proof of the separation property *for Pfister forms (Corollary I.5.7): for $a \in G$,*

(4) (i) $a \in D_G(\varphi) \Leftrightarrow$ For all $h \in X_G$, $h(a) \in D_{\mathbf{3}}(h * \varphi)$.

(ii) $a \in D^t_G(\varphi) \Leftrightarrow$ For all $h \in X_G$, $h(a) \in D^t_{\mathbf{3}}(h * \varphi)$.

Proof. (1) Multiplicativity follows from Theorem IV.5.7 (1,2) (for D and D^t, respectively), since $\varepsilon_G(xy) = \varepsilon_G(x) \,\underline{\triangle}\, \varepsilon_G(y) \leq \varepsilon_G(x) \vee \varepsilon_G(y)$ (in P_G). As for the second assertion, if $x, y \in G$ and $x \leq_G y$, then $\varepsilon_G(x) \leq \varepsilon_G(y)$ (in P_G), and the conclusion follows, again, using IV.5.7 (1) for D, and IV.5.7 (2), for D^t.

(2) Since both $0, 1$ are transversally represented by φ in P_G, Proposition IV.5.2 (3) implies that φ iss universal in P_G. Since φ is Pfister and has entries in G, by Corollary IV.4.7 universality is reflected from P_G down to G.

Let $\psi := \langle a_1, \ldots, a_n\rangle$. Up to a permutation of entries, φ can be written as $\psi \oplus \theta$ for some form θ over G; hence, $D_G(\psi) \subseteq D_G(\varphi)$, and the same inclusion holds in P_G. From I.2.8 (3) we get $-1 \in D^t_{P_G}((-1)^2\varphi) = D^t_{P_G}(\varphi)$. By (1), $D^t_{P_G}(\varphi) = P_G$ and, by Corollary IV.4.7, φ is universal in G.

(i) The implication $(\Rightarrow)$ is clear (any $h \in X_G$ is a RS-morphism).

$(\Leftarrow)$ Assume $a \notin D_g(\varphi)$. By IV.6.5 (1), in P_G we have $\varepsilon_G(a) \not\leq \bigvee_{i=1}^{n} \varepsilon_G(a_i) \vee \underline{\mathbf{c}}$. By Proposition IV.2.5 (i) there is a PA-character σ of P_G such that (in $\mathbf{3}$) $\sigma(\varepsilon_G(a)) > \sigma(\bigvee_{i=1}^{n} \varepsilon_G(a_i)) =$

$\bigvee_{i=1}^{n} \sigma(\varepsilon_G(a_i)) \vee 0$ (σ is a lattice homomorphism). Since **3** itself is a PA, IV.6.5 (1) gives $(-1 =)(\sigma \circ \varepsilon_G)(a) \notin D_{\mathbf{3}}((\sigma \circ \varepsilon_G) * \varphi)$; but $\sigma \circ \varepsilon_G \in X_G$.

(ii) follows from IV.6.5 (2) by an argument similar to that proving (4.i). □

Remark. Item (3) fails with -1 replaced by 0: $0 \in D^t_{P_G}(\langle 0 \rangle)$ but $D_G(\langle\langle 0 \rangle\rangle) = \{1\}$.

Next we deal with representation by multiples of Pfister forms.

Theorem IV.5.9 *Let G be a RS, $x, b, a_1, \ldots, a_n \in G$, and let $\varphi = \langle\langle a_1, \ldots, a_n \rangle\rangle$. Then,*

(1) $x \in D_G(b\varphi) \Leftrightarrow x = b^2 x$ *and* (*in* P_G) $\varepsilon_G(bx) \leq \underline{\mathbf{c}} \vee \bigvee_{i=1}^{n} \varepsilon_G(a_i)$.

(2) $x \in D^t_G(b\varphi) \Leftrightarrow x = b^2 x$ *and* (*in* P_G) $\varepsilon_G(bx) \leq \varepsilon_G(b^2) \vee \bigvee_{i=1}^{n} \Delta(\varepsilon_G(a_i))$.

Proof. (1) is just a restatement of Remark IV.4.6 (ii) using the characterization of the elements represented by Pfister forms in Theorem IV.5.7 (1) (one can also give a proof involving only Post algebra operations). Unfortunately, we can only offer a long proof for item (2).

(2) As in IV.4.6 (ii), we have:

$$x \in D^t_G(b\varphi) \Leftrightarrow x = b^2 x \text{ and } bx \in D^t_G(b^2\varphi).$$

Then, we must prove:

$$bx \in D^t_G(b^2\varphi) \Leftrightarrow \text{ (in } P_G) \ \varepsilon_G(bx) \leq \varepsilon_G(b^2) \vee \bigvee_{i=1}^{n} \Delta(\varepsilon_G(a_i)).$$

By Corollary IV.4.7 (3) we may assume that $G\ (= P)$ is a Post algebra, and hence omit reference to ε_G. Thus, the preceding equivalence boils down to proving that

(i) With C denoting the set of coefficients of φ,

(I) $\bigwedge_{z \in C} \nabla b^2 z \leq \nabla bx \leq (\bigwedge_{z \in C} \nabla b^2 z) \vee (\bigvee_{z \in C} \Delta b^2 z)$, and

(II) $(\bigwedge_{z \in C} \nabla b^2 z) \wedge (\bigvee_{z \in C} \Delta b^2 z) \leq \Delta bx \leq \bigvee_{z \in C} \Delta b^2 z$,

is equivalent to

(ii) $bx \leq b^2 \vee \bigvee_{i=1}^{n} \Delta a_i$.

Proving this equivalence will require to compute explicitly the quantities occurring in (I) and (II). In order to achieve this we first prove:

Lemma IV.5.10 *Let P be a Post algebra, and let $b, x, y \in P$. Recall that $\underline{\mathbf{c}}$ denotes the center of P. Then, we have:*

(*i*) $b^2(x \vee y) = b^2 x \vee b^2 y$; (*ii*) $b^2(x \wedge y) = b^2 x \wedge b^2 y$; (*iii*) $b^2 \leq \underline{\mathbf{c}}$; (*iv*) $b^2 \underline{\mathbf{c}} = \underline{\mathbf{c}}$;

(*v*) $b^2 \leq bx$; (*vi*) $bx \leq y \Rightarrow bx \leq b^2 y$; (*vii*) $\nabla b^2 \wedge \Delta b^2 x = \bot$;

(*viii*) $\nabla b^2 \vee \Delta b^2 x = \nabla b^2 \vee \nabla x$; (*ix*) $\nabla b^2 \leq \nabla bx$.

Proof. (i) is the distributivity of symmetric difference over join, familiar from Boolean algebras. (ii) – (vi) are proved by truth-table checking (see IV.2.6); as an illustration we prove (vi). Given a character h, if $h(b) \neq 0$, then $h(b^2) = 1$, and from the assumption we get $h(b)h(x) \leq h(y) = h(b^2)h(y)$; if $h(b) = 0$, the inequality to be checked holds trivially.

Item (ix) follows by applying ∇ to (v). Items (vii) and (viii) follow from the first identity in Fact IV.5.4 by trivial manipulations, using that elements of the form ∇z are Boolean. $\square$

Returning to the proof of Theorem IV.5.9, since $1 \in C$, from IV.5.10 (ii) we get:

(*) $\quad \bigwedge_{z\in C} \nabla b^2 z = \nabla(\bigwedge_{z\in C} b^2 z) = \nabla(b^2 \cdot \bigwedge_{z\in C} z) = \nabla b^2.$

Note also that $\bigvee_{z\in C} b^2 z = b^2 \cdot \bigvee_{z\in C} z = b^2 \cdot \bigvee_{i=1}^n a_i$ (IV.5.10 (i)). Using the identity $\Delta b^2 z = \Delta z \wedge \neg \nabla b^2$ (IV.5.4 (i)), we obtain:

(**) $\quad \bigvee_{z\in C} \Delta b^2 z = \Delta(\bigvee_{z\in C} b^2 z) = \Delta(b^2 \cdot \bigvee_{i=1}^n a_i) = \neg\nabla b^2 \wedge \bigvee_{i=1}^n \Delta a_i\,.$

Substituting (*) and (**), the inequalities (I) and (II) get, respectively, reduced to:

(I′) $\quad \nabla b^2 \le \nabla bx \le \nabla b^2 \vee (\neg\nabla b^2 \wedge \bigvee_{i=1}^n a_i) = \nabla b^2 \vee \bigvee_{i=1}^n \Delta a_i,$ and

(II′) $\quad \bot \le \Delta bx \le \neg\nabla b^2 \wedge \bigvee_{i=1}^n \Delta a_i\,.$

Since the left inequality in (I′) is valid (IV.5.10 (ix)), (I′) and (II′) boil down to their right-hand side inequalities. Now we prove:

(ii) $\Rightarrow$ (i). Applying the (monotonous) operator ∇ to (ii) and using IV.1.4 (e), gives (I′). Next, apply the operator Δ to (ii) and use $\Delta b^2 = \bot$ to get $\Delta bx \le \bigvee_{i=1}^n \Delta a_i$. From $\nabla b^2 \le \nabla b(-x)$ (IV.5.10 (ix)) we get $\Delta bx \le \neg\nabla b^2$, which proves (II′).

(i) $\Rightarrow$ (ii). It suffices to show that the right-hand side inequalities in (I′) and (II′) imply (ii). Using the identity $z = (\underline{\mathbf{c}} \wedge \nabla z) \vee \Delta z$ (IV.1.4 (i)), (I′) and (II′) yield:

(III)
$$\begin{aligned} bx &= (\underline{\mathbf{c}} \wedge \nabla bx) \vee \Delta bx \le [\,\underline{\mathbf{c}} \wedge (\nabla b^2 \vee \textstyle\bigvee_{i=1}^n \Delta a_i)] \vee (\neg\nabla b^2 \wedge \textstyle\bigvee_{i=1}^n \Delta a_i) \\ &= (\underline{\mathbf{c}} \wedge \nabla b^2) \vee (\underline{\mathbf{c}} \wedge \textstyle\bigvee_{i=1}^n \Delta a_i) \vee (\neg\nabla b^2 \wedge \textstyle\bigvee_{i=1}^n \Delta a_i)\,. \end{aligned}$$

Note that $\Delta b^2 = \bot$ implies $b^2 = \underline{\mathbf{c}} \wedge \nabla b^2$, and that axiom [L3 (ii)] (IV.1.1) yields $b^2 \vee \neg\nabla b^2 = b^2 \vee \neg\, b^2$. Substituting these identities in the last term of (III) and distributing, we obtain:

(IV)
$$\begin{aligned} bx &\le [\,b^2 \vee (\underline{\mathbf{c}} \wedge \textstyle\bigvee_{i=1}^n \Delta a_i) \vee \textstyle\bigvee_{i=1}^n \Delta a_i\,] \wedge [\,b^2 \vee \neg\nabla b^2 \vee (\underline{\mathbf{c}} \wedge \textstyle\bigvee_{i=1}^n \Delta a_i)\,] \\ &= (b^2 \vee \textstyle\bigvee_{i=1}^n \Delta a_i) \wedge [\,b^2 \vee \neg\, b^2 \vee (\underline{\mathbf{c}} \wedge \textstyle\bigvee_{i=1}^n \Delta a_i)\,]\,. \end{aligned}$$

Kleene's inequality IV.1.2 (b) implies $\underline{\mathbf{c}} \le b^2 \vee \neg\, b^2$; thus, (IV) yields:

$$bx \le (b^2 \vee \textstyle\bigvee_{i=1}^n \Delta a_i) \wedge (b^2 \vee \neg\, b^2) = b^2 \vee (\neg\, b^2 \wedge \textstyle\bigvee_{i=1}^n \Delta a_i) \le b^2 \vee \textstyle\bigvee_{i=1}^n \Delta a_i,$$

as required. This completes the proof of Theorem IV.5.9. $\square$

As a corollary to Theorem IV.5.9 we obtain some properties of the value sets of Pfister forms and their multiples in RSs that are weak versions of results well known in the context of fields and reduced special groups.

Corollary IV.5.11 *Let G be a RS, $b, a_1, \ldots, a_n \in G$, and let $\varphi = \langle\langle a_1, \ldots, a_n \rangle\rangle$. Then,*

(1) $b \in D_G(\varphi) \Rightarrow D_G(b\varphi) = D_G(\varphi) \cap \{x \in G \mid x = b^2 x\}$.

(2) $b \in D^t_G(\varphi)$ *and* $x \in D^t_G(b\varphi) \Rightarrow x \in D^t_G(\varphi)$.

(3) $b \in D^t_G(\varphi) \Rightarrow D^t_G(b\varphi) = b \cdot D^t_G(\varphi) = D^t_G(\varphi) \cap \{x \in G \mid x = b^2 x\}$.

The separation properties of IV.5.8 (4) also hold for multiples of Pfister forms:

(4) (*i*) $a \in D_G(b\varphi) \Leftrightarrow$ *For all* $h \in X_G$, $h(a) \in D_{\mathbf{3}}(h(b) \cdot (h * \varphi))$.

(*ii*) $a \in D^t_G(b\varphi) \Leftrightarrow$ *For all* $h \in X_G$, $h(a) \in D^t_{\mathbf{3}}(h(b) \cdot (h * \varphi))$.

Proof. (1) Note that $D(b\psi) = b \cdot D(\psi)$ holds for arbitrary forms ψ (this follows easily from Proposition I.2.8 (2)–(4)). Let $b \in D_G(\varphi)$; since $D_G(\varphi)$ is a subsemigroup of G, $D_G(b\varphi) = b \cdot D_G(\varphi) \subseteq D_G(\varphi)$. If $x \in D_G(\varphi)$, then $bx \in D_G(\varphi)$, and $b^2x \in D_G(b\varphi)$. If in addition $x = b^2x$, we get $x \in D_G(b\varphi)$.

(2) Let $b \in D^t_G(\varphi)$; by IV.5.7 (2), $\varepsilon_G(b) \leq \bigvee_{i=1}^n \Delta\varepsilon_G(a_i)$, in P_G. Note that $\varepsilon_G(b^2) \leq \varepsilon_G(b)$. By IV.5.9 (2), $x \in D^t_G(b\varphi)$ implies $x = b^2x$ and $\varepsilon_G(bx) \leq \varepsilon_G(b^2) \vee \bigvee_{i=1}^n \Delta\varepsilon_G(a_i)$. Hence, $\varepsilon_G(bx) \leq \bigvee_{i=1}^n \Delta\varepsilon_G(a_i)$, and we get $bx \in D^t_G(\varphi)$. Since $D^t_G(\varphi)$ is closed under multiplication, $x = b^2x \in D^t_G(\varphi)$.

(3) The inclusion $b \cdot D^t_G(\psi) \subseteq D^t_G(b\psi)$ holds for arbitrary ψ (I.2.8 (2)). Conversely, if $x \in D^t_G(b\varphi)$, then $x = b^2x$ and $bx \in D^t_G(\varphi)$; hence, $x = b(bx) \in b \cdot D^t_G(\varphi)$. For the second equality, item (2) gives the inclusion $D_G(b\varphi) \subseteq D_G(\varphi) \cap \{x \in G \mid x = b^2x\}$. For the converse, if $x = b^2x$ and $x \in D^t_G(\varphi)$, we get $bx \in D^t_G(\varphi)$, as $D^t_G(\varphi)$ is multiplicatively closed; hence, $x = b(bx) \in b \cdot D^t_G(\varphi)$. □

(4) (i) For the non-trivial implication ($\Leftarrow$), assuming $a \notin D_G(b\varphi)$, IV.5.7 (1) implies $a \neq b^2a$ or, that in P_G we have $\varepsilon_G(ba) \not\leq \bigvee_{i=1}^n \varepsilon_G(a_i) \vee \underline{\mathbf{c}}$. Since X_G separates points, if the first alternative holds, there is $h \in X_G$ so that $h(a) \neq h(b)^2h(a)$, and IV.5.7 (1) applied in **3** yields $h(a) \notin D^t_{\mathbf{3}}(h(b) \cdot (h * \varphi))$. If $a = b^2a$ and the second alternative holds, by IV.2.5 (i) there is a PA-character σ such that (in **3**) $\sigma(\varepsilon_G(ba)) > \sigma(\bigvee_{i=1}^n \varepsilon_G(a_i) \vee \underline{\mathbf{c}}) = \bigvee_{i=1}^n \sigma(\varepsilon_G(a_i)) \vee 0$. From IV.5.7 (1) applied in the PA **3**, we get $(\sigma \circ \varepsilon_G)(a) \notin D_{\mathbf{3}}((\sigma \circ \varepsilon_G) * \varphi)$, with $\sigma \circ \varepsilon_G \in X_G$.

(ii) is proved by a similar argument, using IV.5.7 (2) instead. □

IV.6 Some model theory of Post algebras

In this section we prove some model-theoretic properties of Post algebras. Proposition IV.6.1 — an immediate consequence of the characterization of RS-Post algebras in Theorem IV.3.2 (4)— shows that these structures admit a first-order universal/positive-primitive axiomatisation in the language $\mathcal{L}_{\mathrm{RS}}$ for real semigroups. This automatically yields a number of presevation results for Post algebras under certain algebraic constructions (Proposition IV.6.3), as well as the fact that the canonical embedding of a real semigroup, G, into its Post-hull *is not* pure unless G itself is a Post algebra (Proposition IV.6.5). The remainder of the section is devoted to show that, conversely, any RS-embedding of Post algebras *is pure* (Theorem IV.6.6). □

Proposition IV.6.1 (Axioms for RS-Post algebras) *The class of RS-Post algebras is axiomatized in the first-order language $\mathcal{L}_{\mathrm{RS}}$ for real semigroups by the following sentence, together with the axioms for RSs:*

[RS-PA] $\quad \forall x\, \exists yzw \left[\, y \in D^t(1, -x^2) \wedge xy = 0 \wedge z \in D^t(x^2, -x) \wedge w \in D^t(x^2, x) \wedge zw = 0 \,\right].$ □

Remark IV.6.2 Manifestly, axiom [RS-PA] is of the form $\forall x\, \psi(x)$, where ψ is a **positive-primitive** $\mathcal{L}_{\mathrm{RS}}$-formula, i.e., a formula of the form $\exists \overline{v}\, \theta(x, \overline{v})$, with θ a conjunction of *atomic* $\mathcal{L}_{\mathrm{RS}}$-formulas. We shall refer to the formula ψ as the **positive-primitive matrix** of the formula in [RS-PA]. □

The logical form of the axioms above yields:

Proposition IV.6.3 (1) *The class of RS-Post algebras is closed under the following constructions:*

— *Inductive limits (colimits) over a right-directed index set.*

— *Reduced products*[4] *(in particular, arbitrary products).*

Further,

(2) *Let* $f : G \longrightarrow H$ *be a* surjective *RS-homomorphism, where* G, H *are RSs. If* G *is a RS-Post algebra, so is* H.

In particular,

(3) *Any quotient* $G/\equiv$ *of a RS-Post algebra, modulo a RS-congruence* $\equiv$ *(II.2.1) is a RS-Post algebra.*

Hence,

(4) *Quotients of RS-Post algebras modulo saturated sets (II.3.B) are RS-Post algebras.*

Proof. (1) Closure under inductive limits and reduced products is known to hold for classes of structures (in an arbitrary language) axiomatized by first-order sentences of the form $\forall \overline{v}\,(\varphi_1 \rightarrow \varphi_2)$, where φ_1, φ_2 are positive-primitive formulas (cf. [DM6], Appendix A, Thms. A5, A7 OJO! Correct this ref.). The axioms for RSs and axiom [RS-PA] are of this form.

(2) We check that, for arbitrary structures $\mathfrak{A}$, $\mathfrak{B}$ with language L, say, if $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a *surjective* L-morphism, θ is an L-sentence of the form $\theta : \forall \overline{v}\, \exists \overline{x} \bigwedge_i \varphi_i(\overline{v}, \overline{x})$, with the φ_i atomic L-formulas, and $\mathfrak{A} \models \theta$, then $\mathfrak{B} \models \theta$.

This is routine model-theoretic verification: Let $\overline{b} \in \mathfrak{B}$, and let $\overline{a}$ be a tuple in $\mathfrak{A}$ such that $f(\overline{a}) = \overline{b}$. Since $\mathfrak{A} \models \theta$ there is $\overline{a'} \in \mathfrak{A}$ so that $\mathfrak{A} \models \bigwedge_i \varphi_i[\overline{a}, \overline{a'}]$. Since the φ_i are atomic and f is a L-morphism, $\mathfrak{B} \models \varphi_i[f(\overline{a}), f(\overline{a'})]$ holds for all i, whence $\mathfrak{B} \models \exists \overline{x} \bigwedge_i \varphi_i[\overline{b}, \overline{x}]$. Since $\overline{b}$ is an arbitrary tuple in $\mathfrak{B}$ we have proved $\mathfrak{B} \models \theta$.

(3) is a particular instance of (2) with $f = \pi$, the canonical quotient map $G \longrightarrow G/\equiv$ given by Definition II.2.1, and (4) is a particular case of (3), cf. II.3. □

Remark. Note that item (4) of this Proposition applies, in particular, to the various types of quotients treated in § II.3: quotients modulo saturated subsemigroups, quotients modulo transversally saturated subsemigroups, localizations and residue spaces at saturated prime ideals. □

Recall from Corollary IV.4.7 (3) that the Post-hull embedding $\varepsilon_G : G \longrightarrow P_G$ of a RS is a complete embedding. Using the axiomatization of RS-Post algebras given in Proposition IV.6.1 we show in Proposition IV.6.5 that, in general, this embedding *does not* have the stronger property of purity.

Definition IV.6.4 A RS-homomorphism[5] $f : G \longrightarrow H$ is a **pure embedding** if and only if it reflects positive-primitive (equivalently, positive-existential) $\mathcal{L}_{\mathrm{RS}}$-formulas from H down to G: for every such formula $\varphi(v_1, \ldots, v_n)$ on n variables, and all $a_1, \ldots, a_n \in G$,

$$H \models \varphi[f(a_1), \ldots, f(a_n)] \;\Rightarrow\; G \models \varphi[a_1, \ldots, a_n]. \qquad \square$$

(The converse implication holds automatically because f is a RS-morphism.) For more details on pure embeddings, see [DM1], Ch. 5, § 3, pp. 91–92.

Proposition IV.6.5 *Let* G *be a RS and* P *be a (RS-)Post algebra. Then,*

[4] Cf. [CK], Def. 4.1.6 and § 6.2, or [H], § 9.4.

[5] This notion applies, *mutatis mutandis*, to morphisms of arbitrary first-order structures.

(1) *If $f : G \longrightarrow P$ is a pure embedding of RSs, then G is a Post algebra.*

In particular,

(2) *The canonical Post-hull embedding $\varepsilon_G : G \longrightarrow P_G$* is not *pure unless G itself is a Post algebra. In the latter case, ε_G is an isomorphism of G onto P_G.*

Proof. (1) Assume G is not a Post algebra. Since G is supposed to be a RS, then $G \not\models$ [RS-PA], and there is $a \in G$ such that $G \models \neg\,\psi[a]$, where ψ is the positive-primitive matrix of axiom [RS-PA]. Since $P \models$ [RS-PA], we have $P \models \psi[f(a)]$, implying that f is not pure, contradiction.

(2) follows from (1) ($f = \varepsilon_G$). The last assertion in (2) is proven by chasing the commutative square in IV.4.2 (ii) with the appropriate entries. □

Next we establish the converse to item (1) in the preceding Proposition, namely:

Theorem IV.6.6 *Let P_1, P_2 be Post algebras. Any* injective *RS- (equivalently, PA-) morphism $h : P_1 \longrightarrow P_2$ is a pure embedding (for both the languages of real semigroups and of Post algebras).*

Remark. A similar result for reduced special groups and Boolean algebras was proved in [DM4], Cor. 2.2 (c), p. 951. The proof below follows the same line of argument, replacing the Stone representation theorem for BAs by the constructions explained in Theorem IV.1.6. □

The proof will require some

Preliminaries and Notation IV.6.7 (1) Recall from the Representation Theorem IV.1.7 (ii) that any Post algebra, P, is isomorphic to $\mathcal{C}(X_P, \mathbf{3})$, where X_P is the character space of P.

(2) Let L be a first-order language, let $\mathfrak{M}$ be a L-structure, and let X be a Boolean space. $\mathfrak{M}$ is endowed with the discrete topology; $\mathcal{C}(X, \mathfrak{M})$ is the set of continuous (i.e., locally constant) functions of X into $\mathfrak{M}$. The sets $\mathcal{C}(X, \mathfrak{M})$ and $\mathfrak{M}^X$ are L-structures by pointwise defining the operations and relations of L, and the denotation of any constant c of L as the function with constant value $c^{\mathfrak{M}}$. $\mathcal{C}(X, \mathfrak{M})$ is embedded in $\mathfrak{M}^X$ by sending each $f \in \mathcal{C}(X, \mathfrak{M})$ to its underlying (set-) function, i.e., "forgetting" continuity. We denote this L-embedding by γ. Then, the following holds:

Proposition A. ([DM4], Prop. 2.1, pp. 950-951) *The embedding $\gamma : \mathcal{C}(X, \mathfrak{M}) \longrightarrow \mathfrak{M}^X$ is existentially closed, i.e., it reflects the validity of existential L-sentences (not necessarily positive) with parameters in $\mathcal{C}(X, \mathfrak{M})$, from $\mathfrak{M}^X$ down to $\mathcal{C}(X, \mathfrak{M})$.* □

Note that no restrictions are imposed on the cardinality of $\mathfrak{M}$. We shall use this result when $\mathfrak{M}$ is the real semigroup (and Post algebra) $\mathbf{3}$. Note that both $\mathcal{C}(X, \mathbf{3})$ and $\mathbf{3}^X$ have the structure of a RS and of a PA (by the Representation Theorem IV.1.7 (i) and the fact that the class of PAs is closed under arbitrary products).

(3) We shall also need the following known result from the theory of Post algebras:

Proposition B. ([BD], Thm. X.3.4, p. 196) *A Post algebra is injective iff the Boolean algebra $B(P)$ of Boolean elements of P (IV.1.5) is complete. In particular, (since $B(\mathbf{3}^X) = \mathbf{2}^X$), the Post algebras $\mathbf{3}^X$ are injective.* □

Proof of Theorem IV.6.6. By IV.2.11, RS-morphisms of Post algebras are the same as PA-morphisms. We do the proof for PA-morphisms.

The result will be a consequence of the following facts, proved below:

(i) The map h induces by composition a *continuous surjection* $h^* : X_2 \longrightarrow X_1$: for $y \in X_2$ (i.e., $y : P_2 \longrightarrow \mathbf{3}$ a PA character), $\quad h^*(y) := y \circ h \in X_1$.

(ii) In turn, h^* induces, again by composition, an *injective* PA homomorphism $\widehat{h} : \mathbf{3}^{X_1} \longrightarrow \mathbf{3}^{X_2}$: for $f : X_1 \longrightarrow \mathbf{3}$, $\quad \widehat{h}(f) := f \circ h^* \in \mathbf{3}^{X_2}$.

(iii) The diagram

$$\begin{tikzcd} \mathcal{C}(X_1, \mathbf{3}) = P_1 \arrow[r, "h"] \arrow[d, "\gamma_1"'] \arrow[dr, phantom, "\text{[D]}"] & P_2 = \mathcal{C}(X_2, \mathbf{3}) \arrow[d, "\gamma_2"] \\ \mathbf{3}^{X_1} \arrow[r, "\widehat{h}"'] & \mathbf{3}^{X_2} \end{tikzcd}$$

of Post-algebra homomorphisms, commutes.

Using (i) – (iii), the theorem is proved as follows. Since $\widehat{h}$ is an *injective* PA homomorphism ((ii)) and $\mathbf{3}^{X_1}$ is an injective PA (Proposition B in IV.6.7 (3)), there is a PA homomorphism $g : \mathbf{3}^{X_2} \longrightarrow \mathbf{3}^{X_1}$ such that $g \circ \widehat{h} = \mathrm{id}$ (on $\mathbf{3}^{X_1}$), i.e, g is a retract of $\widehat{h}$; in particular, $\widehat{h}$ is pure. Since the embeddings γ_i ($i = 1, 2$) are pure (Proposition A in IV.6.7 (2)), commutativity of the diagram in (iii) yields at once that h is pure.

Now we prove statements (i) – (iii).

Proof of (i). Continuity. The subbasic clopens for the constructible topology on X_i ($i = 1, 2$) are the sets of the form $[\![\, a = j \,]\!] = \{z \in X_i \mid z(a) = j\}$, with $a \in P_i$ and $j \in \mathbf{3}$. We have,

$$\begin{aligned} (h^*)^{-1}[\, [\![\, a = j \,]\!] \,] &= \{y \in X_2 \mid h^*(y)(a) = j\} = \{y \in X_2 \mid (y \circ h)(a) = j\} = \\ &= \{y \in X_2 \mid \; y(h(a)) \; = j\} = \; [\![\, h(a) = j \,]\!] \,, \end{aligned}$$

proving that the inverse image of a subbasic clopen of X_1 under h^* is subbasic clopen in X_2.

Surjectivity. Let $x \in X_1$ be a PA character. We must find $y \in X_2$ so that $h^*(y) = y \circ h = x$.

Note that injectivity of h entails that $h[x^{-1}[\top, \underline{\mathbf{c}}]]$ is a filter basis of P_2. Otherwise, there are $b_1, \ldots, b_n \in x^{-1}[\top, \underline{\mathbf{c}}]$ such that $h(b_1) \wedge \ldots \wedge h(b_n) = h(b_1 \wedge \ldots \wedge b_n) = \bot$, and injectivity of h yields $\bot = b_1 \wedge \ldots \wedge b_n \in x^{-1}[\top, \underline{\mathbf{c}}]$, contradiction.

Let $\mathcal{P} \subseteq P_2$ be a maximal filter containing $h[x^{-1}[\top, \underline{\mathbf{c}}]]$; then, $h^{-1}[\mathcal{P}] \supseteq x^{-1}[\top, \underline{\mathbf{c}}]$ and, by the maximality of the latter (IV.1.6 (4)), these sets are equal. If $\mathcal{Q}$ is the minimal filter of P_2 under $\mathcal{P}$ (see IV.1.6 (2),(4)), then $h^{-1}[\mathcal{Q}] = x^{-1}[\top]$. By IV.1.6 (5) there is a PA-character $y : P_2 \longrightarrow \mathbf{3}$ such that $y^{-1}[\top, \underline{\mathbf{c}}] = \mathcal{P}$ and $y^{-1}[\top] = \mathcal{Q}$. Hence, $h^{-1}[y^{-1}[\top, \underline{\mathbf{c}}]] = h^{-1}[\mathcal{P}] = x^{-1}[\top, \underline{\mathbf{c}}]$ and $h^{-1}[y^{-1}[\top]] = h^{-1}[\mathcal{Q}] = x^{-1}[\top]$. This clearly entails $y \circ h = x$.

Proof of (ii). Injectivity is routine checking using the surjectivity of h^*. That $\widehat{h}$ is a Post-algebra homomorphism follows from the pointwise definition of the operations in $\mathbf{3}^X$. To illustrate the argument we check that $\widehat{h}$ preserves the nabla operator, i.e., for $f \in \mathbf{3}^{X_1}$,

(*) $\quad \widehat{h}(\nabla_1 f) = \nabla_2(\widehat{h}(f))$,

where ∇_i denotes the nabla operator in $\mathbf{3}^{X_i}$ ($i = 1, 2$). With ∇_3 denoting nabla in $\mathbf{3}$, the pointwise definition of nabla gives, for $x \in X_2$:

$$\widehat{h}(\nabla_1 f)(x) = ((\nabla_1 f) \circ h^*)(x) = (\nabla_1 f)(h^*(x)) = \nabla_3(f(h^*(x))), \text{ and}$$

$$(\nabla_2(\widehat{h}(f)))(x) = (\nabla_2(f \circ h^*))(x) = \nabla_3((f \circ h^*)(x)) = \nabla_3(f(h^*(x))),$$

establishing (*). The remaining verifications are left to the reader.

Proof of (iii). Since the only effect of the maps γ_i is to forget continuity of the functions in $\mathcal{C}(X_i, \mathbf{3})$, we may safely omit them from the notation, and prove:

(I) $h(x) = \widehat{h}(x)$ for all $x \in P_1$.

By the Representation Theorem IV.1.7 (ii), the isomorphism between P and $\mathcal{C}(X_P, \mathbf{3})$ is given by evaluation:

$$x \in P \longmapsto ev_x \in \mathcal{C}(X_P, \mathbf{3}).$$

Hence, for $x \in P_1$ and $\delta \in X_2$ (i.e., $\delta : P_2 \longrightarrow \mathbf{3}$ a PA character), we have:

(II) $h(x)(\delta) = ev_{h(x)}(\delta) = \delta(h(x)) = ev_x(\delta \circ h)$.

Likewise, for $f \in \mathbf{3}^{X_1}$ (i.e., *any* map $f : X_1 \longrightarrow \mathbf{3}$), we have, for $\delta \in X_2$:

(III) $\widehat{h}(f)(\delta) = (f \circ h^*)(\delta) = f(h^*(\delta)) = f(\delta \circ h) = ev_f(\delta \circ h)$.

In particular, when $f = x \in P_1 = \mathcal{C}(X_1, \mathbf{3})$, (II) and (III) prove that diagram [D] commutes, completing the proof of Theorem IV.6.6. □

In the reverse direction to IV.6.5, Theorem IV.6.6 yields:

Corollary IV.6.8 *Any injective RS-morphism from a Post algebra into a real semigroup is a pure embedding (and hence, by Remark IV.7.5 (ii), also a complete embedding). In particular, the canonical embedding of* $\mathbf{3}$ *into any real semigroup is pure (and complete).*

Proof. Let $f : P \longrightarrow H$ be an injective RS-morphism from the Post algebra P into the RS H. In the commutative diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\varepsilon_P"'] \arrow[dr, phantom, "(D)"] & H \arrow[d, "\varepsilon_H"] \\ P_P \arrow[r, "P(f)"'] & P_H \end{tikzcd}$$

(see IV.4.2 (ii)) we have $P_P = P$ and $\varepsilon_P = \mathrm{id}_P$ (IV.4.2 (iv)), whence $\varepsilon_H \circ f = P(f)$. Let $\varphi(v_1, \ldots, v_n)$ be a positive-existential $\mathcal{L}_{\mathrm{RS}}$-formula, and let $a_1, \ldots, a_n \in P$ be such that $H \models \varphi[f(a_1), \ldots, f(a_n)]$. Since ε_H preserves positive-existential formulas, $\varepsilon_H \circ f = P(f)$ yields $P_H \models \varphi[P(f)(a_1), \ldots, P(f)(a_n)]$, wherefrom follows $P \models \varphi[a_1, \ldots, a_n]$, as $P(f)$ is pure by Theorem IV.6.7. □

IV.7 Rings and Post algebras

We now characterize the rings whose associated real semigroup is a Post algebra (IV.7.1), and give concrete examples, notably a certain class of von Neumann-regular rings (Example IV.7.3). We also prove (Theorem IV.7.4) that every Post algebra is "realized" by a ring, i.e., isomorphic to the RS associated to some ring. □

Though proofs are done for the case of RSs of the form $G_A = G_{A, \Sigma A^2}$, where A is a semi-real ring, all relevant results in this section carry over to the more general case of RSs of the form $G_{A,T}$, where T is a preorder of A; their generalization is left as an exercise to the interested reader.

The next Proposition yields a pure ring- (and order-) theoretic characterization of those rings A whose associated RS is a Post algebra.

Proposition IV.7.1 *Let A be a semi-real ring. The following are equivalent:*

(1) G_A *is a Post algebra.*

(2) *a) Any two real prime ideals of A are incomparable under inclusion.*

b) For every real prime ideal P of A, the fraction field K_P of A/P has a unique order.

Proof. To ease notation, write $X = X_{G_A} (= \mathrm{Sper}\,(A))$.

(2) $\Rightarrow$ (1). We check condition (2) of Theorem IV.3.2. Let $h_1, h_2, h_3 \in X$ be such that $h_1 h_2 h_3 \in X$. With $Z(h_i) = \{a \in A \,|\, h_i(\overline{a}) = 0\}$, Lemma II.2.11 (1) shows that there is $i \in \{1,2,3\}$ so that $Z(h_j) \subseteq Z(h_i)$ for all $j \in \{1,2,3\}$. Since these are real prime ideals, assumption (2.a) implies $Z(h_1) = Z(h_2) = Z(h_3)$ $(= P$, say).

We know that each $h \in X$ with $Z(h) = P$ induces a total order of A/P, namely: $\alpha_h = \{a/P \,|\, h(\overline{a}) \in \{0,1\}\}$ (hence also one in K_P). Assumption (2.b) implies, then, $\alpha_{h_1} = \alpha_{h_2} = \alpha_{h_3}$. Since $h(\overline{a}) = 1$ (resp., -1) iff $a/P >_{\alpha_h} 0$ (resp., $<_{\alpha_h} 0$), equality of the orders α_{h_i} entails $h_1 = h_2 = h_3$.

(1) $\Rightarrow$ (2). Item (4) of Theorem IV.3.2 implies that the saturated prime ideals in any RS-Post algebra are an antichain under inclusion. Indeed, assume that $P \subset Q$ are saturated prime ideals of a RS-Post algebra, G, and let $x \in Q \setminus P$. By (4.i) in IV.3.2 there is $y \in D^t_G(1, -x^2)$ so that $xy = 0$. Then, $xy \in P$ and, since $x \notin P$, we get $y \in P \subseteq Q$. We also have $1 \in D^t_G(y, x^2)$ which, by saturatedness of Q, yields $1 \in Q$, contradiction.

In the case of a (semi-real) ring A, the saturated prime ideals of G_A are in a one-one, inclusion-preserving correspondence with the real primes of A (cf. II.1.9 and II.1.10). Hence, if G_A is a Post algebra, the real prime ideals of A also form an antichain for inclusion. This proves (2.a).

As for condition (2.b), if P is a real prime ideal of A, by definition the quotient ring A/P —and hence also K_P— has at least one order. Further, we know that any order α of A/P defines a character h_α of G_A by the rule $h_\alpha(\overline{a}) = \mathrm{sgn}_\alpha(a/P) = \overline{a}(\alpha)$ $(a \in A)$, and, furthermore, $\alpha \neq \beta \Rightarrow h_\alpha \neq h_\beta$.

Suppose now that α, β are total orders of some quotient ring A/P. Since $Z(h_\alpha) = Z(h_\beta) = P$, we clearly have $h^2_\alpha(\overline{a}) = 1$ if $a \notin P$, and $= 0$ if $a \in P$. Then $h^2_\alpha h_\beta = h_\beta$, and the characterization of Post algebras in IV.3.2 (2) entails $h_\alpha = h_\beta$, which in turn implies $\alpha = \beta$. □

Proposition IV.7.1 gives raise to some natural examples of rings whose associated RSs are Post algebras.

Example IV.7.2 Let $A = \prod_{i=1}^n K_i$ be a finite product of fields (i.e., A is a reduced semi-local ring). Assume that at least one of the fields K_i is formally real and that, whenever K_i is formally real, it has a unique order. Then, G_A is a Post algebra.

Proof. It is well-known (or, otherwise, easily verified) that the prime ideals of A are all of the form $P_i = K_1 \times \cdots \times K_{i-1} \times \{0\} \times K_{i+1} \times \cdots \times K_n$ $(i \in \{1, \ldots, n\})$. In particular, they are maximal, and hence pairwise incomparable under inclusion. Thus, condition (2.a) in IV.7.1 is fulfilled. Since $A/P_i = K_i$, the real primes are those P_i such that K_i is formally real.

The assumption that at least one K_i is formally real guarantees that A is semi-real, and the assumption that the K_i's are uniquely ordered whenever formally real ensures that condition (2.b) in IV.7.1 holds. □

Generalizing this example, we have:

Example IV.7.3 Let A be a von Neumann-regular ring. If A is semi-real and all residue fields of A modulo maximal ideals are uniquely ordered, then G_A is a Post algebra.

Note. (Commutative) von Neumann-regular rings ring!von Neumann-regularvon Neumann-regular ring are those in which every principal ideal is generated by an idempotent. For information concerning rings of this type, see [Pi] or [DM5].

Proof. All prime ideals in a von Neumann-regular ring are maximal. In particular, condition (2.a) in IV.7.1 is automatically fulfilled, and $\mathrm{Spec}(A)$ is a Boolean space. In this case, the stalk of the structure sheaf (or affine scheme) of A at a point $P \in \mathrm{Spec}(A)$ is the field A/P. As in the previous example, our assumptions have been made so as to guarantee that A is semi-real and that condition (2.b) in Proposition IV.7.1 is fulfilled. So, G_A is, in fact, a Post algebra. □

Comment. One may think, at first sight, that Post algebras have, in the context of real semigroups (and rings), a behaviour similar to that of Boolean algebras in the context of reduced special groups (and fields). Proposition IV.7.1 and the ensuing examples show that this similarity is not quite complete. In fact, in [DM1], p. 59 and pp. 78 ff, it is shown that the (formally real) fields K whose associated RSG, $G_{\mathrm{red}}(K)$, is a Boolean algebra are exactly the so-called SAP fields; these may have many orders. In contrast, Proposition IV.7.1 shows that the situation is more restrictive in the context of RSs and rings. This different behaviour can be traced to the fact that adding a zero to a Boolean algebra (viewed as a RSG) —as done in I.2.2 (3)— does not produce a Post algebra (except in the case $\mathbb{Z}_2$). Indeed, a Boolean algebra, B, is just the set of invertible elements of the Post hull of the real semigroup $B^* = B \cup \{0\}$ obtained by adding a zero to B. □

A well-known result due to Craven (see [P], Thm. 6.9, pp. 97–98) says that every Boolean algebra is isomorphic, as a special group, to the RSG of some field. Our next result establishes a similar "realizability" result of Post algebras by rings.

Theorem IV.7.4 *Let P be a Post algebra and let $X_P = X$ be its set of characters. Then P is RS-isomorphic to the real semigroup of the ring $\mathcal{C}(X, \mathbb{Z})$ of integer-valued continuous functions on X (discrete topology in $\mathbb{Z}$; pointwise operations). Further, the same result holds replacing $\mathbb{Z}$ by any uniquely ordered field (endowed with the discrete topology).*

Proof. By the Representation Theorem IV.1.7 (ii), $P = \mathcal{C}(X, \mathbf{3})$; let $A = \mathcal{C}(X, \mathbb{Z})$. The elements of A are locally constant functions; that is,

(*) $f \in A$ iff there is a finite partition $\{U_1, \ldots, U_k\}$ of X into non-empty clopens, and $c_1, \ldots, c_k \in \mathbb{Z}$ so that $f = \sum_{i=1}^{k} c_i \chi_{U_i}$,

where χ_U is the characteristic function of U: for $x \in X$,

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

Claim. *A set $I \subseteq A$ is a (proper) real prime ideal if and only if there is $x_0 \in X$ such that $I = \{f \in A \mid f(x_0) = 0\}$.*

[Recall that an ideal I is real iff for all $a_1, \ldots, a_n \in A$, $\sum_{i=1}^n a_i^2 \in I$ implies $a_1, \ldots, a_n \in I$.]

Proof of Claim. ($\Leftarrow$) Clearly, any set of the form $\{f \in A \mid f(x_0) = 0\}$ is a real prime ideal.

($\Rightarrow$) Suppose first that $Z(f) = f^{-1}[0] = \emptyset$ for some $f \in I$. Let $f = \sum_{i=1}^k c_i \chi_{U_i}$, with the clopens U_i as in (*). Since $\chi_U^2 = \chi_U$ and $\chi_{U_i} \cdot \chi_{U_j} = 0$ for $i \neq j$, we have $f^2 = \sum_{i=1}^k c_i^2 \chi_{U_i}$. Since the U_i's are non-empty and $Z(f^2) = \emptyset$, we have $c_i \neq 0$, i.e., $c_i \geq 1$, for all $i = 1, \ldots, k$. As the U_i's form a partition of X, $\sum_{i=1}^k \chi_{U_i} = 1$, and we can write:

$$f^2 = 1 + \textstyle\sum_{i=1}^k (c_i^2 - 1)\, \chi_{U_i}\,,$$

with $c_i^2 - 1 \geq 0$ for all i. Since I is real, $\chi_U^2 = \chi_U$, and $c_i^2 - 1$ is a sum of squares, $f^2 \in I$ entails $1 \in I$, a contradiction. This shows that $Z(f) \neq \emptyset$ for all $f \in I$.

Consider the family $\{Z(f) \mid f \in I\}$ of non-empty closed subsets of X. Since $\bigcap_{i=1}^n Z(f_i) = Z(\sum_{i=1}^k f_i^2)$ for $f_1, \ldots, f_n \in A$, this family has the finite intersection property. By compactness of X, $\bigcap_{f \in I} Z(f) \neq \emptyset$. If x_0 is in this set, $I \subseteq \{f \in A \mid f(x_0) = 0\}$.

To prove the reverse inclusion, assume $f(x_0) = 0$. Representing f in the form $f = \sum_{i=1}^k c_i \chi_{U_i}$, as above, there is a unique index i so that $x_0 \in U_i$; we may assume $i = 1$. Then, $\chi_{U_1}(x_0) = 1$ and $\chi_{U_i}(x_0) = 0$ for $i \in \{2, \ldots, k\}$; in particular, $c_1 = 0$. Since $\chi_{U_i} \cdot \chi_{X \setminus U_i} = 0 \in I$ and I is prime, either $\chi_{U_i} \in I$ or $\chi_{X \setminus U_i} \in I$. If the latter held for some $i \geq 2$, the inclusion $I \subseteq \{f \in A \mid f(x_0) = 0\}$ would lead to the contradiction $x_0 \in U_1 \cap X \setminus U_i$. Hence, $\chi_{U_i} \in I$ for all $i \in \{2, \ldots, k\}$. We conclude that $f = \sum_{i=1}^k c_i \chi_{U_i} \in I$, proving the Claim. □

Remarks and Notation. (a) Clearly, two distinct points of X are separated by a function in A. Hence, there is a unique point $x_0 \in X$ representing a real prime ideal in the form stated in the Claim. For $x \in X$ we write $I_x = \{f \in A \mid f(x) = 0\}$.

(b) For $x \in X$, $A/I_x \approx \mathbb{Z}$ (as rings), via the map $f \mapsto f(x)$, $f \in A$. In particular, A/I_x has a unique order, and $\mathrm{Sper}\,(A)$ gets identified to the set of real prime ideals of A. Recall that, for $f \in A$, $\overline{f} : \mathrm{Sper}\,(A) \longrightarrow \mathbf{3}$ denotes the map giving the sign of (the residue class of) f at each element of the real spectrum (cf. I.1.2 (e)).

(c) In the sequel of this proof $\mathrm{Sper}\,(A)$ is endowed with the constructible topology, having as a subbasis the sets $[\![\, f = \delta \,]\!] = \{I_x \mid \mathrm{sgn}(f(x)) = \delta\}$, for $f \in A$ and $\delta \in \{1, 0, -1\} = \mathbf{3}$. Obviously, the associated function $\overline{f}$ is continuous for this topology ($\mathbf{3}$ discrete).

By the Duality Theorem I.5.1, in order to prove that the real semigroup $G_A = G_{A,\Sigma A^2}$ associated to the ring A (and its preorder $\sum A^2$) is isomorphic to the (RS associated to the) Post algebra $P = \mathcal{C}(X, \mathbf{3})$, it suffices to show that their respective ARSs $(\mathrm{Sper}\,(A), G_A)$ and (X, P) are isomorphic as abstract real spectra, which we do next; cf. [ARS-mor] in the proof of Theorem I.5.1.

Let $\tau : X \longrightarrow \mathrm{Sper}\,(A)$ denote the map $\tau(x) = I_x$. By the discussion in Remark (b) and the Claim above, τ is surjective. We prove:

(1) τ is continuous.

Clear, because $\tau^{-1}[\,[\![f = \delta]\!]\,] = \{x \mid \mathrm{sgn}(f(x)) = \delta\}$ is clopen, as $f : X \longrightarrow \mathbb{Z}$ is continuous.

From (1) we get:

(2) For $f \in A$, $\overline{f} \circ \tau \in \mathcal{C}(X, \mathbf{3}) = P$.

By [ARS-mor] (cf. I.5.1), this shows that τ is a morphism of ARSs.

(3) τ is injective.

If $x, y \in X$, $x \neq y$, there is a clopen U so that $x \in U$, $y \in X \setminus U$, i.e., $\chi_U(x) = 1$, $\chi_U(y) = 0$; since $\chi_U \in A$, we get $\chi_U \in I_y \setminus I_x$, whence, $I_y \neq I_x$.

To complete the proof we must show that the map $\widehat{\tau} : G_A \longrightarrow P$ induced by τ, i.e., $\widehat{\tau}(\overline{f}) = \overline{f} \circ \tau$, is bijective.

(4) $\widehat{\tau}$ is injective.

If $f, g \in A$ are so that $\overline{f} \neq \overline{g}$, there is $x \in X$ such that $\overline{f}(I_x) \neq \overline{g}(I_x)$, i.e., $\overline{f}(\tau(x)) \neq \overline{g}(\tau(x))$, i.e., $\widehat{\tau}(\overline{f}) \neq \widehat{\tau}(\overline{g})$.

(5) $\widehat{\tau}$ is surjective.

Let $\alpha \in P = \mathcal{C}(X, \mathbf{3})$. For $\delta \in \mathrm{Im}(\alpha) \subseteq \mathbf{3}$, set $U_\delta = \alpha^{-1}[\delta]$, a non-empty clopen of X; the sets $\{U_\delta \mid \delta \in \mathrm{Im}(\alpha)\}$ partition X. Set $f = \sum_{\delta \in \mathrm{Im}(\alpha)} \delta \cdot \chi_{U_\delta}$. For $x \in X$ we have:

$$\overline{f}(\tau(x)) = \delta \;\Leftrightarrow\; \overline{f}(I_x) = \delta \;\Leftrightarrow\; \mathrm{sgn}\,(f(x)) = \delta \;\Leftrightarrow\; x \in U_\delta \;\Leftrightarrow\; \alpha(x) = \delta,$$

i.e., $\overline{f} \circ \tau = \alpha$, as required.

The last assertion, i.e., that P is realized by any ring of the form $\mathcal{C}(X, F)$, where F is a uniquely ordered field endowed with the discrete topology, is proved by a slight variant of the preceding argument; as follows. Note first that $F \subseteq \mathcal{C}(X, F)\ (= A)$, via the constant functions. Then, replace in the Claim the words "real prime ideal" by "prime ideal <u>real over F</u>". An ideal $I \subseteq A$ is called <u>real over F</u> iff $\sum_{i=1}^{n} c_i a_i^2 \in I$, with $a_i \in A$ and $c_i \in F^+$ (i.e., $c_i > 0$), implies $a_1, \ldots, a_n \in I$. With this proviso the Claim remains true.

With notation as above, $f^2 = \sum_{i=1}^{k} c_i^2 \chi_{U_i}^2 \in I$ and I real over F imply $\chi_{U_i} \in I$ for $i = 1, \ldots, k$, whence $1 = \sum_{i=1}^{k} \chi_{U_i} \in I$, contradiction (showing, as above, $Z(f) \neq \emptyset$ for all $f \in I$). As in Remark (b) above, uniqueness of the order of F serves to identify $\mathrm{Sper}\,(A)$ with the set of prime ideals of A real over F. The remainder of the proof proceeds as before. □

Remark. The two examples given in the preceding Proposition show that one and the same RS may be realized by very different rings, if at all. For example, in the case above, 2 is not invertible in the ring $\mathcal{C}(X, \mathbb{Z})$, while it is in the ring $\mathcal{C}(X, F)$. Likewise, $\mathcal{C}(X, \mathbb{Z})$ is not von Neumann-regular, while $\mathcal{C}(X, F)$ is. ring!von Neumann-regularvon Neumann-regular ring

Fact IV.7.5 *The ring $\mathcal{C}(X, F)$, with X a Boolean space and F a field with the discrete topology, is von Neumann-regular.*

Proof. It suffices to show that f^2 divides f, for all $f \in \mathcal{C}(X, F)$. As in the proof of Theorem IV.7.4, $f = \sum_{i=1}^{k} c_i \chi_{U_i}$ for some clopen partition $\{U_1, \ldots, U_k\}$ of X, and pairwise different c_i's in F. If 0 occurs among the coefficients c_i, we may assume $c_1 = 0$. Let $c_i' = c_i^{-1}$ if $i \geq 2$, or $i = 1$ and $c_1 \neq 0$, while $c_1' = 0$ if $c_1 = 0$. Let $f' = \sum_{i=1}^{k} c_i' \chi_{U_i}$. Then, $f' \in \mathcal{C}(X, F)$. Since $\chi_{U_i} \cdot \chi_{U_j} = 0$ for $i \neq j$ and $\chi_{U_i} = \chi_{U_i}^2$, we clearly have $f^2 \cdot f' = \sum_{i=1}^{k} c_i^2 c_i' \chi_{U_i} = \sum_{i=1}^{k} c_i \chi_{U_i} = f$, as contended. □

Chapter V

Spectral real semigroups

Introduction

In this chapter we carry out a detailed study of an important class of real semigroups, that we call *spectral*. The abstract real spectra dual to these real semigroups were introduced by Marshall in § 8.8 of [M] under the name "real closed abstract real spectra"[1]; he briefly outlined some of their basic properties. We have adopted the name "spectral" in view of the nowadays standard name for the objects of these RSs, namely *spectral maps* (here with values in $\mathbf{3} = \{1, 0, -1\}$ endowed with the *spectral* topology); cf. [DST] and [KS].

One of our most significant results is that there is a full-fledged topological-algebraic duality —in fact, an anti-equivalence— between the category **HNSS** of hereditarily normal spectral spaces with spectral maps, and the category **SRS** of spectral real semigroups with RS-homomorphisms; see Theorem V.5.4. In fact, our results are finer and show (Theorem V.1.4) that the structures (of language $\mathcal{L}_{\mathrm{RS}} = \{\,\cdot\,, 1, 0, -1, D\}$) dual to *arbitrary* spectral spaces verify all axioms for real semigroups with the possible exception of [RS3b] (i.e., $D^t(\cdot\,,\cdot) \neq \emptyset$, cf. I.2.4), while this axiom is equivalent to the hereditary normality of the space (Theorem V.1.5).

The main thrust in sections V.3, V.4 and V.5 is directed at proving this duality, though many other results are obtained as a by-product. Noteworthy among the latter is that any real semigroup has a natural hull in the category **SRS**, with the required functorial properties; cf. section V.4 and Theorem V.5.3 (ii) (the existence of this spectral hull was observed in [M], p. 177). Further,

(i) The spectral hull of a real semigroup is idempotent: iteration does not produce a larger structure (Theorem V.4.5 and Corollary V.4.6).

(ii) Every RS-character of a real semigroup extends uniquely to its spectral hull (Corollary V.5.5).

Note that similar results hold for the Post hull of a RS; cf. Theorem IV.4.2 (v) and Corollary IV.4.7 (2).

A second type of result stems from the properties of the representation partial order (I.6.2) in spectral RSs (it coincides with their pointwise defined order). Our main result in this direction is Theorem V.6.6, which shows that the spectral RSs are exactly the real semigroups that are *distributive* lattices in the representation partial order; distributivity is the crucial point here. In

[1] A name presumably motivated by the homonymous terminology in the ring case, that we treat in broader generality in section V.10 below.

fact, this property is also equivalent to the existence of a lattice structure in the representation partial order together with the fact that RS-characters are lattice homomorphisms (into **3**). We also prove (Theorem V.6.2) that any real semigroup generates its spectral hull as a lattice.

Another interesting feature is that the lattice operations $\wedge$ and $\vee$ of a spectral RS are first-order definable in terms of the real semigroup product operation and binary representation relation by *positive-primitive* formulas (Theorem V.2.1 and Remark V.7.5). This yields a useful first-order axiomatisation of the class of spectral RSs (Theorem V.7.4), having as a corollary that the class of spectral RS is closed under (right-directed) inductive limits, reduced products —in particular, arbitrary products— and, more significantly, also under quotients modulo RS-congruences (II.2.1); see Proposition V.7.6.

In section V.7 we examine the lattice structure of the spectral RSs together with the involution "$-$" (multiplication with -1). This is done using the framework of the so-called Kleene algebras (see IV.1.9 for definitions). It turns out that spectral RSs are exactly the Kleene algebras verifying two additional requirements, stated in terms of their lattice structure and constants $0, 1$.

Our main result in section V.8 (Theorem V.8.2 and Corollary V.8.4) shows that the RS-congruences of a spectral RS are determined by a constructible subset of its character space (and conversely). Section V.9 is devoted to study the saturated prime ideals of spectral RSs.

In the final section V.10 we prove that the RS associated to any lattice-ordered ring is spectral (Theorem V.10.4), and that the spectral hull of the RS associated to any semi-real ring is canonically isomorphic to the RS of its real closure (Proposition V.10.5); cf. [M], Rmk. (3), p. 178. This result exhibits a huge class of natural examples of spectral RSs among the RSs associated to rings.

V.1 Spectral real semigroups. Basic theory.

V.1.1 Preliminaries and Notation. Basic notions on spectral spaces have been summarily introduced in I.1.16. For general background on spectral spaces the reader is referred to [DST], the notation of which we shall systematically use; certain results therein will be cited here as needed. See also [KS], Kap. III.

(1) If X is a spectral space, the associated constructible topology is denoted by X_{con}.

(2) Recall that a spectral space X is called **hereditarily normal** iff any of the following equivalent conditions are verified:

(i) The specialization order of X is a root-system.

(ii) Every proconstructible subset of X endowed with the induced topology is normal.

(iii) Every open, quasi-compact subset of X endowed with the induced topology is normal.

A proof of the equivalence of these conditions can be found in [DST], Thm. 20.2.2.

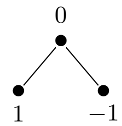
(3) A map $f : X \longrightarrow Y$ between spectral spaces X, Y is called **spectral** iff the preimage of every open and quasi-compact subset of Y under f is, again, open and quasi-compact. Each of the following conditions is equivalent to f being spectral:

(i) f is continuous (for the spectral topologies) and continuous for the constructible topologies of X and Y ([DST], Corol. 3.1.12).

(ii) f is continuous for the constructible topologies and monotone for the specialization orders, of X and Y ([DST], Lemma 5.6.6).

See also [DST], Corol. 4.2.3.

(4) We shall consider two different topologies and two different orders on the set $\{1, 0, -1\}$. The first is the discrete topology and the linear order $1 < 0 < -1$, already used in previous sections, denoted, as above, by $\mathbf{3}$. The second is the spectral topology, where the singletons $\{1\}$ and $\{-1\}$ are a basis of opens, endowed with the specialization partial order:



The set $\{1, 0, -1\}$ endowed with this topology (and order) will be denoted by $\mathbf{3}_{\rm sp}$. Clearly, the singletons $\{\pm 1\}$ are quasi-compact and $\{0\}$ is closed in $\mathbf{3}_{\rm sp}$. Note that $(\mathbf{3}_{\rm sp})_{\rm con}$ is just the discrete topology on $\mathbf{3}$.

Warning. Though $\mathbf{3}$ and $\mathbf{3}_{\rm sp}$ differ by their orders and their topologies, we shall only consider in the sequel the unique representation (and transversal representation) relation on $\{1, 0, -1\}$ that makes it a real semigroup, namely the relations given by Corollary I.2.5. □

Definition V.1.2 Spectral maps $f : X \longrightarrow \mathbf{3}_{\rm sp}$ from a spectral space X into $\mathbf{3}_{\rm sp}$ will be called **spectral characters**. The set of spectral characters on X will be denoted by $\mathrm{Sp}(X)$.□

(5) Clearly, $f : X \longrightarrow \mathbf{3}_{\rm sp}$ is a spectral character iff $f^{-1}[1]$ and $f^{-1}[-1]$ are quasi-compact open in X.

(6) (Product in $\mathrm{Sp}(X)$) $\mathrm{Sp}(X)$ has a product operation: the pointwise defined product of spectral characters $h, g,$ is a spectral character; indeed,

$$(hg)^{-1}[1] = (h^{-1}[1] \cap g^{-1}[1]) \cup (h^{-1}[-1] \cap g^{-1}[-1]),$$
$$(hg)^{-1}[-1] = (h^{-1}[-1] \cap g^{-1}[1]) \cup (h^{-1}[1] \cap g^{-1}[-1]);$$

since X is spectral, the right-hand side of these equalities are quasi-compact open sets.

Obviously, $\mathrm{Sp}(X)$ contains the functions with constant values $1, 0, -1$ (denoted by the same symbols). Then, $\mathrm{Sp}(X)$ is a commutative semigroup and, since product is pointwise defined, also a ternary semigroup.

(7) (Representation in $\mathrm{Sp}(X)$) A ternary (representation) relation is pointwise defined: for $h, h_1, h_2 \in \mathrm{Sp}(X)$,

$$h \in D_{\mathrm{Sp}(X)}(h_1, h_2) \;:\Leftrightarrow\; \forall x \in X\,(h(x) \in D_{\mathbf{3}}(h_1(x), h_2(x)))\,.$$

Note that (by the definition of D^t in terms of D), also $D^t_{\mathrm{Sp}(X)}$ is pointwise defined in terms of $D^t_{\mathbf{3}}$. The structure $\langle \mathrm{Sp}(X), \cdot, 1, 0, -1, D_{\mathrm{Sp}(X)} \rangle$ will be denoted by $\mathbf{Sp}(X)$.

(8) (The pointwise partial order of $\mathrm{Sp}(X)$) This order is induced by the total order $1 < 0 < -1$ of $\mathbf{3}$ in the obvious way: for $f, g \in \mathrm{Sp}(X)$,

$$f \leq g \;\Leftrightarrow\; \forall\, x \in X (f(x) \leq g(x)) \quad (\text{in } \mathbf{3}).$$

The structure $\mathbf{Sp}(X)$ is also endowed with a binary relation $\leq_{\mathbf{Sp}(X)}$ defined as in I.6.2: for $f, g \in \mathrm{Sp}(X)$,

$$(\dagger) \qquad f \leq_{\mathbf{Sp}(X)} g \;:\Leftrightarrow\; f \in D_{\mathbf{Sp}(X)}(1, g) \text{ and } -g \in D_{\mathbf{Sp}(X)}(1, -f).$$

Fact V.1.3 *The relation $\leq_{\mathbf{Sp}(X)}$ coincides with the pointwise partial order $\leq$.*

Proof. Using I.2.5 it is easily checked by hand that the representation partial order $\leq_{\mathbf{3}}$ of the RS **3** (I.6.2) is just the order $1 < 0 < -1$. Then, for $f, g \in \mathrm{Sp}(X)$ we have:

$$f \leq g \Leftrightarrow \forall x \in X(f(x) \leq_{\mathbf{3}} g(x)) \Leftrightarrow \forall x \in X\,[\,f(x) \in D_{\mathbf{3}}(1, g(x)) \wedge -g(x) \in D_{\mathbf{3}}(1, -f(x))\,]$$
$$\Leftrightarrow f \in D_{\mathbf{Sp}(X)}(1, g) \wedge -g \in D_{\mathbf{Sp}(X)}(1, -f) \Leftrightarrow f \leq_{\mathbf{Sp}(X)} g\,.$$

(The equivalences are, respectively, the definition of $\leq$ together with the preceding observation, the definition of $\leq_{\mathbf{3}}$, the definition of $D_{\mathbf{Sp}(X)}$ (see (7)), and the definition of $\leq_{\mathbf{Sp}(X)}$.) □

Remark. Note that V.1.3 holds whether or not $\mathbf{Sp}(X)$ is a real semigroup. In spite of this equivalence it will be useful to keep the notational distinction between $\leq$ and $\leq_{\mathbf{Sp}(X)}$. □

(9) (Lattice structure of $\mathbf{Sp}(X)$) $\mathbf{Sp}(X)$ has a lattice structure pointwise induced by the total order $1 < 0 < -1$ of **3**: for $f, g \in \mathrm{Sp}(X)$ and $x \in X$,

$$(f \vee g)(x) := \max\{f(x), g(x)\} \quad (\text{in } \mathbf{3}),$$
$$(f \wedge g)(x) := \min\{f(x), g(x)\} \quad (\text{in } \mathbf{3}).$$

One must check that the maps $f \vee g$, $f \wedge g$ thus defined are spectral; this is clear as

$$(f \vee g)^{-1}[1] = f^{-1}[1] \cap g^{-1}[1]\,, \quad (f \vee g)^{-1}[-1] = f^{-1}[-1] \cup g^{-1}[-1]\,, \text{ and}$$
$$(f \wedge g)^{-1}[1] = f^{-1}[1] \cup g^{-1}[1]\,, \quad (f \wedge g)^{-1}[-1] = f^{-1}[-1] \cap g^{-1}[-1]\,,$$

are quasi-compact open sets.

Since the order in **3** is total, the lattice structure just defined is *distributive*.

We also note that product in $\mathrm{Sp}(X)$ is identical with symmetric difference (defined in terms of the lattice operations and $-$): for $g, h \in \mathrm{Sp}(X)$,

$$g \cdot h = g \,\triangle\, h \;(:= (g \wedge -h) \vee (h \wedge -g))\,.$$

This follows straightforwardly from the validity of "product = symmetric difference" in **3** and the pointwise definition of all operations in $\mathrm{Sp}(X)$.

(10) ($\mathrm{Sp}(X)$ and Post algebras) In this chapter we shall use the results on the real semigroup structure of Post algebras proved in § IV.2. To abridge, we denote by $P(X)$ the Post algebra $\mathcal{C}(X_{\mathrm{con}}, \mathbf{3})$ of continuous functions of X_{con} into **3** (pointwise defined operations) and by $\leq$ its (pointwise defined) order; see IV.1.7 (i). Note that, by item (3.i) above, $\mathrm{Sp}(X) \subseteq P(X)$. Further, since representation is pointwise defined in both these structures, for $f, g, h \in \mathrm{Sp}(X)$ we have

$$h \in D_{\mathbf{Sp}(X)}(f, g) \;\Leftrightarrow\; h \in D_{P(X)}(f, g),$$

and similarly for transversal representation. The same remark applies to the lattice operations, i.e. the infimum and the supremum of elements of $\mathrm{Sp}(X)$ (see (9)) coincide with those same operations performed in $P(X)$; in other words, $\mathbf{Sp}(X)$ is a sublattice and a $\mathcal{L}_{\mathrm{RS}}$-substructure of the Post algebra $P(X)$. □

Initially we will examine which of the axioms for real semigroups are satisfied by $\mathbf{Sp}(X)$ for an arbitrary spectral space X, and the requirements to be imposed on X for $\mathbf{Sp}(X)$ to become a real semigroup.

Reminder. Recall (I.2.4) that the strong associativity axiom [RS3] is equivalent to the conjuntion of the weak associativity axiom [RS3a] and the axiom

[RS3b] For all a, b, $D^t(a, b) \neq \emptyset$. □

Theorem V.1.4 *For every spectral space X, the structure* $\mathbf{Sp}(X)$ *satisfies all axioms for real semigroups —including the weak associativity axiom* [RS3a]*— except, possibly, axiom* [RS3b]*.*

Proof. The validity of the axioms [RS i] for $i \neq 3$ $(0 \leq i \leq 8)$ is straightforward checking, stemming from the following observations:

(i) Product and representation are pointwise defined in $\mathbf{Sp}(X)$, and

(ii) All axioms for RSs, except [RS3], are universal statements in the language $\mathcal{L}_{\mathrm{RS}} = \{\cdot, 1, 0, -1, D\}$.

Details of this verification are left to the reader.

To prove the weak associativity axiom [RS3a] we will use the lattice structure of $\mathrm{Sp}(X)$, see V.1.1 (9).

Let $a, b, c, d, e \in \mathrm{Sp}(X)$ be such that $a \in D_{\mathbf{Sp}(X)}(b, c)$ and $c \in D_{\mathbf{Sp}(X)}(d, e)$. We must find an $f \in \mathrm{Sp}(X)$ so that $f \in D_{\mathbf{Sp}(X)}(b, d)$ and $a \in D_{\mathbf{Sp}(X)}(f, e)$. We claim that

$$(*) \qquad f = (a \wedge -e) \vee (b \wedge d) \vee (a \wedge b) \vee (a \wedge d)$$

does the job. Indeed:

— $f \in D_{\mathbf{Sp}(X)}(b, d)$.

By the pointwise definition of the lattice operations in $\mathrm{Sp}(X)$, this amounts to checking that

$$\text{For all } x \in X, \quad f(x) \neq 0 \Rightarrow f(x) = b(x) \text{ or } f(x) = d(x).$$

If $f(x) = 1$, then all four disjuncts in (*) have value 1 at x; in particular, $(b \wedge d)(x) = 1$. Since evaluation at a point $x \in X$ commutes to the lattice operations, one of $b(x)$ or $d(x)$ is 1.

If $f(x) = -1$, then at least one of the disjuncts in (*) has value -1 at x and, therefore, both the conjuncts occurring in it also have value -1 at x. If any of three last disjuncts in (*) has value -1, we get $b(x) = -1$ or $d(x) = -1$. So, assume $a(x) = -e(x) = -1$. Then, $a \in D_{\mathbf{Sp}(X)}(b, c)$ entails that one of $b(x)$ or $c(x)$ is -1. If $c(x) = -1$, then $c \in D_{\mathbf{Sp}(X)}(d, e)$ implies $d(x) = -1$ or $e(x) = -1$; but the last possibility is excluded, as $e(x) = 1$. Hence, in any case we have $b(x) = -1$ or $d(x) = -1$, as required.

— $a \in D_{\mathbf{Sp}(X)}(f, e)$.

So, let $x \in X$ be such that $a(x) \neq 0$, and we show $a(x) = f(x)$ or $a(x) = e(x)$.

If $a(x) = 1$, from $a \in D_{\mathbf{Sp}(X)}(b, c)$ follows $b(x) = 1$ or $c(x) = 1$. If $b(x) = 1$, noting that every disjunct in (*) contains a or b, we conclude that $f(x) = 1$. If $c(x) = 1$, the assumption $c \in D_{\mathbf{Sp}(X)}(d, e)$ entails $d(x) = 1$ or $e(x) = 1$. In the latter case we are done. So, assume $d(x) = 1$. Since every disjunct in (*) contains either a or d, we conclude again that $f(x) = 1$.

If $a(x) = -1$, we must prove that $f(x) = -1$ or $e(x) = -1$. From $a \in D_{\mathbf{Sp}(X)}(b, c)$ follows $b(x) = -1$ or $c(x) = -1$. In the first case we get $(a \wedge b)(x) = a(x) \wedge b(x) = -1$, whence $f(x) = -1$. So assume $c(x) = -1$. From $c \in D_{\mathbf{Sp}(X)}(d, e)$ we get $e(x) = -1$, and we are done, or $d(x) = -1$, in which case $(a \wedge d)(x) = a(x) \wedge d(x) = -1$, implying $f(x) = -1$, as required to complete the proof of V.1.4. □

Concerning the remaining axiom [RS3b], we have:

See streamlined proof of V.1.5 in [DP3], Thm. 1.8, pp. 370-372. Replace?

Theorem V.1.5 *The following are equivalent for every spectral space X :*

(1) *X is hereditarily normal.*

(2) *The structure* $\mathbf{Sp}(X)$ *verifies axiom* [RS3b].

The following known results concerning spectral spaces will be needed in the proof of V.1.5. Recall that a subset A of a spectral space X is called **generically closed** if it is downward closed for the specialization order of X: for $x, y \in X$, $x \rightsquigarrow y$ and $y \in A$ imply $x \in A$; cf. [DST], 5.1.6.

Proposition V.1.6 *Let X be a spectral space and let D_1, D_2 be disjoint, generically closed, quasi-compact subsets of X. Then,*

(i) D_1, D_2 are contained in disjoint quasi-compact open *subsets of X.*

(ii) Given quasi-compact opens U_1, U_2 such that $D_i \subseteq U_i$ $(i = 1, 2)$ there are disjoint quasi-compact opens V_1, V_2 so that $D_i \subseteq V_i \subseteq U_i$ $(i = 1, 2)$.

Hint of proof. Item (ii) is a consequence of (i): if V_1', V_2' are disjoint quasi-compact opens such that $D_i \subseteq V_i'$, then the sets $V_i = V_i' \cap U_i$ $(i = 1, 2)$ satisfy the conclusion of (ii).

Item (i) is Prop. 6.1.14 (iii) in [DST]. (In fact, (i) follows from the more general, and fundamental, Separation Lemma 6.0.2 in [DST].) $\square$

Fact V.1.7 *Let X be a topological space, and $B, C \subseteq X$. If B is quasi-compact and C is closed, then $B \cap C$ is quasi-compact.* $\square$

Proof of Theorem V.1.5. (1) $\Rightarrow$ (2). In this proof we use the additional operations of the Post algebra $P(X)$ containing $\mathbf{Sp}(X)$, cf. II.6.1(10). Recall (IV.2.7 (i)) that transversal representation in $P(X)$ has the following form: for $f, g, h \in P(X)$,

$$h \in D^t_{P(X)}(f, g) \;\Leftrightarrow\; (f \wedge \nabla g) \vee (g \wedge \nabla f) \leq h \leq (\Delta f \vee g) \wedge (\Delta g \vee f).$$

Furthermore, it is clear that, for functions $f, g : X \longrightarrow \mathbf{3}$,

(†) $\qquad f \leq g \;\Leftrightarrow\; g^{-1}[1] \subseteq f^{-1}[1]$ and $f^{-1}[-1] \subseteq g^{-1}[-1]$.

Now we set :

(*) $\qquad p := (f \wedge \nabla g) \vee (g \wedge \nabla f)$ and $q := (\Delta f \vee g) \wedge (\Delta g \vee f)$,

and observe :

Claim 1. (i) $p^{-1}[1] = f^{-1}[1] \cup g^{-1}[1]$.

(ii) $p^{-1}[-1] = (f^{-1}[-1] \cap g^{-1}[0, -1]) \cup (f^{-1}[0, -1] \cap g^{-1}[-1])$.

(iii) $q^{-1}[1] = (f^{-1}[0, 1] \cap g^{-1}[1]) \cup (g^{-1}[0, 1] \cap f^{-1}[1])$.

(iv) $q^{-1}[-1] = f^{-1}[-1] \cup g^{-1}[-1]$.

Proof of Claim 1. To illustrate the argument we just check item (ii). Since the order in $\mathbf{3}$ is $1 < 0 < -1$, for $x \in X$ we have,

(††) $$\begin{aligned} p(x) = -1 \;&\Leftrightarrow\; (f(x) \wedge \nabla g(x)) \vee (g(x) \wedge \nabla f(x)) = -1 \\ &\Leftrightarrow\; f(x) \wedge \nabla g(x) = -1 \text{ or } g(x) \wedge \nabla f(x) = -1. \end{aligned}$$

Since $\nabla g(x) = 1$ if $g(x) = 1$, and $\nabla g(x) = -1$ if $g(x) \in \{0, -1\}$, we get:

$$f(x) \wedge \nabla g(x) = -1 \;\Leftrightarrow\; f(x) = -1 \text{ and } \nabla g(x) = -1 \;\Leftrightarrow\; f(x) = -1 \text{ and } g(x) \in \{0, -1\}.$$

Thus, (††) yields:

$$p(x) = -1 \;\Leftrightarrow\; (f(x) = -1 \text{ and } g(x) \in \{0, -1\}) \text{ or } (g(x) = -1 \text{ and } f(x) \in \{0, -1\}),$$

which proves (ii). $\square$

Observe that, if $f, g \in \mathrm{Sp}(X)$, then $f^{-1}[\pm 1]$, $g^{-1}[\pm 1]$ are quasi-compact, while $f^{-1}[0,1]$, $g^{-1}[0,1]$, $f^{-1}[0,-1]$, $g^{-1}[0,-1]$ are closed in X. Since the union of two quasi-compact sets is quasi-compact, Fact V.1.7 and items (ii), (iii) of Claim 1 entail that the sets

$$K_1 := q^{-1}[1] \text{ and } K_2 := p^{-1}[-1]$$

are quasi-compact. Let $\mathrm{Gen}\,(K_i) = \{x \in X \mid \text{There is } y \in K_i \text{ such that } x \rightsquigarrow y\}$ denote the *generization* of K_i ($i = 1, 2$) in X, i.e., the downward closure of K_i under the specialization order $\rightsquigarrow$ of X; cf. [DST], Def. 5.0.1. Since opens sets are downward closed under $\rightsquigarrow$, it readily follows that $\mathrm{Gen}\,(K_i)$ is also quasi-compact. We claim:

Claim 2. $\mathrm{Gen}\,(K_1) \cap \mathrm{Gen}\,(K_2) = \emptyset$.

Proof of Claim 2. Assume there is $t \in \mathrm{Gen}\,(K_1) \cap \mathrm{Gen}\,(K_2)$, and let $k_i \in K_i$ be such that $t \rightsquigarrow k_i$ ($i = 1, 2$). Since X is assumed to be hereditarily normal, either $k_1 \rightsquigarrow k_2$ or $k_2 \rightsquigarrow k_1$, say the first. Since $k_2 \in K_2 = p^{-1}[-1] \subseteq f^{-1}[-1] \cup g^{-1}[-1]$ (see (ii), Claim 1), and the latter sets are open, we get $k_1 \in f^{-1}[-1] \cup g^{-1}[-1]$, whence $k_1 \in (f^{-1}[-1] \cup g^{-1}[-1]) \cap q^{-1}[1]$, contrary to item (iii) in Claim 1. Similarly, $k_2 \rightsquigarrow k_1$ also leads to a contradiction, proving Claim 2. $\square$

From items (iii) and (i) in Claim 1 we get $K_1 = q^{-1}[1] \subseteq f^{-1}[1] \cup g^{-1}[1] = p^{-1}[1]$, and since $p^{-1}[1]$ is open, it follows that $\mathrm{Gen}\,(K_1) \subseteq p^{-1}[1]$. Likewise, $\mathrm{Gen}\,(K_2) \subseteq q^{-1}[-1]$. Now, Proposition V.1.6 applied with $D_i = \mathrm{Gen}\,(K_i)$, $U_1 = p^{-1}[1]$ and $U_2 = q^{-1}[-1]$ shows that there are *disjoint* quasi-compact open subsets V_1, V_2 of X such that, for $i = 1, 2$,

$$\mathrm{Gen}\,(K_i) \subseteq V_i \subseteq \begin{cases} p^{-1}[1] & \text{if } i = 1 \\ q^{-1}[-1] & \text{if } i = 2\,. \end{cases}$$

We define a map $h : X \longrightarrow \mathbf{3}$ as follows; for $x \in X$,

$$h(x) = \begin{cases} 1 & \text{if } x \in V_1 \\ -1 & \text{if } x \in V_2 \\ 0 & \text{if } x \notin V_1 \cup V_2\,. \end{cases}$$

Then, $h^{-1}[1] = V_1$, $h^{-1}[-1] = V_2$, which entails $h \in \mathrm{Sp}(X)$. Further,

Claim 3. $p \leq h \leq q$ (in $P(X)$).

Proof of Claim 3. On the one hand we have $h^{-1}[1] = V_1 \subseteq p^{-1}[1]$ and $p^{-1}[-1] = K_2 \subseteq \mathrm{Gen}\,(K_2) \subseteq V_2 = h^{-1}[-1]$, which proves $p \leq h$ (see (†) above).

On the other hand, $q^{-1}[1] = K_1 \subseteq \mathrm{Gen}\,(K_1) \subseteq V_1 = h^{-1}[1]$, and $h^{-1}[-1] = V_2 \subseteq q^{-1}[-1]$, which proves $h \leq q$. $\square$

In view of the definition of p and q (see (*) above), from Theorem IV.2.7 (i) we get $h \in D^t_{P(X)}(f, g)$. As observed in V.1.1 (10), this representation, together with $h \in \mathbf{Sp}(X)$, entails $h \in D^t_{\mathbf{Sp}(X)}(f, g)$, proving that $\mathbf{Sp}(X)$ satisfies axiom [RS3b].

(2) $\Rightarrow$ (1). With $\rightsquigarrow$ denoting the specialization order of X, assume there are $x, y, z \in X$ such that $x \rightsquigarrow y, z$, but $y \not\rightsquigarrow z$ and $z \not\rightsquigarrow y$. Thus, $z \notin \overline{\{y\}}$ and $y \notin \overline{\{z\}}$. Then, there are quasi-compact opens U, V such that $z \in U$, $y \in V$, $y \notin U$ and $z \notin V$. Let f_U, $f_V : X \longrightarrow \mathbf{3}$ be the spectral maps defined by:

$$f_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}, \qquad f_V(x) = \begin{cases} -1 & \text{if } x \in V \\ 0 & \text{if } x \notin V. \end{cases}$$

Since $\mathbf{Sp}(X) \models$ [RS3b], there is $f \in \mathbf{Sp}(X)$ such that $f \in D^t_{\mathbf{Sp}(X)}(f_U, f_V)$. In particular, for $w \in \{y, z\}$ it holds $f(w) \in D^t_{\mathbf{3}}(f_U(w), f_V(w))$. Now, $f_U(y) = 0$, $f_V(y) = -1$ and $f_U(z) = 1$, $f_V(z) = 0$ imply $f(y) = -1$ and $f(z) = 1$ (cf. Corollary I.2.5). Since f is monotone for the order of $\mathbf{3}_{\mathrm{sp}}$, $x \rightsquigarrow y, z$ entails $f(x) = -1$ and $f(x) = 1$, contradiction. □

Definition V.1.8 A real semigroup is called **spectral** if it is of the form $\mathbf{Sp}(X)$ for some spectral space X (necessarily hereditarily normal by the preceding theorem). □

V.2 Definability of the lattice structure.

This section is devoted to prove a result that is the key towards a structural theory of spectral real semigroups and, hence, to many of the results in this chapter. It gives an explicit first-order definition of the lattice operations of any spectral RS in the language $\mathcal{L}_{RS} = \{\cdot, 1, 0, -1, D\}$ for real semigroups. The specific (logical) form of the definition of the lattice operations given in the next theorem entails that the RS-characters of spectral RSs are automatically lattice homomorphisms. This fact plays a crucial role in later results.

Theorem V.2.1 *Let G be a spectral real semigroup. For $a, b, c, d \in G$ we have:*

(i) $\quad a \wedge 0 = c \;\Leftrightarrow\; c \in \mathrm{Id}(G),\ a \cdot c = c$ *and* $-a \in D_G(1, -c)$.

Setting $a^- := a \wedge 0$ and $a^+ := -(-a)^- = a \vee 0$, we have:

(ii) $\quad a \wedge b = d \;\Leftrightarrow\; d \in D_G(a, b),\ d^+ = -a^+ \cdot b^+$ *and* $d^- \in D^t_G(a^-, b^-)$.

Proof. Let $G = \mathbf{Sp}(X)$, X a hereditarily normal spectral space. Owing to Fact V.1.3, the lattice operations in G can interchangeably be taken in the pointwise order $\leq$ or in the representation partial order $\leq_G$ (we shall use both).

(i). ($\Rightarrow$) We check that $c := a \wedge 0$ verifies the three conditions on the right-hand side of (i).

Firstly, $c \leq 0$ implies $c \in \mathrm{Id}(G)$ (I.6.4 (c)) and $c \leq a$ implies $-a \in D_{\mathbf{Sp}(X)}(1, -c)$ (I.6.2). To check $a\,c = c$ recall that $a\,c = a \,\underline{\triangle}\, c$ ($\underline{\triangle}$ = symmetric difference, cf. V.1.1 (9)). Now compute:

$$\begin{aligned} a \,\underline{\triangle}\, (a \wedge 0) &= (a \wedge -(a \wedge 0)) \vee ((a \wedge 0) \wedge -a) = (a \wedge (-a \vee 0)) \vee (a \wedge -a \wedge 0) = \\ &= (a \wedge -a) \vee (a \wedge 0) \vee (a \wedge -a \wedge 0) = (a \wedge -a) \vee (a \wedge 0)\,; \end{aligned}$$

since $a \wedge -a \leq a, 0$ (cf. I.6.5 (8)), the last term equals $a \wedge 0$.

($\Leftarrow$) Since $c \in \mathrm{Id}(G)$, Proposition I.6.4 (c) gives $c = c^2 \leq 0$. By I.2.3 (5) we have $c = c^2 \in D_G(1, a)$. By assumption we also have $-a \in D_G(1, -c)$, whence $c \leq a$.

To prove $c = a \wedge 0$, let $z \in G$ be such that $z \leq 0$ and $z \leq a$, and show $z \leq c$, i.e., $z(x) \leq c(x)$ for all $x \in X$. Otherwise, since $z = z^2$, we must have $z(x) = 0$ and $c(x) = 1$ for some $x \in X$. From $a\,c = c$ we get $a(x) = 1$, contradicting $z(x) \leq a(x)$.

(ii). ($\Rightarrow$) Set $d := a \wedge b$. We check the three conditions on the right-hand side of (ii).

a) $d \in D_G(a, b)$.

By the pointwise definition of D_G, this boils down to $\forall x \in X (d(x) \in D_{\mathbf{3}}(a(x), b(x))$. On the other hand, the pointwise definition of the lattice operations in $G = \mathbf{Sp}(X)$ gives $d(x) =$

$a(x) \wedge b(x)$ in $\mathbf{3}$ $(x \in X)$. Direct inspection of Corollary I.2.5 shows that, for $i, j \in \mathbf{3}$, $i \wedge j \in D_{\mathbf{3}}(i, j)$, as required.

b) $(a \wedge b)^{+} = -a^{+} \cdot b^{+}$.

We compute the right-hand side using the distributive lattice structure of G and that product in G is symmetric difference (V.1.1 (9)). Recall that $z^{+} = z \vee 0$ and $-(z \underline{\triangle} w) = (-z \vee w) \wedge (-w \vee z)$. We have:

$$\begin{aligned} -a^{+} \cdot b^{+} &= -(a \vee 0) \underline{\triangle} (b \vee 0) = (-(a \vee 0) \vee (b \vee 0)) \wedge (-(b \vee 0) \vee (a \vee 0)) = \\ &= ((-a \wedge 0) \vee (b \vee 0)) \wedge ((-b \wedge 0) \vee (a \vee 0)) = (b \vee 0) \wedge (a \vee 0) = \\ &= (a \wedge b) \vee 0 = (a \wedge b)^{+}, \end{aligned}$$

as asserted.

c) $(a \wedge b)^{-} \in D^{t}_{G}(a^{-}, b^{-})$.

As D^{t}_{G} is pointwise defined and $d = a \wedge b$, (c) reduces to: $\forall x \in X(d^{-}(x) \in D^{t}_{\mathbf{3}}(a^{-}(x), b^{-}(x))$. Observe that, for $e \in G = \mathbf{Sp}(X)$ and $x \in X$, $e^{-}(x) = e(x) \wedge 0$, whence $e^{-}(x) \leq_{\mathbf{3}} 0$ and $e^{-}(x) \in \{0, 1\}$. Further, we have:

(*) $\quad e^{-}(x) = 1 \Leftrightarrow e(x) = 1$; equivalently, $e^{-}(x) = 0 \Leftrightarrow e(x) \in \{0, -1\}$.

Thus, we must prove:

— $d^{-}(x) = 1 \Rightarrow a^{-}(x) = 1$ or $b^{-}(x) = 1$.

— $d^{-}(x) = 0 \Rightarrow a^{-}(x) = b^{-}(x) = 0$.

For the first implication, assumption $d^{-}(x) = 1$ implies $d(x) = 1$; together with $d \in D_{G}(a, b)$ this yields $a(x) = 1$ or $b(x) = 1$, which, by (*), entails $a^{-}(x) = 1$ or $b^{-}(x) = 1$.

For the second implication, assumption $d^{-}(x) = 0$ yields $0 = d^{-}(x) = d(x) \wedge 0 = a(x) \wedge b(x) \wedge 0$, which implies $a(x), b(x) \in \{0, -1\}$; (*) implies, then, $a^{-}(x) = b^{-}(x) = 0$, as required.

($\Leftarrow$) Given $d \in G$, we assume $d \in D_{G}(a, b)$, $d^{+} = -a^{+} \cdot b^{+}$, $d^{-} \in D^{t}_{G}(a^{-}, b^{-})$, and prove $d = a \wedge b$.

a) $d \leq_{G} a$ and $d \leq_{G} b$.

The argument being similar in either case, we prove only the first inequality. In view of the pointwise definition of $\leq_{G}$ (see (†) in V.1.1 (8)), it suffices to prove, for $x \in X$:

— $a(x) = 1 \Rightarrow d(x) = 1$,

— $a(x) = 0 \Rightarrow d(x) \in \{0, 1\}$.

For the first implication, $a(x) = 1$ yields $a^{-}(x) = 1$ (see (*) above). From $d^{-} \in D^{t}_{G}(a^{-}, b^{-})$ and $b^{-}(x) \in \{0, 1\}$ we get $d^{-}(x) \in D^{t}_{\mathbf{3}}(a^{-}(x), b^{-}(x)) = D^{t}_{\mathbf{3}}(1, b^{-}(x)) = \{1\}$, which, by (*), gives $d(x) = 1$, as needed.

For the second implication, if $a(x) = 0$ but $d(x) = -1$, we would have $a^{+}(x) = a(x) \vee 0 = 0$ and $d^{+}(x) = d(x) \vee 0 = -1$, contradicting the equality $d^{+} = -a^{+} \cdot b^{+}$ at the point x.

b) For all $z \in G$, $z \leq_{G} a$ and $z \leq_{G} b$ imply $z \leq_{G} d$.

We must check, for all $x \in X$:

— $d(x) = 1 \Rightarrow z(x) = 1$,

— $d(x) = 0 \Rightarrow z(x) \in \{0, 1\}$.

For the first implication, from (*) we have $d^-(x) = 1$. On the other hand, since $a^-(x), b^-(x) \in \{0,1\}$, from $d^- \in D^t_G(a^-, b^-)$ we obtain $1 = d^-(x) \in D^t_G(a^-(x), b^-(x))$. This relation implies that $a^-(x)$, $b^-(x)$ cannot be both 0 (cf. I.2.5). If, e.g., $a^-(x) = 1$, then $a(x) = 1$, and $z \leq_G a$ yields $z(x) = 1$.

For the second implication, suppose $d(x) = 0$; hence $d^+(x) = d(x) \vee 0 = 0$. This and $d^+ = -a^+ \cdot b^+$ imply that one of $a^+(x)$ or $b^+(x)$ is 0, say, e.g., $a^+(x) = 0$. Then, $0 = a^+(x) = a(x) \vee 0$ entails $a(x) \in \{0,1\}$; this, together with $z \leq_G a$, yields $z(x) \in \{0,1\}$, as required.

The proof of Theorem V.2.1 is now complete. $\square$

Remark. First-order definability of the lattice structure of $\mathbf{Sp}(X)$ in $\mathcal{L}_{\mathrm{RS}}$ follows also from Fact V.1.3: it suffices to express

(i) The definition of $\leq_{\mathbf{Sp}(X)}$ in terms of $D_{\mathbf{Sp}(X)}$ (cf. I.6.2), and

(ii) The usual definition of the glb ($\wedge$) and the lub ($\vee$) in terms of the order $\leq_{\mathbf{Sp}(X)}$.

However, the definition of the lattice operations obtained in this way does not guarantee that the next Corollary holds, while that of Theorem V.2.1 does. Though only implicit here, the reason is that the latter is given by a *positive-primitive* $\mathcal{L}_{\mathrm{RS}}$-formula, while the former is not. For more details, see V.7.5. $\square$

Corollary V.2.2 *The RS-characters of a spectral real semigroup are lattice homomorphisms.*

Proof. To begin with we observe that $\mathbf{3}$ is a spectral RS. Indeed, $\mathbf{3} = \mathrm{Sp}(\mathbf{1})$, where $\mathbf{1}$ is the singleton spectral space; the three functions $\mathbf{1} \longrightarrow \mathbf{3}_{\mathrm{sp}}$ map the unique element to $1, 0$ and -1, respectively; clearly, they are pointwise ordered in the right way.

Let G be a spectral RS, $a, b \in G$ and $\sigma \in X_G$; we show that $\sigma(a \wedge b) = \sigma(a) \wedge \sigma(b)$. The equivalences (i) and (ii) of V.2.1 can (and will) be applied to both G and $\mathbf{3}$.

We first treat the case $b = 0$. We know that $c = a \wedge 0$ verifies the conditions in the right-hand side of V.2.1 (i). Since σ is a RS-homomorphism we get $\sigma(c) \in \mathrm{Id}(\mathbf{3}) = \{0,1\}$, $\sigma(a)\,\sigma(c) = \sigma(c)$ and $-\sigma(a) \in D^t_{\mathbf{3}}(1, -\sigma(c))$. Using the implication ($\Leftarrow$) of V.2.1 (i) in $\mathbf{3}$ gives $\sigma(c) = \sigma(a) \wedge 0$, i.e.,

(†) $\sigma(a \wedge 0) = \sigma(a) \wedge 0$ (equivalently, $\sigma(a^-) = \sigma(a)^-$).

Since $\sigma(-(-a)^-) = -\sigma((-a)^-) = -(\sigma(-a))^- = -(-\sigma(a))^-$, we also get

(††) $\sigma(a^+) = \sigma(a)^+$.

Next, for arbitrary $b \in G$, applying item (ii) of V.2.1 with $d = a \wedge b$, taking into account that σ is a RS-homomorphism, and using (†) and (††) above, we get $\sigma(d) \in D_{\mathbf{3}}(\sigma(a), \sigma(b))$, $\sigma(d)^+ = -\sigma(a)^+ \cdot \sigma(b)^+$ and $\sigma(d)^- \in D^t_{\mathbf{3}}(\sigma(a)^-, \sigma(b)^-)$. On the other hand, $x = \sigma(a) \wedge \sigma(b)$ exists in $\mathbf{3}$ and verifies $x \in D_{\mathbf{3}}(\sigma(a), \sigma(b))$, $x^+ = -\sigma(a)^+ \cdot \sigma(b)^+$ and $x^- \in D^t_{\mathbf{3}}(\sigma(a)^-, \sigma(b)^-)$. That is, both x and $\sigma(d)$ verify in $\mathbf{3}$ the conditions of the right-hand side of (ii) in V.2.1. This implies $x = \sigma(d)$, i.e., $\sigma(a) \wedge \sigma(b) = \sigma(a \wedge b)$, as asserted. $\square$

Remark. A similar argument shows that RS-homomorphisms between spectral RSs are automatically homomorphisms of the corresponding lattice structures. $\square$

V.3 The functor Sp.

We begin here the study of the correspondence $X \longmapsto \mathbf{Sp}(X)$ assigning to each hereditarily normal spectral space, X, the real semigroup $\mathbf{Sp}(X)$.

In this and the next two sections we set the stage to prove that this correspondence establishes an anti-equivalence of the category **HNSS** of hereditarily normal spectral spaces with spectral maps, with the category **SRS** of spectral real semigroups with real semigroup morphisms, a goal to be attained in Theorem V.5.4 below. $\square$

To begin with, the correspondence $X \longmapsto \mathbf{Sp}(X)$ extends to spectral maps, and hence defines a contravariant functor of the first category into the second; namely:

Definition and Notation V.3.1 Given a spectral map $\varphi : X \longrightarrow Y$ between spectral spaces X, Y we define a dual map $\mathrm{Sp}(\varphi) : \mathrm{Sp}(Y) \longrightarrow \mathrm{Sp}(X)$ by composition: for $f \in \mathrm{Sp}(Y)$ we set,

$$\mathrm{Sp}(\varphi)(f) := f \circ \varphi.$$

Being a composition of spectral maps, we have $\mathrm{Sp}(\varphi)(f) \in \mathrm{Sp}(X)$. $\square$

Proposition V.3.2 *Let $\varphi : X \longrightarrow Y$ be a spectral map between spectral spaces. Then $\mathrm{Sp}(\varphi)$ is a homomorphism of $\mathcal{L}_{\mathrm{RS}}$-structures.*

The proof is routine verification from the fact that product and representation in both $\mathrm{Sp}(Y)$ and $\mathrm{Sp}(X)$ are pointwise defined. We omit it. Note that it is not required that X, Y be hereditarily normal. $\square$

As a beginning step in proving that this functor is an anti-equivalence of categories we show that any hereditarily normal spectral space, X, is isomorphic in the category of spectral spaces to the abstract real spectrum $X_{\mathbf{Sp}(X)}$ of the real semigroup $\mathbf{Sp}(X)$. The proof of this requires a fine touch. First, we observe:

Fact V.3.3 *Let X be a spectral space. The evaluation map at a point $x \in X$, $ev_x : \mathbf{Sp}(X) \longrightarrow \mathbf{3}$, given by $ev_x(f) = f(x)$ for $f \in \mathrm{Sp}(X)$, is a character of $\mathcal{L}_{\mathrm{RS}}$-structures., i.e., $ev_x \in X_{\mathbf{Sp}(X)}$.* $\square$

The proof is straightforward and hence omitted.

Let $\mathrm{ev} : X \longrightarrow X_{\mathbf{Sp}(X)}$ be the map $\mathrm{ev}(x) = ev_x$ $(x \in X)$.

Proposition V.3.4 $\mathrm{ev} : X \longrightarrow X_{\mathbf{Sp}(X)}$ *is injective and spectral.*

Proof. (1) ev is injective.

This amounts to showing that $\mathrm{Sp}(X)$ separates points in X: for $x \neq y$ in X, there is $g \in \mathrm{Sp}(X)$ so that $g(x) \neq g(y)$, i.e., $ev_x(g) \neq ev_y(g)$, whence $ev_x \neq ev_y$, i.e., $\mathrm{ev}(x) \neq \mathrm{ev}(y)$.

Since X is T_0, if $x \neq y$, there is a quasi-compact open $U \subseteq X$ so that $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$; say the first. Let $g : X \longrightarrow \mathbf{3}$ be defined by: $g \lceil U = 1$, $g \lceil (X \setminus U) = 0$. Since $g^{-1}[1] = U$, $g^{-1}[-1] = \emptyset$ are quasi-compact open, $g \in \mathrm{Sp}(X)$ and, clearly, $g(x) = 1$, $g(y) = 0$. The case $y \in U$ and $x \notin U$ is symmetrical.

(2) ev is spectral.

By definition, the family $\{[\![f = 1]\!] \mid f \in \mathrm{Sp}(X)\}$, where $[\![f = 1]\!] = \{\sigma \in X_{\mathbf{Sp}(X)} \mid \sigma(f) = 1\}$ is a subbasis for the spectral topology on $X_{\mathbf{Sp}(X)}$, cf. I.1.17. It suffices, then, to show that $\mathrm{ev}^{-1}[\,[\![f = 1]\!]\,]$ is quasi-compact open in X, for $f \in \mathrm{Sp}(X)$. For $x \in X$ we have:

$$x \in \mathrm{ev}^{-1}[\llbracket f = 1 \rrbracket] \Leftrightarrow \mathrm{ev}(x) = ev_x \in \llbracket f = 1 \rrbracket \Leftrightarrow ev_x(f) = f(x) = 1,$$

i.e., $\mathrm{ev}^{-1}[\llbracket f = 1 \rrbracket] = f^{-1}[1]$, a quasi-compact open set, as claimed. $\square$

The surjectivity of ev is the key to establish the anti-equivalence of categories announced above.

Theorem V.3.5 *Let X be a hereditarily normal spectral space. Then, $\mathrm{ev} : X \longrightarrow X_{\mathbf{Sp}(X)}$ is surjective.*

Proof. Given $\sigma \in X_{\mathbf{Sp}(X)}$ we have to find $x \in X$ so that $\mathrm{ev}(x) = ev_x = \sigma$. To accomplish this we shall use Stone's representation theorem of spectral spaces by bounded distributive lattices (cf. [DST], §§ 1,2). This fundamental result proves the existence of a (functorial) bijective correspondence between the points of a spectral space, X, and the prime filters of the bounded distributive lattice $\overline{\mathcal{K}}(X)$ of **closed constructible** —i.e., complements of quasi-compact open— subsets of X; cf. [DST], Thm. 2.1.7 for a precise statement). We shall construct a prime filter $\mathfrak{p}$ of $\overline{\mathcal{K}}(X)$ such that, if x_0 is the unique point of $\bigcap \mathfrak{p}$, then $\sigma = ev_{x_0}$. Recall that the pointwise order coincides with the representation partial order in $\mathbf{Sp}(X)$ (V.1.3).

Since $\sigma : \mathbf{Sp}(X) \longrightarrow \mathbf{3}$ is a lattice homomorphism (V.2.2), the set $\mathfrak{q} = \sigma^{-1}[0, -1]$ is a prime filter of the lattice $\mathrm{Sp}(X)$. For $A \in \overline{\mathcal{K}}(X)$ we define maps $c_A, d_A : X \longrightarrow \mathbf{3}$ as follows: for $x \in X$,

$$c_A(x) = \begin{cases} 0 & \text{if } x \in A \\ -1 & \text{if } x \notin A, \end{cases} \qquad d_A(x) = -c_A(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A. \end{cases}$$

Since $X \setminus A$ is quasi-compact open, we have $c_A, d_A \in \mathrm{Sp}(X)$. Further, since $d_A \leq 0 \leq c_A$ and σ is monotone, we get $\sigma(d_A) \in \{0, 1\}$ and $\sigma(c_A) \in \{0, -1\}$. Now set:

$$\mathfrak{p} := \{A \in \overline{\mathcal{K}}(X) \mid d_A \in \mathfrak{q}\} = \{A \in \overline{\mathcal{K}}(X) \mid \sigma(d_A) = 0\}^{\cdot}$$

Claim 1. $\mathfrak{p}$ is a prime filter of $\overline{\mathcal{K}}(X)$.

<u>Proof of Claim 1</u>. (a) $A \subseteq B$ and $A \in \mathfrak{p}$ imply $B \in \mathfrak{p}$.

Clearly, $A \subseteq B \Rightarrow d_A \leq d_B$. Since σ is monotone, $0 \leq \sigma(d_A) \leq \sigma(d_B)$, and from $\sigma(d_B) \in \{0, 1\}$, it follows $\sigma(d_B) = 0$, i.e., $B \in \mathfrak{p}$.

(b) $A, B \in \mathfrak{p} \Rightarrow A \cap B \in \mathfrak{p}$.

It suffices to check that $d_{A \cap B} = d_A \wedge d_B$, and use that σ is a lattice homomorphism and $\sigma(d_C) \in \{0, 1\}$ for $C \in \overline{\mathcal{K}}(X)$.

(c) $\emptyset \notin \mathfrak{p}$.

Clear, since $d_\emptyset = 1$.

(d) $A \cup B \in \mathfrak{p} \Rightarrow A \in \mathfrak{p}$ or $B \in \mathfrak{p}$.

Check that $d_{A \cup B} = d_A \vee d_B$, and argue as in (b).

Claim 2. For $f \in \mathrm{Sp}(X)$, $f \in \mathfrak{q} \Leftrightarrow f^{-1}[0, -1] \in \mathfrak{p}$.

<u>Proof of Claim 2</u>. Set $A := f^{-1}[0, -1] \in \overline{\mathcal{K}}(X)$. Observe first:

(i) $A \in \mathfrak{p} \Rightarrow d_A \leq f$.

This is clear, since $\mathrm{Im}(d_A) = \{0, 1\}$ and, for $x \in X$, $d_A(x) = 0 \Rightarrow x \in A \Rightarrow f(x) \in \{0, -1\}$.

($\Leftarrow$) If $A \in \mathfrak{p}$, then $d_A \in \mathfrak{q}$, and hence $f \in \mathfrak{q}$ ($\mathfrak{q}$ is a prime filter).

For the converse, observe that

(ii) $d_A = c_A \wedge f$.

Indeed, for $x \in X$ we have:

— $d_A(x) = 0 \Rightarrow x \in A \Rightarrow c_A(x) = 0$ and $f(x) \in \{0, -1\}$, whence $(c_A \wedge f)(x) = 0$;

— $d_A(x) = 1 \Rightarrow x \notin A \Rightarrow f(x) = 1 \Rightarrow (c_A \wedge f)(x) = 1$.

($\Rightarrow$) Assume $A \notin \mathfrak{p}$; then, $\sigma(d_A) = 1$. Since σ is a lattice homomorphism, (ii) yields: $1 = \sigma(d_A) = \sigma(c_A) \wedge \sigma(f)$; since $\sigma(c_A) \in \{0, -1\}$, this equality entails $\sigma(f) = 1$, i.e., $f \notin \mathfrak{q}$.

Now, with $x_0 =$ the unique point in $\bigcap \mathfrak{p}$, we have:

Claim 3. $ev_{x_0} = \sigma$.

Proof of Claim 3. By Claim 2, for $f \in \mathrm{Sp}(X)$ we have:

$$f \in \mathfrak{q} \Leftrightarrow f^{-1}[0,-1] \in \mathfrak{p} \Leftrightarrow x_0 \in f^{-1}[0,-1] \Leftrightarrow f(x_0) \in \{0,-1\}.$$

Using this equivalence we argue by cases according to the values of $\sigma(f)$:

— $\sigma(f) = 0 \Rightarrow f \in \mathfrak{q}$ and $-f \in \mathfrak{q} \Rightarrow f(x_0), -f(x_0) \in \{0,-1\} \Rightarrow f(x_0) = 0$.

— $\sigma(f) = -1 \Rightarrow \sigma(-f) = 1 \Rightarrow -f \notin \mathfrak{q} \Rightarrow -f(x_0) = 1 \Rightarrow f(x_0) = -1$.

— $\sigma(f) = 1 \Rightarrow f \notin \mathfrak{q} \Rightarrow f(x_0) = 1$.

This completes the proof of Claim 3 and, hence, of Theorem V.3.5. $\square$

Corollary V.3.6 *For a hereditarily normal spectral space X,* $\mathrm{ev} : X \longrightarrow X_{\mathbf{Sp}(X)}$ *is a homeomorphism of X_{con} onto $(X_{\mathbf{Sp}(X)})_{\mathrm{con}}$.*

Proof. Immediate consequence of Proposition V.3.4 and Theorem V.3.5, using that ev is a continuous bijection between the Hausdorff spaces X_{con} and $(X_{\mathbf{Sp}(X)})_{\mathrm{con}}$, hence a homeomorphism. $\square$

To prove that ev is an isomorphism between X and $X_{\mathbf{Sp}(X)}$ in the category of spectral spaces we show:

Proposition V.3.7 *Let X be a hereditarily normal spectral space. Then,*

(1) *The map* $\mathrm{ev}^{-1} : X_{\mathbf{Sp}(X)} \longrightarrow X$ *is spectral.*

Hence,

(2) ev *is an isomorphism in the category of spectral spaces. In particular, it is a homeomorphism of the spectral spaces X and $X_{\mathbf{Sp}(X)}$.*

Proof. (1) By the characterization of spectral maps mentioned in V.1.1 (3.ii) above, and the preceding Corollary V.3.6 it only remains to prove that, for $\sigma_1, \sigma_2 \in X_{\mathbf{Sp}(X)}$,

$$\sigma_1 \rightsquigarrow_{X_{\mathbf{Sp}(X)}} \sigma_2 \Rightarrow \mathrm{ev}^{-1}(\sigma_1) \underset{X}{\rightsquigarrow} \mathrm{ev}^{-1}(\sigma_2).$$

Since $\mathrm{ev}^{-1}(\sigma_i)$ is the unique $x_i \in X$ such that $ev_{x_i} = \sigma_i$, this is equivalent to

$$ev_{x_1} \rightsquigarrow_{X_{\mathbf{Sp}(X)}} ev_{x_2} \Rightarrow x_1 \underset{X}{\rightsquigarrow} x_2 .$$

Assume $x_1 \not\rightsquigarrow_X x_2$, i.e., $x_2 \notin \overline{\{x_1\}}$. Then, there is a quasi-compact open $U \subseteq X$ such that $x_2 \in U$ and $x_1 \notin U$. Define $f : X \longrightarrow \mathbf{3}$ by:

$$f(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U\,. \end{cases}$$

Since $f^{-1}[1] = U$, $f^{-1}[-1] = \emptyset$ are quasi-compact open, $f \in \mathrm{Sp}(X)$, and, clearly, $f(x_2) = 1$, $f(x_1) = 0$, which shows $f \in (ev_{x_2})^{-1}[1] \setminus (ev_{x_1})^{-1}[1]$. Hence, $(ev_{x_2})^{-1}[1] \not\subseteq (ev_{x_1})^{-1}[1]$. From Lemma I.1.18 it follows that $ev_{x_1} \not\rightsquigarrow_{X_{\mathbf{Sp}(X)}} ev_{x_2}$, as required.

(2) is an immediate consequence of (1). □

V.4 The spectral hull of a real semigroup. Idempotency.

We now take on a reverse tack, consisting in applying the construction of the spectral real semigroup $\mathbf{Sp}(X)$ to the case where X is the character space X_G of a given real semigroup G. The result will be a real semigroup $\mathrm{Sp}(G)$ extending G and having the functorial properties of a hull for G. This *spectral hull* is at least as interesting as the Post hull introduced in § IV.4, insofar it gives an algebraic counterpart of properties of the spectral topology of X_G, not only of its constructible topology, as the Post hull does; therefore it is more tightly connected than the latter with the given RS. As the Post hull, the spectral hull construction turns out to be *idempotent*, i.e., its iteration does not produce a larger RS. □

Definition and Notation V.4.1 Let G be a RS and let X_G be its character space. By [M], Prop. 6.4.1, p. 114, X_G is a hereditarily normal spectral space.

(*i*) We define $\mathrm{Sp}(G)$ to be the real semigroup $\mathbf{Sp}(X_G)$ (see V.1.5).

(*ii*) We denote by η_G the map of G into $\mathbf{3}^{X_G}$ defined by evaluation at elements $g \in G$:

$$\eta_G(g) := \mathrm{ev}_g : X_G \longrightarrow \mathbf{3} \text{ where, for } \sigma \in X_G,\ \mathrm{ev}_g(\sigma) := \sigma(g). \qquad \square$$

Proposition V.4.2 *Let G be a RS.*

(*i*) *For all $g \in G$, ev_g is a spectral map, i.e., $\eta_G(g) \in \mathrm{Sp}(G)$.*

(*ii*) *η_G is a real semigroup homomorphism.*

(*iii*) *η_G is obtained from the Post hull $\varepsilon_G : G \longrightarrow P_G$ of G (§ IV.4) by restriction of the counterdomain to* $\mathrm{Sp}(G)$.

In particular,

(*iv*) *η_G is a complete embedding (cf. IV.4.3).*

(*v*) *η_G is injective.*

Remark. Concerning (iii), $\eta_G(g)$ considers the evaluation ev_g at $g \in G$ as a spectral map $X_G \longrightarrow \mathbf{3}_{\mathrm{sp}}$, while ε_G considers it as a continuous map $(X_G)_{\mathrm{con}} \longrightarrow \mathbf{3}$; cf. V.1.1 (3.i).

Proof. (i) Same argument as for item (2) in Proposition V.3.4.

(ii) This is straightforward checking using the fact that the constants, product and representation in $\mathrm{Sp}(G)$ are pointwise defined. Details are left to the reader.

(iii) It suffices to compare the definitions of η_G and ε_G.

(iv) We must show (IV.4.3) that, for forms φ, ψ over G,

(*) $\quad \eta_G * \varphi \cong_{\mathrm{Sp}(G)} \eta_G * \psi \;\Rightarrow\; \varphi \cong_G \psi\,.$

Since $\eta_G = \varepsilon_G$ and

(**) $\cong_{\mathrm{Sp}(G)}$ is identical to $\cong_{P_G}$ on forms over G,

the assumption of (*) implies $\varepsilon_G * \varphi \cong_{P_G} \varepsilon_G * \psi$. By Corollary IV.4.7 (3) this, in turn, yields $\varphi \cong_G \psi$, as required.

Proof of $(**)$. By the definition of $\cong$, see I.2.7 (c), the assertion follows by straightforward computation from:

— $\mathrm{X}_{\mathrm{Sp}(G)}$ is in bijective correspondence with X_G (by the map ev, see V.3.4 and V.3.5);

— X_{P_G} is in bijective correspondence with X_G (by the map ε_G^*, see IV.4.7 (1)).

Note. Alternatively (and equivalently) we may have used Corollary V.5.5 below.

(v) Complete embeddings are injective (IV.4.4 (c)). $\square$

Definition and Notation V.4.3 (a) The map $\eta_G : G \longrightarrow \mathrm{Sp}(G)$ —or, more precisely, the pair $(\mathrm{Sp}(G), \eta_G)$— will be called the **spectral hull** of G.

In the sequel we consider the effect of the spectral hull on RS-morphisms.

(b) Any RS-homomorphism $f : G \longrightarrow H$ gives raise, by composition, to a dual map $f^* : X_H \longrightarrow X_G$: for $\gamma \in X_H$,

$$f^*(\gamma) := \gamma \circ f : G \longrightarrow \mathbf{3}\,.$$

Clearly, $f^*(\gamma) \in X_G$.

Fact V.4.4 *$f^* : X_H \longrightarrow X_G$ is a spectral map.*

Proof. It suffices to check that, for all $g \in G$, $f^{*-1}[\,[\![g = 1]\!]\,]$ is quasi-compact open in X_H. For $\gamma \in X_H$ we have,

$$\begin{aligned} \gamma \in f^{*-1}[\,[\![g = 1]\!]\,] \;&\Leftrightarrow\; f^*(\gamma) \in [\![g = 1]\!] \;\Leftrightarrow\; \gamma \circ f \in [\![g = 1]\!] \;\Leftrightarrow\; (\gamma \circ f)(g) = 1 \\ &\Leftrightarrow\; \gamma(f(g)) = 1 \;\Leftrightarrow\; \gamma \in [\![f(g) = 1]\!]\,, \end{aligned}$$

i.e., $f^{*-1}[\,[\![g = 1]\!]\,] = [\![f(g) = 1]\!]$, as needed. $\square$

(c) In the preceding setup we define, as in V.3.1, a map $\mathrm{Sp}(f) : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(H)$ again by composition: for $g \in \mathrm{Sp}(G)$, $\mathrm{Sp}(f)(g) := g \circ f^*$. By V.4.4, $g \circ f^* \in \mathrm{Sp}(H)$. $\square$

Next we show that the operator $\mathbf{Sp}$ is idempotent.

Theorem V.4.5 (Idempotency of $\mathbf{Sp}$) *Let X be a hereditarily normal spectral space. Then, $\eta_{\mathbf{Sp}(X)} : \mathbf{Sp}(X) \longrightarrow \mathrm{Sp}(\mathbf{Sp}(X))$ is an isomorphism of real semigroups.*

Proof. With Proposition V.4.2 applied with $G = \mathbf{Sp}(X)$, it only remains to prove:

(a) $\eta_{\mathbf{Sp}(X)}$ is surjective.

(b) $\eta^{-1}_{\mathbf{Sp}(X)}$ is a RS-homomorphism.

Proof of (a). Since Sp is a contravariant functor (V.3.1, V.3.2), $\mathrm{ev}^{-1} \circ \mathrm{ev} = \mathrm{id}_X$, and $\mathrm{ev} \circ \mathrm{ev}^{-1} = \mathrm{id}_{\mathbf{Sp}(X)}$ imply $\mathrm{Sp}(\mathrm{ev}^{-1} \circ \mathrm{ev}) = \mathrm{Sp}(\mathrm{ev}) \circ \mathrm{Sp}(\mathrm{ev}^{-1}) = \mathrm{id}_{\mathbf{Sp}(X)}$ and $\mathrm{Sp}(\mathrm{ev} \circ \mathrm{ev}^{-1}) = \mathrm{Sp}(\mathrm{ev}^{-1}) \circ \mathrm{Sp}(\mathrm{ev}) = \mathrm{id}_{\mathrm{Sp}(\mathbf{Sp}(X))}$. Hence, for $f \in \mathrm{Sp}(\mathbf{Sp}(X))$,

(*) $f = \mathrm{Sp}(\mathrm{ev}^{-1})(\mathrm{Sp}(\mathrm{ev})(f)) = \mathrm{Sp}(\mathrm{ev}^{-1})(f \circ \mathrm{ev})$,

cf. V.3.1. Then, it suffices to show:

(c) $\eta_{\mathbf{Sp}(X)} = \mathrm{Sp}(\mathrm{ev}^{-1})$.

Proof of (c). We must show, for $b \in \mathrm{Sp}(X)$:

$$\mathrm{Sp}(\mathrm{ev}^{-1})(b) = b \circ \mathrm{ev}^{-1} = \eta_{\mathbf{Sp}(X)}(b) = ev_b\,,$$

i.e., $(b \circ \mathrm{ev}^{-1})(\gamma) = ev_b(\gamma) = \gamma(b)$, for all $\gamma \in X_{\mathbf{Sp}(X)}$.

By definition, $\mathrm{ev}^{-1}(\gamma) =$ the unique $x \in X$ such that $\mathrm{ev}(x) = ev_x = \gamma$. Then, $\gamma(b) = ev_x(b) = b(x)$, and $b(x) = b(\mathrm{ev}^{-1}(\gamma))$, i.e., $(b \circ \mathrm{ev}^{-1})(\gamma) = \gamma(b)$, as required. □

Proof of (b). In the proof of (a) we noted that

$$\mathrm{Sp}(\mathrm{ev})^{-1} = \mathrm{Sp}(\mathrm{ev}^{-1}) \quad \text{and} \quad \mathrm{Sp}(\mathrm{ev}^{-1})^{-1} = \mathrm{Sp}(\mathrm{ev})\,.$$

This, together with (c), gives:

$$\eta^{-1}_{\mathbf{Sp}(X)} = \mathrm{Sp}(\mathrm{ev}^{-1})^{-1} = \mathrm{Sp}(\mathrm{ev})\,.$$

Since $\mathrm{Sp}(\varphi)$ is a RS-homomorphism for any spectral map φ (V.3.2), Proposition V.3.4 yields that $\eta^{-1}_{\mathbf{Sp}(X)}$ is a RS-homomorphism, proving (b) and Theorem V.4.5. □

Theorem V.4.5 can be restated as follows:

Corollary V.4.6 *Let G be a* spectral *RS (V.1.8). Then, the map $\eta_G : G \longrightarrow \mathrm{Sp}(G)$ is an isomorphism. In other words, every spectral RS is canonically isomorphic to its spectral hull (the converse is obviously true).* □

An easy consequence of V.3.6 is:

Corollary V.4.7 *Let G be a real semigroup. Then, G and $\mathrm{Sp}(G)$ have the same Post hull: $P_G \simeq P_{\mathrm{Sp}(G)}$.*

Proof. Corollary V.3.6 tells that $\mathrm{ev} : X_G \longrightarrow X_{\mathrm{Sp}(G)}$ is a homeomorphism for the respective constructible topologies. For $f \in P_{\mathrm{Sp}(G)}$, i.e., $f : X_{\mathrm{Sp}(G)} \longrightarrow \mathbf{3}$ continuous in $(X_{\mathrm{Sp}(G)})_{\mathrm{con}}$, set $\theta(f) := f \circ \mathrm{ev}$. Clearly $\theta(f) \in P_G$, and it is a routine exercise to check that $\theta : P_{\mathrm{Sp}(G)} \longrightarrow P_G$ is an isomorphism of Post algebras. □

V.5 An anti-equivalence of categories.

Our main result in this section is the anti-equivalence of the categories **HNSS** and **SRS** (Theorem V.5.4). The commutativity of diagrams required for this result are proven in V.5.1 and V.5.3. These results also have further important consequences, such as:

(i) The duality of the functors * and **Sp**, and

(ii) Uniqueness of the extension $\mathrm{Sp}(f) : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(H)$ (V.4.3) of any RS-homomorphism $f : G \longrightarrow H$ ($G, H \models \mathrm{RS}$); in particular, unique extension of any RS-character of G to $\mathrm{Sp}(G)$.

Example V.5.6 gives a simple illustration of how a RS sits inside its spectral hull.

Finally, V.5.7 is an analog to Theorem IV.4.5 for spectral real semigroups. □

Proposition V.5.1 *Let X, Y be hereditarily normal spectral spaces and let $\varphi : X \longrightarrow Y$ be a spectral map. Let $\mathrm{ev}_X : X \longrightarrow X_{\mathbf{Sp}(X)}$ and $\mathrm{ev}_Y : Y \longrightarrow X_{\mathbf{Sp}(Y)}$ denote the bijections given by V.3.4 and V.3.5. The following diagram is commutative:*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\mathrm{ev}_X"'] & Y \arrow[d, "\mathrm{ev}_Y"] \\ X_{\mathbf{Sp}(X)} \arrow[r, "\mathrm{Sp}(\varphi)^*"'] & X_{\mathbf{Sp}(Y)} \end{tikzcd}$$

Proof. Recall that, for $x \in X$, $\mathrm{ev}_X(x) = ev_x : \mathrm{Sp}(X) \longrightarrow \mathbf{3}$; likewise, $\mathrm{ev}_Y(\varphi(x)) = ev_{\varphi(x)} : \mathrm{Sp}(Y) \longrightarrow \mathbf{3}$. By the definition of * (V.4.3 (b)),

$$(\mathrm{Sp}(\varphi)^* \circ \mathrm{ev}_X)(x) = \mathrm{Sp}(\varphi)^*(ev_x) = ev_x \circ \mathrm{Sp}(\varphi),$$

and

$$(\mathrm{ev}_Y \circ \varphi)(x) = \mathrm{ev}_Y(\varphi(x)) = ev_{\varphi(x)} .$$

Thus, we must check that $ev_{\varphi(x)} = ev_x \circ \mathrm{Sp}(\varphi)$. Let $b \in \mathrm{Sp}(Y)$; then, $\mathrm{Sp}(\varphi)(b) = b \circ \varphi$, and we get:

$$(ev_x \circ \mathrm{Sp}(\varphi))(b) = ev_x(\mathrm{Sp}(\varphi)(b)) = ev_x(b \circ \varphi) = (b \circ \varphi)(x) = b(\varphi(x)) = ev_{\varphi(x)}(b)\,,$$

as required. □

Next we fix a $G \models \mathrm{RS}$ and set $X = X_G$. We rebaptize $\mathrm{ev}_G : X_G \longrightarrow X_{\mathrm{Sp}(G)}$ the map ev_{X_G} considered above, i.e., $\mathrm{ev}_G(\sigma) := ev_\sigma$, for $\sigma \in X_G$. Then,

Fact V.5.2 *With notation as above, the following identities hold:*

(i) $\mathrm{ev}_G^{-1} = \eta_G^*$.

(ii) $\mathrm{Sp}(\eta_G^*) = \eta_{\mathrm{Sp}(G)}$.

Hence,

(iii) η_G^* *is injective.*

Proof. (i) Fix $\gamma \in X_{\mathrm{Sp}(G)}$. Since $\mathrm{ev}_G^{-1}(\gamma) =$ the unique $\sigma \in X_G$ so that $\gamma = \mathrm{ev}_G(\sigma) = ev_\sigma$, and $\eta_G^*(\gamma) = \gamma \circ \eta_G$, the identity to be proved boils down to showing that $ev_\sigma \circ \eta_G = \sigma$, for $\sigma \in X_G$. Let $g \in G$; since $\eta_G(g) = ev_g$, we get

$$ev_\sigma(\eta_G(g)) = \eta_G(g)(\sigma) = ev_g(\sigma) = \sigma(g)\,,$$

as wanted.

(ii) From item (c) in the proof of idempotency (V.4.5) with $X = X_G$, we have

$$\eta_{\mathrm{Sp}(G)} = \eta_{\mathbf{Sp}(X_G)} = \mathrm{Sp}(\mathrm{ev}_G^{-1})\,,$$

whence the result follows at once from (i).

(iii) Clear, from V.3.4, V.3.5 and item (i). [Alternatively, one may invoke Proposition V.4.2 (iv) and Theorem V.5.7.] □

Next we prove an analog of Thm. 4.17 of [DM1] (and of Theorem IV.4.2 above), a result of crucial importance:

Theorem V.5.3 *(i) Let $f : G \longrightarrow H$ be a homomorphism of real semigroups. Then* $\mathrm{Sp}(f)$ *(defined in V.4.3 (c)) is the unique RS-homomorphism $F : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(H)$ making the following diagram commute:*

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "\eta_G"'] \arrow[dr, phantom, "\text{[D]}"] & H \arrow[d, "\eta_H"] \\
\mathrm{Sp}(G) \arrow[r, "F"'] & \mathrm{Sp}(H)
\end{tikzcd}$$

(*ii*) *Let* G *be a RS. Then* $\mathrm{Sp}(G)$ *is a hull for* G *in the category* **SRS** *of spectral real semigroups. That is, every RS-morphism* $f : G \longrightarrow \mathbf{Sp}(X)$, X *a hereditarily normal spectral space, factors* uniquely *through* $\mathrm{Sp}(G)$, *i.e., there is a unique RS-morphism* $h : \mathrm{Sp}(G) \longrightarrow \mathbf{Sp}(X)$ *making the following diagram commute:*

$$\begin{tikzcd}
G \arrow[rr, "\eta_G"] \arrow[dr, "f"'] & & \mathrm{Sp}(G) \arrow[dl, "h"] \\
& \mathbf{Sp}(X) &
\end{tikzcd}$$

Proof. (i) We first show that with $F = \mathrm{Sp}(f)$ diagram [D] commutes, i.e., $\eta_H(f(g)) = \mathrm{Sp}(f)(\eta_G(g))$, for all $g \in G$. By the definition of η_G, η_H, and with f^* defined in V.4.3 (b) , this amounts to $ev_{f(g)} = ev_g \circ f^*$. For $\gamma \in X_H$ we have $ev_{f(g)}(\gamma) = \gamma(f(g))$, and $(ev_g \circ f^*)(\gamma) = ev_g(f^*(\gamma)) = ev_g(\gamma \circ f) = \gamma(f(g))$.

For uniqueness, let $F_1, F_2 : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(H)$ be RS-homomorphisms making diagram [D] commute. Applying the functor * to this square we get a commutative diagram

$$\begin{tikzcd}
X_G & X_H \arrow[l, "f^*"'] \\
X_{\mathrm{Sp}(G)} \arrow[u, "\eta_G^*"] & X_{\mathrm{Sp}(H)} \arrow[u, "\eta_H^*"'] \arrow[l, shift right, "F_1^*"'] \arrow[l, shift left, "F_2^*"]
\end{tikzcd}$$

whence $\eta_G^* \circ F_1^* = \eta_G^* \circ F_2^*$ $(= f^* \circ \eta_H^*)$. Since η_G^* is injective (V.5.2 (iii)), $F_1^* = F_2^*$, and we show this entails $F_1 = F_2$.

In fact, if $F_1(g) \neq F_2(g)$ for some $g \in \mathrm{Sp}(G)$, since $X_{\mathrm{Sp}(H)}$ separates points in $\mathrm{Sp}(H)$, there is $\gamma \in X_{\mathrm{Sp}(H)}$ so that $(\gamma \circ F_1)(g) \neq (\gamma \circ F_2)(g)$, i.e., $\gamma \circ F_1 \neq \gamma \circ F_2$. By definition $F_i^*(\gamma) = \gamma \circ F_i$, so we get $F_1^*(\gamma) \neq F_2^*(\gamma)$, contradiction.

(ii) Use the commutative square [D] of (i) with $H = \mathbf{Sp}(X)$ and $f : G \longrightarrow \mathbf{Sp}(X)$ the given map. By the idempotency theorem V.4.5, $\eta_H : H \longrightarrow \mathrm{Sp}(H)$ is an isomorphism of real semigroups. Setting $h := \eta_H^{-1} \circ \mathrm{Sp}(f)$ proves the commutativity of the triangle in (ii). Uniqueness is clear from that in (i). $\square$

Putting together some of the preceding results we obtain the anti-equivalence of the categories **HNSS** and **SRS**. This is expressed in rather compact form, using category-theoretic language, by the following:

Theorem V.5.4 (Anti-equivalence theorem) *The functor* $\mathbf{Sp} : \mathbf{HNSS} \longrightarrow \mathbf{SRS}$ *assigning to each hereditarily normal spectral space* X *the real semigroup* $\mathbf{Sp}(X)$ *is an anti-equivalence of categories. Its quasi-inverse is the functor* $\mathbf{ARS} : \mathbf{SRS} \longrightarrow \mathbf{HNSS}$ *assigning to each* $G \in \mathbf{SRS}$ *its associated abstract real spectrum* X_G. *The natural transformations establishing this anti-equivalence are as follows:*

(1) *The isomorphism* $\mathrm{Id}_{\mathbf{HNSS}} \longmapsto \mathbf{ARS} \circ \mathbf{Sp}$ *is the natural transformation that sends* $X \in \mathbf{HNSS}$ *to the homeomorphism* $\mathrm{ev} : X \longrightarrow \mathbf{Sp}(X)$.

(2) *The isomorphism* $\mathrm{Id}_{\mathbf{SRS}} \longmapsto \mathbf{Sp} \circ \mathbf{ARS}$ *is the natural transformation that sends a spectral RS, G, to the isomorphism* $\eta_G : G \longrightarrow \mathrm{Sp}(G)$.

Proof. (1) Given a spectral map $\varphi : X \longrightarrow Y$, with $X, Y \in \mathbf{HNSS}$, commutativity of the diagram in Proposition V.5.1 and the fact that the maps $\mathrm{ev}_X : X \longrightarrow X_{\mathbf{Sp}(X)}$ and $\mathrm{ev}_Y : Y \longrightarrow X_{\mathbf{Sp}(Y)}$ are homeomorphisms of spectral spaces (V.3.7 (2)), prove this assertion.

(2) Given a RS-morphism $f : G \longrightarrow H$, where $G, H \in \mathbf{SRS}$, commutativity of diagram [D] in Theorem V.5.3 (i) together with the fact that the canonical embeddings η_G, η_H are isomorphisms (V.4.6), prove this assertion. □

Corollary V.5.5 *Let G be a RS. Then every $\sigma \in X_G$ extends* uniquely *to a RS-character of* $\mathrm{Sp}(G)$.

Proof. Follows from V.5.3 (ii) by taking $X = \mathbf{1}$ (= the singleton spectral space) and observing that $\mathrm{Sp}(\mathbf{1}) = \mathbf{3}$ (see proof of Corollary V.2.2).

Explicitly, the extension $\widehat{\sigma} : \mathrm{Sp}(G) \longrightarrow \mathbf{3}$ of a RS-character $\sigma \in X_G$ is defined by evaluation at σ: for $f \in \mathrm{Sp}(G)$, $\widehat{\sigma}(f) := f(\sigma)$. The reader can readily check that $\widehat{\sigma} \in X_{\mathrm{Sp}(G)}$ and $\widehat{\sigma} \circ \eta_G = \sigma$ (i.e., $\widehat{\sigma} \lceil G = \sigma$ with G canonically embedded into $\mathrm{Sp}(G)$ via η_G). □

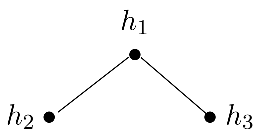
Remark. The uniqueness statements in Theorem V.5.3 and Corollary V.5.5 indicate that a real semigroup "generates" its spectral hull. Below (Theorem V.6.2) we will show that it generates it as a lattice. □

Example V.5.6 Here is a simple example illustrating the way in which a real semigroup sits inside its spectral hull. We compute the spectral hull of the "free" fan on one generator, presented in VI.3.2. A:

$$F = \{1, 0, -1, x, -x, x^2, -x^2\}.$$

— Firstly, F is represented by the seven elements of the form ev_a, $a \in F$.

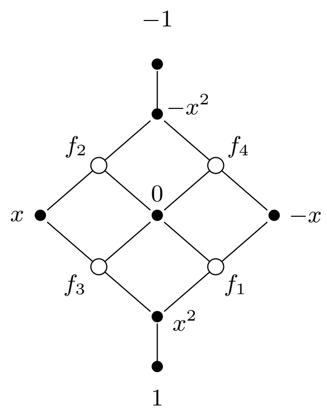
— Besides these, $\mathrm{Sp}(F)$ contains four other elements. Indeed, X_F has the shape:



where $h_1(x) = 0$, $h_2(x) = 1$, $h_3(x) = -1$ (the order being specialization). Since the constructible topology is discrete, the spectral characters are the maps of X_F into $\mathbf{3}_{\mathrm{sp}}$ that preserve the specialization order. Note, further, that if a spectral character sends h_2 or h_3 to 0, then it must also send h_1 to 0. Verification by hand shows, then, that $\mathrm{Sp}(F)$ (= $\mathrm{Sp}(X_F)$) contains exactly the following additional maps:

$$f_1 : \begin{cases} h_1 \mapsto 0 \\ h_2 \mapsto 0 \\ h_3 \mapsto 1, \end{cases} \qquad f_2 : \begin{cases} h_1 \mapsto 0 \\ h_2 \mapsto 0 \\ h_3 \mapsto -1, \end{cases} \qquad f_3 : \begin{cases} h_1 \mapsto 0 \\ h_2 \mapsto 1 \\ h_3 \mapsto 0, \end{cases} \qquad f_4 : \begin{cases} h_1 \mapsto 0 \\ h_2 \mapsto -1 \\ h_3 \mapsto 0, \end{cases}$$

and looks as follows:



Note that, while $\mathrm{Sp}(F)$ has 11 elements, the Post hull of F has $3^3 = 27$. $\square$

Our last result in this section is an analog to Theorem IV.4.5; it gives, in the context of spectral real semigroups, several characterizations of the surjectivity of the dual map f^*.

Theorem V.5.7 *Let G, H be real semigroups, and let $f : G \longrightarrow H$ be a RS-morphism. With f^* denoting the dual of f and $\mathrm{Sp}(f)$ its spectral extension (V.4.3(b), (c)), the following are equivalent:*

(1) *f^* is surjective.*

(2) *$\mathrm{Im}(f^*)$ is dense in $(X_G)_{\mathrm{con}}$ (the constructible topology of X_G).*

(3) *$\mathrm{Sp}(f)$ is injective.*

(4) *For every Pfister form φ over G and every $a \in G$,*

$$f(a) \in D_H(f * \varphi) \;\Rightarrow\; a \in D_G(\varphi)\,.$$

(5) *f is a complete embedding (cf. IV.4.3).*

Sketch of proof. The proof is similar to that of Theorem IV.4.5; we only indicate the modifications needed therein. Recall f^* is a spectral map (V.4.4). (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). By Cor. 6.0.2 of [DST], $\mathrm{Im}(f^*)$ is a proconstructible subset of X_G, i.e., closed in $(X_G)_{\mathrm{con}}$. This, together with (2), implies (1) at once.

(1) $\Rightarrow$ (3). Assume there are $g_1, g_2 \in \mathrm{Sp}(X_G)$ so that $g_1 \neq g_2$ but $\mathrm{Sp}(f)(g_1) = \mathrm{Sp}(f)(g_2)$, i.e., $g_1 \circ f^* = g_2 \circ f^*$ and there is $\sigma \in X_G$ such that $g_1(\sigma) \neq g_2(\sigma)$. Since f^* is assumed surjective, there is $\gamma \in X_H$ such that $f^*(\gamma) = \sigma$. Then,

$$(g_1 \circ f^*)(\gamma) = g_1(f^*(\gamma)) = g_1(\sigma) \neq g_2(\sigma) = (g_2 \circ f^*)(\gamma),$$

contradiction.

(3) $\Rightarrow$ (1). The proof is as that of (3) $\Rightarrow$ (1) in Theorem IV.4.5, upon observing that:

— "surjective" = "epic" holds in the category $\mathtt{Spec}$ of spectral spaces with spectral maps; cf. [DST], Cor. 10.0.4;

— For a spectral map $\rho : X_G \longrightarrow X$ into a spectral space X, replace the map $\overline{\rho}$ in the proof of IV.4.5 by the map $\mathrm{Sp}(\rho)$ defined in V.3.1.

— To complete the proof, in order to show, for spectral maps $\rho_1, \rho_2 : X_G \longrightarrow X$, the implication $\mathrm{Sp}(\rho_1) = \mathrm{Sp}(\rho_2) \Rightarrow \rho_1 = \rho_2$, argue as follows: if $\rho_1 \neq \rho_2$, then $\rho_1(\sigma) \neq \rho_2(\sigma)$ for some $\sigma \in X_G$.

Since X is T_0, there is a quasi-compact open $U \subseteq X$ so that, say, $\rho_1(\sigma) \in U$ and $\rho_2(\sigma) \notin U$ (or the other way around). Let $h : X \longrightarrow \mathbf{3}$ be given by $h\lceil U = 1$ and $h\lceil(X \setminus U) = 0$; h is spectral, i.e., $h \in \mathrm{Sp}(X)$, and $h(\rho_1(\sigma)) = 1 \neq 0 = h(\rho_2(\sigma))$, i.e., $\mathrm{Sp}(\rho_1)(h) \neq \mathrm{Sp}(\rho_2)(h)$, contradiction.

The proofs of (1) $\Rightarrow$ (4), (1) $\Rightarrow$ (5), (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (2) are similar to those of (1) $\Rightarrow$ (5), (1) $\Rightarrow$ (6), (5) $\Rightarrow$ (2) and (6) $\Rightarrow$ (2) in IV.4.5, respectively. □

Corollary V.5.8 *Any injective RS-morphism $f : G \longrightarrow H$ of spectral RSs, G, H, is a complete embedding.*

Proof. We use the commutative diagram [D] of Theorem V.5.3 (i) with $F = \mathrm{Sp}(f)$. Since G, H are spectral, the embeddings η_G and η_H are isomorphisms (V.4.6); hence $\mathrm{Sp}(f)$ is injective. By V.5.7, f is a complete embedding. □

V.6 The distributive lattice structure of spectral real semigroups.

In this section we prove two results concerning the (pure) lattice structure of spectral RSs: firstly, that any real semigroup generates its spectral hull as a lattice (Theorem V.6.2) ; secondly, that the spectral real semigroups are exactly those real semigroups whose representation partial order (I.6.2) is a *distributive* lattice (Theorem V.6.6). □

We start by checking the following simple, but important property of the lattice structure of spectral real semigroups:

Proposition V.6.1 *Let G, H be RSs, and let $f : G \longrightarrow H$ be a RS homomorphism. Then the spectral extension* $\mathrm{Sp}(f) : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(H)$ *of f is a lattice homomorphism: for $g_1, g_2 \in \mathrm{Sp}(G)$,*

$$\mathrm{Sp}(f)(g_1 \wedge g_2) = \mathrm{Sp}(f)(g_1) \wedge \mathrm{Sp}(f)(g_2) \quad \textit{and} \quad \mathrm{Sp}(f)(g_1 \vee g_2) = \mathrm{Sp}(f(g_1) \vee \mathrm{Sp}(f)(g_2)\,.$$

Proof. Recall that $\mathrm{Sp}(f)(g) = g \circ f^*$ for $g \in \mathrm{Sp}(G)$. Hence, for $\gamma \in X_H$ we have $(g \circ f^*)(\gamma) = g(f^*(\gamma)) = g(\gamma \circ f)$, and:

$$((g_1 \vee g_2) \circ f^*)(\gamma) = (g_1 \vee g_2)(\gamma \circ f) = \max\{\, g_1(\gamma \circ f),\, g_2(\gamma \circ f)\} = (g_1 \circ f^*)(\gamma) \vee (g_2 \circ f^*)(\gamma)\,,$$

proving the second equality in the statement. Similar argument for the first. □

Remark. Obviously, the constants $1, 0, -1$ of $\mathrm{Sp}(G)$ and $\mathrm{Sp}(H)$ correspond to each other under $\mathrm{Sp}(f)$; so, $\mathrm{Sp}(f)$ is a homomorphism of bounded lattices. □

We shall now prove that any real semigroup generates its spectral hull as a lattice.[2]

Theorem V.6.2 *Let G be a RS. Then, for every $f \in \mathrm{Sp}(G)$ there is a finite collection $\{F_i \mid i \in I\}$ of finite subsets $F_i \subseteq G$ so that $f = \bigvee_{i \in I} \bigwedge_{g \in F_i} \eta_G(g)$; i.e., $\mathrm{Sp}(G)$ is generated as a lattice by $\mathrm{Im}\,(\eta_G)$.*

Proof. Recall that the family $\{[\![\, x = 1 \,]\!] \mid x \in G\}$ is a subbasis for the spectral topology of X_G. Throughout this proof "basis" means the basis generated by this subbasis, i.e., the collection of all finite intersections of sets of the form $[\![\, x = 1 \,]\!]$ with $x \in G$. To ease notation we write $\widehat{g}$ for $\eta_G(g) = ev_g$ $(g \in G)$.

Let L denote the sublattice of $\mathrm{Sp}(G)$ generated by $\mathrm{Im}\,(\eta_G)$, and fix $f \in \mathrm{Sp}(G)$. We split the proof that $f \in L$ into two cases.

[2] For a similar result concerning the Boolean hull of a reduced special group, see [DM1], Prop. 4.10 (b).

Case I. Both $f^{-1}[1]$ and $f^{-1}[-1]$ are basic opens of X_G.

That is,

(*) $$f^{-1}[1] = \bigcap_{i=1}^{k} [\![a_i = 1]\!] \quad \text{and} \quad f^{-1}[-1] = \bigcap_{j=1}^{n} [\![b_j = 1]\!],$$

for some $a_1, \ldots, a_k, b_1, \ldots, b_n \in G$.

Since $D^t_G(\cdot,\cdot) \neq \emptyset$ (Proposition I.2.3 (14)) for each $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,n\}$ we pick an element $t_{ij} \in D^t_G(a_i, -b_j)$ and consider the following element of L:

(**) $$p := \left[\left(\bigvee_{i=1}^{k} \widehat{a_i}\right) \wedge \widehat{0}\right] \vee \left[\left(\bigwedge_{j=1}^{n} \widehat{-b_j}\right) \wedge \bigvee_{i=1}^{k} \bigwedge_{j=1}^{n} \widehat{t_{ij}}\right].$$

Claim. $p = f$. Hence, $f \in L$.

Proof of Claim. To abridge we set $r := \left(\bigvee_{i=1}^{k} \widehat{a_i}\right) \wedge \widehat{0}$ and $s := \left(\bigwedge_{j=1}^{n} \widehat{-b_j}\right) \wedge \bigvee_{i=1}^{k} \bigwedge_{j=1}^{n} \widehat{t_{ij}}$ in (**). The proof proceeds by cases according to the values of f. Let $\sigma \in X_G$.

— $f(\sigma) = 1$.

By (*), $\sigma(a_i) = \widehat{a_i}(\sigma) = 1$ for all $1 \leq i \leq k$, whence $r(\sigma) = 1$. On the other hand, since $f^{-1}[1]$ and $f^{-1}[-1]$ are disjoint, by (*) there is a $j_0 \in \{1,\ldots,n\}$ so that $\sigma(b_{j_0}) \in \{0,-1\}$. Fix $i \in \{1,\ldots,k\}$. Since $t_{ij_0} \in D^t_G(a_i, -b_{j_0})$ and $-\sigma(b_{j_0}) \in \{0,1\}$, we have $\sigma(t_{ij_0}) \in D^t_{\mathbf{3}}(\sigma(a_i), -\sigma(b_{j_0})) = D^t_{\mathbf{3}}(1, -\sigma(b_{j_0})) = \{1\}$ (cf. I.2.5); that is, $\widehat{t_{ij_0}}(\sigma) = \sigma(t_{ij_0}) = 1$ for all i. It follows that $s(\sigma) = 1$, and hence $p(\sigma) = 1$.

— $f(\sigma) = -1$.

By (*), $\sigma(b_j) = \widehat{b_j}(\sigma) = 1$ for all $1 \leq j \leq n$, i.e., $\bigwedge_{j=1}^{n} \widehat{-b_j}(\sigma) = -1$. Since the sets in (*) are disjoint, there is a $i_0 \in \{1,\ldots,k\}$ so that $\sigma(a_{i_0}) \in \{0,-1\}$. Fix $j \in \{1,\ldots,n\}$. Since $t_{i_0j} \in D^t_G(a_{i_0}, -b_j)$ and $-\sigma(b_j) \in \{0,-1\}$, we get $\sigma(t_{i_0j}) \in D^t_{\mathbf{3}}(\sigma(a_{i_0}), -\sigma(b_j)) = D^t_{\mathbf{3}}(\sigma(a_{i_0}), -1) = \{-1\}$ (cf. I.2.5); that is, $\widehat{t_{i_0j}}(\sigma) = \sigma(t_{i_0j}) = -1$ for all $j \in \{1,\ldots,n\}$, which shows that $s(\sigma) = -1$, and hence $p(\sigma) = -1$.

— $f(\sigma) = 0$.

In this case, $\sigma \notin f^{-1}[1] \cup f^{-1}[-1]$, and (*) implies there are indices $i_0 \in \{1,\ldots,k\}$ and $j_0 \in \{1,\ldots,n\}$ so that $\sigma(a_{i_0}) \neq 1$ and $\sigma(b_{j_0}) \neq 1$. Then, we have $\sigma(a_{i_0}) \in \{0,-1\}$, whence $(\widehat{a_{i_0}} \wedge \widehat{0})(\sigma) = 0$, and therefore $r(\sigma) = 0$. Likewise, $\sigma(b_{j_0}) \in \{0,-1\}$ yields $\widehat{-b_{j_0}}(\sigma) \in \{0,1\}$, whence $(\bigwedge_{j=1}^{n} \widehat{-b_j})(\sigma) \leq 0$, which in turn gives $s(\sigma) \leq 0$. These evaluations together entail $p(\sigma) = 0$, ending the proof of the Claim. □

Case II. $f^{-1}[\pm 1]$ are arbitrary quasi-compact opens.

Then, there are basic quasi-compact opens $V_1, \ldots, V_k, U_1, \ldots, U_n$, so that

$$f^{-1}[1] = \bigcup_{i=1}^{k} V_i \quad \text{and} \quad f^{-1}[-1] = \bigcup_{j=1}^{n} U_j.$$

For each pair of indices $i \in \{1,\ldots,k\}$, $j \in \{1,\ldots,n\}$ we define a map $f_{ij} : X_G \longrightarrow \mathbf{3}$ by: for $\sigma \in X_G$,

$$f_{ij}(\sigma) = \begin{cases} 1 & \text{if } \sigma \in V_i \\ -1 & \text{if } \sigma \in U_j \\ 0 & \text{if } \sigma \notin V_i \cup U_j \,. \end{cases}$$

Since $V_i \cap U_j = \emptyset$, f_{ij} is well defined; clearly, $f_{ij} \in \mathrm{Sp}(X_G)$ $(= \mathrm{Sp}(G))$. Since V_i, U_j are basic opens, Case I proves that each of the functions f_{ij} is in L. On the other hand, straightforward checking according to the values of f shows that $f = \bigvee_{j=1}^{n} \bigwedge_{i=1}^{k} f_{ij}$, entailing $f \in L$. This completes the proof of Theorem V.6.2. $\square$

Remark. In connection with the foregoing theorem, recall that the Post hull P_G of a real semigroup G is generated by G *as a Post algebra*, cf. IV.4.1 (iv), i.e., using the additional operations Δ and ∇, but in general *not as a lattice.* $\square$

Recalling that the lattice operations of $\mathrm{Sp}(G)$ are definable in the language $\mathcal{L}_{\mathrm{RS}}$ for real semigroups (V.2.1), we obtain:

Corollary V.6.3 *Let G be a RS. Then,* $\mathrm{Sp}(G) \subseteq \mathrm{dcl}_{\mathrm{RS}}(G, \mathbf{3}^{X_G})$, *the definitional closure of G in $\mathbf{3}^{X_G}$ for the language $\mathcal{L}_{\mathrm{RS}}$ (and $\mathbf{3}^{X_G}$ endowed with the pointwise defined $\mathcal{L}_{\mathrm{RS}}$-structure). In particular,* $\mathrm{Sp}(G)$ *is* rigid *over G: every $\mathcal{L}_{\mathrm{RS}}$-automorphism of $\mathbf{3}^{X_G}$ which pointwise fixes G is the identity on* $\mathrm{Sp}(G)$. $\square$

Remark. For the notion of *definitional closure* of a structure, see [H], pp. 134 ff.

Our last result in this section characterizes the spectral real semigroups as those real semigroups for which the representation partial order is a *distributive* lattice.

Warning. The essential point here is *distributivity.* In fact, there are other classes of real semigroups for which the representation partial order is a lattice (necessarily non-distributive); for example, the RS-fans, a class to be introduced and studied in Chapter VI, have this property; see Theorem VI.3.5 and Remark VI.3.6 (a). $\square$

As a preliminary step we prove:

Lemma V.6.4 *Let G be a RS. Assume that the representation partial order $\leq_G$ is a* distributive *lattice. Then, product in G coincides with symmetric difference: for $a, b \in G$,*

$$a \cdot b = (a \wedge -b) \vee (b \wedge -a) \ (:= a \,\underline{\triangle}\, b).$$

Proof. To ease notation we write $\leq$ for $\leq_G$. Since the RS-characters of G preserve representation and $-$, they are monotonous for $\leq$. The separation theorem I.5.4, and I.6.2 imply, for $x, y \in G$,

(*) $$x \leq y \;\Leftrightarrow\; \forall \sigma \in X_G\,(\sigma(x) \leq_{\mathbf{3}} \sigma(y)).$$

(a) $a \wedge -b,\ b \wedge -a \ \leq\ a \cdot b$. Hence, $a \,\underline{\triangle}\, b \leq a \cdot b$.

By symmetry it suffices to prove the first inequality. Using (*) we prove $\sigma(a \wedge -b) \leq \sigma(a\,b)$, for $\sigma \in X_G$.

— Suppose $\sigma(a \wedge -b) = 0$. By monotonicity, $\sigma(a), \sigma(-b) \geq 0$, i.e., $\sigma(a) \geq 0$ and $\sigma(b) \leq 0$. This implies that $\sigma(a\,b) \neq 1$, for $\sigma(a\,b) = 1$ implies that both $\sigma(a), \sigma(b)$ are either 1 or -1. Then, $\sigma(a \wedge -b) = 0 \leq \sigma(a\,b)$.

— If $\sigma(a \wedge -b) = -1$, by monotonicity, $\sigma(a) = \sigma(-b) = -1$, i.e., $\sigma(a) = -1$ and $\sigma(b) = 1$. Hence $\sigma(a\,b) = -1 = \sigma(a \wedge -b)$.

(b) $a\,b \le a \vee b,\ -a \vee -b$.

Let $\sigma \in X_G$. If $\sigma(a\,b) = 0$, then at least one of $\sigma(a), \sigma(b)$ is 0, say $\sigma(a) = 0$; by monotonicity, $\sigma(a\,b) = 0 = \sigma(a) \le \sigma(a \vee b)$ and $\sigma(a\,b) = 0 = \sigma(-a) \le \sigma(-a \vee -b)$. If $\sigma(a\,b) = -1$, then, say $\sigma(a) = -1$ and $\sigma(b) = 1$ (or the other way round). By monotonicity, $-1 = \sigma(a) \le \sigma(a \vee b)$ and $-1 = \sigma(-b) \le \sigma(-a \vee -b)$, as required.

The Lemma follows from (b). Indeed, using distributivity we have:

$$(**) \qquad a\,b \le (a \vee b) \wedge (-a \vee -b) = (a \wedge -a) \vee (a \wedge -b) \vee (b \wedge -a) \vee (b \wedge -b).$$

By I.6.5 (8) we have $x \wedge -x \le y \vee -y$ for all $x, y \in G$ (Kleene inequality). Using distributivity again:

$$a \wedge -a = (a \wedge -a) \wedge (b \vee -b) = (a \wedge -a \wedge b) \vee (a \wedge -a \wedge -b) \le (-a \wedge b) \vee (-b \wedge a),$$

and, similarly, $b \wedge -b \le (-a \wedge b) \vee (-b \wedge a)$. The last term in (**) then equals $(-a \wedge b) \vee (-b \wedge a) = a \,\underline{\triangle}\, b$, proving $a\,b \le a \,\underline{\triangle}\, b$, as asserted. $\square$

Corollary V.6.5 *Under the assumptions of Lemma V.6.4, the following holds for $a, b \in G$:*

(i) $a \cdot (a \vee b),\ b \cdot (a \vee b) \le a\,b$.

(ii) For all $x \in G$, $x \in D_G(a^2, b^2) \Leftrightarrow x = x^2$ and $a^2 \wedge b^2 \le x$.

(iii) $(a \vee b)^2 \in D_G(a^2, b^2)$.

Proof. (i) By the Lemma,

$$\begin{aligned} a \cdot (a \vee b) &= a \,\underline{\triangle}\, (a \vee b) = (a \wedge -(a \vee b)) \vee ((a \vee b) \wedge -a) = \\ &= (a \wedge -a \wedge -b)) \vee (a \wedge -a) \vee (b \wedge -a) = (a \wedge -a) \vee (b \wedge -a)\,. \end{aligned}$$

The Kleene inequality $a \wedge -a \le b \vee -b$ implies that $a \wedge -a \le a \,\underline{\triangle}\, b = a\,b$; indeed, by distributivity

$$\begin{aligned} a \wedge -a = (a \wedge -a) \wedge (b \vee -b) &= (a \wedge -a \wedge b)) \vee (a \wedge -a \wedge -b)) \le \\ &\le (-a \wedge b) \vee (a \wedge -b) = a \,\underline{\triangle}\, b\,. \end{aligned}$$

Since, clearly, $b \wedge -a \le a \,\underline{\triangle}\, b$, our contention follows.

The other inequality in (i) holds by symmetry.

Next, we prove, for $\sigma \in X_G$:

(*) $\sigma(a^2 \wedge b^2) = \sigma(a^2) \wedge \sigma(b^2)$.

Recall from Proposition I.6.8 (2) that $\{a^2 \wedge b^2\} = D^t_G(a^2, b^2)$ (any G). Since σ preserves D^t,

$$\sigma(a^2 \wedge b^2) \in D^t_{\mathbf{3}}(\sigma(a^2), \sigma(b^2)) = D^t_{\mathbf{3}}(\sigma(a)^2, \sigma(b)^2) = \{\sigma(a)^2 \wedge \sigma(b)^2\},$$

and (*) follows.

(ii) ($\Rightarrow$) Clearly, $x \in D_G(a^2, b^2)$ implies $x = x^2$. To show $a^2 \wedge b^2 \le x$ we check that $\sigma(a^2 \wedge b^2) \le \sigma(x)$ for all $\sigma \in X_G$; by (*) it suffices to verify $\sigma(a^2) \wedge \sigma(b^2) \le \sigma(x^2)$. But $\sigma(x^2) = 1$ implies $\sigma(x) \ne 0$, whence, from $x = x^2 \in D_G(a^2, b^2)$ follows $1 = \sigma(x^2) = \sigma(a^2)$ or $1 = \sigma(x^2) = \sigma(b^2)$, and hence $\sigma(a^2) \wedge \sigma(b^2) = 1$.

($\Leftarrow$) Conversely, from $x = x^2$ we get $\sigma(x) = \sigma(x^2) \in \{0, 1\}$ for $\sigma \in X_G$. If $\sigma(x) = 1$, then $a^2 \wedge b^2 \le x$ and (*) give $\sigma(a^2 \wedge b^2) = \sigma(a^2) \wedge \sigma(b^2) = 1$, and hence one of $\sigma(a^2)$ or $\sigma(b^2)$ is 1, proving that $x \in D_G(a^2, b^2)$.

(iii) By (ii) it suffices to prove $(a \vee b)^2 \geq a^2 \wedge b^2$. Invoking Lemma V.6.4 and using distributivity, we get:

$$\begin{aligned}(a \vee b)^2 &= (a \vee b) \,\underline{\triangle}\, (a \vee b) = (a \vee b) \wedge -(a \vee b) = (a \vee b) \wedge (-a \wedge -b) = \\ &= (a \wedge -a \wedge -b) \vee (b \wedge -a \wedge -b) \geq (a \wedge -a) \wedge (b \wedge -b) = \\ &= (a \,\underline{\triangle}\, a) \wedge (b \,\underline{\triangle}\, b) = a^2 \wedge b^2,\end{aligned}$$

as needed. □

Theorem V.6.6 *Let G be a RS and let $\leq_G$ denote its representation partial order. Assume that $(G, \leq_G)$ is a lattice. The following are equivalent:*

(1) *$(G, \leq_G)$ is a distributive lattice.*

(2) *The RS-characters of G are lattice homomorphisms of $(G, \leq_G)$ into* **3** *(under the order $1 < 0 < -1$).*

(3) *The canonical embedding $\eta_G : G \longrightarrow \mathrm{Sp}(G)$ is a surjective lattice homomorphism. Hence, $G \simeq \mathrm{Sp}(G)$.*

Each of these conditions is equivalent to:

(4) *G is a spectral RS.*

Proof. (3) ⇒ (4) is clear and (4) ⇒ (1) has been observed in V.1.1 (9). Note also that (3) ⇔ (4) was proved in V.4.6 and (4) ⇒ (2) in V.2.2.

(1) ⇒ (2). We show that every $\sigma \in X_G$ preserves suprema. This is enough, since σ also preserves "$-$" and the De Morgan laws hold in G, i.e., $-(a \wedge b) = -a \vee -b$ and dually. It suffices to prove $\sigma(a \vee b) \leq \sigma(a) \vee \sigma(b)$, as the reverse inequality is immediate from the monotonicity of σ. We argue by cases:

— $\sigma(a \vee b) = 0$.

From $a \cdot (a \vee b) \leq a\, b$ (V.6.5 (i)) comes $\sigma(a)\, \sigma(a \vee b) = 0 \leq \sigma(a)\, \sigma(b)$. This shows that $\sigma(a), \sigma(b)$ cannot both be 1, i.e., $\sigma(a) \geq 0$ or $\sigma(b) \geq 0$, whence $\sigma(a) \vee \sigma(b) \geq 0 = \sigma(a \vee b)$.

— $\sigma(a \vee b) = -1$.

Suppose first that $\sigma(a) = \sigma(b) = 0$. From $(a \vee b)^2 \in D_G(a^2, b^2)$ (V.6.5 (iii)) we get $\sigma(a \vee b)^2 \in D_{\mathbf{3}}(\sigma(a)^2, \sigma(b)^2) = D_{\mathbf{3}}(0, 0) = \{0\}$, whence $\sigma(a \vee b) = 0$, contradiction. So, one of $\sigma(a)$ or $\sigma(b)$ is $\neq 0$. If, say, $\sigma(a) = 1$, as above we get $-1 = \sigma(a)\, \sigma(a \vee b) \leq \sigma(a)\, \sigma(b) = \sigma(b)$, and hence $\sigma(b) = -1$. So, one of $\sigma(a)$ or $\sigma(b)$ is -1, and we get $\sigma(a) \vee \sigma(b) = -1 = \sigma(a \vee b)$.

(2) ⇒ (3). (i) η_G is a lattice homomorphism. This follows from (2) by direct computation: for $a, b \in G$ and $\sigma \in X_G$,

$$\eta_G(a \wedge b)(\sigma) = ev_{(a \wedge b)}(\sigma) = \sigma(a \wedge b) = \sigma(a) \wedge \sigma(b) = ev_a(\sigma) \wedge ev_b(\sigma) = \eta_G(a)(\sigma) \wedge \eta_G(b)(\sigma),$$

and similarly for $\vee$.

(ii) η_G is surjective. This is clear from Theorem V.6.2 and (i). □

Corollary V.6.7 *The set of invertible elements of a spectral real semigroup (with induced product operation, representation relation and constants $1, -1$) is a Boolean algebra and, hence, a reduced special group.*

Proof. Recalling I.6.5 (7), for $G \models \mathrm{RS}$ and $g \in G$, in the representation partial order $\leq_G$ we have $g \wedge -g = g^2$ and $g \vee -g = -g^2$; hence:

$$g \text{ invertible in } G \;\;\Leftrightarrow\;\; g^2 = 1 \;\;\Leftrightarrow\;\; g \wedge -g = 1 \;\text{ and }\; g \vee -g = -1.$$

If G is spectral, by the preceding theorem $\leq_G$ is a distributive lattice, and this shows that $-g$ is the Boolean complement of g. [With the terminology of the next section, this just means that the set of Boolean elements of a Kleene algebra form a Boolean algebra.] □

Example V.6.8 This example (pointed out by F. Miraglia) shows that one may have an injective RS-morphism $f : G \longrightarrow L$ of a RS, G, with values in a spectral RS, L, so that its spectral extension $\mathrm{Sp}(f) : \mathrm{Sp}(G) \longrightarrow L$ $(= \mathrm{Sp}(L))$ *is not injective.*

In [DM1], 5.10, pp. 83-84, an example was constructed of *reduced special groups*, $F \subseteq B$, such that F is a (RSG-)fan, B is a Boolean algebra, and F is not a complete subgroup of B. Indeed, it was shown that there are forms $\varphi = \langle a_1, a_2, a_3 \rangle$, $\psi = \langle b_1, b_2, b_3 \rangle$ of dimension 3 over F such that $\varphi \equiv_B \psi$ but $\varphi \not\equiv_F \psi$, where $\equiv$ denotes isometry. By Pfister's local-global principle ([DM1], Prop. 3.7, p. 51) we have

$$(*) \qquad \mathrm{sgn}_\sigma(\varphi) = \textstyle\sum_{i=1}^{3} \sigma(a_i) = \mathrm{sgn}_\sigma(\psi) = \sum_{i=1}^{3} \sigma(b_i),$$

for every (± 1-valued) character $\sigma \in X_B$ (= the Stone space of B), while this equality fails for some character $\gamma \in X_F$ (which, of course, *does not* extend to B).

Now, add a 0 to both F and B, and extend their respective representation relations in the manner described in I.2.2 (3), to get real semigroups F^* and B^*. Observe that, if G is a RSG and $G^* = G \cup \{0\}$ is the corresponding RS obtained in this manner, then X_{G^*} consists exactly of the extensions of the ± 1-valued characters of G that send 0 to 0; hence, there is an obvious one-one correspondence between X_{G^*} and X_G.

Let P denote the Post hull of B^* (a lattice-ordered RS in the representation partial order). The canonical embedding $\varepsilon : B^* \longrightarrow P$ is complete (I.4.12 (3)) and gives, by restriction, a RS-embedding $\varepsilon \lceil F^* : F^* \longrightarrow P$. However, $\varepsilon \lceil F^*$ cannot be a complete embedding; otherwise, the bijection between X_{F^*} and X_F indicated above, together with (*), would imply that the inclusion $F \subseteq B$ is complete, contrary to the choice of F and B. By V.5.7, $\mathrm{Sp}(\varepsilon \lceil F^*) : \mathrm{Sp}(F^*) \longrightarrow \mathrm{Sp}(P)$ $(= P)$ is not injective. □

V.7 Spectral real semigroups as Kleene algebras.

The arguments in the preceding section and in section V.2 suggest that the spectral real semigroups should be treated as distributive lattices *together* with the involution "$-$" (i.e., product with -1). The appropriate framework to deal with structures of this type is that of Kleene algebras, defined in IV.1.9.

From the preceding results and those in I.6 we have:

Fact V.7.1 *Let X be a spectral space, and let $\leq$, $\wedge$, $\vee$ denote its pointwise defined partial order and lattice operations (cf. V.1.1 (8),(9)). Then, the structure $(\mathbf{Sp}(X), \wedge, \vee, -, 1, 0)$ is a Kleene algebra; in fact, a Kleene subalgebra of the Post algebra $P(X) = \mathcal{C}(X_{\mathrm{con}}, \mathbf{3})$.*

Proof. The validity of axiom [K1] for Kleene algebras (see IV.1.9) was observed in V.1.1 (9). That of [K2.a] and [K3] is obvious. The De Morgan law [K2.b] is checked by a routine argument, using the (pointwise) definitions of $\wedge$ and $\vee$ (V.1.1 (9)), and the obvious fact that $f \leq g \Leftrightarrow -g \leq -f$ $(f, g \in \mathbf{Sp}(X))$. The Kleene inequality [K2.c] is also routine, by the pointwise definition of $\leq$ and of the lattice operations, using the fact that it holds in $\mathbf{3}$.

For the last assertion, recall that spectral maps are continuous for the constructible topologies of domain and codomain (V.1.1 (3.i)) and that the lattice operations are pointwise defined in $P(X)$. □

As our main result in this section we prove that the spectral real semigroups $\mathbf{Sp}(X)$ are Kleene algebras of a special type; Theorem V.7.2 gives an exact algebraic characterization of them.

Remark. The proof of Proposition IV.1.11 yields the following additional information: if X is a spectral space, the Kleene algebra $\mathbf{Sp}(X)$ is isomorphic to $K(\mathcal{K}(X))$, where $\mathcal{K}(X)$ is the distributive lattice of quasi-compact opens of X; the isomorphism is given by the map $h(f) = (f^{-1}[1], f^{-1}[-1])$, for $f \in \mathrm{Sp}(X)$. $\square$

Theorem V.7.2 *Let K be a Kleene algebra with center $\underline{\mathbf{c}}$. The following are equivalent:*

(1) *There is a hereditarily normal spectral space X such that $K \simeq \mathbf{Sp}(X)$.*

(2) *K verifies condition* [dec] *in IV.1.11, and*

[cn][3] *For all $a, b \in K$ such that $a, b \leq \underline{\mathbf{c}}$, there are $x, y \in K$ so that $a \wedge x \leq b$, $b \wedge y \leq a$ and $x \vee y = \underline{\mathbf{c}}$.*

Proof. (1) $\Rightarrow$ (2). Since $\mathbf{Sp}(X) \simeq K(\mathcal{K}(X))$, see Remark above, then $\mathbf{Sp}(X)$ verifies condition [dec]. We check it also verifies [cn].

Let $f, g \in \mathrm{Sp}(X)$ be such that $f, g \leq 0$. Hence $f^{-1}[-1] = g^{-1}[-1] = \emptyset$. Let $K_1 := f^{-1}[1] \cap g^{-1}[0]$ and $K_2 := g^{-1}[1] \cap f^{-1}[0]$. By Fact V.1.7, K_1, K_2 are quasi-compact. Since open sets are downwards closed under specialization, the generizations $\mathrm{Gen}\,(K_i)$ of K_i $(i = 1, 2)$ are also quasi-compact, see proof of Theorem V.1.5.

Claim 1. $\mathrm{Gen}\,(K_1) \cap \mathrm{Gen}\,(K_2) = \emptyset$.

Proof of Claim 1. Assume there is an element t in this intersection; hence, for $i = 1, 2$, there is $x_i \in K_i$ so that $t \rightsquigarrow x_i$. Since X is hereditarily normal, either $x_1 \rightsquigarrow x_2$ or $x_2 \rightsquigarrow x_1$, say the first. Since $g(x_1) = 0$, from $x_1 \rightsquigarrow x_2$ follows $g(x_2) = 0$, contradicting $x_2 \in K_2$. $\square$

By Proposition V.1.6 (i) there are disjoint quasi-compact opens $U_1, U_2 \subseteq X$ such that $\mathrm{Gen}\,(K_i) \subseteq U_i$ $(i = 1, 2)$. We define spectral characters $h_i : X \longrightarrow \mathbf{3}_{\mathrm{sp}}$ by:

$$h_i(x) = \begin{cases} 1 & \text{if } x \in U_i \\ 0 & \text{if } x \notin U_i\,. \end{cases}$$

Then, $h_1^{-1}[1] \cap h_2^{-1}[1] = U_1 \cap U_2 = \emptyset$, whence $h_1 \vee h_2 = 0$. We prove:

— $f \wedge h_2 \leq g$. Assume $g(x) = 1$. If $f(x) = 0$, then $x \in K_2$, whence $x \in U_2$; therefore $h_2(x) = 1$, which implies $(f \wedge h_2)(x) = 1$. Since g only takes on the values $0, 1$, this proves the asserted inequality.

Likewise, $g \wedge h_1 \leq f$, ending the verification of [cn].

(2) $\Rightarrow$ (1). As a first step in the proof, let K be any Kleene algebra and let $X = \mathrm{Hom}_{Kl}(K, \mathbf{3})$ be the set of Kleene-algebra homomorphisms of K into $\mathbf{3}$. With the topology generated by the sets $[\![\, a = 1 \,]\!] = \{\sigma \in X \mid \sigma(a) = 1\}$, for $a \in K$, as a (sub)basis of opens, X is a spectral space where the sets of this form are quasi-compact (cf. I.1.17). Since K is a lattice, the family of these sets is closed under unions and intersections: $\bigcup_{i=1}^{n} [\![\, a_i = 1 \,]\!] = [\![\, (\bigwedge_{i=1}^{n} a_i) = 1 \,]\!]$, and dually for $\bigcap$; therefore it is a basis for the topology. Note that specialization in X is as follows: for $\sigma, \gamma \in X$,

$$\sigma \rightsquigarrow \gamma \;\Leftrightarrow\; \gamma \in \overline{\{\sigma\}} \;\Leftrightarrow\; \forall a \in K\, (\gamma \in [\![\, a = 1 \,]\!] \Rightarrow \sigma \in [\![\, a = 1 \,]\!]) \;\Leftrightarrow\; \gamma^{-1}[1] \subseteq \sigma^{-1}[1]\,,$$

[3] [cn] = “completely normal”; cf. Remark V.7.3 (b) below.

(and hence also $\gamma^{-1}[-1] \subseteq \sigma^{-1}[-1]$).

We observe:

Claim 2. If K verifies [cn], then X is hereditarily normal.

Proof of Claim 2. Assume, towards a contradiction, that there are $\sigma, \gamma_1, \gamma_2 \in X$ such that $\sigma \rightsquigarrow \gamma_i$ $(i = 1, 2)$, but $\gamma_1 \not\rightsquigarrow \gamma_2$ and $\gamma_2 \not\rightsquigarrow \gamma_1$. Then, there are $a, b \in K$ so that $\gamma_1(a) = 1, \gamma_2(a) \neq 1, \gamma_2(b) = 1$ and $\gamma_1(b) \neq 1$.

By [cn] applied with $a \wedge \underline{\mathbf{c}}$ and $b \wedge \underline{\mathbf{c}}$, there are $x, y \in K$ such that $a \wedge x \wedge \underline{\mathbf{c}} \leq b \wedge \underline{\mathbf{c}}$, $b \wedge y \wedge \underline{\mathbf{c}} \leq a \wedge \underline{\mathbf{c}}$ and $x \vee y = \underline{\mathbf{c}}$. Then, $\gamma_2(b) = 1$ and the first inequality give $\gamma_2(a \wedge x \wedge \underline{\mathbf{c}}) = \gamma_2(a) \wedge \gamma_2(x) \wedge 0 = 1$; since $\gamma_2(a) \neq 1$, we get $\gamma_2(x) = 1$. Likewise, the second inequality and $\gamma_1(a) = 1$ imply $\gamma_1(y) = 1$. Since $\gamma_i^{-1}[1] \subseteq \sigma^{-1}[1]$ for $i = 1, 2$, we get $\sigma(x) = \sigma(y) = 1$, contradicting $x \vee y = \underline{\mathbf{c}}$. □

The evaluation maps $ev_a : X \longrightarrow \mathbf{3}_{\mathrm{sp}}$ $(a \in K)$ are spectral, since $ev_a^{-1}[1] = [\![a = 1]\!]$ and $ev_a^{-1}[-1] = [\![\neg a = 1]\!]$. Since homomorphisms into $\mathbf{3}$ separate points of K, the map $\mu : K \longrightarrow \mathrm{Sp}(X)$ defined by $\mu(a) = ev_a$ is a well-defined, injective homomorphism of Kleene algebras. To show it is an isomorphism it remains to prove:

Claim 3. μ is surjective.

Proof of Claim 3. Let $f \in \mathrm{Sp}(X)$. As the sets $f^{-1}[\pm 1]$ are quasi-compact open, there are $a, b \in K$ such that

(*) $f^{-1}[-1] = [\![a = -1]\!]$ and $f^{-1}[1] = [\![b = -1]\!]$.

These sets being disjoint, we get $a \wedge b \leq \underline{\mathbf{c}}$. Condition [cn] applied with $\neg a \wedge \underline{\mathbf{c}}$ and $\neg b \wedge \underline{\mathbf{c}}$ yields the existence of $x, y \in K$ such that $\neg a \wedge x \wedge \underline{\mathbf{c}} \leq \neg b \wedge \underline{\mathbf{c}}$, $\neg b \wedge y \wedge \underline{\mathbf{c}} \leq \neg a \wedge \underline{\mathbf{c}}$ and $x \vee y = \underline{\mathbf{c}}$, whence $\neg x \wedge \neg y = \underline{\mathbf{c}}$. Next we apply [dec] to this situation, to get $t \in K$ so that $t \vee \underline{\mathbf{c}} = \neg y$ and $\neg t \vee \underline{\mathbf{c}} = \neg x$, i.e., $t \wedge \underline{\mathbf{c}} = x$. Let

(**) $s = (t \wedge a) \vee (x \wedge \underline{\mathbf{c}}) \vee (\neg b \wedge \underline{\mathbf{c}})$.

We assert:

Claim 4. $ev_s = f$.

Proof of Claim 4. By (*) and (**), Claim 4 amounts to showing that, for all $\sigma \in X$:

(i) $\sigma(a) = -1 \;\Leftrightarrow\; \sigma(s) = -1$;

(ii) $\sigma(b) = -1 \;\Leftrightarrow\; \sigma(s) = 1$.

Proof of (i). ($\Rightarrow$) If $\sigma(a) = -1$, the inequality $\neg b \wedge y \wedge \underline{\mathbf{c}} \leq \neg a \wedge \underline{\mathbf{c}}$ entails $\sigma(\neg b \wedge y \wedge \underline{\mathbf{c}}) = 1$, which in turn implies $\sigma(\neg b) = 1$ or $\sigma(y) = 1$. However, the first alternative gives $\sigma(a) = \sigma(b) = -1$, contradicting $a \wedge b \leq \underline{\mathbf{c}}$; then, $\sigma(y) = 1$. Since $t \vee \underline{\mathbf{c}} = \neg y$, we conclude $\sigma(t) = -1$, whence $\sigma(t \wedge a) = -1$, and $\sigma(s) = -1$.

($\Leftarrow$) Conversely, if $\sigma(s) = -1$, then $\sigma(t \wedge a) = -1$ (as the other disjuncts in (**) are $\leq \underline{\mathbf{c}}$), and hence $\sigma(a) = -1$.

Proof of (ii). ($\Leftarrow$) Assuming $\sigma(s) = 1$, in particular we have $\sigma(\neg b \wedge \underline{\mathbf{c}}) = 1$, whence $\sigma(\neg b) = 1$, i.e., $\sigma(b) = -1$.

($\Rightarrow$) Conversely, if $\sigma(b) = -1$, from $\neg a \wedge x \wedge \underline{\mathbf{c}} \leq \neg b \wedge \underline{\mathbf{c}}$ we get $\sigma(\neg a) = 1$ or $\sigma(x) = 1$. The first alternative yields $\sigma(a) = \sigma(b) = -1$, contradicting $a \wedge b \leq \underline{\mathbf{c}}$; then, $\sigma(x) = 1$. Since

$t \wedge \underline{\mathbf{c}} = x$, it follows that $\sigma(t) = 1$. Then, σ takes value 1 on all three disjuncts of (**), showing that $\sigma(s) = 1$, and completing the proof of Theorem V.7.2. $\square$

Remarks V.7.3 (a) It should be clear from our results that, with notation as in the proof of Theorem V.7.2, the correspondence $K \longmapsto \mathbf{Sp}(\mathrm{Hom}_{Kl}(K, \mathbf{3}))$, $f \longmapsto \mathrm{Sp}(f)$ (f a Kleene-algebra homomorphism) establishes an equivalence between the category of Kleene algebras verifying conditions [dec] and [cn] (with Kleene-algebra homomorphisms) and the category **SRS** of spectral real semigroups with RS-homomorphisms. We leave it to the reader to work out the precise statement.

(b) A bounded distributive lattice is called **completely normal** if the space Spec(L) —the set of its prime filters with the standard spectral topology, cf. [DST], § 1.4— is hereditarily normal. Motivation for our proof of Theorem V.7.2 came from the following result of A. Monteiro [Mon]: *L is completely normal if and only if for all $a, b \in L$ there are $x, y \in L$ so that $a \wedge x \leq b$, $b \wedge y \leq a$ and $x \vee y = \top$.*

It is clear from the preceding arguments that, if L is a bounded, completely normal distributive lattice, the Kleene algebra $K(L)$ verifies conditions [dec] and [cn], and hence is a spectral real semigroup. $\square$

Summarizing the preceding results and those of section V.2, we have:

Theorem V.7.4 (Axioms for spectral real semigroups)

(1) *The following statements, together with the axioms for real semigroups (I.2.1), give a first-order axiomatization for the class of spectral real semigroups in the language* $\mathcal{L}_{\mathrm{RS}} = \{\cdot, 1, 0, -1, D\}$*:*

[SRS1] $\quad \forall a\, \exists c\, (c = c^2 \wedge a\, c = c \wedge -a \in D(1, -c))$.

Setting:

$$a^- := \textit{the unique } c \textit{ verifying } [\mathrm{SRS1}] \ (\textit{see V.2.1 (i)}), \quad \textit{and} \quad a^+ := -((-a)^-),$$

[SRS2] $\quad \forall a\, b\, \exists d\, (d \in D(a, b) \wedge d^+ = -a^+ b^+ \wedge d^- \in D^t(a^-, b^-))$.

(2) *The axioms for Kleene algebras verifying conditions* [dec] *and* [cn] *—see IV.1.9, IV.1.11, and V.7.2— constitute an alternative first-order axiomatization for spectral real semigroups in the language* $\{\wedge, \vee, \neg, \bot, \underline{\mathbf{c}}\}$. $\square$

Proof. That the spectral RSs verify axioms [SRS1] and [SRS2] is Theorem V.2.1.

Conversely, axioms [SRS1] and [SRS2] define a lattice structure on a given real semigroup, G, in the language $\mathcal{L}_{\mathrm{RS}}$. Since the characters of G preserve the constants, operation and relation in $\mathcal{L}_{\mathrm{RS}}$, these axioms ensure that they are lattice homomorphisms (see Corollary V.2.2). Thus, condition (2) of Theorem V.6.6 is verified, implying that G is spectral. $\square$

Remark V.7.5 Axioms [SRS1] and [SRS2] are of the form $\forall a\, \psi_1(a)$ and $\forall ab\ \psi_2(a, b)$, where ψ_1, ψ_2 are *positive-primitive* $\mathcal{L}_{\mathrm{RS}}$-formulas, i.e., of the form $\exists x\, \theta_1(a, x)$, $\exists \overline{y}\, \theta_2(a, b, \overline{y})$, with θ_1, θ_2 conjunctions of *atomic* $\mathcal{L}_{\mathrm{RS}}$-formulas and $\overline{y}$ a tuple of variables of suitable length. This is clear for [SRS1] $\Big[\theta_1(z, w) : w = w^2 \wedge z\, w = w \wedge -z \in D(1, -w)\Big]$. For [SRS2] (using uniqueness) replace z^- by $\exists z_1\, \theta_1(z, z_1)$ for $z \in \{d, a, b\}$ and, similarly, $z^+ := -((-z)^-)$ by $\exists z_2\, \theta_1(-z, z_2)$. Explicitly, in new variables d_i, a_i, b_i ($i = 1, 2$) (corresponding to d, a, b, respectively), with $\overline{y} = \langle d, d_1, d_2, a_1, a_2, b_1, b_2 \rangle$,

$$\theta_2(a, b, \overline{y}) \,:\, d \in D(a, b) \wedge \bigwedge\nolimits_{z \in \{d,a,b\}} \theta_1(z, z_1) \wedge \theta_1(-z, z_2) \wedge d_2 = a_2\, b_2 \wedge d_1 \in D^t(a_1, b_1). \quad \square$$

These manipulations yield analogs to Propositions IV.6.3 and IV.6.5 for spectral real semigroups:

Proposition V.7.6 (1) *The class of spectral real semigroups is closed under the following constructions:*

— *Inductive limits (colimits) over a right-directed index set.*

— *Reduced products*[4] *(in particular, arbitrary products).*

Further,

(2) *Let $f : G \longrightarrow H$ be a* surjective *RS-homomorphism, where G, H are RSs. If G is spectral, so is H.*

In particular,

(3) *Any quotient $G/\equiv$ of a spectral RS, G, modulo a RS-congruence $\equiv$ (II.2.1) is a spectral RS.*

Hence,

(4) *Quotients of spectral RSs modulo saturated sets* (**??**) *are spectral RSs.*

Proof. The proof is similar to that of Proposition IV.6.3 in view of the logical form (universal quantification of positive-primitive formulas) of the axioms [SRS1], [SRS2] for spectral RSs, and hence omitted. □

Remark. Note that item (4) of this Proposition applies, in particular, to the various types of quotients treated in § II.3: quotients modulo saturated subsemigroups, modulo transversally saturated subsemigroups, localizations and residue spaces at saturated prime ideals. □

Proposition V.7.7 *Let G be a RS and H be a spectral RS. Then,*

(1) *If $f : G \longrightarrow H$ is a pure embedding of RSs, then G is a spectral RS.*

In particular,

(2) *The canonical embedding $\eta_G : G \longrightarrow \mathrm{Sp}(G)$ of G into its spectral hull* is not *pure unless G itself is spectral. In the latter case, η_G is an isomorphism of G onto* $\mathrm{Sp}(G)$.

Proof. (1) Let $\psi_1(v)$, $\psi_2(v_1, v_2)$ denote the positive-primitive matrices of [SRS1], [SRS2] (see IV.6.2). Assume $G \not\models$ SRS. Since G is supposed to be a RS, one of [SRS1] or [SRS2] fails in G, say $G \not\models$ [SRS2]. Then, there are $a, b \in G$ so that $G \models \neg\,\psi_2\,[a, b]$. On the other hand, since $H \models$ [SRS2] by assumption, we have $H \models \psi_2\,[f(a), f(b)]$. Since ψ_2 is positive-primitive, f is not pure, contradiction.

(2) follows from (1) with $f = \eta_G$. The last assertion is Corollary V.4.6. □

V.8 Quotients of spectral real semigroups.

In line with preceding results (see Proposition V.7.6 (2)–(4)), we prove in this section that quotients of spectral real semigroups —in the sense of Definition II.2.1— are determined by the proconstructible subsets of its character space endowed with the spectral topology. As a corollary we obtain a characterization of the spectral hull of any RS-quotient. □

[4]Cf. [CK], Def. 4.1.6 and § 6.2, or [H], § 9.4.

V.8.1 Notation and Preliminaries. (i) Given a hereditarily normal spectral space, X, and a proconstructible subset, Y, we define an equivalence relation on the real semigroup $\mathbf{Sp}(X)$ as follows: for $a, b \in \mathrm{Sp}(X)$,

$$a \equiv_Y b \;\Leftrightarrow\; \forall y \in Y(a(y) = b(y)) \;\Leftrightarrow\; a \lceil Y = b \lceil Y.$$

This is, clearly, a congruence of ternary semigroups. Similarly, we define a ternary relation in the quotient set $\mathbf{Sp}(X)/\equiv_Y$, by pointwise evaluation at elements of Y: for $a, b, c \in \mathrm{Sp}(X)$ and with $\pi_Y : \mathbf{Sp}(X) \longrightarrow \mathbf{Sp}(X)/\equiv_Y$ canonical,

$$\pi_Y(a) \in D_{\mathbf{Sp}(X)/\equiv_Y}(\pi_Y(b), \pi_Y(c)) \;:\Leftrightarrow\; \forall y \in Y(a(y) \in D_{\mathbf{3}}(b(y), c(y)).$$

Routine checking shows:

$$\pi_Y(a) \in D^t_{\mathbf{Sp}(X)/\equiv_Y}(\pi_Y(b), \pi_Y(c)) \;\Leftrightarrow\; \forall y \in Y(a(y) \in D^t_{\mathbf{3}}(b(y), c(y)).$$

To ease notation we shall write D_Y for $D_{\mathbf{Sp}(X)/\equiv_Y}$, and similarly for transversal representation.

(ii) It is a general and well-known fact that the spectral subspaces of a spectral space X are exactly the proconstructible subsets $Y \subseteq X$ with the induced topology. To avoid possible confusion, we shall denote by Y_{sp} the proconstructible subset Y endowed with the (spectral) topology induced from X. The quasi-compact opens of Y_{sp} are exactly the intersections of quasi-compact opens of X with Y, and similarly for the closed constructible subsets of Y_{sp}. These results imply that the specialization order of Y_{sp} is just the restriction of specialization in X. See [DST], Thm. 3.3.1. Hence, it is clear that if, in addition, X is hereditarily normal, then so is Y_{sp}. To ease notation the real semigroup $\mathbf{Sp}(Y_{\mathrm{sp}})$ will be denoted by $\mathbf{Sp}(Y)$. □

Our first result is:

Theorem V.8.2 *Let X be a hereditarily normal spectral space, and let Y be a proconstructible subset of X. Then,*

(1) $(\mathbf{Sp}(X)/\equiv_Y, D_Y)$ *is a real semigroup.*

(2) $(\mathbf{Sp}(X)/\equiv_Y, D_Y)$ *is isomorphic to* $(\mathbf{Sp}(Y), D_{\mathbf{Sp}(Y)})$.

Hence,

(3) *The character space* $X_{\mathbf{Sp}(X)/\equiv_Y}$ *is homeomorphic to* Y_{sp} *(spectral topologies).*

(4) *Every RS-congruence of* $\mathbf{Sp}(X)$ *is of the form* $\equiv_Y$ *for a suitable proconstructible set* $Y \subseteq X$.

Proof. (1) follows from (2) and V.8.1 (ii), see Theorem V.1.5. Here is a proof of (1) without invoking (2).

By Theorem II.2.16 (with $G = \mathbf{Sp}(X), X_{\mathbf{Sp}(X)}$ and $\mathcal{H} = Y$), $(\mathbf{Sp}(X)/\equiv_Y, D_Y)$ verifies all axioms for RSs except, possibly, [RS3a]. Now, the proof of Theorem V.1.4 shows that this structure also verifies axiom [RS3a]: the witness required for weak associativity is $\pi_Y(f)$, where, for given elements $a, b, c, d, e \in \mathrm{Sp}(X)$ whose images under π_Y verify the assumptions of [RS3a] in $\mathbf{Sp}(X)/\equiv_Y$, the element f is defined as in the proof of V.1.4.

(2) The required isomorphism is the map

$$a/\equiv_Y = \pi_Y(a) \overset{\varphi}{\longmapsto} a \lceil Y \quad (a \in \mathrm{Sp}(X)).$$

By the definition of $\equiv_Y$ it is clear that φ is a well-defined, injective TS-homomorphism. Since representation is pointwise defined in both $\mathbf{Sp}(X)/\equiv_Y$ and $\mathbf{Sp}(Y)$, we have

$$\begin{aligned}\pi_Y(a) \in D_{\mathbf{Sp}(X)/\equiv_Y}(\pi_Y(b), \pi_Y(c)) &\Leftrightarrow \forall y \in Y\,(a(y) \in D_{\mathbf{3}}(b(y), c(y)) \\ &\Leftrightarrow a \restriction Y \in D_{\mathbf{Sp}(Y)}(b \restriction Y, c \restriction Y),\end{aligned}$$

showing that both φ and $\varphi^{-1} \restriction \mathrm{Im}(\varphi)$ preserve representation.

The proof that φ is surjective is more delicate; it boils down to:

Claim. Every map $f \in \mathrm{Sp}(Y)$ extends to a map $g \in \mathrm{Sp}(X)$.

Proof of Claim. The argument is similar to (a part of) the proof of Theorem V.1.5, using Proposition V.1.6; we only sketch it.

Recall that $f \in \mathrm{Sp}(Y)$ just means that $f^{-1}[\pm 1]$ are quasi-compact opens in Y_{sp}. To get a spectral map $g : X \longrightarrow \mathbf{3}_{\mathrm{sp}}$ extending f it suffices to construct disjoint quasi-compact opens U_i ($i \in \{\pm 1\}$) of X so that $f^{-1}[i] \subseteq U_i$, and set:

$$g \restriction U_i = i \ \ (i \in \{\pm 1\}) \ \text{ and } \ g \restriction (X \setminus (U_1 \cup U_{-1})) = 0.$$

For $i \in \{\pm 1\}$ let $\mathrm{Gen}_X(f^{-1}[i]) = \{x \in X \mid \exists y \in f^{-1}[i]\,(x \underset{X}{\rightsquigarrow} y)\}$ be the generization of $f^{-1}[i]$ in X. Arguments similar to those in the proof of V.1.5 show:

(i) $\mathrm{Gen}_X(f^{-1}[i])$ is quasi-compact in X ($i \in \{\pm 1\}$).

(ii) $\mathrm{Gen}_X(f^{-1}[1]) \cap \mathrm{Gen}_X(f^{-1}[-1]) = \emptyset$.

By Proposition V.1.6 there are disjoint quasi-compact open subsets U_i of X such that $\mathrm{Gen}_X(f^{-1}[i]) \subseteq U_i$ ($i \in \{\pm 1\}$), as required.

(3) follows from (2) by the Duality Theorem I.5.1 and the fact that $X_{\mathbf{Sp}(Y)} \simeq Y$ (Proposition V.3.7 (2)).

(4) is a particular case of Proposition **??**: with notation therein, given a congruence $\equiv$ of $\mathbf{Sp}(X)$, take $Y := \mathcal{H}_{\equiv}$ (proconstructible by **??** (ii)) ; the last line in the statement of **??** (i) shows that $\equiv \, = \, \equiv_Y$. □

To establish that the equivalence relation $\equiv_Y$ is a RS-congruence we still have to prove that the factoring condition II.2.1 (iii) holds. This will follow from the next Proposition, which gives a lifting for the quotient representation relation D_Y.

Proposition V.8.3 *Let X be a hereditarily normal spectral space, let $Y \subseteq X$ be proconstructible, and let $a, b, c \in \mathbf{Sp}(X)$. The following are equivalent:*

(1) $\pi_Y(a) \in D_Y(\pi_Y(b), \pi_Y(c))$.

(2) *There exists $a' \in \mathbf{Sp}(X)$ such that $a' \restriction Y = a \restriction Y$ and $a' \in D_{\mathbf{Sp}(X)}(b, c)$.*

Proof. (2) $\Rightarrow$ (1) is clear from the pointwise definition of both D_Y and $D_{\mathbf{Sp}(X)}$.

(1) $\Rightarrow$ (2). Throughout this proof i stands for ± 1. Let $U_i := a^{-1}[i]$ and $V_i := b^{-1}[i] \cup c^{-1}[i]$, quasi-compact open subsets of X. Assumption (1) amounts to $U_i \cap Y \subseteq V_i \cap Y$. Set $W_i := U_i \cap V_i$; W_i is quasi-compact open, and $W_1 \cap W_{-1} = \emptyset$. We define a map $a' : X \longrightarrow \mathbf{3}$ by:

$$a' \restriction W_i = i \ \text{ for } i \in \{\pm 1\}, \quad \text{and} \quad a' \restriction X \setminus (W_1 \cup W_{-1}) = 0\,.$$

Clearly, $a' \in \mathbf{Sp}(X)$. We have:

— $a' \restriction Y = a \restriction Y$.

Let $y \in Y$; for $i \in \{\pm 1\}$,

$$a(y) = i \Rightarrow y \in U_i \cap Y \subseteq V_i \cap Y \Rightarrow y \in W_i \cap Y \Rightarrow a'(y) = i\,,$$

and

$$a(y) = 0 \Rightarrow y \in X \setminus (U_1 \cup U_{-1}) \subseteq X \setminus (W_1 \cup W_{-1}) \Rightarrow a'(y) = 0\,.$$

— $a' \in D_{\mathbf{Sp}(X)}(b, c)$.

For $x \in X$ and $i \in \{\pm 1\}$ we have:

$$a'(x) = i \;\Rightarrow\; x \in W_i \subseteq V_i \;\Rightarrow\; b(x) = i \text{ or } c(x) = i\,,$$

as required. □

Corollary V.8.4 *Let X be a hereditarily normal spectral space, and let $Y \subseteq X$ be a proconstructible subset. The equivalence relation $\equiv_Y$ verifies the factoring condition of Definition II.2.1 (iii).*

Proof. Given a RS-morphism $f : \mathbf{Sp}(X) \longrightarrow H$ into a RS, H, such that $a \equiv_Y b \Rightarrow f(a) = f(b)$ for $a, b \in \mathbf{Sp}(X)$, it suffices to show that the map $\widehat{f} : \mathbf{Sp}(X)/{\equiv_Y} = \mathbf{Sp}(Y) \longrightarrow H$ defined by $\widehat{f} \circ \pi = f$ preserves representation, i.e., for $a, b, c \in \mathbf{Sp}(X)$,

$$\pi_Y(a) \in D_Y(\pi_Y(b), \pi_Y(c)) \;\Rightarrow\; f(a) \in D_H(f(b), f(c)).$$

By Proposition V.8.3, the antecedent implies that $a' \in D_{\mathbf{Sp}(X)}(b, c)$ for some $a' \in \mathbf{Sp}(X)$ such that $a' \lceil Y = a \lceil Y$, i.e., $a' \equiv_Y a$. By the assumption on f we have $f(a) = f(a')$ and, since f is a RS-morphism, $f(a) = f(a') \in D_H(f(b), f(c))$, as required. □

V.8.5 The spectral hull of a RS-quotient.

As an application of the foregoing results we prove that formation of the spectral hull commutes with the operation of taking quotients under arbitrary RS-congruences. The result is:

Theorem V.8.6 *Let $\equiv$ be a RS-congruence of a real semigroup G. Let $Y := \mathcal{H}_{\equiv} \subseteq X_G$ denote the (proconstructible) set of characters defined by $\equiv$ (cf. Proposition **??**) and let $\equiv_Y$ denote, as above, the RS-congruence of* $\mathrm{Sp}(G)$ *induced by Y. Then we have* $\mathrm{Sp}(G/{\equiv}) \simeq \mathrm{Sp}(G)/{\equiv_Y}$.

Proof. Recall that in Theorem V.5.3 (i) we proved that —under identification of the RS G in its spectral hull $\mathrm{Sp}(G)$ via the map η_G— any RS-homomorphism $f : G \longrightarrow H$ extends *uniquely* to a RS-morphism $\mathrm{Sp}(f) : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(H)$. The map $\mathrm{Sp}(f)$ is defined by $\mathrm{Sp}(f)(g) = g \circ f^*$ ($g \in \mathrm{Sp}(G)$), where the spectral map $f^* : X_H \longrightarrow X_G$ dual to f is given by right-composition with f: for $\gamma \in X_H$, $f^*(\gamma) = \gamma \circ f$; see V.4.3 (b) and V.4.4. We shall use this with $H = G/{\equiv}$ and $f = \pi_G =$ the canonical quotient map $G \longrightarrow G/{\equiv}$.

As in Theorem V.8.2, $\pi_Y : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(G)/{\equiv_Y}$ denotes the corresponding quotient map.

We note first:

(a) For $a, b \in \mathrm{Sp}(G)$, $a \equiv_Y b \;\Rightarrow\; \mathrm{Sp}(\pi_G)(a) = \mathrm{Sp}(\pi_G)(b)$.

Proof of (a). This implication can be rephrased as

$$a \lceil Y = b \lceil Y \;\Rightarrow\; a \circ \pi_G^* = b \circ \pi_G^*\,.$$

That is, we must show that, for $\sigma \in X_{G/\equiv}$, $(a \circ \pi_G^*)(\sigma) = (b \circ \pi_G^*)(\sigma)$; equivalently, $a(\sigma \circ \pi_G) = b(\sigma \circ \pi_G)$. Now, $\sigma \circ \pi_G \in \mathcal{H}_{\equiv} = Y$ (cf. **??**); since $a \lceil Y = b \lceil Y$, we conclude $a(\sigma \circ \pi_G) = b(\sigma \circ \pi_G)$, as required.

Since $\equiv_Y$ is a RS-congruence of $\mathrm{Sp}(G)$ (V.8.2 (1) and V.8.4), (a) entails that the map $\mathrm{Sp}(\pi_G) : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(G/\equiv)$ induces a RS-morphism $\widehat{\mathrm{Sp}(\pi_G)} : \mathrm{Sp}(G)/\equiv_Y \longrightarrow \mathrm{Sp}(G/\equiv)$ such that $\widehat{\mathrm{Sp}(\pi_G)} \circ \pi_Y = \mathrm{Sp}(\pi_G)$. We show that $\widehat{\mathrm{Sp}(\pi_G)}$ is the required RS-isomorphism.

(b) $\widehat{\mathrm{Sp}(\pi_G)}$ is injective.

Proof of (b). This is just the converse implication to (a): for $a, b \in \mathrm{Sp}(G)$,

$$\mathrm{Sp}(\pi_G)(a) = \mathrm{Sp}(\pi_G)(b) \;\Rightarrow\; a \lceil Y = b \lceil Y, \quad \text{i.e.,}$$

$$(*) \qquad a \circ \pi_G^* = b \circ \pi_G^* \;\Rightarrow\; a \lceil Y = b \lceil Y.$$

Let $p \in Y = \mathcal{H}_\equiv$. By the definition of $\mathcal{H}_\equiv$ (**??**) there is $\sigma \in X_{G/\equiv}$ such that $p = \sigma \circ \pi_G$. Then, from the antecedent of (*) comes

$$a(p) = a(\sigma \circ \pi_G) = (a \circ \pi_G^*)(\sigma) = (b \circ \pi_G^*)(\sigma) = b(\sigma \circ \pi_G) = b(p);$$

since p is an arbitrary element of Y, (*) is proved.

(c) $\widehat{\mathrm{Sp}(\pi_G)}$ is surjective.

Proof of (c). We show $\mathrm{Sp}(\pi_G)$ is surjective. Let $f \in \mathrm{Sp}(G/\equiv)$, i.e., $f : X_{G/\equiv} \longrightarrow \mathbf{3}_{\mathrm{sp}}$ is a spectral map. With $\varphi : Y = \mathcal{H}_\equiv \longrightarrow X_{G/\equiv}$ denoting the (spectral) homeomorphism given by Proposition **??** (iii), we have $f \circ \varphi \in \mathrm{Sp}(Y)$. The Claim in the proof of Theorem V.8.2 shows that $f \circ \varphi$ extends to a map $g \in \mathrm{Sp}(X_G) = \mathrm{Sp}(G)$, i.e., $g \lceil Y = f \circ \varphi$. Inspection of the definition of φ (proof of **??** (iii)) shows that $\varphi = \pi_G^{*\,-1}$. Hence the last equality yields $f = g \circ \pi_G^* = \mathrm{Sp}(\pi_G)(g)$, proving (c).

(d) $\widehat{\mathrm{Sp}(\pi_G)}$ reflects representation.

Proof of (d). This amounts to proving, for $a, b, c \in \mathrm{Sp}(G)$,

$$\mathrm{Sp}(\pi_G)(a) \in D_{\mathrm{Sp}(G/\equiv)}(\mathrm{Sp}(\pi_G)(b), \mathrm{Sp}(\pi_G)(c)) \;\Rightarrow\; \pi_Y(a) \in D_{\mathrm{Sp}(G)/\equiv_Y}(\pi_Y(b), \pi_Y(c)), \quad \text{i.e.,}$$

$$(**) \qquad a \circ \pi_G^* \in D_{\mathrm{Sp}(G/\equiv)}(b \circ \pi_G^*, c \circ \pi_G^*) \;\Rightarrow\; a \lceil Y \in D_{\mathbf{Sp}(Y)}(b \lceil Y, c \lceil Y).$$

For $z \in \mathrm{Sp}(G)$ and $\sigma \in X_{G/\equiv}$ we have $(z \circ \pi_G^*)(\sigma) = z(\sigma \circ \pi_G)$. So, the antecedent of (**) translates as

$$(***) \qquad \forall\, \sigma \in X_{G/\equiv} \left[a(\sigma \circ \pi_G) \in D_{\mathbf{3}}(b(\sigma \circ \pi_G), c(\sigma \circ \pi_G)) \right].$$

To end the proof of (**), let $p \in Y = \mathcal{H}_\equiv$, i.e., $p = \sigma \circ \pi_G$ for some $\sigma \in X_{G/\equiv}$. Then, (***) yields $a(p) \in D_{\mathbf{3}}(b(p), c(p))$. Since p is an arbitrary element of Y, the conclusion of (**) follows. □

V.9 Saturated prime ideals of spectral real semigroups.

Recall that $\mathrm{Spec}_{\mathrm{sat}}(G)$ denotes the space of saturated prime ideals of a real semigroup G, defined in I.6.19.

Theorem V.9.1 *Let X be a hereditarily normal spectral space. Then, X is homeomorphic to* $\mathrm{Spec}_{\mathrm{sat}}(\mathbf{Sp}(X))$. *Hence, all three spaces X, $X_{\mathbf{Sp}(X)}$ and* $\mathrm{Spec}_{\mathrm{sat}}(\mathbf{Sp}(X))$ *are homeomorphic. In particular,* $\mathrm{Spec}_{\mathrm{sat}}(\mathbf{Sp}(X))$ *is hereditarily normal.*

Proof. The Theorem is proved by the five claims that follow.

Claim 1. Let P be a prime ideal of $\mathbf{Sp}(X)$. Then, the set $\bigcap\{Z(f)\,|\,f\in P\}$ is a chain under the specialization order of X $(Z(f)=f^{-1}[0])$.

Proof of Claim 1. Set $S:=\bigcap\{Z(f)\,|\,f\in P\}$. Assume S contains elements x,y incomparable under $\rightsquigarrow$. Then we have:

(a) $\mathrm{Gen}\,(x)\cap\mathrm{Gen}\,(y)=\emptyset$.

This is clear by hereditary normality: if $z\rightsquigarrow x,y$ for some $z\in X$, then x and y are $\rightsquigarrow$-comparable.

(b) $\mathrm{Gen}\,(x)$ and $\mathrm{Gen}\,(y)$ are quasi-compact.

This holds because $\{x\}$, $\{y\}$ are (obviously) quasi-compact and open sets are generically closed (cf. proof of Theorem V.1.5.)

By Proposition V.1.6 (i) there are disjoint, quasi-compact opens U,V such that $\mathrm{Gen}\,(x)\subseteq U$, $\mathrm{Gen}\,(y)\subseteq V$. Define maps $f,g:X\longrightarrow\mathbf{3}$ by:

$$f\lceil V=1\,,\ f\lceil(X\setminus V)=0\quad\text{and}\quad g\lceil U=1\,,\ g\lceil(X\setminus U)=0\,.$$

Since U,V are quasi-compact open, $f,g\in\mathrm{Sp}(X)$. Also:

$$x\in\mathrm{Gen}\,(x)\subseteq U\subseteq X\setminus V\ \Rightarrow\ f(x)=0\ \text{ and }\ g(x)=1\,;$$

$$y\in\mathrm{Gen}\,(y)\subseteq V\subseteq X\setminus U\ \Rightarrow\ g(y)=0\ \text{ and }\ f(y)=1\,.$$

Since $x,y\in S$, $h(x)=h(y)=0$ for all $h\in P$. This and the preceding lines entail $f,g\notin P$. On the other hand, $U\cap V=\emptyset$ implies $X=(X\setminus U)\cup(X\setminus V)=Z(g)\cup Z(f)=Z(fg)$, i.e., $fg=0\in P$, contradicting the assumption that P is prime. □

Set $S':=S\cap\bigcap_{g\notin P}[\![\,g^2=1\,]\!]$, where $[\![\,g^2=1\,]\!]=\{x\in X\,|\,g^2(x)=1\}$. Since $S'\subseteq S$, S' is a specialization chain of X.

Claim 2. $S'\neq\emptyset$ (hence, $S\neq\emptyset$).

Proof of Claim 2. $S'\neq\emptyset$ means that, there is $x\in X$ such that for all $f\in P$ and $g\notin P$, $f(x)=0$ and $g(x)\neq 0$. We first note:

(a) For all $f\in P$ and $g\notin P$, there is $x\in X$ so that $f(x)=0$ and $g(x)\neq 0$.

Otherwise, $Z(f)\subseteq Z(g)$. By I.6.5 (1) applied with $G=\mathbf{Sp}(X)$, this yields $g=f^2\,g\in P$, absurd.

(b) Now, let $f_1,\dots,f_n\in P\,,g_1,\dots,g_m\notin P$ be arbitrary finite sets. We show:

$$\bigcap_{i=1}^{n}Z(f_i)\cap\bigcap_{j=1}^{m}[\![\,g_j^2=1\,]\!]\neq\emptyset\,.$$

In fact, set $g:=\prod_{j=1}^{m}g_j$; clearly, $g\notin P$ and $\bigcap_{j=1}^{m}[\![\,g_j^2=1\,]\!]=[\![\,g^2=1\,]\!]$. On the other hand, let f be the unique element of $D^t_{\mathbf{Sp}(X)}(f_1^2,\dots,f_n^2)$ (cf. IV.5.3 (i)). We have $Z(f)=\bigcap_{i=1}^{n}Z(f_i)$ and, since P is saturated, $f=f^2\in P$. Thus, $\bigcap_{i=1}^{n}Z(f_i)\cap\bigcap_{j=1}^{m}[\![\,g_j^2=1\,]\!]=Z(f)\cap[\![\,g^2=1\,]\!]$, and, by (a), this set is non-empty.

Next, observe that $Z(f)$ and $[\![\,g^2=1\,]\!]$ are both closed in X_{con}. Then, $\{Z(f)\cap[\![\,g^2=1\,]\!]\,|\,f\in P,g\notin P\}$ is a family of closed sets in X_{con} with (by (b)) the finite intersection property. By compactness,

$$S' = \bigcap \{ Z(f) \cap [\![g^2 = 1]\!] \mid f \in P, g \notin P \} \neq \emptyset \,,$$

proving Claim 2. $\square$

Thus, S' is a non-empty, proconstructible specialization chain of X. By [DST], Thm. 5.2.6 (iii), S' has an infimum in the specialization order, say x_0 (which belongs to S'). We claim:

Claim 3. For $f \in \mathbf{Sp}(X)$, $f \in P \Leftrightarrow f(x_0) = 0$.

Proof of Claim 3. The implication $\Rightarrow$ is obvious, since $x_0 \in S' \subseteq Z(f)$.

($\Leftarrow$) Assume $f \notin P$. Since $x_0 \in S' \subseteq [\![f^2 = 1]\!]$, we have $f^2(x_0) = 1$, i.e., $f(x_0) \neq 0$. $\square$

This shows that the map $\theta : X \longrightarrow \mathrm{Spec}_{\mathrm{sat}}(\mathbf{Sp}(X))$ given by $x \longmapsto \{f \in \mathbf{Sp}(X) \mid f(x) = 0\}$ $(:= P_x)$ is surjective (cf. [M], Prop. 6.5.1, p. 117). We also have:

Claim 4. θ is injective.

Proof of Claim 4. Let $x, y \in X, x \neq y$. Since X is T_0, there is a quasi-compact open set U such that, say, $x \in U, y \notin U$. With $f : X \longrightarrow \mathbf{3}$ defined by $f \lceil U = 1, f \lceil (X \setminus U) = 0$ we have $f \in \mathbf{Sp}(X)$ and $f(y) = 0$, but $f(x) = 1$, i.e., $f \in P_y \setminus P_x$, whence $P_y \neq P_x$. $\square$

Finally we prove:

Claim 5. θ is a homeomorphism.

Proof of Claim 5. (a) Let $f \in \mathbf{Sp}(X)$ and let $D(f) = \{P \in \mathrm{Spec}_{\mathrm{sat}}(\mathbf{Sp}(X)) \mid f \notin P\}$ be the basic open set of $\mathrm{Spec}_{\mathrm{sat}}(\mathbf{Sp}(X))$ defined by f (cf. I.6.19). Since (by Claim 3) every saturated prime ideal of $\mathbf{Sp}(X)$ is of the form P_x for some $x \in X$, we have:

$$\theta^{-1}[D(f)] = \{x \in X \mid f(x) \neq 0\} = f^{-1}[1] \cup f^{-1}[-1]$$

a quasi-compact open subset of X.

(b) Conversely, given a quasi-compact open subset U of X, with $f \in \mathbf{Sp}(X)$ defined by $f \lceil U = 1, f \lceil (X \setminus U) = 0$, we have $\theta[U] = D(f)$; indeed, for $x \in X$,

$$x \in U \Leftrightarrow f(x) = 1 \Leftrightarrow f(x) \neq 0 \Leftrightarrow f \notin P_x \Leftrightarrow P_x \in D(f) \,.$$

This shows that θ^{-1} establishes a bijection between basic quasi-compact open subsets of $\mathrm{Spec}_{\mathrm{sat}}(\mathbf{Sp}(X))$ and quasi-compact opens of X, which proves Claim 5, completing the proof of Theorem V.9.1. $\square$

Remark. Since any homeomorphism between spectral spaces preserves and reflects specialization, the preceding theorem entails:

$$\text{For all } x, y \in X\,, \quad x \rightsquigarrow y \Leftrightarrow P_x \rightsquigarrow P_y \quad (\text{i.e., } P_x \subseteq P_y)\,. \qquad \square$$

Added December 2011.

Proposition V.9.2 *A quotient of a spectral RS modulo a saturated prime ideal is a quasi Boolean algebra*[5].

Proof. Let G be a spectral RS and P be a saturated prime ideal of G. By Theorem II.3.15 (d) and with notation therein, $G' := (G/P) \setminus \{\pi(0)\}$ is a RSG under the representation $D_{G/P}$. We show that G' is a Boolean algebra.

[5] That is, a Boolean algebra as a reduced special group with an added zero; cf. I.2.2 (3).

Since the equivalence relation $\sim_P$ that determines the quotient G/P $(= G/\sim_P)$ is a RS-congruence, Proposition V.7.6 (3) shows that G/P is a spectral RS. Now, the argument of Corollary V.6.7 shows that, under the representation partial order, G' is a Boolean algebra: any element of G' is of the form $a/\sim_P$ with $a \in G \setminus P$. Since G' is a RSG, we have $a^2/\sim_P = 1$, and I.6.5 (7) gives:

$$a/\sim_P \wedge - a/\sim_P = a^2/\sim_P = 1 \quad \text{and} \quad a/\sim_P \vee - a/\sim_P = -a^2/\sim_P = -1,$$

proving, in fact, that G' is a Boolean algebra. □

V.10 Rings whose associated real semigroups are spectral.

In this section we prove, first, that the real semigroup associated to any lattice-ordered ring is spectral. This exhibits a very extensive class of examples of spectral RSs arising from rings. Use of the axiomatisation given in V.7.4 (1) makes the proof rather simple. A significant consequence of this, together with previous results, is that the spectral hull of the RS G_A associated to *any* semi-real ring A is canonically isomorphic to the RS $G_{\overline{A}}$ associated to the real closure $\overline{A}$ of A (real closure in the sense of Schwartz [Sch], see also Prestel-Schwartz, [PS]); further, the canonical embedding η_{G_A} of G_A into $\mathrm{Sp}(G_A)$ (V.4.1 (ii)) is induced by the inclusion of A into $\overline{A}$ (see also [M], Remark (3), p. 178). □

V.10.1 Preliminaries and Notation. We assume known the basics on lattice-ordered rings (abbreviated **ℓ-rings**), for which the reader is referred to [BKW], Chs. 8, 9.

Throughout we assume A is an ℓ-ring. The underlying partial order of A will be denoted by $\leq$ (not to be confused with the representation partial order of $G_A = G_{A,\leq}$, cf. I.6). Without risk of confusion, the lattice operations in both A and G_A will be denoted by $\wedge, \vee$.

V.10.2 Reminder. For ready reference we recall from I.1.2 (e) that the ARS associated to the real semigroup G_A is

$$X_{G_A} = \mathrm{Sper}(A, \leq) = \{\alpha \in \mathrm{Sper}(A) \mid \alpha \text{ contains } \leq\},$$

and that, for $a \in A$ and $\alpha \in X_{G_A}$,

$$(*) \qquad \overline{a}(\alpha) = \begin{cases} 1 & \Leftrightarrow\ a \in \alpha \setminus (-\alpha) \quad \Leftrightarrow\ \pi_\alpha(a) >_\alpha 0 \ \ (\text{in } (A/\mathrm{supp}(\alpha), \leq_\alpha)), \\ 0 & \Leftrightarrow\ a \in \mathrm{supp}(\alpha) \quad \Leftrightarrow\ \pi_\alpha(a) = 0, \\ -1 & \Leftrightarrow\ a \in -\alpha \setminus \alpha \quad \Leftrightarrow\ \pi_\alpha(a) <_\alpha 0. \end{cases}$$

The lattice operations in A induce binary operations $(\overline{a}, \overline{b}) \mapsto \overline{a \wedge b}$ (resp. $\overline{a \vee b}$) in G_A. □

Fact. *The operations $(\overline{a}, \overline{b}) \mapsto \overline{a \wedge b}$ (resp. $\overline{a \vee b}$) are well-defined: for $a, a', b, b' \in A$,*

$$\overline{a} = \overline{a'} \ \textit{and}\ \overline{b} = \overline{b'} \ \textit{imply}\ \overline{a \wedge b} = \overline{a' \wedge b'} \ \textit{and}\ \overline{a \vee b} = \overline{a' \vee b'}.$$

Sketch of proof. We just sketch the idea of the argument for the case $(\overline{a}, \overline{b}) \mapsto \overline{a \wedge b}$. Obviously it suffices to show: $\overline{a} = \overline{a'} \Rightarrow \overline{a \wedge b} = \overline{a' \wedge b}$.

The assumption is $\overline{a}(\alpha) = \overline{a'}(\alpha)$, for $\alpha \in \mathrm{Sper}(A, \leq)$. By V.10.2 (*), this amounts to the fact that $\pi_\alpha(a)$ and $\pi_\alpha(a')$ have the same strict sign (strictly positive, strictly negative or zero) in the order $\leq_\alpha$ of $A/\mathrm{supp}(\alpha)$. Since $\pi_\alpha : (A, \leq) \longrightarrow (A/\mathrm{supp}(\alpha), \leq_\alpha)$ is a homomorphism of ordered rings and the counterdomain is totally ordered, we have

$$(\dagger) \qquad \pi_\alpha(a \wedge b) = \min_{\leq_\alpha} \{\pi_\alpha(a), \pi_\alpha(b)\}.$$

Now proceed by a case-wise argument according to the values of $(\overline{a \wedge b})(\alpha)$, using that $\pi_\alpha(a)$, $\pi_\alpha(a')$ have the same sign in $\leq_\alpha$. $\square$

The simple observations that follow will be used in the proof of our main result, as well as the characterization of the representation partial order of G_A given in Proposition I.6.4 (d).

Fact V.10.3 *Let A be a ℓ-ring and let G_A be its associated RS. For $a, b \in A$ we have:*

(1) $a \leq b \ \Rightarrow\ \overline{b} \leq \overline{a}$.

(2) $\overline{b} \leq \overline{a} \ \Leftrightarrow$ *There is* $a' \in A$ *so that* $\overline{a'} = \overline{a}$ *and* $a' \leq b$.

(3) $\overline{b} \leq \overline{0} \ \Leftrightarrow$ *For all* $\alpha \in \mathrm{Sper}(A, \leq)$, $\pi_\alpha(b) \geq_\alpha 0$.

(4) $\overline{a \wedge b} = \overline{a} \vee \overline{b}$ *and* $\overline{a \vee b} = \overline{a} \wedge \overline{b}$.

Proof. (1) By I.6.4 (d) , we must show, for all $\alpha \in \mathrm{Sper}(A, \leq)$:

(i) $\overline{a}(\alpha) = 1 \ \Rightarrow\ \overline{b}(\alpha) = 1$;

(ii) $\overline{a}(\alpha) = 0 \ \Rightarrow\ \overline{b}(\alpha) \in \{0, 1\}$.

By V.10.2 (*) above, these conditions are equivalent to:

(i$'$) $\pi_\alpha(a) >_\alpha 0 \ \Rightarrow\ \pi_\alpha(b) >_\alpha 0$;

(ii$'$) $\pi_\alpha(a) = 0 \ \Rightarrow\ \pi_\alpha(b) \geq_\alpha 0$.

Since $\pi_\alpha : (A, \leq) \longrightarrow (A/\mathrm{supp}(\alpha), \leq_\alpha)$ is a homomorphism of ordered rings, $a \leq b$ implies $\pi_\alpha(a) \leq_\alpha \pi_\alpha(b)$, from which (i$'$) and (ii$'$) clearly follow.

(2) The implication ($\Leftarrow$) is clear from (1).

($\Rightarrow$) Assuming $\overline{b} \leq \overline{a}$ (in G_A), the implications (i$'$) and (ii$'$) above hold.

Set $a' := a \wedge b$. It remains to show that $\overline{a}(\alpha) = \overline{a'}(\alpha)$ for all $\alpha \in \mathrm{Sper}(A, \leq)$; we argue by cases according to the values of $\overline{a}(\alpha)$:

— $\overline{a}(\alpha) = 1$.

Then, $\pi_\alpha(a) >_\alpha 0$ (see V.10.2 (*)); by (i$'$), $\pi_\alpha(b) >_\alpha 0$, and we get,

$$0 <_\alpha \min_{\leq_\alpha} \{\pi_\alpha(a), \pi_\alpha(b)\} = \pi_\alpha(a \wedge b) = \pi_\alpha(a'),$$

i.e., $\overline{a'}(\alpha) = 1$.

— $\overline{a}(\alpha) = 0$.

By V.10.2 (*), $\pi_\alpha(a) = 0$; (ii$'$) gives $\pi_\alpha(b) \geq_\alpha 0$; thus, $\pi_\alpha(a') = \pi_\alpha(a \wedge b) = \min_{\leq_\alpha} \{\pi_\alpha(a), \pi_\alpha(b)\}$ $= 0$, i.e., $\overline{a'}(\alpha) = 0$.

— $\overline{a}(\alpha) = -1$. An argument similar to the first case yields $\overline{a'}(\alpha) = -1$.

(3) The characterization of $\overline{b} \leq \overline{a}$ set forth in items (i$'$) and (ii$'$) of the proof of (1) applied with $a = 0$, gives, for $\alpha \in \mathrm{Sper}(A, \leq)$:

— $\pi_\alpha(0) >_\alpha 0 \ \Rightarrow\ \pi_\alpha(b) >_\alpha 0$;

— $\pi_\alpha(0) = 0 \ \Rightarrow\ \pi_\alpha(b) \geq_\alpha 0$.

The first implication is vacuously true since its antecedent is false, and the second implication is equivalent to $\pi_\alpha(b) \geq_\alpha 0$ since its antecedent is true.

(4) We only prove the first equality. Since $a \wedge b \le a, b$ (in A), item (1) gives $\overline{a}, \overline{b} \le \overline{a \wedge b}$, whence $\overline{a} \vee \overline{b} \le \overline{a \wedge b}$. To prove the reverse inequality we proceed by cases, according to the values of $(\overline{a \wedge b})(\alpha)$, $\alpha \in \mathrm{Sper}(A, \le)$.

— If $(\overline{a \wedge b})(\alpha) = 1$ there is nothing to prove.

— $(\overline{a \wedge b})(\alpha) = 0$.

By V.10.2 (*), $\pi_\alpha(a \wedge b) = 0$; equality (†) above implies that one of $\pi_\alpha(a), \pi_\alpha(b)$ is 0 and the other is $\ge_\alpha 0$, i.e., either $\overline{a}(\alpha) = 0$ and $\overline{b}(\alpha) \in \{0, 1\}$ or the other way round. Since the order in **3** is $1 < 0 < -1$, this clearly yields $\overline{a}(\alpha) \vee \overline{b}(\alpha) = 0$.

— $(\overline{a \wedge b})(\alpha) = -1$.

From V.10.2 (*) and (†) we get $\min_{\le_\alpha}\{\pi_\alpha(a), \pi_\alpha(b)\} <_\alpha 0$. Then, at least one of $\pi_\alpha(a)$ or $\pi_\alpha(b)$ is $<_\alpha 0$, i.e., $\overline{a}(\alpha) = -1$ or $\overline{b}(\alpha) = -1$, which yields $\overline{a}(\alpha) \vee \overline{b}(\alpha) = -1$. □

Now we turn to the proof of:

Theorem V.10.4 *Let $(A, \le)$ be a ℓ-ring. The real semigroup G_A associated to A is spectral.*

Proof. We show that G_A verifies axioms [SRS1] and [SRS2] of V.7.4. We shall use, mostly without explicit mention, the results of V.10.3, notably that the order reverses in passing from A to G_A and that, for $a, b \in A$ and $\alpha \in \mathrm{Sper}(A, \le)$,

(†) $$\pi_\alpha(a \wedge b) = \min_{\le_\alpha}\{\pi_\alpha(a), \pi_\alpha(b)\} \quad \text{and} \quad \pi_\alpha(a \vee b) = \max_{\le_\alpha}\{\pi_\alpha(a), \pi_\alpha(b)\}\,;$$

[SRS1]. Fix $a \in A$, and set $c = a \vee 0$. Then, $c \ge 0$, whence $\overline{c} \le \overline{0}$, and, by I.6.4, $\overline{c} \in \mathrm{Id}\,(G_A)$; in particular, $\overline{c}(\alpha) \in \{0, 1\}$, i.e., $\pi_\alpha(c) \ge_\alpha 0$ for $\alpha \in \mathrm{Sper}(A, \le)$. Also, $c \ge a$, i.e., $\overline{c} \le \overline{a}$, whence, by I.6.2, $-\overline{a} \in D_{G_A}(1, -\overline{c})$.

To prove $\overline{a}\,\overline{c} = \overline{c}$, let $\alpha \in \mathrm{Sper}(A, \le)$. The equality holds at α if $\overline{c}(\alpha) = 0$. So, assume $\overline{c}(\alpha) = 1$, i.e., $\pi_\alpha(c) >_\alpha 0$; by (†), $\pi_\alpha(c) = \max_{\le_\alpha}\{\pi_\alpha(a), 0\} >_\alpha 0$ which clearly implies $\pi_\alpha(a) >_\alpha 0$, i.e., $\overline{a}(\alpha) = 1$.

[SRS2]. Given $a, b \in A$, set $d = a \vee b$, i.e., $\overline{d} = \overline{a} \wedge \overline{b}$. We show:

(i) $\overline{d} \in D_{G_A}(\overline{a}, \overline{b})$.

That is, we must prove: $\overline{d}(\alpha) \ne 0 \Rightarrow \overline{d}(\alpha) = \overline{a}(\alpha)$ or $\overline{d}(\alpha) = \overline{b}(\alpha)$, for $\alpha \in \mathrm{Sper}(A, \le)$. Since the order $\le_\alpha$ is total, the second equality in (†) yields $\pi_\alpha(d) = \pi_\alpha(a)$ or $\pi_\alpha(d) = \pi_\alpha(a)$, which obviously entails the required conclusion.

(ii) $\overline{d}^{\,+} = -\overline{a}^{\,+} \cdot \overline{b}^{\,+}$.

Since $\overline{z}^{\,+} = \overline{z} \vee \overline{0} \ge \overline{0}$ $(z \in A)$, we get $\overline{z}^{\,+}(\alpha) \in \{0, -1\}$ for $\alpha \in \mathrm{Sper}(A, \le)$. To prove (ii), we argue by cases according to the values of $\overline{d}^{\,+}(\alpha)$.

— $\overline{d}^{\,+}(\alpha) = -1$.

This means $(\overline{d \wedge 0})(\alpha) = -1$, i.e., $\pi_\alpha(d \wedge 0) <_\alpha 0$; by (†), $\pi_\alpha(d) <_\alpha 0$. Since $d = a \vee b$, the second equality in (†) implies that both $\pi_\alpha(a)$ and $\pi_\alpha(b)$ are $<_\alpha 0$, whence $\overline{a}^{\,+}(\alpha) = \overline{b}^{\,+}(\alpha) = -1$. It follows that $-\overline{a}^{\,+}(\alpha) \cdot \overline{b}^{\,+}(\alpha) = -1 = \overline{d}^{\,+}(\alpha)$.

— $\overline{d}^{\,+}(\alpha) = 0$.

That is, $(\overline{d \wedge 0})(\alpha) = 0$, i.e., $\pi_\alpha(d \wedge 0) = 0$. The first equality in (†) gives $\pi_\alpha(d) \ge_\alpha 0$, and (by $d = a \vee b$) the second equality shows that at least one of $\pi_\alpha(a)$ or $\pi_\alpha(b)$ must be $\ge_\alpha 0$. Then, one of $\pi_\alpha(a\wedge 0)$ or $\pi_\alpha(b\wedge 0)$ equals 0, i.e., either $\overline{a}^{\,+}(\alpha)=0$ or $\overline{b}^{\,+}(\alpha)=0$, proving that (ii) holds at α.

(iii) $(\overline{d})^{\,-}\in D^t_{G_A}((\overline{a})^{\,-},(\overline{b})^{\,-})$.

For $z\in A$ we have $(\overline{z})^{\,-}=\overline{z}\wedge\overline{0}\le\overline{0}$, and hence $(\overline{z})^{\,-}(\alpha)\in\{0,1\}$ for $\alpha\in\mathrm{Sper}(A,\le)$. To establish (iii) we must show:

— $(\overline{d})^{\,-}(\alpha)=0 \;\Rightarrow\; (\overline{a})^{\,-}(\alpha)=(\overline{b})^{\,-}(\alpha)=0,$ and

— $(\overline{d})^{\,-}(\alpha)=1 \;\Rightarrow\; (\overline{a})^{\,-}(\alpha)=1$ or $(\overline{b})^{\,-}(\alpha)=1\,.$

For the first implication, the assumption is $(\overline{d\vee 0})(\alpha)=0$, i.e., $\pi_\alpha(d\vee 0)=0$. The first equality in (†) shows that $\pi_\alpha(d)\le_\alpha 0$, and the second (applied with $d=a\vee b$) yields $\pi_\alpha(a)$, $\pi_\alpha(b)\le_\alpha 0$. We get $\pi_\alpha(a\vee 0)=\pi_\alpha(b\vee 0)=0$, i.e., $(\overline{a})^{\,-}(\alpha)=(\overline{b})^{\,-}(\alpha)=0$.

For the second implication, the assumption amounts to $\pi_\alpha(d\vee 0)>_\alpha 0$, which implies $\pi_\alpha(d)>_\alpha 0$. The conclusion to be proved amounts to $\pi_\alpha(a)>_\alpha 0$ or $\pi_\alpha(b)>_\alpha 0$, which obviously follows from the second equality in (†) applied with $d=a\vee b$. □

As a consequence of Theorem V.10.4 and of previous results in this section, we have:

Proposition V.10.5 *Let A be a semi-real ring, let $\overline{A}$ denote its real closure (in the sense of Prestel-Schwartz* [PS]*), and let $\iota : A\longrightarrow\overline{A}$ be the inclusion map. Then,*

(1) *The spectral hull* $\mathrm{Sp}(G_A)$ *of the real semigroup G_A is canonically isomorphic to $G_{\overline{A}}$, the RS associated to $\overline{A}$.*

(2) *The canonical embedding η_{G_A} of G_A into* $\mathrm{Sp}(G_A)$ *(cf. V.4.1 (ii)) is induced by the RS-morphism $\overline{\iota} : G_A\longrightarrow G_{\overline{A}}$ given by the ring inclusion ι.*

Proof. The result is a consequence of the following observations:

— $\mathrm{Sper}\,(A)=\mathrm{Sper}\,(\overline{A})$ (cf. [PS], p. 264) entails $\mathrm{Sp}(G_A)=\mathrm{Sp}(G_{\overline{A}})$; indeed, both these RSs consist of the spectral characters of the space $X=\mathrm{Sper}\,(A)=\mathrm{Sper}\,(\overline{A})$ into $\mathbf{3}_{\mathrm{sp}}$ (V.1.2).

— By Theorem V.5.3 (i) we have a commutative diagram

$$\begin{array}{ccc} G_A & \xrightarrow{\ \overline{\iota}\ } & G_{\overline{A}} \\ {\scriptstyle \eta_{G_A}}\downarrow & & \downarrow{\scriptstyle \eta_{G_{\overline{A}}}} \\ \mathrm{Sp}(G_A) & \xrightarrow[\mathrm{Sp}(\overline{\iota})]{} & \mathrm{Sp}(G_{\overline{A}}) \end{array}$$

The previous observation and the uniqueness in V.5.3 (i) entail that $\mathrm{Sp}(\overline{\iota})$ is the identity of $\mathrm{Sp}(G_A)=\mathrm{Sp}(G_{\overline{A}})$ (the reader can easily check that this identity makes the above diagram commute). Since $\overline{A}$ (ordered by $\overline{A}^2$) is a ℓ-ring (in fact, a reduced f-ring), by V.10.4 $G_{\overline{A}}$ is a spectral RS. Corollary V.4.6 entails, then, that $\eta_{G_{\overline{A}}}$ is an isomorphism of RSs.

Let $\varphi : \mathrm{Sp}(G_A)\longrightarrow G_{\overline{A}}$ be the map $\varphi := \eta^{-1}_{G_{\overline{A}}}\circ\mathrm{Sp}(\overline{\iota})$. By the preceding observation, φ is an isomorphism of RSs, which proves (1). Commutativity of the diagram above then gives $\eta_{G_A}=\varphi^{-1}\circ\overline{\iota}$, which proves (2). □

Remark V.10.6 The well-known Delzell-Madden example of a hereditarily normal spectral space that is not homeomorphic to the real spectrum of any ring, [DeMa], also yields an example of a spectral RS not realizable by a ring: if X denotes this space, the duality RS/ARS

(Theorem I.5.1) and Proposition V.3.7 (2) show that $\mathbf{Sp}(X)$ is not isomorphic to G_A for any ring A. Further, in [M], p. 177, Marshall observes, using a dual terminology, that $\mathbf{Sp}(X)$ cannot even be of the form $G_{A,T}$ for a ring A and a preorder T of A. $\square$

Chapter VI

Fans

Introduction

Our aim in this chapter is to introduce a general notion of "fan" in the dual categories of abstract real spectra (**ARS**)index[sym]**ARS**index[sub]category!**ARS** and of real semigroups (**RS**),index[sym]**RS**index[sub]category!**RS** and to study in detail the properties of these structures. The notion of a fan is a well-known and a central notion in the categories **AOS**index[sym]**AOS**index[sub of abstract order spaces, see [M], Ch. 3, and the dual category **RSG**index[sym]**RSG**index[sub]category!**RSG** of reduced special groups, cf. [DM2], Chs. 1, 3. Furthermore, fans are the building blocks necessary to understand the geometry of AOSs as well as the key to many applications in real geometry; cf. [ABR], Chs. 3, 4. However, little or no attention has so far been paid to the role of fans in the categories **ARS** and **RS**: only a very particular case of the general notion introduced below occurs in [M], p. 162.

To motivate the ideas let us briefly review the definition of a fan in the (dual) categories **AOS** and **RSG**:

— A fan in the category **AOS** (henceforth called an **AOS-fan**)index[sub]AOS-fanindex[sub]fan!AOS- is an abstract space of orders (X, G) where "X is biggest possible"; there are two equivalent ways of making sense of this idea:

(1) X consists of all group homomorphisms $h : G \longrightarrow \{\pm 1\}$ such that $h(-1) = -1$.

(2) (X, G) is an AOS and X is closed under the product of any three of its members.

— A fan in the category **RSG** (henceforth an **RSG-fan**)index[sub]RSG-fanindex[sub]fan!RSG- is a reduced special group G whose binary representation relation is "smallest posible"; there is only one way of making sense of this:

$$a \in D_G(b, c) \quad \text{iff} \quad \text{either} \;\; b = -c \;\; \text{or} \;\; (b \neq -c \;\; \text{and} \;\; a \in \{b, c\}).$$

Remarks. While condition (1) above implies that (X, G) is an AOS, the last requirement in (2) alone is not sufficient to guarantee that (X, G) is an AOS; in addition, one must require that:

(i) X separates points in G, i.e., $\bigcap_{\sigma \in X} \ker(\sigma) = \{1\}$.index[sub]separates points

(ii) X verifies the following maximality condition (see [M], axiom [AX2] for AOSs, p. 22): for every group homomorphism $\sigma : G \longrightarrow \{\pm 1\}$, if $\sigma(-1) = -1$ and $a, b \in ker(\sigma) \Rightarrow D_X(a, b) \subseteq ker(\sigma)$, then $\sigma \in X$.

The definition of binary representation given above (together with $1 \neq -1$) implies that G is a

RSG. □

We shall define the notion of a fan in the category **ARS** of abstract real spectra by postulating the analogs of conditions (1) and (2) above, *upon replacing* the underlying notion of a group of exponent 2 with a distinguished element -1 *by that of a ternary semigroup*:

Definition VI.0.1 Given a ternary semigroup G and a non-empty $X \subseteq \mathbf{3}^G$,

(1) (X, G) is a **fan$_1$** iff X consists of all TS-homomorphisms from G to $\mathbf{3} = \{-1, 0, 1\}$. index[sub]fan@fan$_1$

(2) (X, G) is a **fan$_2$** iff X is an ARS and is closed under the product of any three of its members. index[sub]fan@fan$_2$
We shall frequently use in the sequel the following, weaker notion to which we give a name:

(3) (X, G) is a **q-fan** (quasi-fan) iff X is closed under the product of any three of its members and X separates points in G, i.e., for every $a, b \in G$, $a \neq b$, there is $h \in X$ such that $h(a) \neq h(b)$. index[sub]q-fan □

Note. In (2) we allow products of type $h_1^2 h_2$; as opposed to the case of special groups, squaring a TS-homomorphism does not produce a trivial map. Note also that $h^3 = h$, and that the product of any three TS-homomorphisms is again a TS-homomorphism. □

In section VI.2 below we shall prove that both these notions of fan are equivalent; until then we keep the distinction.

As for the dual category **RS**, we shall prove that, under a suitable necessary condition, q-fans automatically produce real semigroups where both binary representation relations D^t and D are smallest possible.

INSERT DESCRIPTION OF CHAPTER'S CONTENTS HERE.

Remark. Recall that $\mathrm{Hom}_{\mathrm{TS}}(G, \mathbf{3})$ denotes the set of all TS-homomorphisms from the ternary semigroup G into $\mathbf{3}$.

VI.1 Preliminaries

Fact VI.1.1 *Let G be a ternary semigroup, let $X \subseteq \mathrm{Hom}_{\mathrm{TS}}(G, \mathbf{3})$, and assume that (X, G) is a q-fan. A necessary condition for (X, G) to be an ARS is that for all $a, b \in G$, either $Z(a) \subseteq Z(b)$ or $Z(b) \subseteq Z(a)$. Here, $Z(a) = \{h \in X \mid h(a) = 0\}$.*

Proof. Assume $(X, G) \models$ARS but there are $a, b \in G$ so that $Z(a) \nsubseteq Z(b)$ and $Z(b) \nsubseteq Z(a)$, i.e., $h_1(a) = 0$, $h_1(b) \neq 0$, $h_2(b) = 0$, $h_2(a) \neq 0$, for some $h_1, h_2 \in X$. Since (X, G) is a q-fan, $h_1^2 h_2 \in X$. By Lemma II.2.11 (1), $Z(h_1)$ and $Z(h_2)$ are comparable under inclusion, contradicting that $a \in Z(h_1) \setminus Z(h_2)$ and $b \in Z(h_2) \setminus Z(h_1)$. □

Our next result gives some alternative characterizations of the necessary condition in Fact VI.1.1.

Proposition VI.1.2 *Let T be a ternary semigroup. The following conditions are equivalent:*

(1) *The family $\{Z(a) \mid a \in T\}$ is totally ordered under inclusion.*

(2) *For all $a, b \in T$, either $a^2 b^2 = a^2$ or $a^2 b^2 = b^2$.*

(3) *Every proper ideal of T is prime.*

(4) *The set of ideals of T is totally ordered under inclusion.*

Proof. First observe:

(*) $Z(a) \subseteq Z(b) \Leftrightarrow a^2b^2 = b^2$.

This is proved by the argument proving the equivalence of items (ii) and (iii) in Proposition I.6.5(1), using the separation theorem for ternary semigroups (Theorem I.1.12), instead of the corresponding result for real semigroups.

The equivalence of (1) and (2) follows immediately from (*).

(2) $\Rightarrow$ (3). Let I be an ideal of T, and suppose $ab \in I$; then $a^2b^2 \in I$ and, by (2), $a^2 \in I$ or $b^2 \in I$, which implies $a \in I$ or $b \in I$ (as $x = x^3 = x^2x$).

(3) $\Rightarrow$ (4). If J_1, J_2 are incomparable ideals, then $J_1 \cap J_2$ is not prime (if $a \in J_2 \setminus J_1$, $b \in J_1 \setminus J_2$ then $ab \in J_1 \cap J_2$ but $a, b \notin J_1 \cap J_2$).

(4) $\Rightarrow$ (2). Given $a, b \in T$, consider the principal ideals I_a, I_b, I_{ab} generated by a, b, ab, respectively ($I_c = \{x \in G \mid c^2x = x\}$, cf. I.1.11). Clearly we have $I_{ab} \subseteq I_a \cap I_b$. Conversely, let $x \in I_a \cap I_b$, i.e., $x = ay = bz$ for some $y, z \in T$. Then, $x = x^3 = xx^2 = (ay)(b^2z^2) = ab(byz^2) \in I_{ab}$. Hence, $I_{ab} = I_a \cap I_b$. By (4), either $I_a \subseteq I_b$ or $I_b \subseteq I_a$; say the first; then, $I_{ab} = I_a$. The result follows from:

Fact. For $x, y \in T$, $I_x = I_y \Leftrightarrow x^2 = y^2$.

Proof of Fact. ($\Leftarrow$) is clear (since $x^2 = y^2 \Rightarrow x = y^2x \in I_y$).

($\Rightarrow$) From $I_x = I_y$ we get $x = yz$ and $y = xw$ for some $z, w \in T$. Thus, $x^2 = y^2z^2, y^2 = x^2w^2$, whence $x^2 = x^2w^2z^2, y^2 = y^2z^2w^2 = x^2w^2w^2z^2 = x^2w^2z^2$, i.e., $x^2 = y^2$. □

VI.2 Fans are real semigroups and abstract real spectra

To prove the results announced in the title, our first order of business is to work out the explicit form of the representation relations corresponding to the notion of "q-fan". Recall that, given $X \subseteq \mathbf{3}^G$, the relations D_X and D^t_X are defined by the clauses [R] and [TR] of **??** (b), respectively.

The main results to be proved in this section are:

Theorem VI.2.1 *Let G be a ternary semigroup verifying*

[Z] $\forall\, a, b \in G\, (a^2b^2 = a^2 \text{ or } a^2b^2 = b^2)$.

*Let $X \subseteq \mathrm{Hom}_{\mathrm{TS}}(G, \mathbf{3})$ be such that $(X, G) \models$ q-fan. With $D = D_X$ and $D^t = D^t_X$ denoting the representation relations defined by [R] and [TR] of **??** (b), for $a, b \in G$ we have:*

$$[D^t] \quad D^t(a,b) = \begin{cases} \{a\} & \text{if } Z(a) \subset Z(b) \\ \{b\} & \text{if } Z(b) \subset Z(a) \\ \{a, b\} & \text{if } Z(a) = Z(b) \text{ and } b \neq -a \\ \{a^2x \mid x \in G\} & \text{if } b = -a. \end{cases}$$

[D] $D(a, b) = a \cdot \mathrm{Id}(G) \cup b \cdot \mathrm{Id}(G) \cup \{x \in G \mid xa = -xb \wedge x = a^2x\}$.

Note. $xa = -xb$ implies $a^2x = b^2x$. Note also that $\{a, b\} \subseteq D^t_X(a, b)$ whenever $Z(b) = Z(a)$.

Theorem VI.2.2 *Let G be a ternary semigroup verifying condition [Z] of Theorem VI.2.1. Then:*

(1) *Conditions [D] and [D^t] in VI.2.1 are interdefinable in the following sense:*

(*a*) *Assuming that a ternary relation D on G is defined as in [D] and the corresponding transver-*

sal representation is given by the clause

$$a \in D^t(b,c) \Leftrightarrow a \in D(b,c) \wedge -b \in D(-a,c) \wedge -c \in D(b,-a).$$

(cf. [t-rep], section I.2), then D^t verifies condition $[D^t]$ of VI.2.1.

(b) Conversely, if D^t is defined as in $[D^t]$ and the associated ternary representation relation D is defined by the stipulation $a \in D(b,c) \Leftrightarrow a \in D^t(a^2b,\, a^2c)$, then D verifies the equality $[D]$ of VI.2.1.

(2) (G,D) is a real semigroup.

Before engaging in the proof of these theorems we draw some important consequences of them.

Corollary VI.2.3 *Let G be a TS verifying condition [Z] of Theorem VI.2.1 and let $X \subseteq \mathrm{Hom}_{\mathrm{TS}}(G,\mathbf{3})$. The following are equivalent:*

(1) $(X,G) \models \mathrm{fan}_1$ (i.e., $X = \mathrm{Hom}_{\mathrm{TS}}(G,\mathbf{3})$).

(2) (X,G) is a q-fan and verifies axiom [AX2] for ARSs ([M], p. 99): for every subsemigroup S of G such that $S \cup -S = G$ and $S \cap -S$ is a prime ideal, there is $h \in X$ such that $S = h^{-1}[0,1]$.

(3) $(X,G) \models \mathrm{fan}_2$.

Remark. In (2), S is automatically a saturated subsemigroup of (G,D), since the other requirements imply that S contains $\mathrm{Id}(G)$; see Corollary VI.2.8 below.

Proof. (1) $\Rightarrow$ (2). With $X = \mathrm{Hom}_{\mathrm{TS}}(G,\mathbf{3})$, the condition of closure under products in the definition of q-fan is clear. The condition on separating points is exactly the content of the separation theorem for ternary semigroups, I.1.12. For the axiom [AX2], with S a saturated subsemigroup as in the hypotheses of this axiom (cf. [M], p. 99), the map given by:

$$h(x) = \begin{cases} 1 & \text{if } x \in S \setminus (-S) \\ 0 & \text{if } x \in S \cap -S \\ -1 & \text{if } x \in (-S) \setminus S. \end{cases}$$

is in $\mathrm{Hom}_{\mathrm{TS}}(G,\mathbf{3}) = X$ and clearly $S = h^{-1}[0,1]$ (cf. Remark I.1.6 ff).

(2) $\Rightarrow$ (3). Theorems VI.2.1 and VI.2.2 show that if (X,G) is a q-fan, then $(G,D_X) \models \mathrm{RS}$; in particular, the strong associativity axiom [RS3] (i.e., axiom [AX3] of [M], p. 100) holds in (G,D_X); by (2), (X,G) is an ARS, i.e., $(X,G) \models \mathrm{fan}_2$.

(3) $\Rightarrow$ (2). Assumption (3) implies that (X,G) is an ARS; in particular, it satisfies axiom [AX2] of [M], p. 99, and separates points of G. It is also closed under products of any three members of X. Hence, (X,G) satisfies condition (2).

(2) $\Rightarrow$(1). We must prove that $\mathrm{Hom}_{\mathrm{TS}}(G,\mathbf{3}) \subseteq X$. Let $g \in \mathrm{Hom}_{\mathrm{TS}}(G,\mathbf{3})$; set $S := g^{-1}[0,1]$. Remark I.1.6 shows that it verifies the assumptions of axiom [AX2]. By (2) there is $h \in X$ so that $S = h^{-1}[0,1]$, whence $g^{-1}[0] = S \cap -S = h^{-1}[0]$. The equalities $g^{-1}[0,1] = h^{-1}[0,1]$ and $g^{-1}[0] = h^{-1}[0]$ entail $g = h$, and hence $g \in X$. □

Definition and Notation VI.2.4 Henceforth we simply write "**fan**" (or "**ARS-fan**") for either of the equivalent conditions $\mathbf{fan}_1$ or $\mathbf{fan}_2$. In using the notation "$(X,G) \models$ fan" we implicitly assume that the underlying ternary semigroup G verifies condition [Z] in Theorem VI.2.1; this assumption is crucial and, in fact, distinguishes fans from most other classes of ARSs. We shall also say "G is a **fan**" (or an "**RS-fan**"), tacitly assuming that its representation relations are those given in Theorem VI.2.1.

Corollary VI.2.5 *Let G be a TS verifying condition [Z] of Theorem VI.2.1. Let H be a real semigroup, and let $f : G \longrightarrow H$ be a homomorphism of ternary semigroups. Then, f preserves the representation relation D defined by clause [D] of VI.2.1, and hence it is a RS-homomorphism from (G, D) into H. In other words,* $\mathrm{Hom}_{\mathrm{RS}}((G,D),H) = \mathrm{Hom}_{\mathrm{TS}}(G,H)$.

Proof. In view of the definition of D the proof boils down to the following obvious facts:

(1) f preserves products and idempotents, hence the clauses defining the relation D.

(2) For a, b, c in an arbitrary RS, H, we have

(i) $a \cdot \mathrm{Id}(H) \subseteq D_H(a,b)$, and (ii) $ca = -cb \wedge c = a^2c \Rightarrow c \in D_H(a,b)$.

For the proof of (2.ii), observe that, for $h \in X_H$, $h(c) \neq 0$ and $c = a^2c$ imply $h(a) \neq 0$. If $h(c) \neq h(a)$, then $ca = -cb$ yields $h(c) = h(b)$. This shows that $h(c) \in D_{\mathbf{3}}(h(a), h(b))$ for all $h \in X_H$; by the Separation Theorem for RSs (Theorem I.5.4 (1)), we conclude that $c \in D_H(a,b)$. □

In particular we have:

Corollary VI.2.6 *Let G be a TS verifying condition [Z] of Theorem VI.2.1. Then,*

(1) $\mathrm{Hom}_{\mathrm{RS}}((G,D),\mathbf{3}) = \mathrm{Hom}_{\mathrm{TS}}(G,\mathbf{3})$.

Hence,

(2) *The ARS dual to the real semigroup (G,D) is* $(\mathrm{Hom}_{\mathrm{TS}}(G,\mathbf{3}), \overline{G})$. [1]

(3) $(\mathrm{Hom}_{\mathrm{TS}}(G,\mathbf{3}), \overline{G})$ *is a* fan_1 *(hence an ARS-fan, see VI.2.4).*

(4) (G,D) *is a RS-fan.*

Proof. (1) is VI.2.5 with $H = \mathbf{3}$, and (2) comes from the definition of the ARS dual to any RS (see proof of Theorem I.5.1). For (3) and (4) see VI.0.1 (1) and VI.2.7. □

Corollary VI.2.7 *Let (G,D) be a RS-fan. Then, the set $G^\times = \{a \in G \mid a^2 = 1\}$ of invertible elements of G with representation induced by restriction of D to $G^\times$, is a RSG-fan, i.e., a fan in the category of reduced special groups.*

Proof. It readily follows from axiom [RS6] that D_G and D^t_G coincide on $G^\times$ for any RS, G. Since $Z(a) = \emptyset$ for $a \in G^\times$, only the last two clauses in the characterization of D^t_G given by Theorem VI.2.1 apply, whenever $a, b \in G^\times$, and we have:

$$D^t_G(a,b) = \begin{cases} \{a,b\} & \text{if } \ b \neq -a\,, \\ G & \text{if } \ b = -a\,. \end{cases}$$

But this is exactly the definition of representation in a RSG-fan, cf. [DM1], Lemma 1.8, p. 9. Since $D^t_G(a,b) \cap G^\times = D_G(a,b) \cap G^\times$, our contention is proved. □

Corollary VI.2.8 *Let G be a TS verifying condition [Z] of Theorem VI.2.1 and let D be the ternary relation on G defined by clause [D] of that Theorem. Then,*

(1) *Every TS-ideal of G is a saturated ideal of the real semigroup (G,D).*

(2) *A TS-subsemigroup S of G is saturated in (G,D) iff it contains* $\mathrm{Id}(G) = \{x^2 \mid x \in G\}$ *and $S \cap -S$ is an ideal.*

---

[1] Recall that $\overline{G} = \{\overline{a} \mid a \in G\}$, where $\overline{a} \in \mathbf{3}^{\mathrm{Hom}_{\mathrm{TS}}(G,\mathbf{3})}$ is the map "evaluation at a" (see item (I) in proof of I.5.1.

Proof. (1) Straightforward verification.

(2) The implication ($\Rightarrow$) is obvious. For the converse, write $I = S \cap -S$; I is a prime ideal (VI.1.2(3)). Let $a, b \in S$ and $c \in D(a,b) = a\cdot\mathrm{Id}(G) \cup b\cdot\mathrm{Id}(G) \cup \{\, x \in G \mid xa = -xb \wedge x = a^2x \,\}$. If $c \in a\cdot\mathrm{Id}(G)$, then $c = ax^2$, whence $c \in S$, since both a and x^2 are in S. The case $c \in b\cdot\mathrm{Id}(G)$ is similar. If $ca = -cb$ and $c = a^2c$, then $c^2a = -c^2b$, which implies $c^2a \in I$. Since I is prime, either a or c are in I; if $a \in I$, then $c = a^2c \in I$; in both cases we have $c \in I \subseteq S$. $\square$

Proof of Theorem VI.2.1. First we prove:

(A) $(X, G) \models$ q-fan implies that the relation D^t_X (defined by clause [TR] in I.3.2 (b)) verifies condition $[D^t]$ in the statement of the Theorem. Note that:

(1) $Z(a) \subseteq Z(b) \Rightarrow a \in D^t_X(a,b)$.

(immediate verification). Next we prove:

(2) $Z(a) \subseteq Z(b) \wedge b \neq -a \;\Rightarrow\; D^t_X(a,b) \subseteq \{a,b\}$.

Proof of (2). Suppose there is $c \in D^t_X(a,b)$ such that $c \neq a$ and $c \neq b$. Since X separates points, these inequalities, together with $b \neq -a$ give TS-characters $h_1, h_2, h_3 \in X$ whose images at the points a, b, c verify the corresponding inequalities in **3**. By assumption, $h = h_1h_2h_3 \in X$, and h contradicts $c \in D^t_X(a,b)$; more precisely, h verifies either

(*) $\;h(c) = 0\;$ and $\;h(a) \neq -h(b),\;$ or

(**) $\;h(c) \neq 0,\;\; h(c) \neq h(a)\;$ and $\;h(c) \neq h(b)$.

(I) Since $c \neq a$, there is $h_1 \in X$ so that $h_1(c) \neq h_1(a)$. According to the values of $h_1(c) \in \{0, 1, -1\}$, conditions $c \in D^t_X(a,b)$ and $Z(a) \subseteq Z(b)$ yield the following alternatives:

I.a: $\quad h_1(c) = 0\;$ and $\;h_1(a)h_1(b) = -1$.

I.b: $\quad h_1(c) = h_1(b) = 1\;$ and $\;h_1(a) = -1$.

I.c: $\quad h_1(c) = h_1(b) = -1\;$ and $\;h_1(a) = 1$.

(II) Assumption $c \neq b$, yields a character $h_2 \in X$ so that $h_2(c) \neq h_2(b)$. An analysis similar to that of (I) narrows the possible values of h_2 at the points a, b, c down to:

II.a: $\quad h_2(c) = 0\;$ and $\;h_2(a)h_2(b) = -1$.

II.b: $\quad h_2(c) = h_2(a) = 1\;$ and $\;h_2(b) \in \{\,0, -1\}$.

II.c: $\quad h_2(c) = h_2(a) = -1\;$ and $\;h_2(b) \in \{\,0, 1\}$.

(III) The hypothesis $b \neq -a$ gives an $h_3 \in X$ such that $h_3(b) \neq h_3(-a)$. An argument similar to that of (I) and (II), using the assumptions $c \in D^t_X(a,b)$ and $Z(a) \subseteq Z(b)$, shows that h_3 can only take the following combination of values at a, b, c:

III.a: $\quad h_3(a) = h_3(b) = h_3(c) \in \{\pm 1\}$.

III.b: $\quad h_3(b) = 0\;$ and $\;h_3(a) = h_3(c) \in \{\pm 1\}$.

With these data, a long and tedious, but straightforward checking of all possible combinations of the values of the characters h_i ($i = 1, 2, 3$) at the points a, b, c, shows that $h = h_1h_2h_3$ has the properties (*) and (**), contradicting $c \in D^t_X(a,b)$. This proves item (2).

Next we show:

(3) $Z(a) \subset Z(b) \Rightarrow D^t_X(a,b) = \{a\}$.

Proof of (3). $b \in D^t_X(a,b)$ implies $Z(b) \subseteq Z(a)$ (immediate verification); hence $b \notin D^t_X(a,b)$. Since $Z(a) \subset Z(b)$ implies $b \neq -a$, items (1) and (2) give the conclusion.

The assertions (1) through (3) yield at once:

(4) If $b \neq -a$, then $D^t_X(a,b) = \{a,b\} \Leftrightarrow Z(a) = Z(b) \Leftrightarrow a^2 = b^2$.

(5) $c \in D^t_X(a,-a) \Leftrightarrow c = a^2c \Leftrightarrow c = a^2x$ for some $x \in G$.

Proof of (5). The last equivalence is obvious: $c = a^2x$ implies $a^2c = a^2(a^2x) = a^2x = c$. As for the first, we have:

($\Leftarrow$) Let $h \in X$. Obviously, $h(a) = -h(-a)$. The equality $c = a^2c$ implies $Z(a) \subseteq Z(c)$; hence $h(c) \neq 0$ implies $h(a) \neq 0$, and $h(c)$ equals either $h(a)$ or $h(-a)$, proving $c \in D^t_X(a,-a)$.

($\Rightarrow$) For $h \in X$, $c \in D^t_X(a,-a)$ and $h(c) \neq 0$ imply $h(c) = h(a)$ or $h(c) = -h(a)$; hence $h(a) \neq 0$. This shows that $Z(a) \subseteq Z(c)$, which implies $a^2c^2 = c^2$ (cf. (*) in the proof of VI.1.2); scaling by c gives $c = a^2c$.

This completes the proof of statement (A).

Next we deal with the identity $[D]$. We shall, in fact, prove the assertion (1.b) of Theorem VI.2.2, i.e.,

(B) If the ternary relation D^t is defined as in $[D^t]$ of VI.2.1 and the equivalence

($\dagger$) $\quad c \in D(a,b) \Leftrightarrow c \in D^t(c^2a,\, c^2b)$

holds for all $a,b,c \in G$, then D verifies the equality $[D]$. Remark that this equivalence is readily checked for the relations D_X and D^t_X using their definitions [R] and [TR] in I.3.2 (b). We write Id for $\mathrm{Id}(G) = \{x^2 \mid x \in G\}$.

(6) $a \cdot \mathrm{Id} \subseteq D(a,b)$.

Proof of (6). Let $x \in G$. By ($\dagger$) it suffices to show:

($\dagger\dagger$) $\quad ax^2 \in D^t(ax^2,\, a^2x^2b)$.

Since $Z(ax^2) \subseteq Z(a^2x^2b)$, in case $a^2x^2b \neq -ax^2$, the first and third clauses of $[D^t]$ give ($\dagger\dagger$), and in case $a^2x^2b = -ax^2$ the last clause in $[D^t]$ proves ($\dagger\dagger$).

(7) $ca = -cb \wedge c = a^2c \Rightarrow c \in D(a,b)$.

Proof of (7). Since $c^2a = -c^2b$ and $c = a^2c = (c^2a)(ac)$, the last clause in $[D^t]$ yields $c \in D^t(c^2a,\, c^2b)$, whence, by ($\dagger$), $c \in D(a,b)$.

Items (6) and (7) prove the inclusion $\supseteq$ in $[D]$. Conversely, assuming $c \in D(a,b)$, we have $c \in D^t(c^2a,\, c^2b)$, by ($\dagger$). An analysis according to the inclusions of the zero-sets of c^2a and c^2b gives:

(8) If $Z(c^2a) \subseteq Z(c^2b)$ and $c^2a \neq -c^2b$, then $c \in D^t(c^2a,\, c^2b) \subseteq \{c^2a,\, c^2b\}$, implying $c \in a \cdot \mathrm{Id} \cup b \cdot \mathrm{Id}$.

(9) If $c^2a = -c^2b$, scaling by c gives $ca = -cb$ and (by the last clause in $[D^t]$) $c = c^2a^2x$ for some $x \in G$; this proves $a^2c = a^2(c^2a^2x) = c^2a^2x = c$, as required. $\square$

Proof of Theorem VI.2.2. Item (1.b) has just been proved ((B), proof of VI.2.1).

Proof of (1.a). Assume that G is as in the statement, that the ternary relation D is defined by $[D]$ of VI.2.1, and that D^t is given by:

$$c \in D^t(a,b) \Leftrightarrow c \in D(a,b) \wedge -a \in D(-c,b) \wedge -b \in D(a,-c).$$

The right-hand side of this equivalence amounts to:

(I) $\quad c \in a \cdot \mathrm{Id}(G) \cup b \cdot \mathrm{Id}(G) \cup \{x \in G \mid xa = -xb \wedge x = a^2x\}$.

(II) $\quad -a \in -c \cdot \mathrm{Id}(G) \cup b \cdot \mathrm{Id}(G) \cup \{x \in G \mid xc = xb \wedge x = c^2x\}$.

(III) $\quad -b \in a \cdot \mathrm{Id}(G) \cup -c \cdot \mathrm{Id}(G) \cup \{x \in G \mid xa = xc \wedge x = a^2x\}$.

As above we write Id for $\mathrm{Id}(G)$. Remark that

(*) $\quad x \in y \cdot \mathrm{Id} \Leftrightarrow x = yx^2, \qquad$ and $\qquad$ (**) $\quad xy = xz \Rightarrow xy^2 = xz^2$.

We argue by cases, according to the various clauses in $[D^t]$.

(1) $Z(a) \subset Z(b) \Rightarrow c = a$.

The clauses $-a \in b \cdot \mathrm{Id}$ and $-a = c^2(-a) = b^2(-a)$ (see (**)) in (II) imply $Z(b) \subseteq Z(a)$, and hence are excluded; thus, (II) reduces to $a \in c \cdot \mathrm{Id}$. The following cases arise from (I) and (II):

(1.i) $\; c \in a \cdot \mathrm{Id}$ and $a \in c \cdot \mathrm{Id}$.

By (*), $c = ac^2$ and $a = ca^2$. Hence, $c = ac^2 = (ca^2)c^2 = ca^2 = a$.

(1.ii) $\; c \in b \cdot \mathrm{Id}$ and $a \in c \cdot \mathrm{Id}$.

Then, $c = bc^2$ and $a = ca^2$, implying $a = ca^2 = a^2c^2b$; it follows that $Z(b) \subseteq Z(a)$, contrary to the assumption in (1).

(1.iii) $\; ca = -cb \wedge c = a^2c = b^2c \wedge a \in c \cdot \mathrm{Id}$.

The middle equality implies $Z(b) \subseteq Z(c)$ and the last $Z(c) \subseteq Z(a)$. Hence $Z(b) \subseteq Z(a)$, and this case is also excluded.

(2) $Z(b) \subset Z(a) \Rightarrow c = b$.

Same argument as in (1) interchanging a and b.

(3) $Z(a) = Z(b) \wedge b \neq -a \Rightarrow c \in \{a, b\}$.

The first assumption gives $a^2 = b^2$. Each of the clauses $-a \in b \cdot \mathrm{Id}$ in (II) and $-b \in a \cdot \mathrm{Id}$ in (III) yield $-a = ba^2 = b$ and hence are excluded. From (I) – (III) the following cases arise:

(3.i) $\; c \in a \cdot \mathrm{Id}$ and $a \in c \cdot \mathrm{Id}$.

We have $c = ac^2$ and $a = ca^2$; as in (1.i) we get $c = a$.

(3.ii) $\; c \in a \cdot \mathrm{Id}$, $ac = ab$ and $a = c^2a$.

The first term gives $c = ac^2$; hence $a = c$. The cases

(3.iii) $\; c \in b \cdot \mathrm{Id}$ and $-b \in -c \cdot \mathrm{Id}$, and

(3.iv) $\; c \in b \cdot \mathrm{Id}$, $ab = ac$ and $b = c^2b$,

are similar to (3.i) and (3.ii) —with b replacing a—, and yield $c = b$.

(3.v) $\; ca = -cb$, $c = a^2c = b^2c$ and $-a \in -c \cdot \mathrm{Id}$.

As in (3.ii) this gives $c = a^2c = a$ (see (*)).

(3.vi) $ca = -cb$, $c = a^2c = b^2c$ and $-b \in -c \cdot \mathrm{Id}$.

As in (3.v) we obtain $c = b$.

(3.vii) $ca = -cb$, $c = a^2c = b^2c$, $ac = ab$, $a = c^2a$, $ab = bc$ and $b = c^2b$ (the last disjunct from (I), (II) and (III)).

We have $ac = -bc$, $ac = ab$ and $ab = bc$; hence $bc = -bc$. Scaling by b, $b^2c = -b^2c$, whence $c = -c$. It follows that $c = 0$, which clearly implies $a = b = 0$, i.e., a, b, c are all 0.

(4) $b = -a \Rightarrow c = a^2c = b^2c$.

Each disjunct in (I) implies $c = a^2c$. The third disjunct contains this condition. If $c \in a \cdot \mathrm{Id}$, then $c = ac^2$ (by (*)); hence $a^2c = a^2(ac^2) = ac^2 = c$. Likewise, $c \in b \cdot \mathrm{Id}$, implies $c = b^2c = a^2c$.

Proof of (2). Checking that (G, D) verifies the axioms for real semigroups is straightforward, except for [RS3], [RS4] and [RS7].

[RS4] $e \in D(c^2a, d^2b) \Rightarrow e \in D(a, b)$.

The antecedent means:

$$e \in ac^2 \cdot \mathrm{Id} \cup bd^2 \cdot \mathrm{Id} \cup \{\, x \in G \,|\, xc^2a = -xd^2b \wedge x = c^2a^2x = d^2b^2x \,\}.$$

Obviously $xy^2 \cdot \mathrm{Id} \subseteq x \cdot \mathrm{Id}$, and hence the first two disjuncts give $e \in a \cdot \mathrm{Id} \cup b \cdot \mathrm{Id}$. So, assume the last clause holds:

(i) $eac^2 = -ebd^2$ and (ii) $e = a^2c^2e = b^2d^2e$,

and prove the required conclusion: $ea = -eb$ and $e = a^2e = b^2e$.

— Scaling (ii) by a^2 and by b^2 gives $ea^2 = ea^2c^2$ and $eb^2 = eb^2d^2$; by (ii) again, $e = a^2e = b^2e$.

— From (ii) and (i) we get: (iii) $e = ea^2c^2 = (eac^2)a = -(ebd^2)a$,

which, in turn, yields: (iv) $ea = -(ebd^2)a^2 = -(eba^2)d^2$.

Succesively scaling by b in $e = a^2e$ (already proved) and in (ii) gives $eb = eba^2$ and $eb = ebd^2$; substituting in (iv), yields $ea = -(eba^2)d^2 = -ebd^2 = -eb$, as required.

[RS7] $D^t(a, -b) \cap D^t(b, -a) \neq \emptyset \Rightarrow a = b$.

Invoking item (1.b), we can use the expression for D^t given by $[D^t]$ in Theorem VI.2.1, and analyze according to the inclusions of the zero-sets of a and b.

(i) $Z(a) \subset Z(b)$.

In this case $[D^t]$ shows that $D^t(a, -b) = \{a\}$ and $D^t(-a, b) = \{-a\}$. The hypothesis of [RS7] implies $a = -a$, which yields $a = 0$. It follows that $Z(a) = \mathrm{Hom}_{\mathrm{TS}}(G, \mathbf{3})$, contradicting (i).

(ii) $Z(b) \subset Z(a)$.

Same argument as in (i).

(iii) $Z(b) = Z(a)$.

If $b \neq a$, the third clause in $[D^t]$ gives $D^t(a, -b) = \{a, -b\}$ and $D^t(-a, b) = \{-a, b\}$; the hypothesis of [RS7] entails, then, that $a = -a$, and hence $a = 0$. It follows that $Z(a) = \mathrm{Hom}_{\mathrm{TS}}(G, \mathbf{3}) = Z(b)$; the separation theorem for ternary semigroups I.1.12 implies $b = 0$, contradicting $b \neq a$.

[RS3] We shall instead check the equivalent statement

[RS3′] $\forall\, a, b, c, d\, (D^t(a, b) \cap D^t(c, d) \neq \emptyset \ \Rightarrow\ D^t(a, -c) \cap D^t(-b, d) \neq \emptyset)$,

of Proposition I.2.10. This makes the proof shorter than a direct verification of [RS3]. To abridge, we shall refer to the hypothesis of [RS3′] as (*) and to its conclusion as (**). The characterization of transversal representation for fans in $[D^t]$ of Theorem VI.2.1 will be of constant use and will be simply referred to as "$[D^t]$".

We consider three cases and therein argue according to the mutual inclusions of the zero-sets of a, b, c, d.

(1) $a = -b$.

(1.i) $Z(c) \subset Z(a)$.

By the last clause in $[D^t]$ and (*) there is $x \in G$ so that $a^2 x \in D^t(c, d)$. If $Z(c) \subset Z(d)$, we would have $D^t(c, d) = \{c\}$, whence $a^2 x = c$, which implies $Z(a) \subseteq Z(c)$, contrary to (1.i). Hence, $Z(d) \subseteq Z(c) \subset Z(a)$, and this yields $a^2 x \neq c, d$, implying $D^t(c, d) \not\subseteq \{c, d\}$. By $[D^t]$ we then have $d = -c$, and it follows that $D^t(a, -c) \cap D^t(-b, d) = D^t(a, -c) = \{-c\} \neq \emptyset$.

(1.ii) $Z(a) \subseteq Z(c)$.

From $[D^t]$ we have $a \in D^t(a, -c)$, and show that $a \in D^t(-b, d) = D^t(a, d)$. Otherwise, by $[D^t]$ again, we must have $Z(d) \subset Z(a) \subseteq Z(c)$, and assumption (*) gives an $x \in G$ such that $a^2 x \in D^t(c, d) = \{d\}$, implying $Z(a) \subseteq Z(d)$, a contradiction.

(2) $c = -d$. Argument similar to that of case (1).

(3) $a \neq -b$ and $c \neq -d$.

The first three clauses of $[D^t]$ show that $D^t(a, b) \subseteq \{a, b\}$ and $D^t(c, d) \subseteq \{c, d\}$. We consider the following subcases:

(3.i) $Z(a) \subset Z(b)$ and $Z(c) \subset Z(d)$.

In this case $[D^t]$ and (*) imply $a = c$, and (**) reduces to $D^t(a, -a) \cap D^t(-b, d) \neq \emptyset$. Since $Z(a) = Z(c) \subset Z(b), Z(d)$, we get $b = a^2 b$ and $d = a^2 d$ (cf. Proposition I.6.5 (1)). If $b \neq d$, the first three clauses of $[D^t]$ show that one of $-b$ or d is in $D^t(-b, d)$, and the preceding equalities give $a^2 x \in D^t(-b, d)$ for some $x \in G$. By the last clause of $[D^t]$ this also holds if $b = d$, proving that $D^t(a, -a) \cap D^t(-b, d) \neq \emptyset$, as required.

(3.ii) $Z(a) \subset Z(b)$ and $Z(d) \subseteq Z(c)$.

From (*) we have $a \in D^t(c, d)$, whence $a = c$ or $a = d$. In either case, $Z(d) \subset Z(b)$, whence $D^t(-b, d) = \{d\}$, and we are reduced to prove $d \in D^t(a, -c)$.

In case $a = c$ we must show that $a^2 x = c^2 x = d$ for some $x \in G$. If $Z(d) = Z(a)$ this holds with $x = d$ by I.6.5 (1). If $Z(d) \subset Z(a) = Z(c)$, (*) gives $a \in D^t(c, d) = \{d\}$, leading to $a = c = d$, a contradiction.

Finally, in case $a = d$, since $D^t(-b, d) = \{d\}$, we are reduced to prove $d \in D^t(d, -c)$. Since we may assume $d \neq c$, this follows from $Z(d) \subseteq Z(c)$, using $[D^t]$.

(3.iii) $Z(a) = Z(b)$ and $Z(c) \subset Z(d)$.

Assumptions (*) and (3) imply $c = a$ or $c = b$, and we have $Z(a) = Z(b) = Z(c)$. Then, the conclusion (**) boils down to $-b \in D^t(a,-c)$. If $c = b$ this follows from $Z(a) = Z(-b)$ by the third clause of $[D^t]$. If $c = a$, then $Z(a) = Z(-b)$ implies $-b = a^2(-b)$, and the conclusion holds as well.

(3.iv) $Z(a) = Z(b)$ and $Z(d) \subseteq Z(c)$.

If $Z(d) \subset Z(c)$, $[D^t]$ and the assumptions (*) and (3) give $d \in D^t(a,b) = \{a,b\}$. If $d = a$, the desired conclusion boils down to $a \in D^t(-b,a)$, as $D^t(a,-c) = \{a\}$. Since $Z(a) = Z(-b)$, this holds by the last two clauses of $[D^t]$. If $d = b$ (and $a \neq b$), conclusion (**) reduces to $a \in D^t(-b,b)$, since $D^t(a,-c) = \{a\}$. But $Z(a) = Z(b)$ implies $a = b^2 a \in D^t(-b,b)$.

If $Z(d) = Z(c)$, assumptions (*) and (3) give $\{a,b\} \cap \{c,d\} \neq \emptyset$. It follows that $Z(a) = Z(b) = Z(c) = Z(d)$, and then $\{a,-c\} \subseteq D^t(a,-c)$, $\{-b,d\} \subseteq D^t(-b,d)$. If $a = c$, then $Z(a) = Z(c) = Z(d)$ entails $d \in D^t(a,-c)$, and (**) follows. If $d = b$, the same argument shows $a \in D^t(-b,d)$. If $a \neq c$ and $d \neq b$, then $a = d$ and $b = c$; this obviously implies $\{a,-c\} \cap \{-b,d\} \neq \emptyset$, whence $D^t(a,-c) \cap D^t(-b,d) \neq \emptyset$.

(3.v) $Z(b) \subset Z(a)$ and $Z(c) \subset Z(d)$.

In this case we have $D^t(a,b) = \{b\}$ and $D^t(c,d) = \{c\}$, whence $b = c$, by (*). Therefore $Z(c) \subset Z(a)$, $Z(b) \subset Z(d)$, which yields $D^t(a,-c) = \{-c\}$, $D^t(-b,d) = \{-b\}$, and hence (**).

(3.vi) $Z(b) \subset Z(a)$ and $Z(d) \subseteq Z(c)$.

The first clause of $[D^t]$ gives $D^t(a,b) = \{b\}$, and hence $b = c$ or $b = d$, by (*) and (3). In the latter case we must show that $b^2 x \in D^t(a,-c)$ for some $x \in G$. If $a \in D^t(a,-c)$, since $b^2 a = a$ (as $Z(b) \subset Z(a)$), it suffices to take $x = a$. If $a \notin D^t(a,-c)$, then $Z(c) \subseteq Z(a)$, and the second and fourth clauses of $[D^t]$ yield $-c \in D^t(a,-c)$; since $b^2(-c) = -c$ ($Z(b) \subseteq Z(c)$), we can take $x = -c$.

Finally, if $b = c$, then $D^t(a,-c) = \{-c\}$. If $-c \notin D^t(-c,d) = D^t(-b,d)$, then $Z(d) \subset Z(c) = Z(b) \subset Z(a)$, and (*) yields $b = d$, a contradiction. Thus, $-c \in D^t(-b,d)$, verifying (**) and completing the proof of Theorem VI.2.2. □

VI.2.9 A digression on q-fans.index[sub]q-fan—(

The results proved above use in a crucial way the auxiliary —but nonetheless important— notion of a q-fan, introduced in VI.0.1 (3). In Corollary VI.2.3 we proved that q-fans verifying Marshall's axiom [AX2] for ARSs are the same thing as fans. We shall now examine to what extent this notion is genuinely weaker than that of a fan.

Proposition VI.2.10 *Let G be a ternary semigroup and let $X \subseteq X_G = \mathrm{Hom}_{\mathrm{TS}}(G,\mathbf{3})$ be a non-empty set of TS-characters closed under product of any three of its members. Then,*

$$(X,G) \text{ is a q-fan} \Leftrightarrow X \text{ is dense for the constructible topology of } X_G.$$

In particular, if X is proconstructible (e.g., if it is finite, then $X = X_G$, and hence (X,G) is a fan.

Sketch of proof. ($\Rightarrow$) The sets of the form

$$U = U(a_1) \cap \ldots \cap U(a_n) \cap Z(b_1) \cap \ldots \cap Z(b_k),$$

with $a_1,\ldots,a_n,b_1,\ldots,b_k \in G$, form a basis for the constructible topology of X_G, see I.1.17. We must show that $U \neq \emptyset \Rightarrow U \cap X \neq \emptyset$.

This is proved by exactly the same inductive argument used in the proof of item (1) in Theorem I.1.27. This argument only uses the defining properties of q-fans: X separates points in G, and X is closed under products of any three elements. We leave the details to the interested reader. [Note. The set $\mathcal{H}_2$ in the proof of I.1.27 (1) is to be replaced by X, and the congruence $\equiv_{\mathcal{H}_2}$ by equality.]

($\Leftarrow$) Since X is assumed to be closed under products of any three of its members, we need only show that X separates points in G.

Let $a \neq b$. By the separation theorem for ternary semigroups (I.1.12), $h(a) \neq h(b)$ for some $h \in X_G$, i.e., the set $\{h \in X_G \mid h(a) \neq h(b)\}$ is non-empty. This set is also open in the constructible topology of X_G (cf. proof of **??** (ii)), and hence it intersects X, i.e., there is $h \in X$ such that $h(a) \neq h(b)$. $\square$

Example VI.2.11 A q-fan that is not a fan.

Our example relies on the construction, in **??**, of a ternary semigroup out of a pair of 2-semigroups G, H, and a 2-semigroup isomorphism between them. Here we choose G to be the closed real interval $[-r, 0]$, where r is a positive real, with product = the maximum of the two factors in the order of the real line, $H\,(= -G)$ the symmetric interval $[0, r]$ with product = the minimum of the factors, the isomorphism being the flipping map $x \mapsto -x$ ($x \in G$). The unit $1_G = 1_T$ is $-r$ and the absorbent element $0_G = 0_T$ is 0. Thus, with obvious identifications, $T = [-r, r]$, $-1_T = r$, and the semigroup operation of T —which we denote by $\cdot$ instead of the $*$ used in **??**— is as follows: for $x, y \in T$,

$$(*) \qquad x \cdot y = \begin{cases} \max\{x, y\} & \text{if } x, y \in G = [-r, 0] \\ -\min\{x, y\} & \text{if } x, y \in -G = [0, r] \\ -\max\{-x, y\} & \text{if } y \in G = [-r, 0], x \in -G = [0, r] \\ -\max\{-y, x\} & \text{if } x \in G = [-r, 0], y \in -G = [0, r]. \end{cases}$$

With this definition T is a TS (Proposition **??**), and we have:

Lemma VI.2.12 *The TS-characters of T are exactly the monotone functions from $[-r, r]$ with the order of the real line into* **3** *with the order $1 < 0 < -1$, that map $1_T = -r$, $0_T = 0$ and $-1_T = r$ onto $1, 0, -1$, respectively.*

Proof. First note that with the order $1 < 0 < -1$, multiplication in **3** is also given by $(*)$.

To illustrate the argument we prove the asserted equivalence for the first clause of $(*)$, i.e., when $x, y \in G = [-r, 0]$; the remaining cases are similar.

($\Rightarrow$) $x \leq y \;\Rightarrow\; h(x) \leq h(y)$ for $h \in X_T$.

Our assumptions give: $x \cdot y = \max\{x, y\} = y$, whence $h(xy) = h(y)$ and then $h(x)h(y) = h(y)$ since h is a semigroup homomorphism. Since product in **3** is given by the law $(*)$, $h(x)h(y) = \max\{h(x), h(y)\} = h(y)$, whence $h(x) \leq h(y)$.

($\Leftarrow$) Assume h is monotone, $h(i_T) = i$ for $i \in \{1, 0, -1\}$, and $x, y \in [-r, 0]$. If, for instance, $x \leq y$, we have $x \cdot y = \max\{x, y\} = y$, $h(x) \leq h(y)$, and hence $h(x)h(y) = \max\{h(x), h(y)\} = h(y)$. Thus, $h(xy) = h(y) = h(x)h(y)$, showing that h preserves product. $\square$

Corollary VI.2.13 *The correspondence $h \mapsto h^{-1}[-1]$ ($h \in X_T$) is a bijection between X_T and the set of intervals in $\mathbb{R}$ of the forms $(a, r]$ with $0 \leq a < r$, and $[a, r]$ with $0 < a \leq r$.*

Proof. Let $h \in X_T$, and let $a = \inf h^{-1}[-1]$. Monotonicity of h (Lemma VI.2.12) implies that $h^{-1}[-1]$ is either $(a, r]$ or $[a, r]$. Since $h(0_T) = 0$ we clearly have $0 < a \leq r$ if $h^{-1}[-1] = [a, r]$, and $0 \leq a < r$ if $h^{-1}[-1] = (a, r]$. Note also that $h^{-1}[0]$ is a symmetric interval of form $(-a, a)$ or $[-a, a]$, since $h(x) = 0 \Rightarrow h(-x) = 0$. Obviously, the correspondence $h \mapsto h^{-1}[-1]$

is injective. It also is surjective (onto the set of intervals of the statement) for, given $0 < a \leq r$, the map

$$h(x) = \begin{cases} 1 & \text{if } x \in [-r, -a] \\ 0 & \text{if } x \in (-a, a) \\ -1 & \text{if } x \in [a, r], \end{cases}$$

is non-decreasing and carries $1_T = -r$, $0_T = 0$ and $-1_T = r$ onto $1, 0, -1$, respectively; by the preceding Lemma it defines a TS-character of T. An obvious variant of this argument takes cares of the case of left-open intervals $(a, r]$. □

Now we define $X \subseteq X_T$ to be the set of all $h \in X_T$ such that the left endpoint, a, of the interval $h^{-1}[-1]$ is rational. Density of $\mathbb{Q}$ obviously entails:

(i) X separates points in T.

Further,

(ii) X is stable under the product of any three elements.

Let $h_1, h_2, h_3 \in X$ and let $a_i = \inf h_i^{-1}[-1] \in \mathbb{Q} \cap [-r, r]$ $(i = 1, 2, 3)$. Since $h_i \lceil (-a_i, a_i) = 0$ we have $\inf (h_1 h_2 h_3)^{-1}[-1] = \max \{a_1, a_2, a_3\} \in \mathbb{Q}$, and hence $h_1 h_2 h_3 \in X$.

Then, X is a q-fan. However, since the left endpoint of the interval $h^{-1}[-1]$ can also take irrational values, we have $X \neq X_T$. □index[sub]q-fan—)

VI.3 Examples

With the aim of illustrating the notions introduced above, we present in this section some examples of (alas, finite) fans based on ternary semigroups with up to three generators. For each example we shall draw both the root-system of an ARS-fan ordered under specialization and the representation partial order of its dual real semigroup.

In order to determine the representation partial order in the examples below, we will need the following supplement to Propositions I.6.4 and I.6.5, valid in the case of fans.

Lemma VI.3.1 *Let G be a RS-fan, and let $\leq$ denote its representation partial order (§I.6). For a non-invertible element $x \in G$, and an invertible $w \in G$, we have:*

(1) *If $x \leq w$, then $w = -1$.* (2) *If $w \leq x$, then $w = 1$.*

In other words, x and w are $\leq$-incomparable, unless $w \in \{\pm 1\}$.

(3) *If $v \in G$ is invertible, $v \neq w$ and $v, w \notin \{\pm 1\}$, then v and w are $\leq$-incomparable.*

Proof. By assumption, $x^2 \neq 1$. Hence, there is $h_0 \in X_G$ so that $h_0(x^2) \neq 1$, whence, $h_0(x) = 0$ (Theorem I.1.12). Note also that $h(w) \neq 0$ for all $h \in X_G$.

(1) Assume $w \neq -1$; there is $h_1 \in X_G$ such that $h_1(w) = 1$. If $h_0(w) = 1$, we have $h_0(x) = 0 >_{\mathbf{3}} 1 = h_0(w)$. If $h_0(w) = -1$, let $h = h_0^2 h_1 \in X_G$; then, $h(x) = 0 >_{\mathbf{3}} 1 = h(w)$. By I.6.4(d) this proves $x \not\leq w$.

(2) Apply (1) to $-x, -w$ instead of x, w.

(3) By the Separation Theorem I.1.12, there are characters $h_0, h_1, h_2, h'_1, h'_2 \in X_G$ such that:

$$h_0(v) \neq h_0(w), \ \ h_1(v) = 1, \ \ h_2(v) = -1, \ \ h'_1(w) = 1, \ \ h'_2(w) = -1.$$

To fix ideas, let us suppose that $h_0(v) = 1$ and $h_0(w) = -1$; the reverse case is similar and left to the reader. Then, $h_0(w) >_{\mathbf{3}} h_0(v)$, and this already shows that $w \not\leq v$ (I.6.4(d)). To prove $v \not\leq w$ we must get a character $h \in X_G$ so that $h(v) = -1$ and $h(w) = 1$. If $h_2(w) = 1$ take $h = h_2$, and if $h'_1(v) = -1$, put $h = h'_1$. In the remaining case where $h_2(w) = -1$ and $h'_1(v) = 1$, take $h = h_0 h'_1 h_2$. □

VI.3.2 The examples. Recall that condition $[Z]$ in Theorem VI.2.1,

(*) $\forall\, ab\, (a^2b^2 \in \{a^2, b^2\})$,

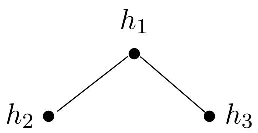
is a necessary condition to obtain a fan. Once this is fulfilled, the representation relations defined in VI.2.1 automatically turn the underlying ternary semigroup into an RS-fan.

Example VI.3.2. A. Ternary semigroups on one generator.

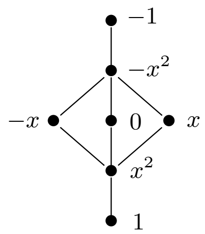
Call x the generator. We treat first the case where there are no additional relations ("free" case). The corresponding TS is:

$$F_1 = \{1, 0, -1, x, -x, x^2, -x^2\}.$$

The necessary condition (*) is trivially verified. Characters are determined by their value on x, and any value 1, 0 and -1 is possible; hence the dual ARS, X_{F_1}, consists of three characters given by: $h_1(x) = 0$, $h_2(x) = 1$, $h_3(x) = -1$. Clearly, $h_1 = h_1^2 \cdot h_i$, whence $h_i \preceq h_1$, for $i = 2, 3$ (Lemma I.1.18). So we get the specialization root-system below left.



Specialization root-system of X_{F_1}

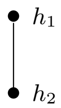


Representation partial order of F_1

Figure 1

The representation partial order of the real semigroup F_1 —illustrated in Figure 1, right— is computed straightforwardly from Proposition I.6.4.

Remark. Barring the case where the generator x becomes invertible (i.e., $x^2 = 1$, which gives a four element RSG-fan with an added 0, cf. I.2.2(3)), the only possible additional relation is $x^2 = x$, which eliminates the character h_3. Thus, we get the following diagrams for the specialization order (left) and the representation order (right):



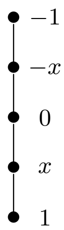


Figure 2

More interesting examples are given in the sequel.

Example VI.3.2. B. Ternary semigroups on three generators.

Generators: x, y, z. Condition (*) above gives raise to the following possible relations:

1. $x^2 = y^2 = z^2$. [(*) is automatically verified in this case.]

2. $x^2 = y^2 \neq z^2$ and $x^2 z^2 = y^2 z^2 \in \{x^2, z^2\}$.

The two identities obtained from the last clause give raise to non-isomorphic cases, and, upon permutation, all cases where two of the three generators have equal squares (i.e., equal zero-sets) are isomorphic to these.

3. x^2, y^2, z^2 are different, and $x^2y^2 \in \{x^2, y^2\}$, $x^2z^2 \in \{x^2, z^2\}$, $y^2z^2 \in \{y^2, z^2\}$.

A case-by-case analysis of all eight combinations of these values shows that, up to isomorphism by permutation, the only surviving case is where $x^2y^2 = x^2z^2 = x^2$ and $y^2z^2 = y^2$.

In order to abridge we shall only analyze some of the configurations arising in case (B.2).

(a) $x^2 = y^2 \neq z^2$ and $x^2z^2 = y^2z^2 = x^2$.

This amounts to $Z(z) \subset Z(x) = Z(y)$ (I.1.19). We consider three alternatives:

i) No relations other than the above.

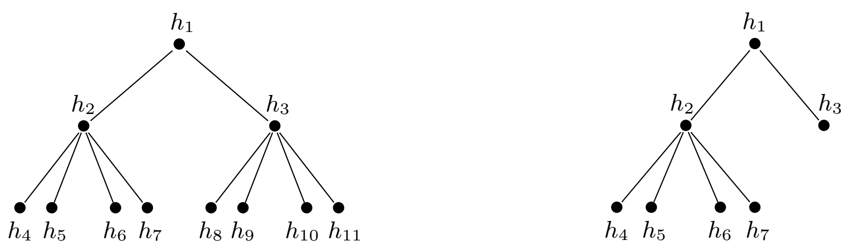
Routine checking shows that the following are all possible characters:

— h_1 sends all three generators to 0;

— h_2, h_3 send x, y to 0 and, say, $h_2(z) = 1$, $h_3(z) = -1$;

— $h_4, \ldots, h_{11}$ assign to the generators all possible combinations of values ± 1, with, say, $h_4, \ldots, h_7$ sending z to 1, and $h_8, \ldots, h_{11}$ sending z to -1.

Call F_2 the TS corresponding to this case. Using Lemma I.1.18 one sees at once that the specialization root-system of the ARS dual to F_2 looks as in Figure 3 below left.



Specialization root-system of X_{F_2} Specialization root-system of $X_{F_2'}$

Figure 3

Since X_{F_2} has 11 elements, by Corollary VI.6.18 we must have $\text{card}\,(F_2) = 23$; the reader is invited to check that:

$$F_2 = \{1, 0, -1, x, -x, y, -y, z, -z, x^2, -x^2, z^2, -z^2, xy, -xy, xz, -xz,$$
$$yz, -yz, x^2z, -x^2z, xyz, -xyz\}.$$

The Hasse diagram of the representation partial order of F_2 is drawn in Figure 4 below. Propositions I.6.4 and I.6.5 are used in the computation of this diagram. For example, item (2) of the latter shows that $x^2 \leq x\,w \leq -x^2$ for all w. One uses item (d) in I.6.4 to prove incomparability of elements of F_2 as shown in Figure 4; for instance, to prove that xyz and y are $\leq$-incomparable, direct inspection of the characters of F_2 given in (a.i) above, shows that there are $h, h' \in X_{F_2}$ such that $h(y) = 1$, $h(xz) \in \{0, -1\}$ —whence $h(xyz) >_{\mathbf{3}} h(y)$, i.e., $xyz \not\leq y$—, and $h'(y) = -1$, $h'(xz) \in \{0, -1\}$ —hence $h'(y) >_{\mathbf{3}} h(xyz)$, i.e., $y \not\leq xyz$. Further details are left to the reader.

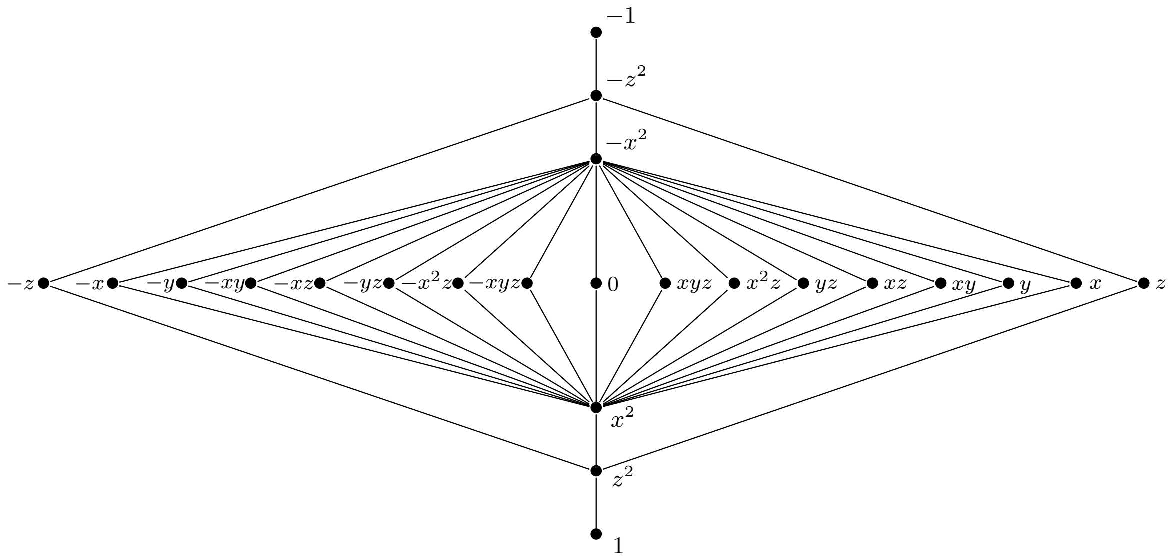


Figure 4. Representation partial order of F_2.

One may also consider fans arising from additional relations between generators, such as:

ii) The additional relation $xz = x$.

Under this relation, each character sending z to -1 must also send x to 0. Thus, the characters $h_8, \ldots, h_{11}$ in the preceding example disappear; the order of specialization of the resulting ARS, $X_{F_2'}$, is illustrated in Figure 3 above right. The RS-fan F_2' consists of:

$$F_2' = \{1, 0, -1, x, -x, y, -y, z, -z, x^2, -x^2, z^2, -z^2, xy, -xy\}.$$

The diagram of its representation partial order is computed in much the same way as in the preceding example; details are left to the reader.

Other relations are also possible. An interesting example is:

iii) The additional relation $z^2 = 1$.

This makes the generator z invertible, and hence forbids characters sending z to 0, i.e., with notation as in item (a.i), the character h_1. Specialization among the remaining characters does not change, but the characters h_2, h_3 have now become "disconnected"; we obtain a "two-component" root-system:

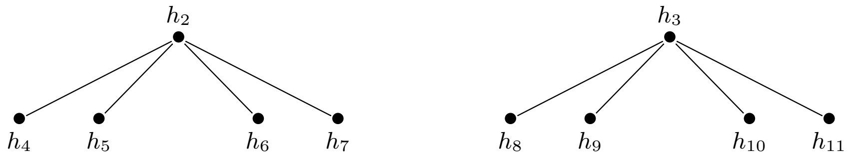


Figure 5. Specialization root-system of $X_{F_2''}$

The corresponding RS-fan, F_2'', has 21 elements; in fact, $F_2'' = F_2 \setminus \{z^2, -z^2\}$. Its representation partial order has a diagram similar to that of F_2, omitting $z^2, -z^2$, with z and $-z$ "linked" only to 1 and -1 (Lemma VI.3.1).

The notion of connectedness for root-systems —with due attention to the case of ARS-fans— will be examined in §VI.7 (cf. items VI.7.12 – VI.7.18). In §VI.10 we shall prove that the phenomenon made apparent by the preceding example holds in full generality: the connected components of a ARS-fan are uniquely determined by the invertible elements of its dual RS-fan.

Remark. Adding both the relations $xz = x$ and $z^2 = 1$ produces a two-component root-system as follows (notation as in (a.i)):

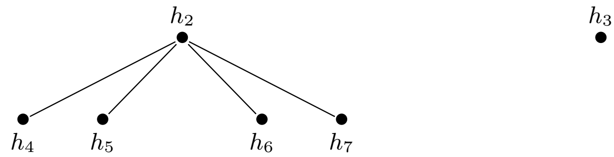


Figure 6. Specialization root-system of $X_{F_2'''}$

The dual RS-fan is: $F_2''' = \{1, 0, -1, x, -x, y, -y, z, -z, x^2, -x^2, xy, -xy\}$, with representation partial order:

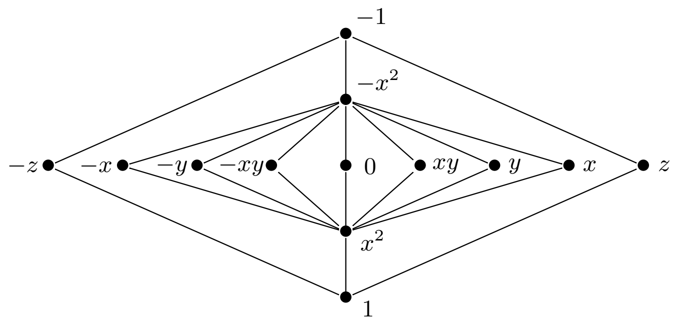


Figure 7. Representation partial order of F_2'''

The other situation arising in case (B.2) is as follows:

(b) $x^2 = y^2 \neq z^2$ and $x^2z^2 = y^2z^2 = z^2$.

This amounts to $Z(x) = Z(y) \subset Z(z)$. For this case we consider several alternatives:

i) No relations other than the above.

Here we have the following TS-characters:

— h_1 sends x, y, z to 0;

— h_2, h_3 h_4, h_5 send z to 0 and x, y to ± 1;

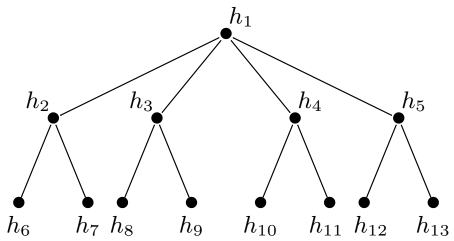
— $h_6, \ldots, h_{13}$ assign all combinations of values ± 1 to the generators.

Clearly, $h_2, \ldots, h_5 \preceq h_1$ (cf. Lemma I.1.18). To determine the specialization relations among the remaining characters we observe that, for $i \in \{2, \ldots, 5\}$, $j \in \{6, \ldots, 13\}$ we have:

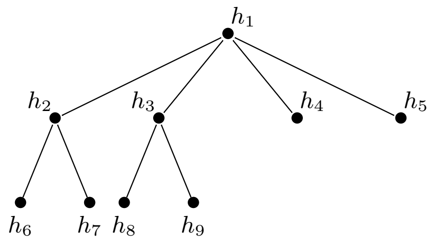
$$h_j \preceq h_i \;\Leftrightarrow\; h_i(x) = h_j(x) \text{ and } h_i(y) = h_j(y).$$

(This is because $h_j \preceq h_i \;\Leftrightarrow\; \forall\, a \in G\,(h_i(a) \neq 0 \Rightarrow h_j(a) = h_i(a))$, cf. Lemma I.1.18(4)). Hence, with suitable indexing of the h_i's (e.g., $h_i(z) = 1$ for $i = 6, 8, 10, 12$), the corresponding

ARS, X_{F_3}, looks as in Figure 8 below, left:



Specialization root-system of X_{F_3}



Specialization root-system of $X_{F_3'}$

Figure 8

The RS-fan F_3 has cardinality 27:

$$F_3 = \{1, 0, -1, x, -x, y, -y, z, -z, x^2, -x^2, z^2, -z^2, xy, -xy, xz, -xz, yz, -yz, xz^2, -xz^2, yz^2, -yz^2, xyz, -xyz, xyz^2, -xyz^2\}.$$

We leave it to the reader to compute the Hasse diagram of the representation partial order of F_3; it is somewhat similar to that of F_2 (Figure 4), and is computed using the same technique, relying on Propositions I.6.4 and I.6.5.

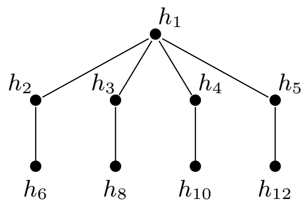
Next, we consider variants of this example obtained by adding various types of relations among the generators.

ii) <u>The additional relation $xz = z$</u>.

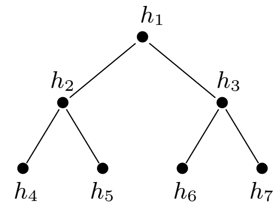
Here, any character sending x to -1 must send z to 0. With notation as in the previous example, this amounts to cutting off, in X_{F_3}, the characters $h_{10}, \ldots, h_{13}$. The resulting ARS-fan, called $X_{F_3'}$, is drawn in Figure 8, right. As an exercise the reader may compute the corresponding RS-fan F_3' and its representation partial order.

iii) <u>The additional relation $z^2 = z$</u>.

This relation forbids the characters sending z to -1. With notation as in Example (b.i), only survive the characters $h_1, \ldots h_5$, and h_i for $i = 6, 8, 10, 12$ (which send z to 1 and x, y to ± 1). The specialization root-system of the corresponding ARS-fan, $X_{F_3''}$, is shown in Figure 9 below, left.



Specialization root-system of $X_{F_3''}$



Specialization root-system of $X_{F_3'''}$

Figure 9

This example suggests that a relation of type $z^2 = z$ (for a suitable generator z) produces specialization "without branching" at the appropriate level. In Lemma VI.10.7 we shall prove in full generality that this is the case for finite ARS-fans.

The RS-fan F_3'' is:

$$F''_3 = \{1, 0, -1, x, -x, y, -y, z, -z, x^2, -x^2, xy, -xy, xz, -xz, yz, -yz, xyz, -xyz\}.$$

Exercises. (1) Prove that the relation $x^2 = x$ gives the configuration in Figure 9, right.

(2) Compute the specialization root-system of the free TS on two generators x, y, with $Z(x) \subset Z(y)$.

(3) Find explicitly an isomorphism between the RS-fan of Exercise (2) and the RS-fan F'''_3 whose dual ARS appears in Figure 9, right. □

Summarizing a common feature of all the examples presented above, we shall prove that, under the representation partial order $\leq$, see § I.6, every RS-fan is a bounded lattice.

We recall from Propositions I.6.4 and I.6.5:

(1) In any RS, G,

(i) For all $a, b \in G$, $a^2 \leq 0 \leq -b^2$ (I.6.4 (c));

(ii) For all $a \in G$, $a^2 \leq \pm a \leq -a^2$ (I.6.5 (2)).

Further, since the zero-sets of elements of a fan are totally ordered under inclusion (VI.1.1), Proposition I.6.5 (1) yields:

(2) In a RS-fan, the set $\mathrm{Id}(F) \cup -\mathrm{Id}(F)$ is totally ordered by $\leq$.

Lemma VI.3.3 *Let F be a RS-fan and let $a, b \in F$. If $a, b \notin \mathrm{Id}(F) \cup -\mathrm{Id}(F)$ —i.e., $\pm a$ and $\pm b$ are not squares—, and $a \neq b$, then a, b are incomparable under $\leq$.*

Proof. Assume, towards a contradiction, that a, b are comparable, say $a \leq b$, i.e., $a \in D(1, b)$ and $-b \in D(1, -a)$ (I.6.2). This implies $a \in D^t(a^2, a^2b)$ and $-b \in D^t(b^2, -b^2a)$, which in turn gives:

(1) $Z(a) = Z(b)$.

If $Z(a) \subset Z(b)$, then $Z(a^2) \subset Z(a^2b)$, and the first clause in the definition of $[D^t]$ in VI.2.1 gives $a = a^2$, contrary to assumption. Likewise, the second transversal representation precludes $Z(b) \subset Z(a)$. Since the zero-sets of elements of a RS-fan are totally ordered under inclusion, (1) is proved.

It follows that $Z(a^2) = Z(a^2b)$ and $Z(b^2) = Z(-b^2a)$. We consider two cases:

(2.a) $a^2 \neq -a^2b$.

By the third clause in $[D^t]$ of Theorem VI.2.1, $D^t(a^2, a^2b) = \{a^2, a^2b\}$; since $a \neq a^2$, we get $a = a^2b$. We have two alternatives:

— $b^2 \neq b^2a$.

Since $-b \neq b^2$, VI.2.1 leads to $b = b^2a$. We get $a^2 = a^2b^2 = b^2$, which gives $a = a^2b = b^3 = b$, contrary to assumption.

— $b^2 = b^2a$.

In this case $b = ba$, and then $a = a^2b = a^2(ba) = ab = b$, a contradiction again.

(2.b) $a^2 = -a^2b$.

— If $b^2 \neq b^2a$, the third clause in $[D^t]$ of VI.2.1 gives $b = b^2a$. Hence, $a = -ab = -b^2a^2 \in -\mathrm{Id}(F)$, contrary to assumption.

— If $b^2 = b^2 a$, then $b = ba$, so $a = -ab = -b$, and we get $a = -ab = a^2 \in \mathrm{Id}(F)$, contradiction. $\square$

Lemma VI.3.4 *Let F be a RS-fan and let $x, b \in F$. If $b \notin \mathrm{Id}(F)$ (i.e., $b \neq b^2$), then $b \leq -x^2 \leq -b^2$ implies $b^2 = x^2$. That is, $-b^2$ is the smallest $y \in F$ such that $b \leq -y^2$. Dually, b^2 is the largest $y \in F$ such that $y^2 \leq b$.*

Proof. Assume $b \leq -x^2 < -b^2$. Since $b^2 \leq -x^2$ (I.6.4 (c)), from Proposition I.6.5 (1) follows $Z(b) \subseteq Z(x)$, i.e., $b^2 x^2 = x^2$ (I.1.19 (1)). On the other hand, $b \leq -x^2$ yields $b \in D(1, -x^2)$, and then $b \in D^t(b^2, -b^2x^2) = D^t(b^2, -x^2)$. If $Z(b) \subset Z(x)$, the first clause of VI.2.1 gives $b = b^2$, contrary to assumption. So, $Z(b) = Z(x)$, and I.1.19 (1) gives $b^2 = x^2$. The dual assertion is obvious. $\square$

Theorem VI.3.5 *Let F be a RS-fan and let $\leq$ denote its representation partial order (I.6). Then, $(F, \leq)$ is a lattice with smallest element 1 and largest element -1.*

Notation. For elements a, b in a RS, G, we write $a \perp b$ to mean that a and b are incomparable under the representation partial order of G. $\square$

Proof. We must show that every pair of elements $a, b \in F$ has a least upper bound, $\vee$, and a greatest lower bound, $\wedge$, for the order $\leq$. If a, b are comparable under $\leq$ there is nothing to prove; so, we may assume $a \perp b$.

Since F is a RS-fan, the zero-sets of a and b are comparable under inclusion. This, together with $a \perp b$, implies that one of a or b is not in $\mathrm{Id}(F) \cup -\mathrm{Id}(F)$; indeed:

— If $Z(a) \subseteq Z(b)$, I.6.5 (1) yields $a^2 \leq b \leq -a^2$; hence, $a \perp b$ implies $a \notin \mathrm{Id}(F) \cup -\mathrm{Id}(F)$.

— Likewise, $Z(b) \subseteq Z(a)$ implies $b \notin \mathrm{Id}(F) \cup -\mathrm{Id}(F)$.

Since $-a^2$ and $-b^2$ are $\leq$-comparable, cf. I.6.5 (1), we may assume without loss of generality that $-a^2 \leq -b^2$. Further, Lemma VI.3.4 shows

(*) $$-b^2 = \text{least } x \in -\mathrm{Id}(F) \text{ such that } a, b \leq x.$$

Claim. $-b^2 = a \vee b$.

Proof of Claim. By assumption, $a, b \leq -b^2$, so we only need prove:

$$\forall c \in F\, (c \geq a, b \;\Rightarrow\; c \geq -b^2).$$

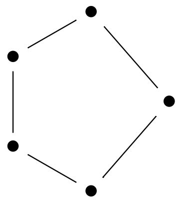
Note that $c \geq a, b$ and $a \perp b$ imply $c \neq a, b$. If $c \notin \mathrm{Id}(F) \cup -\mathrm{Id}(F)$, since one of a, b —say a— is not in $\mathrm{Id}(F) \cup -\mathrm{Id}(F)$, then, by Lemma VI.3.3, $c \neq a$ implies $c \perp a$, absurd; hence $c \in \mathrm{Id}(F) \cup -\mathrm{Id}(F)$. If $c \in \mathrm{Id}(F)$, then $c \geq a, b$, implies $a, b \in \mathrm{Id}(F)$, whence a, b are $\leq$-comparable, contradiction. So, $c \in -\mathrm{Id}(F)$, and (*) gives $c \geq -b^2$, as claimed.

Under the current assumptions $-a^2 \leq -b^2$ and $a \perp b$, upon observing that

$$b^2 = \text{largest } y \in \mathrm{Id}(F) \text{ such that } y \leq a, b,$$

a similar argument yields $b^2 = a \wedge b$. $\square$

Remarks VI.3.6 (a) Examination of the examples presented above shows that the lattices $(F, \leq)$ are not modular —hence not distributive either— except in very special cases. In fact, most of these lattices contain the configuration



as a sublattice (cf. [B], Ch. V, § 2, Thm. 2, p. 66); see Figures 4 and 7 above. Example VI.3.2. A is modular but not distributive.

(b) Since $\mathrm{Id}(F) \cup -\mathrm{Id}(F)$ is a totally ordered subset of $(F, \leq)$, the proof of Theorem VI.3.5 shows that the lattice operations in $(F, \leq)$; satisfy the following identities:

$$a \wedge b \;=\; \begin{cases} \min_{\leq}\{a^2, b^2\} & \text{if } \ a \perp b \\ \min_{\leq}\{a, b\} & \text{if } \ a, b \text{ are } \leq\text{-comparable,} \end{cases}$$

and

$$a \vee b \;=\; \begin{cases} \max_{\leq}\{-a^2, -b^2\} & \text{if } \ a \perp b \\ \max_{\leq}\{a, b\} & \text{if } \ a, b \text{ are } \leq\text{-comparable.} \end{cases}$$

Note that, if $a \perp b$, then $a \wedge b$, $a \vee b \in \mathrm{Id}(F) \cup -\mathrm{Id}(F)$.

(c) The operation $x \mapsto -x$ $(x \in F)$ *is not* a complement in the lattice-theoretic sense, but it verifies:

(c_1) The Kleene inequality IV.1.2 (b):

$$a \wedge -a \;\leq\; b \vee -b.$$

(A particular case of Proposition I.6.5 (7),(8).)

(c_2) The De Morgan laws:

$$\text{(i)} \quad -(a \wedge b) = -a \vee -b\,; \qquad\qquad \text{(ii)} \quad -(a \vee b) = -a \wedge -b\,.$$

This is clear if a, b are comparable under $\leq$. If $a \perp b$, assuming without loss of generality that $-a^2 \leq -b^2$, (i.e., $b^2 \leq a^2$), from (b) we get:

(i) $-(a \wedge b) = -(a^2 \vee b^2) = -a^2$, and $-a \vee -b = -(-a)^2 \wedge -(-b)^2 = -a^2 \wedge -b^2 = -a^2$.

(ii) $-(a \vee b) = -(-a^2 \wedge -b^2) = -(-a^2) = a^2$, and $-a \wedge -b = (-a)^2 \vee (-b)^2 = a^2 \vee b^2 = a^2$. $\square$

VI.4 Characterizations of fans

New section; Jan. 2014.

The main purpose of this section is to prove the characterization of RS-fans given in Theorem VI.4.2 below. To make the statement understandable we summarize the following

Notation VI.4.1 Let G be a real semigroup and let $X_G = \mathrm{Hom}_{\mathrm{RS}}(G, \mathbf{3})$ denote the set of RS-characters of G.

(i) $\rightsquigarrow$ stands for the specialization relation of X_G endowed with the spectral topology (see I.1.16, I.1.17 and I.1.18 for details).

(ii) For $h \in X_G$, $Z(h) = \{a \in G \mid h(a) = 0\}$ denotes the zero-set of h.

(iii) The notion of RS-congruence of a real semigroup is defined, and its basic properties developed in section II.2. Congruences determined by saturated prime ideals are studied in paragraph (F) of section II.3, where in Theorem II.3.15 (d) we prove that the structure $G_I := (G/I) \setminus \{\pi_I(0)\}$ obtained by omitting "zero" $(= \pi_I(0))$ from the quotient G/I, endowed with the natural quotient representation $D_{G_I} := D_{G/I} \lceil G_I$ (cf. II.3 (F)), is a reduced special group.

(iv) For the notion of a fan in the dual categories of reduced special groups (RSG-fan) and of abstract order spaces (AOS-fan) the reader is referred to [DM1], pp. 8-9, 89-90, and [M], Ch. III (Check ref.) respectively. $\square$

The characterization of RS-fans alluded to is:

Theorem VI.4.2 *For a real semigroup G, the following are equivalent:*

(1) *G is a RS-fan.*

(2) *G satisfies the following conditions:*

- (i) $\forall a, b \in G\,(a^2b^2 = a^2$ *or* $a^2b^2 = b^2)$.
- (ii) *Given $g, h \in X_G$ such that $Z(g) \subseteq Z(h)$, there is $h' \in X_G$ such that $Z(h) = Z(h')$ and $g \rightsquigarrow h'$.*
- (iii) *For every saturated prime ideal I of G, the quotient reduced special group (G_I, D_{G_I}) is a RSG-fan.*

For the implication (2) $\Rightarrow$ (1) we seem to need that (X_G, G) is a q-fan. Check!!

Remark. The proof is organized as follows: Lemma VI.4.3, Proposition VI.4.5 and Corollary VI.11.2 below show, respectively, that any RS-fan satisfies conditions (2.i) – (2.iii), i.e., altogether they prove (1) $\Rightarrow$ (2). The converse implication (2) $\Rightarrow$ (1) is proved in Proposition VI.4.10 below.

Lemma VI.4.3 *Any RS-fan verifies condition* $(2.i)$ *of Theorem VI.4.2.*

Proof. This follows at once from Fact VI.1.1 and Proposition VI.1.2. $\square$

Omit Reminder.

VI.4.4 Reminder. Proposition VI.1.2 shows that the following are equivalent to condition $(2.i)$ of Theorem VI.4.2, for any *ternary* semigroup G:

- The collection $\{Z(a) \mid a \in G\}$ of zero-sets of elements of G is totally ordered under inclusion.
- The set of all ideals of G is totally ordered under inclusion.
- Every proper (TS)-ideal of G is prime. $\square$

The implication (1) $\Rightarrow$ (2.i) in Theorem VI.4.2 is item (2) of the following Proposition.

Proposition VI.4.5 *Let G be a RS-fan. Then:*

(1) *For all elements $g, h \in X_G$ such that $g \rightsquigarrow h$ (hence $Z(g) \subseteq Z(h)$) and every ideal I such that $Z(g) \subseteq I \subseteq Z(h)$ there is $f \in X_G$ such that $g \rightsquigarrow f \rightsquigarrow h$ and $Z(f) = I$.*

(2) *For every $g \in X_G$ and every ideal $I \supseteq Z(g)$ there is a (necessarily unique) $f \in X_G$ such that $g \rightsquigarrow f$ and $Z(f) = I$.*

(3) *For every ideal I of F there is an $f \in X_G$ such that $Z(f) = I$.*

Proof. Since G is a RS-fan, every TS-character $f : G \longrightarrow \mathbf{3}$ is a RS-homomorphism. Thus, it suffices to construct TS-homomorphisms $f : G \longrightarrow \mathbf{3}$ verifying (1) – (3) of the statement.

First we prove (1); the same proof, omitting item (c) below, also proves (2). Let $f : G \longrightarrow \mathbf{3}$ be defined by:

$$f \lceil I = 0 \quad \text{and} \quad f \lceil (G \setminus I) = g \lceil (G \setminus I).$$

(a) $Z(f) = I$.

By construction, $I \subseteq Z(f)$. Since $Z(g) \subseteq I$, $f(x) = g(x) \neq 0$ for $x \in G \setminus I$, i.e., $Z(f) \subseteq I$.

(b) $g \rightsquigarrow f$.

Clear, from (a) and Lemma I.1.18,

(c) $f \rightsquigarrow h$.

If $f(a) = 0$, then $a \in I \subseteq Z(h)$, and $h(a) = 0$.

If $h(a) \neq 0$, then $a \notin I$; since $g \rightsquigarrow h$, then $g(a) = h(a)$. Hence, $f(a) = g(a) = h(a)$, and we get $f \rightsquigarrow h$ by Lemma I.1.18.

(d) f is a TS-homomorphism.

Clearly $f(0) = 0$ and $f(\pm 1) = g(\pm 1) = \pm 1$. Let $a, b \in G$. If one of a, b is in I, so is ab, and we have $f(a)f(b) = 0 = f(ab)$. If $a, b \notin I$, then $ab \notin I$, and f and g take the same value on a, b and ab; the result follows from the fact that g is a TS-character. Since G is a fan, $f \in X_G$.

(3) This is Lemma I.1.7 (alternatively, Lemma I.4.8) □

Remark VI.4.6 The element f such that $g \rightsquigarrow f$ and $Z(f) = I$ in VI.4.5 (2) can also be obtained by taking any $h \in X_G$ with $Z(h) = I$ (VI.4.5 (3)) and setting $f = h^2 g$. □

OJO! Duplication with Remark VI.6.7; check!

The following Proposition VI.11.1 and its Corollary VI.11.2 prove that *every* quotient of a RS-fan is a RS-fan; in particular, the implication (1) ⇒ (2.iii) holds in Theorem VI.4.2.

OJO! Duplication with Proposition VI.11.1 and Corollary VI.11.2; check!

Proposition VI.4.7 *Let G be a RS-fan and let $\mathcal{H}$ be a proconstructible subset of X_G stable under product of any three of its elements. Then $\equiv_{\mathcal{H}}$ is a RS-congruence, the quotient $G/\mathcal{H}$ is a RS-fan, and the spectral spaces $X_{G/\mathcal{H}}$ and $\mathcal{H}$ are homeomorphic; in particular, the (Boolean) spaces $(X_{G/\mathcal{H}})_{\mathrm{con}}$ and $\mathcal{H}_{\mathrm{con}}$ are homeomorphic.*

Proof. Follows closely the proof of Theorem I.1.27; we shall use notation therein.

The quotient structure $G/\mathcal{H}$ is a ternary semigroup and $X_{G/\mathcal{H}} = \mathrm{Hom}_{\mathrm{TS}}(G/\mathcal{H}, \mathbf{3})$ is its set of (TS-) characters. The proof of item (3) in Theorem I.1.27 shows that, under our hypotheses on $\mathcal{H}$, the map $\theta : X_{G/\mathcal{H}} \longrightarrow X_G$ given by $\theta(g) = g \circ \pi$ $(g \in X_{G/\mathcal{H}})$ is a homeomorphism between the spectral spaces $X_{G/\mathcal{H}}$ and $\mathcal{H}$, as asserted.

According to equality (***) in the proof of Theorem I.1.27 (with U replaced by Z) we have $\theta^{-1}[Z(a) \cap \mathcal{H}] = Z(\pi(a))$ for $a \in G$. Since G verifies condition (2.i) in Theorem VI.4.2, this equality implies that the zero-sets of elements of $G/\mathcal{H}$ are also totally ordered by inclusion; Corollary VI.2.3 implies, then, that $(X_{G/\mathcal{H}}, G/\mathcal{H})$ is a fan. □

Observe that *all* RS-congruences of a fan are obtained in the way given by the preceding Proposition:

Corollary VI.4.8 *Let G be a RS-fan and let $\equiv$ be a RS-congruence of G. Then:*

(a) $\equiv\, = \,\equiv_{\mathcal{H}}$ for some proconstructible set $\mathcal{H} \subseteq X_G$ stable under product of any three elements. Hence,

(*b*) $G/\!\equiv$ *is a RS-fan.*

(*c*) *The correspondence* $\mathcal{H} \longmapsto \equiv_{\mathcal{H}}$ *establishes an inclusion-reversing bijection between proconstructible subsets of* X_G *stable under product of any three elements, and the set* $Con(G)$ *of RS-congruences of* G*.*

Proof. (a) The set $\mathcal{H} = \mathcal{H}_{\equiv}$ is given by Proposition **??**. Items (b) and (c) follow, respectively, from Proposition VI.11.1 and Theorem I.1.27. □

Alternatively, we may use the following, less general but more direct result, that appears as Proposition V.5.1 of our monograph.

Proposition VI.4.9 *Let* F *be a RS-fan. Let* I *be a proper ideal of* F*. Let* $\pi = \pi_I : F \longrightarrow F/I$ *denote the canonical quotient map. Then,* $F_I = (F/I) \setminus \{\pi(0)\}$ *is a RSG-fan.*

Proof. In Theorem II.3.15 and with notation therein, it was shown that if G is a RS and I is a saturated prime ideal of G, then $\langle (G/I) \setminus \{\pi(0)\}, \cdot, \pi(-1), D_{G/I} \rangle$ is a RSG. We must prove: given $a, b \in F$ so that $\pi(a)$, $\pi(b) \neq 0$ and $\pi(a) \neq \pi(-1)$,

$$\pi(b) \in D_{F/I}(\pi(1), \pi(a)) \ \Rightarrow\ \pi(b) = \pi(1) \vee \pi(b) = \pi(a).$$

By the characterization of $D_{F/I}$ in Theorem II.3.15 (b), there are $x \in F \setminus I$ and $i \in I$ such that $bx^2 \in D_F(i, 1, a)$. Hence, there is $c \in F$ such that $bx^2 \in D_F(i, c)$ and $c \in D_F(1, a)$. From the characterization of representation for fans (Theorem VI.2.1) we get:

$$\text{(A)} \qquad bx^2 \in D_F(i,c) \quad \Leftrightarrow \quad \begin{array}{ll} \text{(i)} & bx^2 = iy^2 \text{ for some } y \in F, \text{ or} \\ \text{(ii)} & bx^2 = cy^2 \text{ for some } y \in F, \text{ or} \\ \text{(iii)} & bx^2 i = -cbx^2 \text{ and } bx^2 = i^2bx^2. \end{array}$$

$$\text{(B)} \qquad c \in D_F(1,a) \quad \Leftrightarrow \quad \begin{array}{ll} \text{(i)} & c = z^2 \text{ for some } z \in F, \text{ or} \\ \text{(ii)} & c = az^2 \text{ for some } z \in F, \text{ or} \\ \text{(iii)} & c = -ac \text{ and } c = a^2c. \end{array}$$

Since $b, x \notin I$, we have $bx^2 \notin I$, which clearly excludes cases (A.i) and (A.iii), and entails $c \notin I$ in (A.ii), whence $\pi(c^2) = \pi(1)$. Case (B.iii) yields:

$$c = -ac \ \Rightarrow\ \pi(c) = -\pi(a)\pi(c) \ \Rightarrow\ \pi(c)^2 = \pi(1) = -\pi(a)\pi(c)^2 = -\pi(a),$$

and hence $\pi(a) = \pi(-1)$, contrary to assumption; thus, case (B.iii) is excluded as well. In the remaining cases we have:

(1) $bx^2 = cy^2$ and $c = z^2$ for some $y, z \in F$.

Hence, $bx^2 = (yz)^2$. Since $bx^2 \notin I$, it follows $y, z \notin I$, and then $\pi(b) = \pi(b)\pi(x^2) = \pi((yz)^2) = \pi(1)$.

(2) $bx^2 = cy^2$ and $c = az^2$ for some $y, z \in F$.

Thus, $bx^2 = a(yz)^2$. As in case (1) we have $yz \notin I$. Then, $\pi(b) = \pi(b)\pi(x^2) = \pi(a)\pi((yz)^2) = \pi(a)\pi(1) = \pi(a)$. □

The next Proposition proves the implication (2) $\Rightarrow$ (1) in Theorem VI.4.2.

Proposition VI.4.10 *Let G be a real semigroup verifying conditions $(2.i)-(2.iii)$ of Theorem VI.4.2. Then, G is a RS-fan.*

Proof. From VI.4.4 we know that condition (2.i) implies that every (TS)-ideal of G is prime and that the set of all (prime) ideals of G is totally ordered under inclusion. By Theorem VI.2.1, we must show, for $a, b, c \in G$:

(I) $c \in D^t_G(a,b)$ and $Z(a) \subset Z(b)$ imply $c = a$.

(II) $c \in D^t_G(a,b)$, $Z(a) = Z(b)$ and $a \neq -b$ imply $c = a$ or $c = b$.

Proof of (I). We first observe that the assumptions of (I) imply $Z(a) = Z(c)$.

Let $h \in X_G$. If $h(a) = 0$, then $h(b) = 0$ (as $Z(a) \subseteq Z(b)$), and $c \in D^t_G(a,b)$ yields $h(c) = 0$; hence, $Z(a) \subseteq Z(c)$.

If $Z(c) \subseteq Z(b)$, then $c \in D^t_G(a,b)$ yields $-a \in D^t_G(-c,b)$, and so $Z(c) \subseteq Z(a)$. If $Z(b) \subseteq Z(c)$, then $-a \in D^t_G(-c,b)$ entails $Z(b) \subseteq Z(a)$, contrary to assumption. Hence, $Z(c) \subseteq Z(a)$, and $Z(a) = Z(c)$.

In order to prove $c = a$, let $h \in X_G$. If $h(b) = 0$, then $h(c) \in D_{\mathbf{3}}(h(a), 0) = \{h(a)\}$, whence $h(c) = h(a)$. Henceforth, assume $h(b) \neq 0$. Since $Z(a) \subset Z(b)$, there is $g \in X_G$ so that $g(b) = 0$ and $g(a) \neq 0$. Since the set of ideals of G is totally ordered under inclusion, $h(b) \neq 0$ and $g(b) = 0$, we have $Z(h) \subset Z(g)$. By (2.ii), there is $g' \in X_G$ so that $Z(g') = Z(g)$ and $h \rightsquigarrow g'$. Then, $g'(b) = 0$; from $c \in D^t_G(a,b)$ and $g(a) \neq 0$ comes $g'(a) = g'(c) \neq 0$. From $h \rightsquigarrow g'$ we infer $h(a) = g'(a)$ and $h(c) = g'(c)$ (Lemma I.1.18 (4)), and from $g'(a) = g'(c)$ we conclude $h(a) = h(c)$, and hence $a = c$.

Proof of (II). Assume $c \in D^t_G(a,b)$, $Z(a) = Z(b)$ and $a \neq -b$; then, there is $g \in X_G$ so that $g(b) = g(a) \neq 0$. First we claim:

Claim 1. Under the assumptions of (II), $Z(c) = Z(a) = Z(b)$.

Proof of Claim 1. In fact, $c \in D^t_G(a,b)$ yields $Z(a) = Z(b) \subseteq Z(c)$. Assume, towards a contradiction, that there is $h \in X_G$ such that $h(c) = 0$ and $h(a) \neq 0$. From $c \in D^t_G(a,b)$ and $g(b) = g(a)$ we get $g(c) = g(b) = g(a) \neq 0$. Since the set of ideals of G is totally ordered under inclusion, this and $h(c) = 0$ imply $Z(g) \subseteq Z(h)$. By (2.ii), there is $h' \in X_G$ such that $Z(h') = Z(h)$ and $g \rightsquigarrow h'$; it follows that $h'(a) \neq 0$ and, since $Z(a) = Z(b)$, $h'(b) \neq 0$. Invoking Lemma I.1.18 (4), we get $h'(a) = g(a)$ and $h'(b) = g(b)$; from $g(b) = g(a)$ we obtain $h'(b) = h'(a)$. On the other hand, $c \in D^t_G(a,b)$ and $h'(c) = h(c) = 0$ entail $h'(a) = -h'(b)$, whence $h'(a) = h'(b) = 0$, contradiction. This proves $Z(c) = Z(a) = Z(b)$, as asserted.

If one of a or b is 0, the equality of zero-sets in Claim 1 implies $c = a = b = 0$. So, assume, e.g., $b \neq 0$. Let I be an ideal of G —necessarily prime and saturated— maximal for $b \notin I$. Let $\sim_I$ be the congruence relation on G determined by I, namely, for $x, y \in G$,

$$x \sim_I y \;\Leftrightarrow\; h(x) = h(y) \text{ for all } h \in X_G \text{ such that } Z(h) = I. \qquad \text{(Cf. II.4 (F))}$$

Note that the equality of zero-sets established in Claim 1, together with $b \notin I$, implies that none of a, b, c is in I.

Claim 2. $a \not\sim_I -b$.

Proof of Claim 2. Assume that $a \sim_I -b$. Since $g(b) \neq 0$, i.e., $b \notin Z(g)$, maximality of I entails $Z(g) \subseteq I$. By (2.ii), there is $h \in X_G$ such that $Z(h) = I$ and $g \rightsquigarrow h$. Since $h(b), h(a) \neq 0$, the specialization $g \rightsquigarrow h$ yields $h(a) = g(a)$ and $h(b) = g(b)$ (I.1.18 (4)), which, by $g(a) = g(b)$, entails $h(a) = h(b)$. On the other hand, $a \sim_I -b$ and $Z(h) = I$ implies $h(a) = -h(b)$. Altogether, these equalities imply $h(a) = h(b) = 0$, a contradiction, showing that $a \not\sim_I -b$.

By assumption (2.iii), the reduced special group (quasi-RSG) G_I is a RSG-fan. With $\pi_I : G \longrightarrow G/I$ denoting the canonical quotient map, we have $\pi_I(a) \neq \pi_I(-b) = -\pi_I(b)$. Note also that $a, b, c \notin I$ implies $\pi_I(a), \pi_I(b), \pi_I(c) \neq \pi_I(0)$. From $c \in D^t_G(a, b)$ it follows $\pi_I(c) \in D^t_{G/I}(\pi_I(a), \pi_I(b))$ which implies (since $G_I = (G/I) \setminus \{\pi_I(0)\}$ is a RSG-fan) $\pi_I(c) = \pi_I(a)$ or $\pi_I(c) = \pi_I(b)$.

Claim 3. $\pi_I(c) = \pi_I(a) \;\Rightarrow\; c = a$.

Proof of Claim 3. Assumption $\pi_I(c) = \pi_I(a)$ means $c \sim_I a$.

Let $h \in X_G$. Since the saturated prime ideals of G are an inclusion chain, we consider two cases:

— $Z(h) \subseteq I$.

Invoking (2.ii), let $h' \in X_G$ be such that $Z(h') = I$ and $h \rightsquigarrow h'$. Since $a, c \notin I$, the specialization $h \rightsquigarrow h'$ entails $h(a) = h'(a)$ and $h(c) = h'(c)$; further, $c \sim_I a$ gives $h'(c) = h'(a)$, whence $h(c) = h(a)$, for all $h \in X_G$ such that $Z(h) \subseteq I$.

— $Z(h) \supset I$.

The maximality of I implies $a, b, c \in Z(h)$, i.e., $h(a) = h(b) = h(c) = 0$.

These two cases prove that $h(c) = h(a)$, for all $h \in X_G$, i.e., $c = a$.

A similar argument proves that $\pi_I(c) = \pi_I(b) \;\Rightarrow\; c = b$, completing the proof of (II), of Proposition VI.4.10, and of Theorem VI.4.2. □

The next two corollaries of Theorem VI.4.2 give stylized (abstract) versions of the notion of a *trivial fan*, a basic concept in the theory of (pre-)orders on fields (see [L2], Prop. 5.3, p. 39). Their translation in the case of preordered rings is given in Theorem **??** below, where it will be obvious that in the case fields they boil down to the notion of a trivial fan defined in [L2]. Add precise ref.

Corollary VI.4.11 *Let G be a real semigroup such that the character space X_G is totally ordered under specialization. Then, G is a RS-fan.*

Proof. We check that conditions (2.i) – (2.iii) of Theorem VI.4.2 hold.

Since every saturated prime ideal of G is the zero-set of some character (I.4.9) and $g \rightsquigarrow h \Rightarrow Z(g) \subseteq Z(h)$ for $g, h \in X_G$ (I.1.18 (4)), the set of saturated prime ideals of G is an inclusion chain, i.e., item (2.i) of VI.4.2 holds.

Further, every saturated prime ideal is the zero-set of a *unique* character: if $h_1, h_2 \in X_G$ are such that $Z(h_1) = Z(h_2)$, then $h_1^2 = h_2^2$ (I.1.19 (1)); if, say, $h_1 \rightsquigarrow h_2$, by Lemma I.1.18 (5), $h_2 = h_2^2 h_1 = h_1^2 h_1 = h_1$. It follows that, for every saturated prime ideal I the quotient G/I has a unique character, and hence $G/I \cong \mathbf{3}$, which is a RS-fan, showing that condition (2.iii) of VI.4.2 holds.

Finally, to check item VI.4.2 (2.ii), observe that the linearity assumption and the uniqueness proved in the preceding paragraph yield $Z(g) \subseteq Z(h) \Rightarrow g \rightsquigarrow h$. □

In the sequel we present an example showing that conditions (2.i) and (2.iii) of Theorem VI.4.2 alone are not sufficient to guarantee that a real semigroup is a RS-fan.

Example VI.4.12 Let $G = \{\pm 1, \pm c\}$ be a four-element group of exponent 2, and let $x \notin G$. We set $G[x] := G \cup \{0, \pm x\}$, subject to the relations $x \neq 0$, $x^2 = x$ and $cx = -x$. It is easily checked that $G[x]$ is a ternary semigroup whose product extends that of G and whose ideals are $\{0\}$ and $\{0, \pm x\}$ (both being prime). For $i = 1, 2$ we define TS-characters $h_i : G[x] \longrightarrow \mathbf{3}$, as follows:

$$h_1(y) = \begin{cases} 0 & \text{if } y = 0 \\ 1 & \text{if } y \in \{1, x, -c\} \\ -1 & \text{if } y \in \{-1, -x, c\} \end{cases} \qquad h_2(y) = \begin{cases} 0 & \text{if } y \in \{0, \pm x\} \\ 1 & \text{if } y \in \{1, c\} \\ -1 & \text{if } y \in \{-1, -c\} \end{cases}$$

Direct inspection shows:

(i) $Z(h_1) = \{0\} \subset \{0, \pm x\} = Z(h_2)$ (hence VI.4.2 (2.i) holds).

(ii) The set $\mathcal{H} := \{h_1, h_2\}$ separates points in $G[x]$.

(iii) $h_1 \not\rightsquigarrow h_2$ (I.1.18 (4)).

It follows from Theorem I.3.3 and Proposition I.3.4 that the ternary relation $D_{\mathcal{H}}$ defined by $\mathcal{H}$ (see clause $[D]_{\mathcal{H}}$ in Definition I.3.1), endows the ternary semigroup $G[x]$ with the structure of a real semigroup. Note that the corresponding transversal representation relation $D^t_{\mathcal{H}}$ is given by:

$$\text{For } p, q, r \in G[x],\ \ p \in D^t_{\mathcal{H}}(q, r) \ \text{ iff } \ h_i(p) \in D^t_{\mathbf{3}}(h_i(q), h_i(r)) \ \text{ for } i = 1, 2;$$

(cf. I.3.2 [TR]). In particular,

(iv) $c \in D^t_{\mathcal{H}}(1, -x)$.

Claim. $X_{G[x]} := \mathrm{Hom}_{\mathrm{RS}}((G[x], D_{\mathcal{H}}), \mathbf{3}) = \mathcal{H}$.

Proof of Claim. Let $h \in X_{G[x]}$; then, $Z(h) = \{0\}$ or $Z(h) = \{0, \pm x\}$.

— If $Z(h) = \{0\}$, then $h(x), h(c) \in \{\pm 1\}$. Since $x^2 = x$, we have $h(x) \in \{0, 1\}$; hence, $h(x) = 1$. Since $cx = -x$, we have $h(c) = -1$. So, $h = h_1$, as they coincide on the generators c, x of $G[x]$.

— If $Z(h) = \{0, \pm x\}$, from (iv) we get $h(c) \in D^t_{\mathbf{3}}(1, 0) = \{1\}$, whence $h = h_2$, since they coincide on the generators c, x. □

The Claim shows:

(1) Condition VI.4.2 (2.ii) fails in $(G[x], D_{\mathcal{H}})$: this real semigroup has no (RS-)character specializing h_1 and having $\{0, \pm x\}$ as zero-set.

(2) Condition VI.4.2 (2.iii) holds in $(G[x], D_{\mathcal{H}})$: both (saturated, prime) ideals I of $G[x]$ are the zero-sets of exactly one RS-character; then, the quotient $G[x]/I$ has exactly one character, whence $G[x]/I \cong \mathbf{3}$, a RS-fan. □

VI.5 Fans and preordered rings

New section; Jan. 2014. Replaces old section "Fans and valuation rings".

The aim of this section is to deal with a number of situations and exhibit some examples of semi-real rings and preordered rings (hereafter **p-rings**) whose associated real semigroups are fans.

A. Basic properties of p-rings whose associated real semigroup is a fan. Throughout this subsection we assume that $\langle A, T \rangle$ is a p-ring whose associated real semigroup $G_{A,T}$ is a RS-fan.

VI.5.1 Reminder and Notation. (a) From VI.1.2 and VI.2.8 we know that all ideals of $G_{A,T}$ are prime and saturated; further, the family $\mathcal{I} = \mathcal{I}(A, T)$ of all ideals of $G_{A,T}$ is totally ordered by inclusion (VI.1.2 (4)).

(b) For $J \in \mathcal{I}$, let $\widehat{J} = \{a \in A \,|\, \overline{a} \in J\}$. By Fact II.1.1, the $\widehat{J}$'s are prime ideals of A and the collection $\{\widehat{J} \,|\, J \in \mathcal{I}\}$ is totally ordered under inclusion. Since $J \in \mathcal{I}$ is saturated, $\widehat{J}$ is a T-radical ideal of A (Theorem II.1.12), hence T-convex (II.1.4 or [BCR], Prop. 4.2.5, p. 87).

(c) We denote by $\mathcal{T}(\mathcal{A},\mathcal{T})$ the set of all T-convex prime ideals of A.

Fact VI.5.2 *With notation as above,*
(i) $\{\widehat{J}\,|\,J\in\mathcal{I}\}=\mathcal{T}(A,T)$.
(ii) The map $J\mapsto\widehat{J}$ $(J\in\mathcal{I})$ *is bijective and order-preserving.*

Proof. (i) Let $I\in\mathcal{T}(A,T)$. By II.1.11, $\overline{I}$ is a prime ideal of $G_{A,T}$, and by Lemma II.1.8 (b), $I=\widehat{\overline{I}}$ $(=\sqrt[T]{I})$.

(ii) By (i) the map $J\mapsto\widehat{J}$ is onto $\mathcal{T}$. By II.1.1 (vi) it is also injective. That it preserves inclusion is proved in II.1.1 (v). □

Remarks VI.5.3 (i) Since $\{0\}$ is the smallest element of $\mathcal{I}$, the proof of VI.5.2 (i) (or II.1.8) shows that $\sqrt[T]{(0)}$ is the smallest element of $\mathcal{T}$.

(ii) The maximal element of $\mathcal{T}$ is:

$$\mathfrak{M}=\text{set of non-invertible elements of } G_{A,T}=\{\overline{x}\,|\,x\in A\text{ and }\overline{x}^2\neq 1\}.$$

Then, we have:

$$\begin{aligned}M=\widehat{\mathfrak{M}}&=\{a\in A\,|\,\overline{a}\in\mathfrak{M}\}=\{a\in A\,|\,\overline{a}^2\neq 1\}=\\&=\{a\in A\,|\,\exists\alpha\in\operatorname{Sper}(A,T)\text{ such that }\overline{a}(\alpha)=0\}=\\&=\{a\in A\,|\,\exists\alpha\in\operatorname{Sper}(A,T)\text{ such that }a\in\operatorname{supp}(\alpha)\}=\\&=\bigcup\{\operatorname{supp}(\alpha)\,|\,\alpha\in\operatorname{Sper}(A,T)\}.\end{aligned}$$

(iii) Warning. Even though the ideal M is maximal in $\mathcal{T}$, and prime, it *may not* be a maximal ideal of A; however, it is maximal in some important cases, e.g., when $\langle A,T\rangle$ is a *bounded inversion ring* (BIR). (Add Ref. here) □

VI.5.4 In case G_A (A a semi-real ring) (resp. $G_{A,T}$, $\langle A,T\rangle$ a p-ring) is a fan, the elements of Sper (A,T) have a simpler characterization, coming from the characterization of ARS-fans given in VI.2.3, namely

> A real semigroup G is a RS-fan if and only if the set of its prime ideals is totally ordered under inclusion and every character of *ternary* semigroups $G\longrightarrow\mathbf{3}$ preserves representation.

Since the RS-characters of G_A (resp. $G_{A,T}$) are, by any another name, the elements of Sper (A) (resp., Sper (A,T)), see I.5.5, our task is to characterize the TS-characters of G_A (resp. $G_{A,T}$) in terms of the ring A (resp., the p-ring $\langle A,T\rangle$). We begin by the simpler case of semi-real rings.

Proposition VI.5.5 *Let A be a semi-real ring and let X_{G_A} (or X_A for short) denote the set of all ternary semigroup (TS-)characters of G_A into* $\mathbf{3}$.

(i) The correspondence

$$h\in X_A\ \longmapsto\ \{a\in A\,|\,h(\overline{a})\in\{0,1\}\}=\{a\in A\,|\,h(\overline{a})=h(\overline{a^2})\}$$

establishes a bijection from X_A onto the family of all subsets of $S\subseteq A$ satisfying the following conditions:

[MSO-1] $\sum A^2\subseteq S$;

[MSO-2] *S is closed under product;*

[MSO-3] $-1\notin S$;

[MSO-4] $S \cup -S = A$;

[MSO-5] *$S \cap -S$ is prime: for all $x, y \in A$, $xy \in S \cap -S \Rightarrow x \in S \cap -S$ or $y \in S \cap -S$.*

[MSO-6] *For every $n \geq 1$ and $a_1, \ldots, a_n \in A$, $\sum_{i=1}^n a_i^2 \in S \cap -S \Rightarrow a_1, \ldots, a_n \in S \cap -S$.*

(ii) This correspondence is an order isomorphism from X_A, ordered by specialization (in the spectral topology), onto the set of $S \subseteq A$ satisfying [MSO-1] – [MSO-6], *ordered by inclusion.*

Remark. Note that [MSO-2] and [MSO-4] imply that $S \cap -S$ is closed under multiplication by arbitrary elements of A.

Proof. For $h \in X_A$, the set $S_h = \{a \in A \,|\, h(\overline{a}) \in \{0, 1\}\}$ satisfies [MSO-1] – [MSO-6]:

— [MSO-2] holds because h preserves products.

— [MSO-3] comes from $h(\overline{-1}) = -1$ (h is a TS-homomorphism).

— [MSO-4] If $a \in A \setminus S_h$, then $h(\overline{a}) = -1$, whence $h(-\overline{a}) = 1$, and $a \in -S_h$.

— [MSO-5] Follows from $S_h \cap -S_h = \{a \in A \,|\, h(\overline{a}) = 0\}$ and the fact that $xy = 0 \Rightarrow x = 0$ or $y = 0$, for $x, y \in \mathbf{3}$.

In order to prove [MSO-1] and [MSO-6] we recall:

Fact 1. *Given $\alpha \in \mathrm{Sper}\,(A)$, let $\pi_\alpha : A \longrightarrow A/\mathrm{supp}\,(\alpha)$ denote the canonical ring homomorphism and let $\leq_\alpha$ denote the (total) order of $A/\mathrm{supp}\,(\alpha)$ determined by α, i.e., $\pi_\alpha(x) \geq_\alpha 0 \Leftrightarrow x \in \alpha$ $(x \in A)$. Then, for $a \in A$, $\overline{a}(\alpha) = \mathrm{sgn}_{\leq_\alpha}(\pi_\alpha(a))$.* Give ref.: [BCR]. $\square$

Claim. *(i)* If $a \in \sum A^2$, then $\overline{a} = \overline{a^2}$.
(ii) If $a = \sum_{i=1}^n a_i^2$ and $k \in \{1, \ldots, n\}$, then $\overline{a} \cdot \overline{a_k^2} = \overline{a_k^2}$.

Proof of Claim. (i) Let $\alpha \in \mathrm{Sper}\,(A)$. Since $\sum A^2 \subseteq \alpha$, Fact 1 implies $\overline{a}(\alpha) \in \{0, 1\}$, and then $\overline{a}(\alpha) = \overline{a^2}(\alpha)$, since $x^2 = x$ for $x \in \{0, 1\} \subseteq \mathbf{3}$.

(ii) It suffices to prove $\overline{a_k^2}(\alpha) = 1 \Rightarrow \overline{a}(\alpha) = 1$, for $\alpha \in \mathrm{Sper}\,(A)$. Using Fact 1, since π_α is a ring homomorphism, from $\pi_\alpha(a_k^2) >_\alpha 0$ and $\pi_\alpha(x^2) \geq_\alpha 0$ for all $x \in A$, we get $\pi_\alpha(a) = \sum_{i=1}^n \pi_\alpha(a_i^2) \geq_\alpha \pi_\alpha(a_k^2) >_\alpha 0$, as required. $\square$

— [MSO-1] follows at once from item (i) in the Claim.

— [MSO-6] comes from the already noted fact that $S_h \cap -S_h = \{a \in A \,|\, h(\overline{a}) = 0\}$ and item (ii) in the Claim: if $\sum_{i=1}^n a_i^2 \in S_h \cap -S_h$, then $h\left(\overline{\sum_{i=1}^n a_i^2}\right) = 0$, and (by Claim) $h\left(\overline{a_k^2}\right) = 0$ $(= h(\overline{a_k}))$, whence $\overline{a_k} \in S_h \cap -S_h$, for $k \in \{1, \ldots, n\}$.

Conversely, we must prove that any set $S \subseteq A$ satisfying [MSO-1] – [MSO-6] determines a TS-character $h_S \in X_A$ such that $S = S_{h_S}$. For $a \in A$, set

$$h_S(\overline{a}) = \begin{cases} 1 & \text{if } a \in S \setminus (-S) \\ 0 & \text{if } a \in S \cap -S \\ -1 & \text{if } a \in (-S) \setminus S\,. \end{cases}$$

In order to show that h_S is well defined, i.e., $\overline{a} = \overline{b} \Rightarrow h_S(\overline{a}) = h_S(\overline{b})$, we will need the following ring-theoretic characterization of "equality of bars":

Fact 2. ([M], Cor. 5.4.3, p. 94) *Given a p-ring $\langle A, T \rangle$ and $a, b \in A$,*

$\overline{a}_T = \overline{b}_T \Leftrightarrow$ *There exists $s, t \in T$ and $k \geq 0$ such that $sab = (a^2 + b^2)^k + t$.* $\square$

— h_S is well defined. (In this proof, $T = \sum A^2$.) Assume $\overline{a} = \overline{b}$, and suppose first that

$h_S(\overline{a}) = 0$, i.e., $a \in S \cap -S$; by the Remark following the statement, $sab \in S \cap -S$, and by Fact 2, $h_S\left(\overline{(a^2+b^2)^k+t}\right) = 0$. Since $t \in \sum A^2$, the sum $(a^2+b^2)^k + t$ is in $\sum A^2$. Then, [MSO-6] yields $b \in S \cap -S$, i.e., $h_S(\overline{b}) = 0$. Interchanging a and b, we have $h_S(\overline{a}) = 0 \Leftrightarrow h_S(\overline{b}) = 0$.

Next, suppose $a, b \notin S \cap -S$ and that $h_S(\overline{a}), h_S(\overline{b})$ have different signs, e.g., $a \in S \setminus (-S)$, $b \in (-S) \setminus S$. Note that $s \in \sum A^2 \subseteq S$ ([MSO-1]); therefore $sab \in -S$. On the other hand, the expression in the right-hand side of the equality in Fact 2 is a sum of squares, and hence is in S ([MSO-1]); it follows that $sab \in S \cap -S$, whence $(a^2+b^2)^k + t \in S \cap -S$ (Fact 2) and, by [MSO-6], $a, b \in S \cap -S$, contradiction. Conclusion: $h_S(\overline{a}) = h_S(\overline{b})$, as required.

Next, note that $\mathrm{dom}(h_S) = A$, by [MSO-4]. That h_S preserves product is proved by cases, using [MSO-5]; details are left to the reader. That $h_S(\overline{i}) = i$ for $i \in \{0, \pm 1\}$ is routine, using [MSO-3], $1 = 1^2 \in S$ and $0 \in S \cap -S$ ([MSO-5]).

Finally, $S = (S \setminus (-S)) \cup (S \cap -S)$ entails $a \in S \Leftrightarrow h_S(\overline{a})) \in \{0, 1\}$, i.e., $S = S_{h_S}$.

(ii) follows at once from the equivalence of (1) and (3) in Lemma I.1.18. □

Reminder. We emphasize that the sets $S \subseteq A$ satisfying conditions [MSO-1] – [MSO-6] *may not* be additively closed. Those that are additively closed belong to Sper (A). □

Remark VI.5.6 The case of $G_{A,T}$ requires more care. With notation as in the proof of VI.5.5, for $h \in X_{A,T}$ —i.e., h a TS-character of $G_{A,T}$— we clearly have:

$$T \subseteq S_h \;\Leftrightarrow\; h \lceil \overline{T} \subseteq \{0, 1\},$$

where $\overline{T} = \{\overline{t} \,|\, t \in T\}$. Even though this condition holds for $\sum A^2$ ([MSO-1]), it is not automatically fulfilled by arbitrary preorders of A.

Example VI.5.7 Let X be a completely regular topological space, and let $\emptyset \neq K \subsetneq X$ be a closed subset; with $C(X)$ denoting the ring of real-valued continuous functions on X, set

$$T_K = \{f \in C(X) \,|\, f \lceil K \geq 0\}.$$

T_K is a preorder of $C(X)$. Recall that Spec $(C(X))$ is canonically homeomorphic to Sper $(C(X))$ by the map $\alpha \mapsto \alpha \cap -\alpha$ ($\alpha \in$ Sper $(C(X))$); in other words, for every prime ideal $P \subseteq C(X)$, the quotient domain $C(X)/P$ has a unique (total) order, denoted $\leq_P$. For $f \in C(X)$ and $P \in$ Spec $(C(X))$,

$$\overline{f}(P) = \mathrm{sgn}_{\leq_P}(f/P).$$

For every $x \in X$, the evaluation

$$h_x(\overline{f}) = \mathrm{sgn}(f(x)) \quad (\text{in } \mathbb{R}),$$

is a TS-character of $G_{C(X)}$. By complete regularity, if $x_0 \in X \setminus K$, there is $t \in C(X)$ so that $t \lceil K = 0$ and $t(x_0) = -1$; hence, $t \in T_K$ and $h_{x_0}(\overline{t}) = -1$. □

However, we have:

Proposition VI.5.8 *Let $\langle A, T \rangle$ be a p-ring. Then, $h \lceil \overline{T} \subseteq \{0, 1\}$ (i.e., with notation as in the proof of VI.5.5, $T \subseteq S_h$) for every TS-character $h : G_{A,T} \longrightarrow \mathbf{3}$ which preserves representation. In particular, if $G_{A,T}$ is a RS-fan, this inclusion holds for every $h \in X_{A,T}$.*

Proof. For $t \in T$, we have $t = t \cdot 1 + t \cdot 0$, and hence $\overline{t} \in D_{G_{A,T}}(\overline{1}, \overline{0})$ (cf. [M], Add ref.). Since h preserves representation, $h(\overline{t}) \in D_{\mathbf{3}}(h(\overline{1}), h(\overline{0})) = D_{\mathbf{3}}(1, 0)$; this entails $h(\overline{t}) \in \{0, 1\}$. □

The analog of Proposition VI.5.5 for p-rings is:

Proposition VI.5.9 *Let $\langle A, T \rangle$ be a p-ring and let $X_{A,T}$ denote the set of all ternary semi-group $(TS\text{-})$characters of $G_{A,T}$ into $\mathbf{3}$. To keep matters straight we denote by $\overline{a_T} = \overline{a} \restriction \mathrm{Sper}\,(A, T) : \mathrm{Sper}\,(A, T) \longrightarrow \mathbf{3}$ $(a \in A)$, the elements of $G_{A,T}$.*

(i) The correspondence

$$h \in X_{A,T} \ \longmapsto\ \{a \in A \,|\, h(\overline{a_T}) \in \{0, 1\}\} = \{a \in A \,|\, h(\overline{a_T}) = h(\overline{a^2_T})\},$$

establishes a bijection from $X_{A,T}$ onto the family of all subsets of $S \subseteq A$ satisfying the following conditions:

[MSO_T−1] $T \subseteq S$;

[MSO-2] – [MSO-5] *as in Proposition VI.5.5.*

[MSO_T−6] *For all $t_1, t_2 \in T$, if $t_1 + t_2 \in S \cap -S$, then $t_1, t_2 \in S \cap -S$.*[2]

(ii) This correspondence is an order isomorphism from $X_{A,T}$, ordered by specialization (in the spectral topology), onto the set of $S \subseteq A$ satisfying [MSO_T−1]–[MSO_T−6]*, ordered by inclusion.*

Proof. We only indicate the modifications to be done in the proof of Proposition VI.5.5, i.e., we prove only [MSO_T−1] and [MSO_T−6]. As before, for $h \in X_{A,T}$, we set $S_h = \{a \in A \,|\, h(\overline{a_T}) \in \{0, 1\}\} = \{a \in A \,|\, h(\overline{a_T}) = h(\overline{a^2_T})\}$.

— [MSO_T−1] holds because $T \subseteq \alpha$ for all $\alpha \in \mathrm{Sper}\,(A, T)$: if $t \in T \subseteq \alpha$, then $\overline{t_T}(\alpha) \in \{0, 1\}$, and hence $\overline{t_T} = \overline{t^2_T}$, which in turn gives $h(\overline{t_T}) = h(\overline{t^2_T})$.

— [MSO_T−6], is an immediate consequence of the following analog to item (ii) of the Claim in the proof of VI.5.5, and $h(\overline{a_T}) = 0$ iff $a \in S_h \cap -S_h$.

Claim. For $t_1, t_2 \in T$ and $t = t_1 + t_2$, we have $\overline{t} \cdot \overline{t_i} = \overline{t_i}$ $(i = 1, 2)$.
Proof of Claim. It suffices to show, for $\alpha \in \mathrm{Sper}\,(A, T)$: $\overline{t_i}(\alpha) = 1 \Rightarrow \overline{t}(\alpha) = 1$. Fix $i = 1$. The antecedent means $\pi_\alpha(t_1) >_\alpha 0$. Since $t_2 \in T$ implies $\pi_\alpha(t_2) \geq_\alpha 0$, we get $\pi_\alpha(t) = \pi_\alpha(t_1) + \pi_\alpha(t_2) \geq_\alpha \pi_\alpha(t_1) >_\alpha 0$, whence $\overline{t}(\alpha) = 1$. □

Conversely, any set $S \subseteq A$ satisfying [MSO_T−1]– [MSO_T−6], determines a map $h_S : G_{A,T} \longrightarrow \mathbf{3}$ defined as in the proof of VI.5.5; the proof of well-definedness and of the required properties of h_S is a minor modification of that in VI.5.5, replacing $\sum A^2$ by T.

The proof of (ii) is similar to that of the corresponding item (ii) in Proposition VI.5.5 □

Remark VI.5.10 Proposition VI.5.9 shows that, if $G_{A,T}$ is a RS-fan, then every set $S \subseteq A$ satisfying conditions [MSO_T− 1]–[MSO_T− 6] is automatically closed under addition, i.e., an element of $\mathrm{Sper}\,(A, T)$. This is a ring-theoretic analog of the definition of a fan (as a preorder) in a field, due to [BK]; cf. [L2], Def. 5.1, p. 39. Below we prove that the converse holds as well. □

It will be convenient to give a name to the objects at hand. Following the terminology in [BK], set

Definition VI.5.11 For a semi-real ring A and a preorder T of A, we call:

(i) **Multiplicative semi-ordering** (or **multiplicative prime cone**) any set $S \subseteq A$ satisfying conditions [MSO-1] – [MSO-6] in Proposition VI.5.5 (i).

[2] In other words, the *set* $S \cap -S$ (not necessarily an ideal!) is T-convex.

(ii) **Multiplicative T-semi-ordering** (or **multiplicative T-prime cone**) any set $S \subseteq A$ satisfying conditions $[\mathrm{MSO}_T-1]$–$[\mathrm{MSO}_T-6]$ in Proposition VI.5.9 (i).

(iii) $\mathcal{M}(A)$ and $\mathcal{M}(A,T)$ will respectively denote the families of multiplicative semi-orderings and multiplicative T-semi-orderings of A. □

As a next step we show that the condition $\mathcal{M}(A,T) = \mathrm{Sper}\,(A,T)$, necessary for $G_{A,T}$ to be a RS-fan, entails that the set $\mathcal{T}(A,T)$ of T-convex prime ideals of A is totally ordered under inclusion; see VI.1.1.

Proposition VI.5.12 *(i) Let $\langle A,T\rangle$ be a p-ring. With notation as above, condition $\mathcal{M}(A,T) =$ $\mathrm{Sper}\,(A,T)$ implies that the set $\mathcal{T}(A,T)$ of T-convex prime ideals of A is totally ordered under inclusion.*
(ii) A similar statement holds for semi-real rings: $\mathcal{M}(A) = \mathrm{Sper}\,(A)$ implies that the set of real prime ideals of A is a chain under inclusion.

Proof. We only prove (i). Let $I, J \in \mathcal{T}(A,T)$; let $\alpha \in \mathrm{Sper}\,(A,T)$ be such that $I = \mathrm{supp}\,(\alpha)$ (cf. [BCR], Prop. 4.3.8, p. 90). Set $S = J \cup \alpha$.

We first observe that $S \in \mathcal{M}(A,T)$. Conditions $[\mathrm{MSO}_T-1]$, [MSO-2] and [MSO-3] are obvious.

— [MSO-4]. Since $-S = J \cup -\alpha$ and $\alpha \cup -\alpha = A$, we have $S \cup -S = J \cup \alpha \cup -\alpha = A$.

— [MSO-5]. By the previous item we have $S \cap -S = (J \cup \alpha) \cap (J \cup -\alpha) = J \cup (\alpha \cap -\alpha) = J \cup I$. Since both I, J are prime (ideals), we get $xy \in S \cap -S$ implies $x \in S \cap -S$ or $y \in S \cap -S$.

— $[\mathrm{MSO}_T-6]$. Again, since $S \cap -S = J \cup I$ and both I, J are T-convex, we get the desired conclusion.

By assumption, S is additively closed. Assume, towards a contradiction, that there are $a, b \in A$ such that $a \in I \setminus J$ and $b \in J \setminus I$. In particular, $a \in I \subseteq \alpha \subseteq S$ and $b \in J \subseteq S$, whence $a + b \in S$. If $a + b \in J$, since $-b \in J$ we get $a = (a + b) + (-b) \in J$, contradiction. Then, $a + b \in \alpha$, and from $a \in I \subseteq -\alpha$, we get $b = (a + b) + (-a) \in \alpha$. Next, since $-a \in I \setminus J$ and $-b \in J \setminus I$, the preceding argument can be carried out with a, b replaced with $-a, -b$, respectively, to conclude that $-b \in \alpha$. Thus, $b \in \alpha \cap -\alpha = I$, contradiction. □

Summarizing the preceding arguments, we state:

Corollary VI.5.13 *Let A be a semi-real ring and let T be a preorder of A.*

(i) Are equivalent:
- *(a) G_A is a RS-fan.*
- *(b) Every multiplicative semi-ordering of A is automatically closed under addition, i.e., $\mathcal{M}(A) = \mathrm{Sper}\,(A)$.*

(ii) Are equivalent:
- *(a) $G_{A,T}$ is a RS-fan.*
- *(b) Every multiplicative T-semi-ordering of A is automatically closed under addition, i.e., $\mathcal{M}(A,T) = \mathrm{Sper}\,(A,T)$.* □

Proof. We only comment on (b) ⇒ (a) in (ii). The statement

> "Every $S \subseteq A$ satisfying conditions $[\mathrm{MSO}_T–1]$-$[\mathrm{MSO}_T–6]$ in VI.5.9 (i) is closed under addition",

is just a translation of

> "Every TS-character of $G_{A,T}$ is a RS-character",

which, in the terminology of Definition VI.0.1 (1), means $G_{A,T}$ is a "fan$_1$". Since we have just proved (VI.5.12 (i)) that the set $\mathcal{T}(A,T)$ of T-convex prime ideals of A is totally ordered under inclusion, Fact VI.5.2 shows that the set of all (TS-)ideals of $G_{A,T}$ is an inclusion chain; then, by Proposition VI.1.2, the real semigroup $G_{A,T}$ satisfies condition [Z] in Theorem VI.2.1 and, by Corollary VI.2.3 and the subsequent Remark, $G_{A,T}$ is a RS-fan. □

This characterization yields a first batch of natural examples of p-rings whose associated real semigroup is a fan.

Corollary VI.5.14 *Let K be a field and T be a preorder of K which is a fan. Let A be a subring of K whose field of fractions is K. Then, the real semigroup $G_{A,T\cap A}$ is a fan. In particular, if $A = A_v$ is the valuation ring of a T-compatible valuation v of K, the real semigroup $G_{A_v,T\cap A_v}$ is a fan.*

Proof. According to Corollary VI.5.13 (ii) we must check that any mutiplicative semi-ordering $S \in \mathcal{M}(A, T\cap A)$ is closed under addition. Let $S' = \{\frac{a}{b} \mid a, b \in A, b \neq 0 \text{ and } ab \in S\} \subseteq K$. We first show:

— $S' \setminus \{0\}$ is a subgroup of $K^\times$, $T \subseteq S'$ and $-1 \notin S'$.

Clearly, $S \subseteq S'$ and (by [MSO-3]) $-1 \notin S'$. Since K is the field of fractions of A, any element of T can be written as $\frac{a}{b}$, with $a, b \in A, b \neq 0$. Then, $ab = \frac{a}{b} \cdot b^2 \in T \cap A$. Since $T \cap A \subseteq S$ ([$\mathrm{MSO}_T$$-$1]), we get $\frac{a}{b} \in S'$, hence $T \subseteq S'$. Since S is multiplicative ([MSO-2]), it follows that $S' \setminus \{0\}$ is a subgroup of $K^\times$.

According to one of the known characterizations of fans in fields (cf. [L2], Thm. 5.5, p. 40), S' is closed under addition in K, which clearly implies that S is additively closed in A. □

Proposition VI.5.15 *Let $\langle A, T\rangle$ be a p-ring whose associated real semigroup $G_{A,T}$ is a fan. Then* Sper(A,T) *has the following lifting property:*

[Lift] *For every $\alpha \in$ Sper(A,T) and $J \in \mathcal{T}(A,T)$ so that* supp$(\alpha) \subseteq J$*, there is $\beta \in$ Sper(A,T) such that $\alpha \subseteq \beta$ and* supp$(\beta) = J$.

Proof. Assume $G_{A,T}$ is a RS-fan. By VI.5.13 (ii.b), it suffices to prove [Lift] for $\mathcal{M}(A,T)$. The argument is, in fact, a reformulation in the present context of the proof Lemma VI.6.7 (2).

Let $S \in \mathcal{M}(A,T)$ and $J \in \mathcal{T}(A,T)$ be such that supp$(S) := S \cap -S \subseteq J$. Set $\beta = J \cup (S\setminus J)$. Clearly, $S \subseteq \beta$; it remains to prove that supp$(\beta) = J$ and $\beta \in \mathcal{M}(A,T)$ which, under our standing assumption, yields $\beta \in$ Sper(A,T). Clearly, $T \subseteq S \subseteq \beta$ ([$\mathrm{MSO}_T$$-$1]), and $-1 \notin \beta$ ([MSO-3]).

— β is closed under product ([MSO-2]).

Let $a, b \in \beta$. If one of a, b is in J, then $ab \in J \subseteq \beta$. So, suppose $a, b \notin J$; then $a, b \in S \setminus J$ and, since S is closed under product and J is prime, we get $ab \in S \setminus J \subseteq \beta$.

Now, note that $-\beta = J \cup ((-S) \setminus J)$.

— $\beta \cup -\beta = A$ ([MSO-4]).

We have: $\quad \beta \cup -\beta = J \cup (S \setminus J) \cup ((-S) \setminus J) = J \cup ((S \cup -S) \setminus J) \supseteq S \cup -S = A.$

— $\beta \cap -\beta = J$; in particular, since J is a prime ideal, $\beta \cap -\beta$ verifies [MSO-5] and, since J is T-convex, $\beta \cap -\beta$ verifies [$\mathrm{MSO}_T$$-$6].

We compute:

$$\beta \cap -\beta = (J \cup (S \setminus J)) \cap (J \cup ((-S) \setminus J)) = \qquad [J \cap ((\pm S) \setminus J) = \emptyset]$$

$= J \cup ((S \setminus J) \cap ((-S) \setminus J)) = J \cup ((S \cap -S) \setminus J) =$ [since $S \cap -S \subseteq J$]

$= J.$ □

Notation VI.5.16 For $I \in \mathcal{T}(A,T)$ ($\langle A, T \rangle$ a p-ring), we let

- A_I denote the localization of A at I,
- $M_I = I \cdot A_I$ denote the maximal ideal of A_I, and
- $T_I = T \cdot (A \setminus I)^{-2}$ denote the preorder induced by T on A_I. □

Fact VI.5.17 *T_I/M_I is a proper preorder of the field A_I/M_I.*

Proof. Straightforward checking shows that T_I/M_I is a preorder of A_I/M_I. We prove that it is proper.

Assume, on the contrary, that $-1 \in T_I/M_I$, i.e., $-1 = \left(\frac{t}{x^2}\right)/M_I$, with $t \in T$ and $x \in A \setminus I$; that is, $\frac{t}{x^2} + 1 \in M_I = I \cdot A_I$, i.e., $\frac{t + x^2}{x^2} = \frac{i}{y}$, for some $i \in I$ and $y \in A \setminus I$. Since I is prime, we get $y \cdot (t + x^2) \equiv x^2 i \pmod{I}$, whence $y \cdot (t + x^2) \in I$, and $t + x^2 \in I$. Since $t, x^2 \in T$ and I is T-convex, we obtain $x \in I$, contradiction. □

B. Total preorders and trivial fans in rings.

Definition VI.5.18 A **total preorder** in a ring A is a (proper) preorder T such that $T \cup -T = A$. □

Fact VI.5.19 *For a total preorder T of a ring A, $T \cap -T$ is a proper ideal of A.* □

The easy proof is left as an exercise.

Remarks VI.5.20 (i) The ideal $T \cap -T$ *may not* be prime (see Example VI.5.21). When it is, the notion of "total preorder" coincides with "prime cone", i.e., element of Sper(A).

(ii) When $T \cap -T = \{0\}$ the total preorders are just the total orders of A. □

Example VI.5.21 Let $A := \mathbb{R}[X]/(X^2)$; the elements of A are uniquely representable in the form $aX + b$ with $a, b \in \mathbb{R}$. Clearly, the zero ideal of A is not radical, hence not prime either: $X \neq 0$ but $X^2 = 0$. We define a total (pre)order T in A by the stipulation:

$$aX + b \in T \text{ iff } b > 0 \text{ or } (b = 0 \text{ and } a \geq 0).$$

Checking that T is a total (pre)order of A is routine, left to the reader. However, the ideal $T \cap -T = \{0\}$ is not prime. □

Propositions VI.5.22 and **??** below show that total preorders are preserved by localization at and lifting by convex prime ideals.

Proposition VI.5.22 *Let T be a total preorder of a ring A, let I be a T-convex prime ideal of A, and let $T_I = T \cdot (A \setminus I)^{-2}$ be the preorder induced by T on the localization of A at I. Then, T_I is a total preorder of A_I.*

Proof. We already know that T_I is a proper preorder of A_I (cf. proof of Lemma VI.5.17; note that T-convexity of I is needed to prove that T_I is proper). So, we need only show that $T_I \cup -T_I = A_I$. Let $\frac{a}{x} \in A_I$, i.e., $x \notin I$. Then, $\frac{x}{1}$ is invertible in A_I and $\frac{1}{x^2} = \left(\frac{1}{x}\right)^2 \in T_I$. Hence, $\frac{a}{x} = \frac{ax}{1} \cdot \left(\frac{1}{x^2}\right)$. If $ax \in T$, then $\frac{a}{x} \in T_I$, and if $ax \in -T$, then $\frac{a}{x} \in -T_I$; we conclude from $T \cup -T = A$. □

Proposition VI.5.23 *Let I be a prime ideal of the ring A and let Q be a total preorder of A_I. Then, there is a total preorder T of A such that $T_I = Q$. Further,*
(i) $T \cap -T = \iota_I^{-1}[Q \cap -Q]$, where $\iota_I : A \longrightarrow A_I$ is the canonical map $a \mapsto \dfrac{a}{1}$ $(a \in A)$.
(ii) If the maximal ideal M_I of A_I is Q-convex, then I is T-convex.

Proof. Let $T = \iota_I^{-1}[Q] = \{a \in A \mid \iota_I(a) = \frac{a}{1} \in Q\}$. Since ι_I is a ring homomorphism, we have:
— T is closed under under sum and product, and contains A^2.
— T is proper (since $\iota_I(-1) = \dfrac{-1}{1} \notin Q$).
— $T \cup -T = A$.

We show:
— $T_I \subseteq Q$.
Let $z \in T_I$, i.e., $z = \dfrac{t}{x^2}$, with $t \in T, x \notin I$. Then, $\dfrac{t}{1} \in Q$, $\dfrac{x^2}{1}$ is invertible in A_I, and $\dfrac{1}{x^2} \in Q$. It follows that $z = \dfrac{t}{1} \cdot \dfrac{1}{x^2} \in Q$.
— $Q \subseteq T_I$.
Let $z \in Q$; then, $z = \dfrac{x}{y}$, with $x, y \in A, y \notin I$, which implies $z = \dfrac{xy}{y^2}$; this gives $\dfrac{y^2}{1} \cdot z = \dfrac{xy}{1}$. Clearly, $\dfrac{y^2}{1} = \left(\dfrac{y}{1}\right)^2 \in Q$, whence $\dfrac{y^2}{1} \cdot z \in Q$, and $\dfrac{xy}{1} \in Q$, which shows that $xy \in T$. Hence, $z = \dfrac{xy}{y^2} \in T_I$.

(i) It is obvious that $T \cap -T = \iota_I^{-1}[Q \cap -Q]$.
(ii) Let $t_1, t_2 \in T$ be such that $t_1 + t_2 \in I$. Then, $\dfrac{t_i}{1} \in Q$ $(i = 1, 2)$, and $\dfrac{t_1 + t_2}{1} \in I \cdot A_I = M_I$. By the convexity assumption, $\dfrac{t_1}{1}, \dfrac{t_2}{1} \in I \cdot A_I$. For $i = 1, 2$, we have $\dfrac{t_i}{1} = \dfrac{j}{x}$, with $j \in I, x \notin I$. It follows that $xt_i - j \in I$, whence $t_i \in I$, since $x \notin I$, as required. □

Remark VI.5.24 Even if Q is a total order of A_I, T *may not* be a total order of A. In fact,
$$T \cap -T = \iota_I^{-1}[Q \cap -Q] = \iota_I^{-1}[0],$$
which, in general is not $\{0\}$. Note that, for $x \in A$,
$$x \in \iota_I^{-1}[0] \Leftrightarrow \iota_I(x) = 0 \text{ (in } A_I) \Leftrightarrow \exists\, z \notin I\, (zx = 0).$$
□

The following result proves two important properties of total preorders in rings:

Theorem VI.5.25 *(i) Let T be a total preorder of a ring A. Then, the real semigroup $G_{A,T}$ is a fan.*
(ii) Let T_0, T_1 be total preorders of a ring A, and let $T = T_0 \cap T_1$. Assume that the set $\mathcal{T}(\mathcal{A}, \mathcal{T})$ of T-convex prime ideals of A is totally ordered under inclusion. Then, the real semigroup $G_{A,T}$ is a fan.

Remark. In case the ring A is a field, K, a total preorder is just a (total) order of K. Thus, Theorem VI.5.25 is a ring-theoretic analog of the well known fact that the intersection of at most two total orders of a field is a fan, namely a *trivial fan*, cf. [L2], Prop. 5.3, p. 39. □

Proof. (i) By Corollary VI.4.11 it suffices to check that Sper(A, T) $(= X_{G_{A,T}})$ is totally ordered under inclusion (= specialization); the proof is identical to that showing that the real spectrum of a ring is a root system: let $\alpha, \beta \in$ Sper(A, T), and assume that $\alpha \not\subseteq \beta$ and $\beta \not\subseteq \alpha$;

let $a \in \alpha \setminus \beta$ and $b \in \beta \setminus \alpha$; since the preorder T is total, either $a - b \in T \subseteq \beta$ or $b - a \in T \subseteq \alpha$; hence, $a = b + (a - b) \in \beta$ or $b = a + (b - a) \in \alpha$, absurd.

We check that assumptions (1) – (3) of VI.4.12 are verified by $G_{A,T}$.

Assumption (1) holds by hypothesis, as the saturated prime ideals of $G_{A,T}$ are in a bijective, inclusion-preserving correspondence with the T-convex prime ideals of A (cf. Fact VI.5.1 (b)).

Assumption (2) follows from the proof of (i) and:

$(*)$ Sper $(A, T) =$ Sper $(A, T_0) \cup$ Sper (A, T_1).

Proof of $(*)$. Clearly, Sper $(A, T_i) \subseteq$ Sper (A, T) for $i = 0, 1$. Assume there is $\alpha \in$ Sper (A) such that $T \subseteq \alpha$ but $T_0, T_1 \nsubseteq \alpha$, and pick $t_i \in T_i \setminus \alpha$ $(i = 0, 1)$. Then, $-t_0 \in \alpha$ and $t_0 \notin T_1$ (otherwise $t_0 \in T_0 \cap T_1 \subseteq \alpha$). Since T_1 is a total preorder, $t_0 \in -T_1$. Likewise, $-t_1 \in \alpha$ and $t_1 \in -T_0$.

From $t_0 \in T_0$ and $-t_1 \in T_0$ we get $-t_0 t_1 \in T_0$; from $t_1 \in T_1$ and $-t_0 \in T_1$ we get $-t_0 t_1 \in T_1$; hence, $-t_0 t_1 \in T_0 \cap T_1 \subseteq \alpha$. From $-t_0, -t_1 \in \alpha$ comes $t_0 t_1 = (-t_0)(-t_1) \in \alpha$. Hence, $t_0 t_1 \in \alpha \cap -\alpha = \mathrm{supp}(\alpha)$. Since this is a prime ideal, $t_i \in \mathrm{supp}(\alpha) \subseteq \alpha$ for $i = 0$ or $i = 1$, contradiction.

In order to prove assumption (3) of VI.4.12, we first show:

$(**)$ Every T-convex prime ideal I of A is *both* T_0-convex *and* T_1-convex.

Proof of $(**)$. From [BCR], Prop. 4.2.8 (ii), p. 87, we know that I is either T_0-convex or T_1-convex. Assume towards a contradiction that I is T_0-convex but not T_1-convex. Then, there are elements $t_0, t_1 \in T_1$ such that $t_0 + t_1 \in I$, but $t_0, t_1 \notin I$. Since I is T-convex, we have $t_0, t_1 \notin T_0$ and, since T_0 is a total preorder, $-t_0, -t_1 \in T_0$. As we have $-(t_0 + t_1) \in I$, T_0-convexity yields $-t_0, -t_1 \in I$, whence $t_0, t_1 \in I$, contradiction.

Now, [BCR], Prop. 4.3.8, p. 90 finishes the proof: for $i = 0, 1$, there is $\alpha_i \in$ Sper (A, T_i) so that $\mathrm{supp}(\alpha_i) = I$. $\square$

Remark VI.5.26 The following example shows that the requirement in item (ii) of Theorem VI.5.25 does not hold automatically. Let $A = C(\mathbb{R})$ be the ring of real-valued continuous functions on the reals. For $i = 0, 1$, let $T_i = \{f \in A \mid f(i) \geq 0\}$ and $M_i = \{f \in A \mid f(i) = 0\}$. The (maximal) ideal M_i is T_i-convex; hence, with $T = T_0 \cap T_1$, both M_0 and M_1 are T-convex; however, M_0 and M_1 are incomparable under inclusion. $\square$

VI.6 Levels of a ARS-fan

Look at changes in "Fans in Th. of Real Semigps."

The saturated prime ideals of a real semigroup induce a partition of its character space. The pieces are called *levels*: the level corresponding to a saturated prime ideal I of G is the set of all $g \in X_G$ such that $Z(g) = I$. In the case of RS-fans, (proper) ideals —automatically prime (VI.1.2) and saturated (VI.2.8)— are totally ordered under inclusion, a fact of much help in studying the mutual relationship of its levels. This notion, together with that of a *connected component* (VI.6.11), will be the main technical tools employed in the analysis of the fine structure of ARS-fans, initiated in this section and continued in subsequent ones.

We begin by proving that levels have a canonical structure of AOS-fans (VI.6.2), that is, of fans in the category of abstract order spaces (cf. [M], § 3.1, pp. 37 ff.). Inclusion of ideals induces AOS-morphisms between the corresponding levels (REF). The main results proved in this section are:

(i) The connected components of a ARS-fan are complete join-semilattices under the specialization order; we also exhibit interesting relations between the sup operation and product

(VI.6.12).

(ii) A relation between the cardinality of a finite RS-fan and that of its character space, an analog for RSs of a result known to hold for reduced special groups (VI.6.18). □

Move Prop. VI.6.1 before?. To section VI.4?

Proposition VI.6.1 *Let F be a RS-fan. Let I be a proper ideal of F. Let $\pi = \pi_I : F \longrightarrow F/I$ denote the canonical quotient map. Then, $(F/I) \setminus \{\pi(0)\}$ is a RSG-fan.*

Proof. In Theorem II.3.15 and with notation therein, it was shown that if G is a RS and I is a saturated prime ideal of G, then $\langle (G/I) \setminus \{\pi(0)\}, \cdot, \pi(-1), D_{G/I} \rangle$ is a RSG. We must prove: given $a, b \in F$ so that $\pi(a), \pi(b) \neq 0$ and $\pi(a) \neq \pi(-1)$,

$$\pi(b) \in D_{F/I}(\pi(1), \pi(a)) \Rightarrow \pi(b) = \pi(1) \vee \pi(b) = \pi(a).$$

By the characterization of $D_{F/I}$ in Theorem II.3.15 (b), there are $x \in F \setminus I$ and $i \in I$ such that $bx^2 \in D_F(i, 1, a)$. Hence, there is $c \in F$ such that $bx^2 \in D_F(i, c)$ and $c \in D_F(1, a)$. From the characterization of representation for fans (Theorem VI.2.1) we get:

$$\text{(A)} \qquad bx^2 \in D_F(i,c) \quad \Leftrightarrow \quad \begin{array}{ll} \text{(i)} & bx^2 = iy^2 \text{ for some } y \in F, \text{ or} \\ \text{(ii)} & bx^2 = cy^2 \text{ for some } y \in F, \text{ or} \\ \text{(iii)} & bx^2 i = -cbx^2 \text{ and } bx^2 = i^2 bx^2. \end{array}$$

$$\text{(B)} \qquad c \in D_F(1,a) \quad \Leftrightarrow \quad \begin{array}{ll} \text{(i)} & c = z^2 \text{ for some } z \in F, \text{ or} \\ \text{(ii)} & c = az^2 \text{ for some } z \in F, \text{ or} \\ \text{(iii)} & c = -ac \text{ and } c = a^2 c. \end{array}$$

Since $b, x \notin I$, we have $bx^2 \notin I$, which clearly excludes cases (A.i) and (A.iii), and entails $c \notin I$ in (A.ii), whence $\pi(c^2) = \pi(1)$. Case (B.iii) yields:

$$c = -ac \Rightarrow \pi(c) = -\pi(a)\pi(c) \Rightarrow \pi(c)^2 = \pi(1) = -\pi(a)\pi(c)^2 = -\pi(a),$$

and hence $\pi(a) = \pi(-1)$, contrary to assumption; thus, case (B.iii) is excluded as well. In the remaining cases we have:

(1) $bx^2 = cy^2$ and $c = z^2$ for some $y, z \in F$.

Hence, $bx^2 = (yz)^2$. Since $bx^2 \notin I$, it follows $y, z \notin I$, and then $\pi(b) = \pi(b)\pi(x^2) = \pi((yz)^2) = \pi(1)$.

(2) $bx^2 = cy^2$ and $c = az^2$ for some $y, z \in F$.

Thus, $bx^2 = a(yz)^2$. As in case (1) we have $yz \notin I$. Then, $\pi(b) = \pi(b)\pi(x^2) = \pi(a)\pi((yz)^2) = \pi(a)\pi(1) = \pi(a)$. □

Remarks and Notation VI.6.2 Given a real semigroup G and a saturated prime ideal I of G, we denote by G_I the RSG $(G/I) \setminus \{\pi(0)\}$. Recall the setup from § II.3 F: Every character $h \in X_G$ such that $Z(h) = I$ induces a map $\widehat{h} : G_I \longrightarrow \{\pm 1\}$ defined by $\widehat{h} \circ \pi_I = h$. The correspondence $h \mapsto \widehat{h}$ is a bijection between the set $L_I(G) = \{h \in X_G \mid Z(h) = I\}$ and the space of orders X_{G_I} of G_I. (L_I stands for "I-th level"; see Definition VI.6.6 (b) below.) Thus, we can identify the set $L_I(G) \subseteq X_G$ with the AOS (X_{G_I}, G_I). We shall systematically use this identification in the sequel, and unambiguously refer to the AOS structure of the set $L_I(G)$. In case G is a RS-fan, the preceding Proposition shows that $L_I(G)$ is an AOS-fan. □

The sets L_I, I an ideal, form a partition of the ARS associated to a given RS. In the context of fans, the correspondence assigning to each ideal I the set L_I is a (functorial) bijection, preserving, in a suitable sense, the (total) order of inclusion between ideals (VI.1.2(4)).

The following fact will be repeatedly used in the sequel:

Lemma VI.6.3 *Let I be an ideal of a RS-fan F. Then, for $a, b \in F \setminus I$:*

$$a \sim_I b \Leftrightarrow \exists z \notin I \; (az = bz).$$

Proof. ($\Rightarrow$) The congruence $\sim_I$ determined by an ideal I was defined in § II.3.F. Theorem II.3.15 (a) proves:

$$a \sim_I b \Leftrightarrow \exists z \notin I \; \exists i \in I \; (i \in D^t_F(az, -bz)).$$

Clause $[D^t]$ in Theorem VI.2.1 gives:

$$D^t_F(az, -bz) = \begin{cases} \{az, -bz\} & \text{if } \; az \neq bz \\ \{a^2z^2x \,|\, x \in F\} & \text{if } \; az = bz. \end{cases}$$

Since $az, -bz \notin I$ (as $a, b, z \notin I$) but $i \in I$, the first alternative is excluded; hence, $az = bz$.

($\Leftarrow$) Assume $az = bz$ for some $z \notin I$. Since $0 \in D^t_F(az, -bz) \cap I$, we conclude that $a \sim_I b$. □

The first step to establish the results mentioned above is: (MAKE MORE PRECISE!)

Proposition VI.6.4 *Let F be a RS-fan and let $I \subseteq J$ be ideals of F. With notation as in Remark VI.6.2, the rule $a/J \mapsto a/I \quad (a \in F \setminus J)$ defines a homomorphism of special groups $\iota_{JI} : F_J \longrightarrow F_I$.*

Proof. (1) ι_{JI} is well-defined.

We must show: $a, b \in F \setminus J \wedge a \sim_J b \Rightarrow a \sim_I b$. Since $I \subseteq J$, this is clear from Lemma VI.6.3.

Clearly, we have:

(2) ι_{JI} is a group homomorphism sending $-1/J$ to $-1/I$.

Since F_J is a fan, ι_{JI} is automatically a SG-homomorphism. □

The map $\iota^*_{JI} : X_{F_I} \longrightarrow X_{F_J}$ dual to ι_{JI} is, then, an AOS-morphism. Via the identification of $L_I(F)$ with X_{F_I}, see Remark VI.6.2, we get:

Fact VI.6.5 *Let F be a RS-fan and let $I \subseteq J$ be ideals of F. The map $\kappa_{IJ} : L_I(F) \longrightarrow L_J(F)$ which assigns to each $g \in L_I(F)$ the unique element $h \in L_J(F)$ such that $g \rightsquigarrow h$ is an AOS-morphism.*

Proof. To be precise, the map κ_{IJ} is $\kappa_{IJ} = (\varphi_J)^{-1} \circ \iota^*_{JI} \circ \varphi_I$, where φ_I denotes the bijection $g \mapsto \widehat{g}$ $(g \in L_I(F))$, identifying $L_I(F)$ with X_{F_I} (VI.6.2), and similarly for $L_J(F)$. It only remains to be proved that $g \rightsquigarrow \kappa_{IJ}(g)$, for $g \in L_I(F)$. To ease notation, write $h = \kappa_{IJ}(g)$. According to Lemma I.1.18 we must show $Z(g) \subseteq Z(h)$ and $a \notin Z(h) \Rightarrow g(a) = h(a)$. The inclusion of the zero-sets is $I \subseteq J$. Let $a \notin Z(h) = J$. Since:

$$\varphi_J(h) = \varphi_J(\kappa_{IJ}(g)) = \iota^*_{JI}(\varphi_I(g)) = \varphi_I(g) \circ \iota_{JI}, \quad \varphi_I(g)(a/I) = g(a) \quad \text{and}$$
$$\varphi_J(h)(a/J) = h(a),$$

(cf. VI.6.2), we get,

$$h(a) = (\varphi_I(g))(\iota_{JI}(a/J)) = \varphi_I(g)(a/I) = g(a),$$

as required. $\square$

Definition and Notation VI.6.6 Let F be a RS-fan.

(a) We denote by $\mathrm{Spec}\,(F)$ the set of all (necessarily prime (VI.1.2 (3)) and saturated (VI.2.8)) proper ideals of F.

(b) For $I \in \mathrm{Spec}\,(F)$ the set $L_I(F) = \{h \in X_F \mid Z(h) = I\}$ is called the $\boldsymbol{I}$**-th level** of X_F.

(c) For $f \in X_F$, the **depth** of f, denoted $d(f)$, is the order type of the set $\{g \in X_F \mid f \rightsquigarrow g\}$ under the order of specialization. (Since $(X_F, \rightsquigarrow)$ is a root-system, the order $\rightsquigarrow$ is total on this set.)

(d) For $I \in \mathrm{Spec}\,(F)$, the order type of the set $\{J \in \mathrm{Spec}\,(F) \mid J \supseteq I\}$ under the (total) order of inclusion will be called the **depth** of I, denoted $d(I)$.

(e) The **length** of X_F, denoted $\ell(X_F)$, is the order type of the (totally ordered) set $\mathrm{Spec}\,(F)$. $\square$

Remark. It is clear that the union and the intersection of an inclusion chain of (proper) prime ideals in any ternary semigroup is a (proper) prime ideal. In particular, if F is a fan, the totally ordered set Spec(F) is (Dedekind) **complete**. Its smallest element is $\{0\}$ (a prime ideal by VI.1.2 (3)). Any ternary semigroup has a (unique) largest prime ideal, namely, the set of non-invertible elements. See also Theorem VI.6.12 (c) below. $\square$

Next we shall prove that the depth of an ideal in a fan is the same as the depth of any element in the corresponding level; in particular, elements of the same depth belong to the same level. We shall need:

Lemma VI.6.7 *Let F be a RS-fan. Then:*

(1) *For all elements $g, h \in X_F$ such that $g \rightsquigarrow h$ (hence $Z(g) \subseteq Z(h)$) and every ideal I such that $Z(g) \subseteq I \subseteq Z(h)$ there is $f \in X_F$ such that $g \rightsquigarrow f \rightsquigarrow h$ and $Z(f) = I$.*

(2) *For every $g \in X_F$ and every ideal $I \supseteq Z(g)$ there is a (necessarily unique) $f \in X_F$ such that $g \rightsquigarrow f$ and $Z(f) = I$.*

(3) *For every ideal I of F there is an $f \in X_F$ such that $Z(f) = I$.*

Remark. The statement obtained by reversing both order relations in (2) is false, in general.

Proof. We first prove (1); the same proof, omitting item (c) below, also proves (2). Let $f : F \longrightarrow \mathbf{3}$ be defined by:

$$f \lceil I = 0 \quad \text{and} \quad f \lceil (F \setminus I) = g \lceil (F \setminus I).$$

(a) $Z(f) = I$.

By construction, $I \subseteq Z(f)$. Since $Z(g) \subseteq I$, $f(x) = g(x) \neq 0$ for $x \in F \setminus I$, i.e., $Z(f) \subseteq I$.

(b) $g \rightsquigarrow f$.

Clear, from (a) and Lemma I.1.18,

(c) $f \rightsquigarrow h$.

If $f(a) = 0$, then $a \in I \subseteq Z(h)$, and $h(a) = 0$.

If $h(a) \neq 0$, then $a \notin I$; since $g \rightsquigarrow h$, then $g(a) = h(a)$. Hence, $f(a) = g(a) = h(a)$, and we get $f \rightsquigarrow h$ by Lemma I.1.18.

(d) f is a TS-homomorphism.

Clearly $f(0) = 0$ and $f(\pm 1) = g(\pm 1) = \pm 1$. Let $a, b \in F$. If one of a, b is in I, so is ab, and we have $f(a)f(b) = 0 = f(ab)$. If $a, b \notin I$, then $ab \notin I$, and f and g take the same value on a, b and ab; the result follows from the fact that g is a TS-character. Since F is a fan, $f \in X_F$.

(3) This is Lemma I.1.7 (alternatively, Lemma I.4.8) $\square$

Remark VI.6.8 The element f such that $g \rightsquigarrow f$ and $Z(f) = I$ in VI.6.7(2) can also be obtained by taking any $h \in X_F$ with $Z(h) = I$ (VI.6.7(3)) and setting $f = h^2 g$. $\square$

Proposition VI.6.9 *Let F be a RS-fan. For $f \in X_F$ we have $d(f) = d(Z(f))$; equivalently, the sets $\{g \in X_F \,|\, f \rightsquigarrow g\}$ (ordered under specialization) and $\{J \in \mathrm{Spec}(F) \,|\, J \supseteq Z(f)\}$ (ordered under inclusion) are order-isomorphic.*

Proof. To ease notation, set $f\!\uparrow\, = \{g \in X_F \,|\, f \rightsquigarrow g\}$ and $I\!\uparrow\, = \{J \in \mathrm{Spec}(F) \,|\, J \supseteq I\}$ ($I \in \mathrm{Spec}(F)$).

We prove that the map $Z : f\!\uparrow\, \longrightarrow Z(f)\!\uparrow$ assigning to each $g \in f\!\uparrow$ its zero-set, is the required order isomorphism. That

(a) Z is increasing, and (b) Z is surjective,

is clear, from $g \rightsquigarrow h \Rightarrow Z(g) \subseteq Z(h)$ and Lemma VI.6.7(2), respectively.

(c) Z is injective.

Let $g, h \in f\!\uparrow$ be such that $Z(g) = Z(h)$. Since $(X_F, \rightsquigarrow)$ is a root-system, either $g \rightsquigarrow h$ or $h \rightsquigarrow g$; say the first. Then we must check that $Z(g) = Z(h) \wedge g \rightsquigarrow h \Rightarrow g = h$. If $a \in Z(g)$, then $g(a) = h(a) = 0$. If $a \notin Z(g)$, then $g(a), h(a) \neq 0$; $g \rightsquigarrow h$ yields $h = h^2 g$ (Lemma I.1.18), i.e., $h(a) = h(a)^2 g(a) = g(a)$, since $h(a)^2 = 1$. $\square$

A trivial variant of the proof of VI.6.9 gives:

Proposition VI.6.10 *Let F be a RS-fan. Given $f_1, f_2 \in X_F$ such that $f_1 \rightsquigarrow f_2$, the intervals $\{g \in X_F \,|\, f_1 \rightsquigarrow g \rightsquigarrow f_2\}$ (ordered under specialization) and $\{J \in Spec(F) \,|\, Z(f_1) \subseteq J \subseteq Z(f_2)\}$ (ordered under inclusion) are order-isomorphic.* $\square$

Revise $\downarrow$; put in agreement with Section I.7.

Our next result shows that each connected component of an ARS-fan, endowed with the specialization order $\rightsquigarrow$, is a complete join-semilattice. Further, there are interesting algebraic relations between the sup operation and the product of characters. Recall:

Definition VI.6.11 Let $(X, \preceq)$ be a root-system, and let $g_1, g_2 \in X$. Define:

$$g_1 \equiv_C g_2 \quad \text{iff} \quad g_1,\ g_2 \text{ have a common } \preceq\text{-upper bound.}$$

$\equiv_C$ is an equivalence relation; its classes are called **connected components** of $(X, \preceq)$. $\square$

Theorem VI.6.12 *Let F be a RS-fan, and let X_F be its dual character space. Let $\{h_i \,|\, i \in I\} \subseteq X_F$ be a non-empty family of characters belonging to a single connected component of X_F (i.e., having a common upper bound under $\rightsquigarrow$). Then,*

(a) $\{h_i \mid i \in I\}$ *has a least upper bound for the specialization order. In other words, each connected component of* X_F *is a complete join-semilattice.*

In particular:

(b) *Any two elements in the same connected component have a least upper bound for* $\rightsquigarrow$.

(c) *Any ARS-fan is an order-complete root-system, that is, any non-empty chain under specialization bounded above* (*resp., below*) *has a least upper bound* (*resp., a greatest lower bound*).

(d) *For* $g_1, g_2 \in X_F$, $(\bigvee_{i\in I} h_i)\, g_1 g_2 = \bigvee_{i\in I} (h_i\, g_1 g_2)$.

Proof. Let $J = \bigcap\{Z(g) \mid g \in X_F \text{ and } h_i \rightsquigarrow g \text{ for all } i \in I\}$. By assumption this definition makes sense, and by VI.2.8 (1) and VI.1.2 (3), J is a saturated prime ideal of F.

(a) Since $h_i \rightsquigarrow g$ implies $Z(h_i) \subseteq Z(g)$ $(i \in I)$, we have $\bigcup_{i\in I} Z(h_i) \subseteq J$. By VI.6.7 (2), for each $i \in I$ there is an $f_i \in X_F$ such that $h_i \rightsquigarrow f_i$ and $Z(f_i) = J$. We first show:

(a.i) $f_i = f_j$ for $i, j \in I$.

Let $a \in F$. If $a \in J$, then $f_i(a) = f_j(a) = 0$. Assume $a \notin J$. By the definition of J there is a $g \in X_F$ such that $h_i \rightsquigarrow g$ for all $i \in I$ and $a \notin Z(g)$. By Lemma I.1.18 this implies $h_i(a) = g(a)$, and $h_i \rightsquigarrow f_i$, $f_i(a) \neq 0$ imply $f_i(a) = h_i(a)$; hence $f_i(a) = g(a)$ for all $i \in I$, whence $f_i(a) = f_j(a)$. This proves (a.i).

Set $f = f_i$ (any $i \in I$). We claim:

(a.ii) f is the l.u.b. of $\{h_i \mid i \in I\}$ for the specialization order.

By the choice of f we have $h_i \rightsquigarrow f$ for all $i \in I$. Let $g \in X_F$ be such that $h_i \rightsquigarrow g$ for all $i \in I$. We show that $f \rightsquigarrow g$.

Firstly, we have $Z(f) = J \subseteq Z(g)$. Let $a \in F$ be such that $g(a) \neq 0$. By I.1.18 (4), $h_i \rightsquigarrow g$ implies $g(a) = h_i(a)$. Since we also have $f(a) \neq 0$, from $h_i \rightsquigarrow f$ follows $f(a) = h_i(a)$, whence $g(a) = f(a)$. Lemma I.1.18 (4) implies, then, that $f \rightsquigarrow g$.

(c) Let $\mathcal{C} \subseteq X_F$ be a non-empty $\rightsquigarrow$ - chain. If $\mathcal{C}$ is bounded above, its l.u.b. is given by (a): $\{h_i \mid i \in I\} = \mathcal{C}$ meets the assumptions. If $\mathcal{C}$ is bounded below, its g.l.b. is obtained by applying (a) with $\{h_i \mid i \in I\}$ = the (non-empty) set $\{h \in X_F \mid h \rightsquigarrow g \text{ for all } g \in \mathcal{C}\}$ of lower bounds of $\mathcal{C}$. Since $\mathcal{C} \neq \emptyset$, the family $\{h_i \mid i \in I\}$ has a common upper bound under $\rightsquigarrow$, namely any element of $\mathcal{C}$. Routine checking shows that the l.u.b. of $\{h_i \mid i \in I\}$ is the g.l.b. of $\mathcal{C}$.

(d) Set $f = \bigvee_{i\in I} h_i$. Fact VI.7.4 (a) shows that, for fixed $i \in I$, $h_i \rightsquigarrow f$ implies $h_i\, g_1 g_2 \rightsquigarrow f g_1 g_2$. Thus, $f g_1 g_2$ is a common $\rightsquigarrow$ -upper bound for the family $\{h_i\, g_1 g_2 \mid i \in I\}$. By (a), $\bigvee_{i\in I} h_i\, g_1 g_2$ exists; call it f'; we have proved that $f' \rightsquigarrow f g_1 g_2$.

To prove $f g_1 g_2 \rightsquigarrow f'$, by I.1.18 (4) it suffices to show:

(d.i) $Z(f g_1 g_2) \subseteq Z(f')$, and

(d.ii) For $a \in F$, $f'(a) \neq 0 \Rightarrow f'(a) = f(a)\, g_1(a)\, g_2(a)$.

(d.i) We first note that, for $i \in I$ and $g' \in X_F$, $h_i\, g_1 g_2 \rightsquigarrow g' \Rightarrow h_i \rightsquigarrow g' g_1 g_2$. In fact, by Lemma I.1.18 (4) the assumption amounts to $g' = (g')^2 g_1 g_2 h_i$; scaling by $g_1 g_2$ gives $g' g_1 g_2 = (g' g_1 g_2)^2\, h_i$, i.e., $h_i \rightsquigarrow g' g_1 g_2$.

In particular, since $h_i\, g_1 g_2 \rightsquigarrow f'$, we get $h_i \rightsquigarrow f' g_1 g_2$ for all $i \in I$, which proves that $f = \bigvee_{i\in I} h_i \rightsquigarrow f' g_1 g_2$. By I.1.18,

(*) $\quad Z(f) \subseteq Z(f') \cup Z(g_1) \cup Z(g_2)$.

Next, from $h_i\, g_1 g_2 \rightsquigarrow f'$ we get $Z(h_i\, g_1 g_2) \subseteq Z(f')$ for all $i \in I$, which gives

$$\bigcup_{i\in I} Z(h_i) \cup Z(g_1) \cup Z(g_2) \subseteq Z(f').$$

In particular, $Z(g_1) \cup Z(g_2) \subseteq Z(f')$ which, together with (*), gives $Z(f g_1 g_2) \subseteq Z(f')$.

(d.ii) Let $f'(a) \neq 0$ $(a \in F)$. From $h_i\, g_1 g_2 \rightsquigarrow f'$ and I.1.18 (4) comes

(**) $\quad f'(a) = h_i(a)\, g_1(a)\, g_2(a) \neq 0 \quad (i \in I)$.

By (d.i) we also have $f(a) \neq 0$, and from $h_i \rightsquigarrow f$ we get $f(a) = h_i(a)$ for all $i \in I$. Together with (**) this yields $f'(a) = f(a)\, g_1(a)\, g_2(a)$, proving (d.ii), and completing the proof of the theorem. $\square$

Remark VI.6.13 Further algebraic relations between the sup operation and product in a ARS-fan follow from item (d) of the preceding Theorem. For example, with $h_0, h_1, g_0, g_1 \in X_F$ and h_0, h_1 in the same connected component, we have:

$$g_i \rightsquigarrow h_i \ (i = 0, 1) \ \Rightarrow \ h_0 \vee h_1 = h_0\, g_0\, g_1 \ \vee \ h_1\, g_0\, g_1 \qquad ;$$

in particular, with $g_i = h_i$:

$$h_0 \vee h_1 = h_0^2\, h_1 \ \vee \ h_1^2\, h_0.$$

Proof. Theorem VI.6.12 (d) gives:

$$(h_0 \vee h_1)\, g_0\, g_1 = h_0\, g_0\, g_1 \ \vee \ h_1\, g_0\, g_1.$$

On the other hand, $g_i \rightsquigarrow h_i$ $(i = 0, 1)$ implies $(h_0 \vee h_1)\, g_0\, g_1 = h_0 \vee h_1$. This follows from

(†) $\qquad g_0, g_1 \rightsquigarrow f \ \Rightarrow \ f\, g_0\, g_1 = f \qquad (g_0, g_1, f \in X_F)$,

and $g_0, g_1 \rightsquigarrow h_0 \vee h_1$.

To prove (†), the assumption and I.1.18 (4) yield $Z(g_i) \subseteq Z(f)$ $(i = 0, 1)$, whence $Z(f) = Z(f\, g_0\, g_1)$, and also $f(a) \neq 0 \Rightarrow f(a) = g_0(a) = g_1(a)$, which in turn implies $f(a)\, g_0(a)\, g_1(a) = f(a)^3 = f(a)$. $\square$

Revise ↑.

Our last result in the section, Corollary VI.6.18, shows that if F is a finite RS-fan and xF its character space, then $\mathrm{card}(F) = 2 \cdot \mathrm{card}(X_F) + 1$. This identity is the analog of a well known result relating the cardinalities of a finite RSG-fan and its space of orders ([ABR], p. 75). The result follows from a more general observation, valid for RS-fans of arbitrary cardinality.

Proposition VI.6.14 *Let $I \subset J$ be consecutive ideals of a RS-fan (with, possibly, $J = F$). Then,*

(i) Under product induced by F, $J \setminus I$ is a group of exponent 2 with unit x^2 for any $x \in J \setminus I$ (and distinguished element $-1 = -x^2$).

(ii) The restriction of the quotient map $\pi_I \lceil (J \setminus I) : J \setminus I \longrightarrow F_I = F/I \setminus \{\pi_(0)\}$ is a group isomorphism preserving the distinguished element -1.

Proof. Since I is prime, $J \backslash I$ is closed under product. Given $x, y \in J \backslash I$, we must prove $x^2 = y^2$ (which implies $x^2 = y^3 = y$). By the separation theorem I.1.12 it suffices to show $h(x^2) = h(y^2)$ for all $h \in X_F$. If $J \subseteq Z(h)$, then $h(x^2) = h(y^2) = 0$. If $Z(h) \subseteq I$, then $h(x), h(y) \neq 0$, whence $h(x^2) = h(y^2) = 1$.

(ii) Clearly, $\pi_I(x) \neq \pi_I(0)$, i.e., $\pi_I(x) \in F_I$, for all $x \in J \setminus I$, and π_I preserves product.

— $\pi_I \lceil (J \setminus I)$ is injective.

Suppose $\pi_I(x) = \pi_I(y)$, i.e., $x \sim_I y$, with $x, y \in J \setminus I$. By Lemma VI.6.3, $xz = yz$ for some $z \notin I$. To prove $x = y$, let $h \in X_F$. If $J \subseteq Z(h)$, then $h(x) = h(y) = 0$. If $Z(h) \subseteq I$, then $h(z) \neq 0$, and we get $h(x) = h(y)$.

— $\pi_I(x^2) = \pi_I(1)$, for $x \in J \setminus I$.

Clear, for $Z(h) = I$ implies $h(x^2) = 1$. In particular, π_I preserves -1.

— $\pi_I \lceil (J \setminus I)$ is onto F_I.

Let $p \in F_I$; then, $p = \pi_I(q)$ with $q \notin I$. Taking $z \in J \setminus I$, we have $qz^2 \in J \setminus I$, whence $\pi_I(qz^2) = \pi_I(q)\pi_I(1) = \pi_I(q) = p$. □

Notation VI.6.15 Let F be a finite RS-fan, and let

$$\{0\} = I_n \subset I_{n-1} \subset \cdots \subset I_2 \subset I_1 \subset F = I_0$$

be the set of all its ideals; thus, for $1 \leq d \leq n$, I_d is the ideal of depth d. With notation as in Proposition VI.6.1, we set $F_d = F_{I_d} = (F/I_d) \setminus \{\pi_d(0)\}$, where $\pi_d : F \longrightarrow F/I_d$ denotes the canonical quotient map. We also write L_d for L_{I_d}; cf. VI.6.6 (b). □

Clearly, $F \setminus \{0\} = \bigcup_{d=1}^{n} (I_{d-1} \setminus I_d)$ (disjoint union), whence card $(F) = \sum_{d=1}^{n}$ card $(I_{d-1} \setminus I_d) + 1$. Further, since the levels L_I are a partition of X_F, Remark VI.6.2 yields:

Fact VI.6.16 *For any finite RS-fan F, card* $(X_F) = \sum_{d=1}^{n}$ card $(L_d) = \sum_{d=1}^{n}$ card (X_{F_d}). □

Lemma VI.6.17 *With notation as in VI.6.15, for $1 \leq d \leq n$ we have* card (F_d) = card $(I_{d-1} \setminus I_d)$.

Proof. The Lemma follows from the next two assertions, proved below. For $1 \leq d \leq n$,

(1) card (F_d) = card $((I_{d-1} \setminus I_d)/I_d)$, where $(I_{d-1} \setminus I_d)/I_d = \{\, x/I_d \mid x \in I_{d-1} \setminus I_d \,\}$.

(2) card $((I_{d-1} \setminus I_d)/I_d)$ = card $(I_{d-1} \setminus I_d)$.

Proof of (1). It suffices to prove:

(*) For all $x \in F \setminus I_d$ there is $y \in I_{d-1} \setminus I_d$ such that $x/I_d = y/I_d$.

If $x \in I_{d-1}$, just take $y = x$. If $x \notin I_{d-1}$, pick any $z \in I_{d-1} \setminus I_d$ and set $y = xz^2$; clearly $y \in I_{d-1} \setminus I_d$. That $x/I_d = y/I_d$, i.e., $x \sim_{I_d} y$, follows from Lemma VI.6.3, since $yz = (xz^2)z = xz$ and $z \notin I_d$.

Proof of (2). It suffices to show:

(**) For $x, y \in I_{d-1} \setminus I_d$ $(x/I_d = y/I_d \Rightarrow x = y)$.

By Lemma VI.6.3 the antecedent of this implication amounts to $xz = yz$ for some $z \notin I_d$. Since X_F separates points in F, to prove the conclusion $x = y$ it suffices to check that $h(x) = h(y)$ for all $h \in X_F$. If $Z(h) \supseteq I_{d-1}$, then $x, y \in I_{d-1}$ yield $h(x) = h(y) = 0$. If $Z(h) \subseteq I_d$, then $h(x)$, $h(y)$ and $h(z)$ are all $\neq 0$. Taking images under h in $xz = yz$ gives $h(x) = h(y)$, as required. □

Corollary VI.6.18 *For a finite RS-fan, F,* $\operatorname{card}(F) = 2 \cdot \operatorname{card}(X_F) + 1$.

Proof. Since the F_d are finite RSG-fans (Proposition VI.6.1), we know that $\operatorname{card}(F_d) = 2 \cdot \operatorname{card}(X_{F_d})$ for $1 \le d \le n$. The result follows, then, from the cardinality identities observed in (and before) Fact VI.6.16, and from Lemma VI.6.17. □

VI.7 Involutions of ARS-fans

Notation VI.7.1 In addition to the notation introduced in Definition VI.6.6 of the previous section, for $J \subseteq I$ in $\operatorname{Spec}(F)$ we define the sets:

$$S^I_J = \{\, h \in L_I \mid \exists g \in X_F\, (\, g \rightsquigarrow h \wedge Z(g) \subseteq J\,)\}.$$

$$C^I_J = \{\, h \in L_I \mid \exists g \in X_F\, (\, g \rightsquigarrow h \wedge Z(g) = J\,) \wedge \forall g' \in X_F\, (\, g' \rightsquigarrow h \Rightarrow J \subseteq Z(g'))\}.$$

That is, S^I_J consists of those elements of level I having predecessors of level J <u>or lower</u> in the specialization partial order; C^I_J is the set of elements in L_I having predecessors at level J <u>but not lower</u>. □

Remarks VI.7.2 (i) For $I \in \operatorname{Spec}(F)$, $S^I_{\{0\}} = C^I_{\{0\}}$, and $S^I_I = L_I$. (Recall that $\{0\}$ is the least member of $\operatorname{Spec}(F)$, i.e., the zero-set of the lowest level of X_F.)

(ii) For $J \subseteq I$ in $\operatorname{Spec}(F)$, $S^I_J \neq \emptyset$.

Proof. Let $g \in X_F$ be such that $Z(g) = J$ (exists by Lemma VI.6.7(3)). If h is the unique $\rightsquigarrow$ -successor of g of level I (cf. Lemma VI.6.7(2), or Remark VI.6.8), then $h \in S^I_J$.

(iii) For $J \subseteq I$ in $\operatorname{Spec}(F)$, $S^I_J = \operatorname{Im}(\kappa_{JI})$, where $\kappa_{JI} : L_J(F) \longrightarrow L_I(F)$ is the AOS-morphism defined in Fact VI.6.5.

(iv) For $J \subseteq I$ in $\operatorname{Spec}(F)$, $S^I_J \supseteq \bigcup \{\, C^I_{J'} \mid J' \in \operatorname{Spec}(F)$ and $J' \subseteq J\,\}$. (Note that $C^I_{J'}$ may be empty for some $J' \subseteq J$.)

(v) For $J \subseteq I$ in $\operatorname{Spec}(F)$, $C^I_J = S^I_J \setminus \bigcup \{\, S^I_{J'} \mid J' \in \operatorname{Spec}(F)$ and $J' \subset J\,\}$.

(vi) For $J, J' \subseteq I$ in $\operatorname{Spec}(F)$, $J \neq J'$, we have $C^I_J \cap C^I_{J'} = \emptyset$. □

The preliminary results which follow will be needed later.

Fact VI.7.3 *Let $g_1, \ldots, g_r, h \in X_F$ be so that $\bigcup_{i=1}^r Z(g_i) \subseteq Z(h)$. For $i = 1, \ldots, r$, let $f_i \in X_F$ be such that $g_i \rightsquigarrow f_i$ and $Z(g_i) \subseteq Z(f_i) \subseteq Z(h)$. Then,*

$(*)$ $\quad h\,g_1 \cdot \ldots \cdot g_r = h f_1 \cdot \ldots \cdot f_r$.

<u>Note</u>. The products in (*) may not be in X_F.

Proof. Obviously, (*) holds whenever $x \in Z(h)$. If $x \notin Z(h)$, from the assumptions we get $x \notin \bigcup_{i=1}^r Z(g_i)$ and $x \notin \bigcup_{i=1}^r Z(f_i)$. Since $g_i \rightsquigarrow f_i$, we get $g_i(x) = f_i(x)$ for $i = 1, \ldots, r$, and (*) follows. □

Fact VI.7.4 (*a*) *For $i = 1, \ldots, r$, with r <u>odd</u>, let $g_i, h_i \in X_F$ be such that $g_i \rightsquigarrow h_i$. Then, $g_1 \cdot \ldots \cdot g_r \rightsquigarrow h_1 \cdot \ldots \cdot h_r$.*

(*b*) *Let $h_1, h_2, f, g, k \in X_F$ be such that $f,\, g \rightsquigarrow h_1$, $k \rightsquigarrow h_2$, and $Z(h_1) \subseteq Z(h_2))$. Then, $fg\,k \rightsquigarrow h_2$.*

Note. Here the products are in X_F as the number of factors is odd.

Proof. (a) For $i = 1, \ldots, r$, $h_i^2 = h_i g_i$ (Lemma I.1.18). Multiplying these equalities termwise gives $(h_1 \cdot \ldots \cdot h_r)^2 = (h_1 \cdot \ldots \cdot h_r)(g_1 \cdot \ldots \cdot g_r)$, which proves the assertion.

(b) By Lemma I.1.18 we must prove $h_2^2 = h_2(fgk)$. Obviously, this equality holds at every $x \in Z(h_2)$. If $x \notin Z(h_2)$, then $x \notin Z(h_1)$, and $f, g \rightsquigarrow h_1$ implies $h_1(x) = f(x) = g(x) \neq 0$; also $k \rightsquigarrow h_2$ implies $h_2(x) = k(x) \neq 0$, whence $f(x)g(x) = 1$ and $h_2(x)k(x) = 1$. This yields $(h_2 fgk)(x) = (f(x)g(x))(h_2(x)k(x)) = 1$. On the other hand, $(h_2(x))^2 = 1$, proving that the required identity holds at $x \notin Z(h_2)$ as well. $\square$

In order to make later arguments as transparent as possible, we recall the following simple (and well-known) facts about fans in the categories **RSG** and **AOS**.

Fact VI.7.5 *Let $g : H \longrightarrow G$ be a SG-homomorphism between RSG-fans, and let $g^* : (X_G, G) \longrightarrow (X_H, H)$ denote the AOS-morphism dual to g. Then,*

(1) *With representation induced by that of H, $\mathrm{Im}(g)$ is a RSG-fan, and G is isomorphic to the extension of $\mathrm{Im}(g)$ by the exponent-two group $\Delta = G/\mathrm{Im}(g)$.*

(2) $(\mathrm{Im}(g^*), H/ker(g))$ *is an AOS-fan.*

Remarks. (a) For the definition of extension of a SG by a group of exponent two, see [DM1], Ex. 1.10, p. 10.

(b) By the duality between RSGs and AOSs ([DM1], Ch. 3), the dual statement holds as well: given an AOS-morphism of (AOS-)fans, $\kappa : (X, G) \longrightarrow (Y, H)$, the assertions (1) and (2) hold with $g := \kappa^*$ (the SG-morphism dual to κ), and with $g^* = \kappa$.

Sketch of proof. (1) The first assertion is easily checked. For the second, $\mathrm{Im}(g)$ is a direct summand of the group G. Let $pr : G \longrightarrow \mathrm{Im}(g)$ be the projection onto the factor $\mathrm{Im}(g)$; pr is a SG-morphism (G and $\mathrm{Im}(g)$ are fans), and is the identity on $\mathrm{Im}(g)$. The isomorphism between G and $\mathrm{Im}(g)[\Delta]$ is: for $a \in G$,

$$f(a) = \langle\, pr(a),\, a/\mathrm{Im}(g)\,\rangle.$$

(2) Recall that g^* is defined by composition $g^*(\sigma) = \sigma \circ g$ ($\sigma \in X_G$), and that $\mathrm{Im}(g^*)^{\perp} = \bigcap\{ker(\gamma) \mid \gamma \in \mathrm{Im}(g^*)\} = \bigcap\{ker(\sigma \circ g) \mid \sigma \in X_G\}$. Routine checking from these definitions proves that $\mathrm{Im}(g^*)$ is 3-closed (cf. **??**), and that $\mathrm{Im}(g^*)^{\perp} = ker(g)$ (whence $\mathrm{Im}(g^*) \subseteq X_{H/ker(g)}$).

Clearly, the map $\overline{g} : H/ker(g) \longrightarrow \mathrm{Im}(g)$ induced by g is an SG-isomorphism. Thus, we have a commutative diagram of SG-morphisms:

$$\begin{tikzcd}
H \arrow[r, "g"] \arrow[dr, "\pi"'] & \mathrm{Im}(g) \arrow[r, hook, shift left] & G \arrow[l, "pr", shift left] \overset{f}{\cong} \mathrm{Im}(g)[\Delta] \\
& H/ker(g) \arrow[u, "\overline{g}"'] &
\end{tikzcd}$$

It only remains to be shown that $\mathrm{Im}(g^*) \supseteq X_{H/ker(g)}$. Any SG-character $\gamma : H/ker(g) \longrightarrow \mathbb{Z}_2$ can be lifted to a map $\sigma : G \longrightarrow \mathbb{Z}_2$, via the identification of G with $\mathrm{Im}(g)[\Delta]$, as follows: for each $a \in G$ there is $b \in H$ such that $pr(a) = g(b)$. We set $\sigma(a) = \gamma(b/ker(g)) = \gamma(\pi(b))$. In terms of the diagram above, we have: $\sigma = \gamma \circ (\overline{g})^{-1} \circ pr$. It follows that σ is a well-defined SG-morphism, i.e., $\sigma \in X_G$, and (since $pr \circ g = g$ and $(\overline{g})^{-1} \circ g = \pi$), $g^*(\sigma) = \sigma \circ g = \gamma \circ \pi$. $\square$

This Fact, together with item (iii) in VI.7.2 and Fact VI.6.5, gives:

Corollary VI.7.6 *Let F be a RS-fan, and let $J \subseteq I$ be in* $\mathrm{Spec}(F)$. *The set S^I_J is an AOS-fan. Indeed, it is a sub-fan of $L_I(F)$, when the latter is endowed with its structure of AOS-fan, as indicated in VI.6.2. More generally, if $\mathcal{F} \subseteq L_J(F)$ is an AOS-fan, the set $S^I_J(\mathcal{F}) = \{ h \in L_I \mid \exists g \in \mathcal{F}\,(g \rightsquigarrow h)\}$ is an AOS-subfan of $L_I(F)$.*

Proof. The first assertion is a particular case of the second (with $\mathcal{F} = L_J(F)$). For the latter, just observe that $S^I_J(\mathcal{F}) = \kappa_{JI}[\mathcal{F}] = \mathrm{Im}(\kappa_{JI}\lceil\mathcal{F})$ and use Remark (b) following the statement of VI.7.5. □

The following definition will have a crucial role in the sequel:

Definition VI.7.7 Let F be a RS-fan, let $g_1, g_2 \in X_F$, and fix $I \in \mathrm{Spec}(F)$ so that $Z(g_1)$, $Z(g_2) \subseteq I$. We define a map $\varphi^{g_1,g_2}_I : L_I(F) \longrightarrow L_I(F)$ as follows: for $h \in L_I(F)$,

$$\varphi^{g_1,g_2}_I(h) = h\, g_1 g_2. \qquad \square$$

<u>Note</u>. Since $Z(g_i) \subseteq I = Z(h)$ $(i = 1, 2)$, we have $Z(h\, g_1 g_2) = I$, whence $h\, g_1 g_2 \in L_I$.

Fact VI.7.8 *With notation as in the preceding Definition, let $J \in$* $\mathrm{Spec}(F)$ *be such that $Z(g_1) \cup Z(g_2) \subseteq J \subseteq I$, and for $i = 1, 2$, let g'_i be the unique $\rightsquigarrow$-successor of g_i of level J. Then, $\varphi^{g_1,g_2}_I = \varphi^{g'_1,g'_2}_I$. Thus, in Definition VI.7.7 we may assume $Z(g_1) = Z(g_2)$.*

Proof. Fact VI.7.3 shows that $h\, g_1 g_2 = h\, g'_1 g'_2$ for $h \in L_I$. □

Theorem VI.7.9 *With notation as in Definition VI.7.7, we have:*

(a) φ^{g_1,g_2}_I is an AOS-automorphism of L_I.

(b) φ^{g_1,g_2}_I is an involution: for $h \in L_I$, $\varphi^{g_1,g_2}_I(\varphi^{g_1,g_2}_I(h)) = h$.

(c) For $i = 1, 2$, let h_i be the unique $\rightsquigarrow$-successor of g_i in L_I. Then, $\varphi^{g_1,g_2}_I(h_1) = h_2$.

In particular,

(d) If g_1, g_2, have a common $\rightsquigarrow$-upper bound h at some level $I \supseteq Z(g_1), Z(g_2)$, then h is a fixed point of φ^{g_1,g_2}_I.

(e) Let $J \subseteq I$ be in Spec(F). Assume $Z(g_1), Z(g_2) \subseteq J$, and let $h_1 \in L_J$, $h_2 \in L_I$. Then,

$$h_1 \rightsquigarrow h_2 \;\;\Rightarrow\;\; \varphi^{g_1,g_2}_J(h_1) \rightsquigarrow \varphi^{g_1,g_2}_I(h_2).$$

For the proof of this Theorem we will need an improvement on Remark VI.6.2, valid for fans but not for arbitrary RSs; namely:

Fact VI.7.10 *Let F be a RS-fan, and I be an ideal of F. Any $g \in X_F$ such that <u>$Z(g) \subseteq I$</u> induces a SG-character $\widehat{g} : F_I \longrightarrow \mathbb{Z}_2$, by setting $\widehat{g} \circ \pi_I = g$.*

Proof. The only delicate point is well-definedness: for $a \in F \setminus I$, $a \sim_I 1 \Rightarrow g(a) = 1$. By Lemma VI.6.3, $a \sim_I 1$ means $az = z$ for some $z \notin I$; since $g(z) \neq 0$, taking images under g in this equality yields $g(a) = 1$. □

Proof of Theorem VI.7.9. We begin by proving:

(b) For $h \in L_I(F)$, $\varphi^{g_1,g_2}_I(\varphi^{g_1,g_2}_I(h)) = h\, g_1^2\, g_2^2$. But $h\, g_1^2\, g_2^2 = h$; this is clear if $h(x) = 0$ $(x \in F)$; if $h(x) \neq 0$, then $g_i(x) \neq 0$ (since $Z(g_i) \subseteq Z(h)$), and hence $g_i^2(x) = 1$ $(i = 1, 2)$, proving the stated identity, and item (b).

(a) i) $\varphi_I^{g_1,g_2}$ is an AOS-morphism.

Since F_I is the RSG-fan dual to $L_I(F)$, we must show:

(*) For every $\alpha \in F_I$ there is $\beta \in F_I$ such that $\widehat{\alpha} \circ \varphi_I^{g_1,g_2} = \widehat{\beta}$,

where $\widehat{\alpha} : X_{F_I} \longrightarrow \mathbb{Z}_2$ denotes the map "evaluation at α": for $\sigma \in X_{F_I}$, $\widehat{\alpha}(\sigma) = \sigma(\alpha)$. We claim that $\beta = \alpha\,\widehat{g}_1(\alpha)\,\widehat{g}_2(\alpha)$ does the job. By the Fact above, $\widehat{g}_i(\alpha) \in \mathbb{Z}_2$ $(i = 1, 2)$, whence $\beta \in F_I$. For $h \in L_I(F)$ we have:

$$(\widehat{\alpha} \circ \varphi_I^{g_1,g_2})(h) = \widehat{\alpha}\,(h\,g_1 g_2) = h(\alpha)\,\widehat{g}_1(\alpha)\,\widehat{g}_2(\alpha) = h(\alpha\,\widehat{g}_1(\alpha)\,\widehat{g}_2(\alpha)) = h(\beta) = \widehat{\beta}\,(h),$$

as required. Note that (b) implies

ii) $\varphi_I^{g_1,g_2}$ is bijective.

iii) The dual map $(\varphi_I^{g_1,g_2})^* : F_I \longrightarrow F_I$ is also bijective.

Item (i) proves that, for $\alpha \in F_I$, $(\varphi_I^{g_1,g_2})^*(\alpha) = \alpha\,\widehat{g}_1(\alpha)\,\widehat{g}_2(\alpha)$. For injectivity, assume $\alpha\,\widehat{g}_1(\alpha)\,\widehat{g}_2(\alpha) = 1$; if $\widehat{g}_1(\alpha)\,\widehat{g}_2(\alpha) = -1$, then $\alpha = -1$, whence (as $\widehat{g}_i$ is a SG-character), $\widehat{g}_i(\alpha) = -1$ $(i = 1, 2)$, and $\alpha = 1$, absurd. Thus, $\widehat{g}_1(\alpha)\,\widehat{g}_2(\alpha) = 1$, which entails $\alpha = 1$. For surjectivity, given $\beta \in F_I$, set $\alpha = \beta\,\widehat{g}_1(\beta)\,\widehat{g}_2(\beta)$. Then, $\widehat{g}_1(\alpha) = \widehat{g}_2(\beta)$ and $\widehat{g}_2(\alpha) = \widehat{g}_1(\beta)$, whence $(\varphi_I^{g_1,g_2})^*(\alpha) = \beta$.

(c) We must prove that $h_1 g_1 g_2 = h_2$. This clearly holds at any $x \in Z(h_1) = Z(h_2)$. If $x \notin Z(h_i)$ $(i = 1, 2)$, then $x \notin Z(g_i)$; since $g_i \rightsquigarrow h_i$, it follows $h_i(x) = g_i(x) \neq 0$ (see Lemma I.1.18), and $h_i(x)\,g_i(x) = 1$; hence, $h_1 g_1 g_2(x) = g_2(x) = h_2(x)$.

(e) Invoking Lemma I.1.18 we must prove: $h_2^2 = h_2 h_1 \Rightarrow (h_2 g_1 g_2)^2 = (h_2 g_1 g_2)(h_1 g_1 g_2)$. This is immediate upon multiplying both sides of the antecedent by $g_1^2 g_2^2$. $\square$

By use of these involutions we obtain a number of regularity results concerning the order structure of ARS-fans.

Proposition VI.7.11 *Let F be a RS-fan. For $J \subseteq J_1 \subseteq J_2 \subseteq I$ in* $\mathrm{Spec}(F)$*, and $h \in S_J^I$ set:*

$$B^{J_1,J_2}(h) = \{\, g \in S_{J_1}^{J_2} \mid g \rightsquigarrow h \,\}, \quad \textit{and} \quad A^{J_1,J_2}(h) = \{\, g \in C_{J_1}^{J_2} \mid g \rightsquigarrow h \,\}.$$

Then,

(a) For $h_1, h_2 \in S_J^I$, we have $\mathrm{card}\,(B^{J_1,J_2}(h_1)) = \mathrm{card}\,(B^{J_1,J_2}(h_2))$.

(b) For $h_1, h_2 \in C_J^I$, we have $\mathrm{card}\,(A^{J_1,J_2}(h_1)) = \mathrm{card}\,(A^{J_1,J_2}(h_2))$.

Remark. The assumptions of the Proposition guarantee that the sets $B^{J_1,J_2}(h)$ are non-empty. In fact, given $h \in S_J^I$, there is $u \rightsquigarrow h$ so that $Z(u) \subseteq J$; set $J' = Z(u)$. Since $J' \subseteq J \subseteq J_2$, u has a unique $\rightsquigarrow$-successor g in L_{J_2}. But $u \rightsquigarrow g, h$ and $J_2 = Z(g) \subseteq I = Z(h)$ imply $g \rightsquigarrow h$ (Lemma I.1.19(3)). Since $J' \subseteq J \subseteq J_1$, we conclude that $g \in S_{J_1}^{J_2}$, i.e., $g \in B^{J_1,J_2}(h)$.

The sets $A^{J_1,J_2}(h)$ may be empty for some choices of h and the J_i's. However, if $h \in C_J^I$ and $J_1 = J$, we have $A^{J_1,J_2}(h) \neq \emptyset$. Indeed, in this case the element $g \in S_J^{J_2}$ constructed above is in $C_J^{J_2}$, for if $g \in S_{J'}^{J_2}$ for some $J' \subset J$, then $g \rightsquigarrow h$ would imply $h \in C_{J'}^I$, contrary to the assumption $h \in C_J^I$. $\square$

Proof. (a) With J_1, J_2 as in the statement, write B_i for $B^{J_1,J_2}(h_i)$ $(i = 1, 2)$. The assumption

$h_i \in S^I_J$ implies the existence of elements $u_i \in X_F$ so that $u_i \rightsquigarrow h_i$ and $Z(u_i) \subseteq J$. Replacing u_i by its unique successor of level J we may assume $Z(u_i) = J$ (see VI.7.8). We fix u_i's with these properties throughout the proof, and for $J \subseteq J' \subseteq I$ we denote by $\varphi_{J'}$ the involution $\varphi^{u_1,u_2}_{J'}$ of $L_{J'}$ defined in VI.7.7.

Since the maps $\varphi_{J'}$ are bijective, it suffices to prove $\varphi_{J_2}[B_1] = B_2$. Further, since φ_{J_2} is an involution it suffices just to prove the inclusion $\subseteq$, i.e.,

(*) $\quad g \in S^{J_2}_{J_1}$ and $g \rightsquigarrow h_1 \Rightarrow \varphi_{J_2}(g) \rightsquigarrow h_2$ and $\varphi_{J_2}(g) \in S^{J_2}_{J_1}$.

(i) $\varphi_{J_2}(g) = g\,u_1 u_2 \rightsquigarrow h_2$.

Immediate consequence of Fact VI.7.4(b), since $g, u_1 \rightsquigarrow h_1$ and $u_2 \rightsquigarrow h_2$.

(ii) $\varphi_{J_2}(g) \in S^{J_2}_{J_1}$.

Since $g \in S^{J_2}_{J_1}$, there is a $v \rightsquigarrow g$ so that $Z(v) \subseteq J_1$. Replacing, if necessary, v by a suitable successor of a level containing J, we may assume $Z(v) \supseteq J = Z(u_i)$; thus, v is in the domain of $\varphi_{Z(v)} = \varphi^{u_1,u_2}_{Z(v)}$, and Theorem VI.7.9(e) gives $\varphi_{Z(v)}(v) \rightsquigarrow \varphi_{J_2}(g)$, proving (ii) and item (a).

(b) Write A_i for $A^{J_1,J_2}(h_i)$ $(i = 1, 2)$. As above, it suffices to prove the analogue of (*):

(**) $\quad g \in C^{J_2}_{J_1}$ and $g \rightsquigarrow h_1 \Rightarrow \varphi_{J_2}(g) \rightsquigarrow h_2$ and $\varphi_{J_2}(g) \in C^{J_2}_{J_1}$.

where h_1, h_2 are now assumed to be in C^I_J. In fact, by (*) it only remains to prove:

(iii) There is no $w \in X_F$ such that $Z(w) \subset J_1$ and $w \rightsquigarrow \varphi_{J_2}(g)$.

Otherwise, we would have $w \rightsquigarrow \varphi_{J_2}(g) \rightsquigarrow h_2$ (the last relation holding by (*)). Since $h_2 \in C^I_J$, we get $J \subseteq Z(w)$, and since $Z(u_i) = J$, $\varphi_{Z(w)}(w)$ is defined. Theorem VI.7.9(e) applied to the first of the preceding inequalities yields: $\varphi_{Z(w)}(w) \rightsquigarrow \varphi_{J_2}(\varphi_{J_2}(g)) = g$. This, together with $\varphi_{Z(w)}(w) \in L_{Z(w)}$ and $Z(w) \subset J_1$, contradicts the assumption $g \in C^{J_2}_{J_1}$, proving (iii), and item (b). $\square$

A slight variant of the argument proving Proposition VI.7.11 yields:

Proposition VI.7.12 *Let F be a RS-fan and let $J \subseteq I$ be in Spec(F). For $g_1, g_2 \in X_F$ such that $Z(g_i) \subseteq J$ $(i = 1, 2)$, the map φ^{g_1,g_2}_I is a permutation of S^I_J and of C^I_J.* $\square$

For a RS-fan, F, and $h \in X_F$, we denote by $P_h = \{\, g \in X_F \mid g \rightsquigarrow h \,\}$ the root-system of $\rightsquigarrow$ - predecessors of h. We prove first:

Theorem VI.7.13 (1) *P_h is an ARS-fan.*

(2) *The RS dual to P_h is the quotient F/P_h, where the congruence on F determined by P_h can be characterized as follows: set $T = h^{-1}[1]$ and $\Delta = \bigcap \{P(g) \mid g \in P_h\}$; then, for $a, b \in F$,*

$$a \sim_{P_h} b \quad \textit{iff} \quad \textit{either } (i)\ a, b \in \Delta \cap -\Delta \ (\textit{equivalently, } a \sim_{P_h} 0 \sim_{P_h} b),$$
$$\textit{or } (ii)\ ab \in \Delta \textit{ and there are elements } t \in T \textit{ and } x, y \in \mathrm{Id}(G) \textit{ so that } a^2t^2 = b^2t^2x \textit{ and } b^2t^2 = a^2t^2y.$$

Proof. (1) Lemma I.1.18 shows that $g \rightsquigarrow h$ iff $T = h^{-1}[1] \subseteq g^{-1}[1]$. With notation as in [M], § 6.3, p. 110, and § 6.6, p. 126, the latter condition just means $g \in U(T)$, i.e., P_h is the saturated set $U(T) (= W(T) \cap U(T^2))$. [M], Cor. 6.6.8, p. 126, proves that sets of this form are ARSs. Fact VI.7.4(a) shows that it is 3-closed, hence a fan by the results of § VI.2.

(2) Observe first that $a \in \Delta \cap -\Delta$ iff $a \sim_{P_h} 0$. In fact, since $\Delta \cap -\Delta = \bigcap\{Z(g) \mid g \in P_h\}$, each term in this equivalence just means $a \in \bigcap\{Z(g) \mid g \in P_h\}$.

($\Leftarrow$) Obviously (i) implies $a \sim_{P_h} b$. Assuming (ii) we show that $g(a) = g(b)$ for all $g \in P_h$. For any such g we have $g(t^2) = 1$ and $g(x), g(y) \in \{0, 1\}$. If one of $g(x)$ or $g(y)$ is 0, say $g(x) = 0$, then $g(a)^2 = g(a^2t^2) = g(b^2t^2)g(x)$ implies $g(a) = 0$. But then, the equation $b^2t^2 = a^2t^2y$ yields $g(b) = 0$; thus, $g(a) = g(b) = 0$. If $g(x) = g(y) = 1$, then $g(a)^2 = g(b)^2$. If one of these is 0, then, $g(a) = g(b) = 0$. If both squares are 1, using that $ab \in \Delta$, i.e., $g(ab) \in \{0, 1\}$, we get $g(ab) = 1$, whence $g(a) = g(b)$.

($\Rightarrow$) To prove this implication we apply Theorem II.3.5 to the saturated subsemigroup Δ of F defined in the statement, and the multiplicative set $T_0 = F \setminus (\bigcup \mathcal{J})$, where $\mathcal{J} = \{Z(g) \mid g \in P_h\}$, a family of saturated prime ideals of F. Note that the following are equivalent for $g \in X_F$:

(*) $$g \rightsquigarrow h \;\Leftrightarrow\; P(g) \subseteq P(h) \;\Leftrightarrow\; T \subseteq P(g) \subseteq P(h).$$

By Lemma I.1.18 it suffices to prove $g \rightsquigarrow h \Rightarrow T \subseteq P(g)$; items (3) and (4) of that Lemma prove: $T = h^{-1}[1] \subseteq g^{-1}[1] \subseteq P(g) \subseteq P(h)$.

First, we check that the assumptions of Theorem II.3.5 hold in the present situation.

(a) $\Delta \cap -\Delta = \bigcap \mathcal{J}$ (and hence $\Delta \cap -\Delta \cap T = \emptyset$),

was proved above.

(b) (Condition II.3.4 (C)) $x \in F \setminus (\bigcup \mathcal{J})$ and $a \in F$ imply: $ax^2 \in \Delta \Leftrightarrow a \in \Delta$.

The conclusion just means $g(ax^2) \geq 0 \Leftrightarrow g(a) \geq 0$, for all $g \in P_h$. This clearly holds since, by assumption, $g(x) \neq 0$, i.e., $g(x^2) = 1$, for all $g \in P_h$.

Incidentally, note that

(c) $T_0 = F \setminus (\bigcup \mathcal{J}) = T \cup -T$.

In fact, $x \notin \bigcup \mathcal{J}$ implies $x \notin Z(h)$, i.e., $h(x) = \pm 1$, i.e., $x \in T \cup -T$. Conversely, $x \in T \cup -T$ means $h(x) = \pm 1$; if $g \rightsquigarrow h$, I.1.18(4) shows that $g(x) = h(x) \neq 0$, and hence $x \notin \bigcup \mathcal{J}$.

(d) If, as in II.3.5, $\mathcal{H}_\Delta^{T_0} = \{g \in X_F \mid \Delta \subseteq P(g)$ and $Z(g) \subseteq is\mathcal{J}\}$, then $\mathcal{H}_\Delta^{T_0} = P_h$.

The inclusion $\supseteq$ is clear: if $g \in P_h$, we have $\Delta \subseteq P(g)$ and $Z(g) \in \mathcal{J}$. Conversely, assume $g \in \mathcal{H}_\Delta^{T_0}$. To show $g \rightsquigarrow h$ it suffices to prove $P(g) \subseteq P(h)$ (I.1.18(3)). Otherwise, there is $y \in P(g)$ so that $y \notin P(h)$; then, $h(y) = -1$, i.e., $-y \in T \subseteq \Delta \subseteq P(g)$. Thus, both $g(y)$ and $g(-y)$ are ≥ 0, whence $y \in Z(g)$. Since $Z(g) \subseteq \bigcup \mathcal{J}$, there is $g' \in P_h$ such that $g'(y) = 0$; then, $y \in Z(g') \subseteq Z(h) \subseteq P(h)$, contradiction.

Theorem II.3.5 (a) gives, then, the following characterization of the congruence $\sim_{P_h}$:

$$\begin{array}{lll} a \sim_{P_h} b & \text{iff} & ab \in \Delta \text{ and there are elements } t \in T_0 = T \cup -T \text{ and } d_1, d_2 \in \Delta \\ & & \text{so that } a^2t^2 \in D_F^t(-d_1, a^2b^2t^2) \text{ and } b^2t^2 \in D_F^t(-d_2, a^2b^2t^2). \end{array}$$

Now, it only remains to apply the characterization of D_F^t given by Theorem VI.2.1 to show that the tranversal representation relations above imply the validity of conditions (2.i) or (2.ii) of the statement. This is done by case analysis of the inclusions between the zero-sets of $-d_1$ (resp. $-d_2$) and those of $a^2b^2t^2$.

First, observe that, for $y, z \in F$,

(**) If $y^2 = -z$ and $z \in \Delta$, then $y \in \Delta \cap -\Delta$.

Indeed, for $g \in P_h$ we then have $0 \le g(y)^2 = -g(z) \le 0$, whence $y \in Z(g)$, and so $y \in \bigcap \mathcal{J} = \Delta \cap -\Delta$.

— $Z(d_1) \subset Z(a^2b^2t^2)$. By the first clause in [D^t], Theorem VI.2.1, $a^2t^2 = -d_1$; by (**), $at \in \Delta \cap -\Delta$; since $t \notin \Delta \cap -\Delta$, we get $a \in \Delta \cap -\Delta$, i.e., $a \sim_{P_h} 0$. The assumption $a \sim_{P_h} b$ gives, then, $b \sim_{P_h} 0$. Thus, condition (2.i) holds.

— $Z(a^2b^2t^2) \subset Z(d_1)$. The second clause of [$D^t$], Theorem VI.2.1 implies $a^2t^2 = a^2b^2t^2$; take $x = a^2$.

— $Z(a^2b^2t^2) = Z(d_1)$ and $a^2b^2t^2 \neq d_1$. Then, VI.2.1 gives $a^2t^2 = a^2b^2t^2$ or $a^2t^2 = -d_1$. In the first case take $x = a^2$; by (**), the latter case leads to $a, b \in \Delta \cap -\Delta$.

— $a^2b^2t^2 = d_1$. The last clause in [D^t], Theorem VI.2.1 yields $a^2t^2 = (a^2b^2t^2)^2z$ for some $z \in F$. Then, $a^2t^2 = (a^2t^2)^2 = a^2b^2t^2z^2$; take $x = a^2z^2$.

The same analysis, replacing d_1 by d_2, shows that condition $b^2t^2 \in D^t_F(-d_2, a^2b^2t^2)$ implies either (2.i) or there is $y \in \mathrm{Id}(G)$ so that $b^2t^2 = a^2b^2t^2y$, as asserted. $\square$

Continuing the analysis of the (AOS-)fans of the form P_h, we show:

Theorem VI.7.14 *Let F be a RS-fan and let $J \subseteq I$ be in* Spec(F). *Let $h_1 \in C^I_J$, $h_2 \in S^I_J$. For $i = 1, 2$, we write P_i for P_{h_i}. Then,*

(1) *There is an ARS-embedding φ of P_1 into P_2. Further, $\varphi[P_1] = \{ u \in P_2 \mid J \subseteq Z(u) \}$. In particular, φ is an order-embedding of $(P_1, \rightsquigarrow)$ into $(P_2, \rightsquigarrow)$.*

(2) *If, in addition, $h_2 \in C^I_J$, then φ is an isomorphism of ARSs.*

Proof. Our assumption on the h_i's guarantees the existence of $u_1, u_2 \in L_J$ so that $u_i \rightsquigarrow h_i$ ($i = 1, 2$). For $J \subseteq J' \subseteq I$ in Spec(F) let $\varphi_{J'}$ denote the involution $\varphi^{u_1,u_2}_{J'}$ of $L_{J'}$ (Definition VI.7.7).

(1) We construct $\varphi : P_1 \longrightarrow P_2$ by "collecting together" all the relevant maps $\varphi_{J'}$ $(J \subseteq J' \subseteq I)$: given $g \in L_{J'}$, $g \rightsquigarrow h_1$, we set

$$\varphi(g) = \varphi_{J'}(g).$$

Since the levels $L_{J'}$ are pairwise disjoint, φ is well-defined.

i) $\varphi[P_1] \subseteq P_2$.

By Theorem VI.7.9 (e), $g \rightsquigarrow h_1$ implies $\varphi_{J'}(g) \rightsquigarrow \varphi_I(h_1)$. Since h_i is the unique successor of u_i at level I, VI.7.9 (c) yields $\varphi_I(h_1) = h_2$, whence $\varphi_{J'}(g) \rightsquigarrow h_2$, as required. Note this also gives $J \subseteq J' = Z(\varphi(g))$.

ii) $\{ u \in P_2 \mid J \subseteq Z(u) \} \subseteq \varphi[P_1]$.

Let u be in the left-hand side, with $J' = Z(u)$, say. Set $v = \varphi_{J'}(u)$; then, $\varphi_{J'}(v) = u$ (VI.7.9(b)). By VI.7.4(b), $u_1 \rightsquigarrow h_1$ and $u, u_2 \rightsquigarrow h_2$ imply $u\,u_1\,u_2 = \varphi_{J'}(u) = v \rightsquigarrow h_1$, i.e., $v \in P_1$. Hence $\varphi(v) = u \in \varphi[P_1]$.

iii) φ is injective.

This is clear using VI.7.9(b), since $Z(\varphi(g)) = Z(g)$ for $g \in P_1$.

iv) φ is an ARS-morphism.

The proof is similar to that of item (a) in Theorem VI.7.9. To keep matters straight we make explicit the changes to be made in the latter. Firstly, the RSG-fan in VI.7.9(a) is to be replaced here by the RSs dual to $P_1 = U(T_1)$ and $P_2 = U(T_2)$, where $T_i = h_i^{-1}[1]$ $(i = 1, 2)$; cf. Theorem VI.7.13. The statement to be proved is:

(†) For every $a \in F$ there is $b \in F$ such that $(\widehat{a/T_2}) \circ \varphi = \widehat{b/T_1}$,

where $\widehat{a/T_2} : P_2 \longrightarrow \mathbf{3}$ is the evaluation map: $\widehat{a/T_2}\,(g) = \widehat{g}\,(a/T_2) = g(a)$, for $g \in P_2$, and similarly for $\widehat{b/T_1} : P_1 \longrightarrow \mathbf{3}$. (Note that $g \in P_2 = U(T_2)$ ensures that $\widehat{a/T_2}$ depends only on the congruence class of a modulo T_2.)

The conclusion of (†) can equivalently be written as $\widehat{\varphi(g)}(a/T_2) = \widehat{g}\,(b/T_1)$, i.e., $(u_1 u_2\, g)(a) = g(b)$, or $u_1(a)u_2(a)g(a) = g(b)$. Since $u_i(a) \in \{0, 1, -1\}$, it is clear that the element $b = a\, u_1(a)u_2(a) \in F$ verifies (†); see VI.7.9 (a).

(2) Since $h_2 \in C^I_J$, the preceding construction can be performed with the roles of h_1 and h_2 reversed. Routine verification using VI.7.9(b) shows that the map obtained is φ^{-1}, which then is an ARS-morphism, proving that φ is an ARS-isomorphism. □

Proposition VI.7.11 and Theorem VI.7.14 provide significant information on the structure of the connected components of ARS-fans; see Definition VI.6.11.

Reminder. (a) A connected component of an order-complete root-system contains exactly one top (i.e., maximal) element, and the component is the set of its predecessors. Recall that any ARS is order-complete under the order of specialization, Theorem VI.6.12 (c).

(b) Taking h to be a maximal element in the specialization order, Theorem VI.7.13 (a) shows that the connected components of an ARS-fan are also ARS-fans.

(c) The $\rightsquigarrow$ - top elements of the connected components of an ARS-fan (X, F) have all the same level, namely the level determined by the maximal ideal M of F; this follows from Remark VI.6.8 by taking g to be any element in a given component K, and h to be any element of X such that $Z(h) = M$. Lemma VI.6.7 (1) shows that if $g \in K$, any ideal I of F such that $Z(g) \subseteq I$ is of the form $Z(f)$ for some $f \in K$ (take $h =$ the $\rightsquigarrow$ - top element of K).

(d) Since every connected component of an ARS-fan is itself an ARS-fan, the zero-sets of its elements attain a lowest level, which can be explicitly determined, cf. Proposition VI.7.15 below. However, different components may have different lowest levels; more on this in Corollary VI.7.17. □

Notation. The sets L_I, S^I_J and C^I_J defined in VI.6.6 and VI.7.1 relativize in an obvious way to the connected components of a fan (X, F); if K is such a component and $J \subseteq I$ are in Spec(F) we set:

$$L_I(K) = L_I \cap K, \quad S^I_J(K) = S^I_J \cap K, \quad \text{and} \quad C^I_J(K) = C^I_J \cap K.$$

Note that some (or all) of these sets may be empty, depending on I, J and the component K. Clearly, if h_0 is the $\rightsquigarrow$ - top element of K, we have $L_I(K) = \{\, g \in L_I \mid g \rightsquigarrow h_0 \,\}$, and similarly for $S^I_J(K)$ and $C^I_J(K)$. $L_I(K) \neq \emptyset$ just means that K "reaches at least" the I-th level of X (possibly lower). □

Proposition VI.7.15 *Let K be a connected component of an ARS-fan (X, F). Let h_0 be the $\rightsquigarrow$ - top element of K, and let $T = h_0^{-1}[1]$. Then, the lowest level of K (i.e., the smallest ideal I of F such that $L_I(K) \neq \emptyset$) is $I = \Gamma \cap -\Gamma$, where Γ is the saturated subsemigroup of F generated by* $\mathrm{Id}(F) \cdot T$.

Note. The subsemigroup $\mathrm{Id}(F) \cdot T$ may not be saturated, since $\mathrm{Id}(F) \cdot T \cap -(\mathrm{Id}(F) \cdot T)$ is not, in general, an ideal; see Corollary VI.2.8 (2).

Proof. With notation as in VI.7.13, we have $K = P_{h_0} = U(T) = \{g \in X \mid g \lceil T = 1\} =$ the ARS $X_{F/T}$ (where $F/T = F/\sim_K$, with $\sim_K$ denoting the congruence on F induced by K). Let $\pi_T : F \longrightarrow F/T$ be the quotient map. The lowest level of $X_{F/T}$ is $\{0\}$; with K identified to a subset of X via the map $g \mapsto \widehat{g}$ $(\widehat{g} \circ \pi_T = g)$, the corresponding ideal of F is $\pi_T^{-1}[0] = \{a \in F \mid a \sim_K 0\}$. Then, with the ideal I defined in the statement, we must prove:

$$a \sim_K 0 \iff a \in I \quad (a \in F).$$

($\Leftarrow$) This follows from $I \subseteq Z(g)$ for all $g \in K$. Since $g \lceil T = 1$, we get $\mathrm{Id} \cdot T \subseteq P(g) = g^{-1}[0, 1]$; since $P(g)$ is a saturated subsemigroup, it comes $\Gamma \subseteq P(g)$. Hence, $x \in I = \Gamma \cap -\Gamma$ implies $g(x) = 0$.

($\Rightarrow$) Assume $a \notin I$. In order to prove $a \not\sim_K 0$ we construct a character $g \in X$ such that $g \lceil T = 1$ and $g(a) \neq 0$ (i.e., $g(a^2) = 1$). Applying Lemma I.4.10 (b) to the ideal I, the saturated subsemigroup Γ and the multiplicative set a^2T, condition

($\dagger$) $I[\Gamma] \cap a^2T = \emptyset$,

guarantees the existence of a character $g \in X$ such that $I \subseteq Z(g)$, $\Gamma \subseteq P(g)$ and $g(a^2t) \neq 0$ for all $t \in T$. Since $a^2T \subseteq \Gamma$, we get $g[a^2T] \subseteq \{0, 1\}$, and hence $g \lceil (a^2T) = 1$; clearly, this yields $g(a^2) = 1$ and $g \lceil T = 1$, as required.

To prove ($\dagger$), assume there is $t \in T$ such that $a^2t \in I[\Gamma]$, that is, $-a^2t^2 \in D_F(i, d)$ for some $i \in I$ and $d \in \Gamma$; since Γ is saturated, we get $-a^2t^2 \in \Gamma$, and hence $a^2t^2 \in \Gamma \cap -\Gamma = I$. Since I is prime, either $a \in I$, contrary to assumption, or $t \in I$, which in turn contradicts $T \cap I = \emptyset$ (recall that $h_0 \lceil T = 1$, while $I \subseteq Z(h_0)$). □

Proposition VI.7.11 implies:

Corollary VI.7.16 *Let (X, F) be an ARS-fan and let K_1, K_2 be connected components of (X, F). Then,*

(1) *Let $I \in \mathrm{Spec}(F)$; if $L_I(K_i) \neq \emptyset$ for $i = 1, 2$, then* $\mathrm{card}\,(L_I(K_1)) = \mathrm{card}\,(L_I(K_2))$.

(2) *Let $J \subseteq J'$ be in* $\mathrm{Spec}(F)$, *and assume $L_J(K_i) \neq \emptyset$ $(i = 1, 2)$. Then,* $\mathrm{card}\,(S_J^{J'}(K_1)) = \mathrm{card}\,(S_J^{J'}(K_2))$.

Proof. (1) follows from (2), as $L_I = S_I^I$.

(2) Fix $i \in \{1, 2\}$. Let h_i be the $\rightsquigarrow$-top element of K_i. The assumption $L_J(K_i) \neq \emptyset$ implies that the sets $S_J^{J'}(K_i) = \{\, g \in S_J^{J'} \mid g \rightsquigarrow h_i \,\}$ are non-empty. Now, applying Proposition VI.7.11(a) with $I = M$ (= the maximal ideal of F), $J_1 = J$, $J_2 = J'$ we have $B^{J,J'}(h_i) = \{\, g \in S_J^{J'} \mid g \rightsquigarrow h_i \,\} = S_J^{J'}(K_i)$, and the result follows. □

Remark. Assertion (2) of Corollary VI.7.16 fails, in general, if the sets $S_J^{J'}(K_i)$ are replaced by $C_J^{J'}(K_i)$, even if both sets $C_J^{J'}(K_i)$, $i = 1, 2$, are assumed non-empty. The snag is that $C_J^{J'}(K_i) \neq \emptyset$ <u>does not</u> imply that the $\rightsquigarrow$-top element h_i of K_i belongs to $C_J^M(K_i)$, a condition required for Proposition VI.7.11(b) to apply. It is easy to construct counterexamples. □

Theorem VI.7.14 gives:

Corollary VI.7.17 *Let K_1, K_2 be connected components of the ARS-fan (X, F). Let $I_1, I_2 \in \operatorname{Spec}(F)$ be the lowest levels of K_1, K_2, resp. (cf. VI.7.15). Then,*

(1) If $I_2 \subseteq I_1$, then K_1 endowed with the specialization order is (order-)isomorphic to the root-system obtained by deleting all levels $I \subset I_1$ in K_2.

(2) If $I_1 = I_2$, then K_1, K_2 are order-isomorphic.

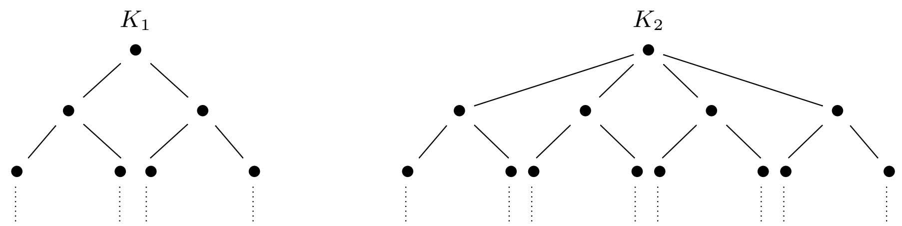
Proof. (1) Use Theorem VI.7.14 (1) with $I = M =$ the maximal ideal of F, $J = I_1$, and h_1, h_2 the $\rightsquigarrow$ - top elements of K_1, K_2, resp. The ARS-embedding $\varphi : K_1 \longrightarrow K_2$ constructed therein verifies $\varphi[K_1] = \{ u \in K_2 \mid I_1 \subseteq Z(u) \}$, which is exactly the statement (1).

(2) follows from Theorem VI.7.14 (2). □

VI.7.18 Some impossible configurations.

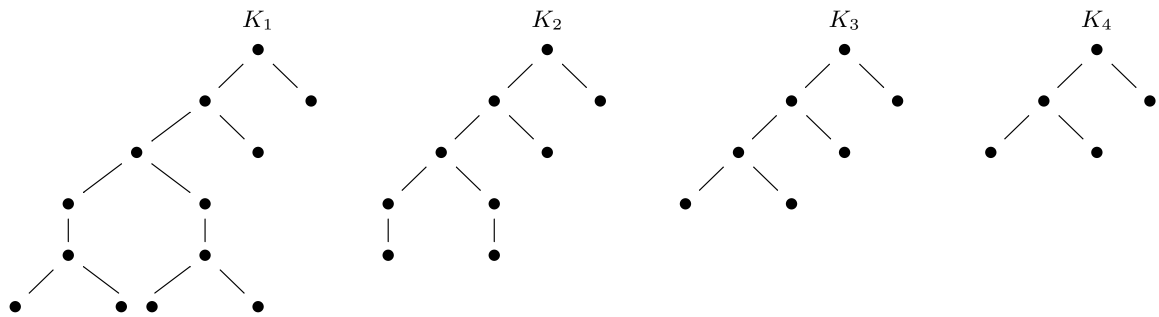
The preceding results show that there are strong constraints on the order structure of ARS-fans, especially when there is more than one connected component. We include a few examples to help the reader visualize the extent of those restrictions.

(1) A configuration like



contradicts Corollary VI.7.16 (1).

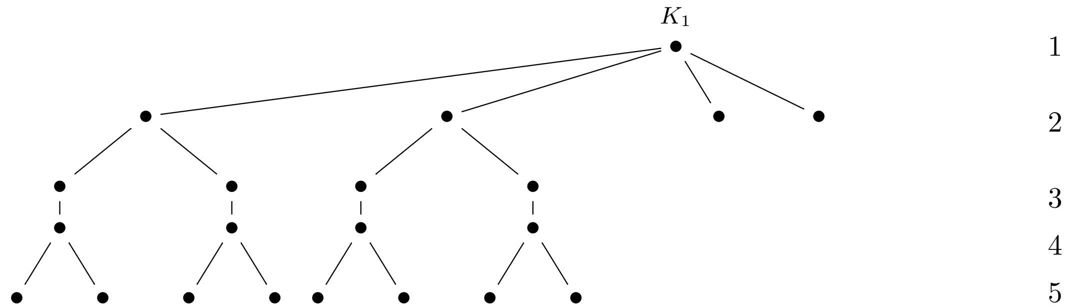
(2) The four-component configuration

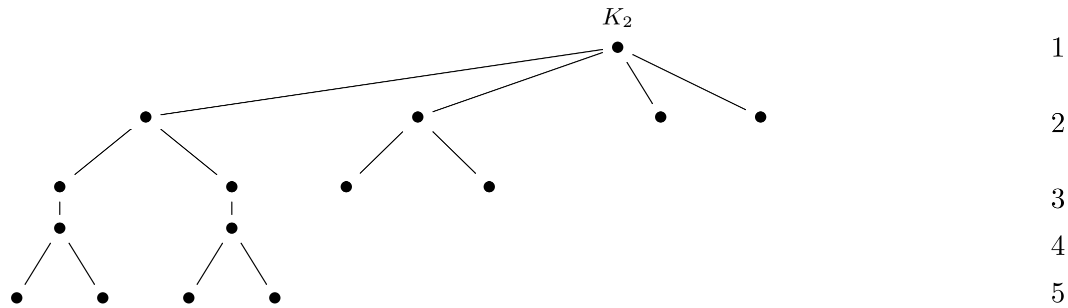


(where the components pairwise verify the conclusion of VI.7.16 (2)) is also impossible: card (S^3_4) $= 3$ is not a power of 2, and hence S^3_4 cannot be an AOS-fan (see Corollary VI.7.6). However, the same configuration with K_3 replaced by another copy of K_4 does not clash with either VI.7.16 or VI.7.17.

Note. Our notation here (and below) follows the convention introduced in VI.6.15 for finite fans. Thus, S^3_4 stands for the set $S^{I_3}_{I_4}$, see VI.7.1 and VI.8.1.

(3) The two-component root-system





contradicts both Corollary VI.7.16 (2) (card $(S_4^3(K_1)) = 4$, but card $(S_4^3(K_2)) = 2$) and Corollary VI.7.17 (K_1 and K_2 have the same "length" but are not order-isomorphic). □

VI.8 The specialization root-system of finite ARS-fans

In this section we mainly deal with finite fans in the categories **ARS** and **RS**. Our main result is Theorem VI.8.9 —the isomorphism theorem for finite ARS-fans— which proves that, in this case, the order of specialization alone determines the isomorphism type. The proof depends on the notion of a "standard generating system" which we introduce in VI.8.4. □

Notation VI.8.1 (a) In this and later sections we shall use systematically the notation introduced in VI.6.15 for finite (ARS- and RS-)fans —rather than that of Section 7 which applies to fans of arbitrary cardinality; that is, the inclusion-decreasing sequence of ideals of a finite RS-fan, F, is numbered in increasing order:

$$\{0\} = I_n \subset I_{n-1} \subset \cdots \subset I_2 \subset I_1 \subset F = I_0;$$

thus, $n = \ell(X_F)$ = the length of the ARS X_F dual to F (see VI.6.6(e)), I_d = the ideal of depth d ($1 \leq d \leq n$), and (hence) $I_1 = M$ = the maximal ideal (i.e., the set of all non-invertible elements) of F. The notation employed in §7 will be adapted in a self-explanatory way; thus, for $1 \leq k \leq j \leq n = \ell(X_F)$, L_k (or $L_k(X_F)$, if necessary), will stand for the level L_{I_k}, S_j^k for the set $S_{I_j}^{I_k}$, etc.

(b) We shall also make constant use of the combinatorial geometric structure —as AOS-fans— of the levels of an ARS-fan X_F and, more generally, of the sets S_J^I ($J \subseteq I$ in Spec(F)); see Remark VI.6.2 and Corollary VI.7.6.

Recall that the AOSs have a combinatorial geometric (matroid) structure; it was introduced in [D1] and [D2] for spaces of orders of fields, and later generalized to abstract order spaces in

[Li]. In general, ARSs do not possess such a structure. Thus, combinatorial geometric notions such as *dependent set, independent set, basis, closed set, closure, dimension*, etc., will always refer, below, to the abovementioned combinatorial geometric structure, and apply only to AOSs. For the definition and the mutual relationships, in the general context of matroid theory, of combinatorial notions such as those just mentioned, the reader is referred to [Wh].

Since the combinatorial geometric structure of any AOS is isomorphic to that of a set of vectors in a (possibly infinite-dimensional) vector space over the two-element field $\mathbb{F}_2$ with the structure induced by linear dependence (cf. [D1], Thm. 3.1, p. 618), the notions above coincide with the corresponding notions over vector spaces. For example, a subset $A \subseteq X$ of an AOS $(X, G, -1)$ (G a group of exponent 2) is dependent iff there are pairwise distinct elements $g, g_1, \ldots, g_r \in A\,(r \geq 2)$, such that $g = g_1 \cdot \ \ldots\ \cdot g_r$ (as characters of G). Observe that, since functions in X send -1 to -1, this functional identity can only hold if r is odd. Likewise, A is closed iff the product of any odd number of members of A belongs to A. □

Lemma VI.8.2 *Let (X, F) be an ARS-fan (not necessarily finite). Let $J \subseteq I$ be in* Spec(F)*, and let $A \subseteq L_J$, $B \subseteq L_I$, be sets such that:*

i) The unique $\rightsquigarrow$ - successor in L_I of each $g \in A$ belongs to B.

ii) Every $h \in B$ has a unique $\rightsquigarrow$ - predecessor in A.

Then, A dependent $\Rightarrow$ B dependent.

Proof. By assumption there are pairwise distinct elements $g, g_1, \ldots, g_r \in A$ such that $g = g_1 \cdot \ \ldots\ \cdot g_r$; as observed above, r is odd ≥ 3. Let $h, h_1, \ldots, h_r$ be the unique successors of $g, g_1, \ldots, g_r$, resp., in B coming from (i); thus, $g \rightsquigarrow h$ and $g_i \rightsquigarrow h_i$, for $i = 1, \ldots, r$. By VI.7.4(a) we have $g = g_1 \cdot \ \ldots\ \cdot g_r \rightsquigarrow h_1 \cdot \ \ldots\ \cdot h_r$. Since $h_1 \cdot \ \ldots\ \cdot h_r \in L_I$ (r is odd) and g has a unique $\rightsquigarrow$ - successor in L_I, we get $h = h_1 \cdot \ \ldots\ \cdot h_r$.

By assumption (ii), every element in A is the *unique* predecessor of an element in B. Since $g_i \neq g_j$, we get $h_i \neq h_j$ for $1 \leq i \neq j \leq r$; likewise, $h \neq h_i$ for $i = 1, \ldots, r$. This proves that h is the product of r *distinct* elements in B, and hence that B is dependent. □

Proposition VI.8.3 (Choice of basis). *Let (X, F) be a finite ARS-fan; let $1 \leq k < n = \ell(X)$. Let $\mathcal{G}$ be an arbitrary AOS-subfan of $L_{k+1} = L_{k+1}(X)$. Let $\mathcal{F} = \{\, h \in L_k \mid$ There is $g \in \mathcal{G}$ such that $g \rightsquigarrow h) \,\}$ be the set of depth-k successors of elements of $\mathcal{G}$; then, $\mathcal{F}$ is an AOS-fan. Assume:*

$(*)\quad \forall\, h, h' \in \mathcal{F},\ \ \mathrm{card}\,(\{g \in \mathcal{G} \mid g \rightsquigarrow h\}) = \mathrm{card}\,(\{g \in \mathcal{G} \mid g \rightsquigarrow h'\})$.

Let $\mathcal{B} = \{h_1, \ldots, h_r\}$ be a basis of $\mathcal{F}$ (as an AOS). Let $\mathcal{C}$ be a basis of the AOS-fan $\{g \in \mathcal{G} \mid g \rightsquigarrow h_1\}$. For $i = 2, \ldots, r$, let $g_i \in \mathcal{G}$ be such that $g_i \rightsquigarrow h_i$. Then, $\mathcal{C} \cup \{g_2, \ldots, g_r\}$ is a basis of $\mathcal{G}$.

Proof. If $r = 1$, then $\mathcal{F} = \mathcal{B} = \{h_1\}$, whence $\mathcal{G} = \{g \in \mathcal{G} \mid g \rightsquigarrow h_1\}$, and the result holds by the choice of $\mathcal{C}$. Henceforth we assume $r \geq 2$. We observe:

— $r = \mathrm{card}(\mathcal{B}) = \mathit{dim}(\mathcal{F})$. Since $\mathcal{F}$ is an AOS-fan, $\mathrm{card}(\mathcal{F}) = 2^{r-1}$.

— For every $h \in \mathcal{F}$, $A_h = \{g \in \mathcal{G} \mid g \rightsquigarrow h\}$ is a AOS-fan; this follows from the assumption that $\mathcal{G}$ is an AOS-fan, since A_h is closed under the product of any three of its elements, cf. Fact VI.7.4(b).

— $A_h \cap A_{h'} = \emptyset$ for $h \neq h'$ in $\mathcal{F}$.

By assumption $(*)$, $\mathrm{card}\,(A_h) = \mathrm{card}\,(A_{h'})$ $(= 2^{p-1}$, say), for $h, h' \in \mathcal{F}$. Since $\mathcal{G} = \bigcup_{h\in\mathcal{F}} A_h$, we get $\mathrm{card}\,(\mathcal{G}) = \mathrm{card}\,(\mathcal{F})\cdot \mathrm{card}\,(A_h)$ (any $h \in \mathcal{F}$), and then $\mathrm{card}\,(\mathcal{G}) = 2^{r-1} \cdot 2^{p-1} = 2^{p+r-2}$; hence $\dim\,(\mathcal{G}) = p + r - 1$. Since $\mathrm{card}\,(\,\mathcal{C} \cup \{g_2, \ldots, g_r\}) = p + r - 1$, it suffices to prove:

$(**)$ $\mathcal{C} \cup \{g_2, \ldots, g_r\}$ is an independent set.

Proof of $(**)$. Assume false.

Case 1. Some g_{i_0}, with $2 \le i_0 \le r$, is dependent on the rest, i.e., there are $\mathcal{C}' \subseteq \mathcal{C}$ and $J \subseteq \{2, \ldots, r\} \setminus \{i_0\}$ so that $g_{i_0} = \prod_{c\in\mathcal{C}'} c \cdot \prod_{j\in J} g_j$, i.e.,

$(+)$ $\prod_{c\in\mathcal{C}'} c = \prod_{j\in J\cup\{i_0\}} g_j$.

— If $\mathrm{card}\,(\mathcal{C}')$ is odd, since A_{h_1} is an AOS-fan, and hence a closed set, the left-hand side of $(+)$ is an element $g' \rightsquigarrow h_1$, and we have $g' \cdot \prod_{j\in J\cup\{i_0\}} g_j = 1$. Setting $A = \{g'\} \cup \{g_j \mid j \in J \cup \{i_0\}\} \subseteq L_{k+1}$ and $B = \{h_1\} \cup \{h_j \mid j \in J \cup \{i_0\}\} \subseteq L_k$, the assumptions of Lemma VI.8.2 are met. Since A is dependent, so is B, contradicting that $B \subseteq \mathcal{B}$ and $\mathcal{B}$ is a basis of $\mathcal{F}$, whence an independent set.

— If $\mathcal{C}' = \emptyset$, the same argument works, yielding a contradiction.

— Assume $\mathrm{card}\,(\mathcal{C}')$ even > 0. Fix $c_0 \in \mathcal{C}'$. Then $\mathrm{card}\,(\mathcal{C}' \setminus \{c_0\}) = \text{odd}$, and $g' = \prod_{c\in\mathcal{C}'\setminus\{c_0\}} c \in L_{k+1}$; also $g' \rightsquigarrow h_1$, and we have:

$$c_0 \cdot g' \cdot \prod_{j\in J\cup\{i_0\}} g_j = 1.$$

Pick any index $j_0 \in J \cup \{i_0\}$ (so, $j_0 \ge 2$). Since $c_0, g' \rightsquigarrow h_1$ and $g_{j_0} \rightsquigarrow h_{j_0}$, Fact VI.7.4(b) yields $g'_{j_0} := c_0\, g'\, g_{j_0} \rightsquigarrow h_{j_0}$, and $g'_{j_0} \cdot \prod_{j\in(J\cup\{i_0\})\setminus\{j_0\}} g_j = 1$. Hence, $A = \{g'_{j_0}\} \cup \{g_j \mid j \in (J \cup \{i_0\}) \setminus \{j_0\}\}$ is a dependent subset of L_{k+1}. Setting $B = \{h_j \mid j \in J \cup \{i_0\}\}$ the assumptions of Lemma VI.8.2 are met, and hence B is also dependent, contradicting that $B \subseteq \mathcal{B}$.

Case 2. Some $c_0 \in \mathcal{C}$ is dependent on the rest.

Then, there are $\mathcal{C}' \subseteq \mathcal{C} \setminus \{c_0\}$ and $J \subseteq \{2, \ldots, r\}$ so that

$(++)$ $c_0 = \prod_{c\in\mathcal{C}'} c \cdot \prod_{j\in J} g_j$.

Note that $J \ne \emptyset$ (otherwise $\mathcal{C}$ would be dependent). Taking J minimal so that $(++)$ holds, and picking $j_0 \in J$, it follows that c_0 is not in the closure ($=$ linear span) of $\mathcal{C}' \cup \{g_j \mid j \in J \setminus \{j_0\}\}$. By the exchange property, g_{j_0} is in the closure of $\mathcal{C}' \cup \{c_0\} \cup \{g_j \mid j \in J \setminus \{j_0\}\}$, contrary to the result of Case 1. □

VI.8.4 Standard generating systems.

For any finite ARS-fan, (X, F), we will construct, by induction on k, $1 \le k \le n = \ell(X)$, a class of bases $\mathcal{B}_k$ of the AOS-fan $L_k(X)$. Each basis $\mathcal{B}_k$ will be required to satisfy the additional condition:

$(*)$ For $k \le j \le n$, $\mathcal{B}_k \cap S^k_j$ is a basis of the AOS-fan S^k_j.

This additional requirement guarantees that the inductive construction of the $\mathcal{B}_k$'s does not get interrupted before the n-th (and last) step. The construction uses Proposition VI.8.3 and the results from §VI.7 above. The set $\mathcal{B} = \bigcup_{k=1}^{n} \mathcal{B}_k$ will be called a **standard generating system** for (X, F).

Level 1. It suffices to observe that a basis $\mathcal{B}_1$ of L_1 satisfying condition $(*)$ exists. Begin by choosing a basis $\mathcal{B}_1(n)$ of the AOS-fan $S^1_n = C^1_n$ (cf. Corollary VI.7.6). S^1_n is a closed subset of the (AOS-)fan $S^1_{n-1} = S^1_n \cup C^1_{n-1}$; hence, $\mathcal{B}_1(n)$ is an independent subset of S^1_{n-1}; choose $\mathcal{B}_1(n-1)$ to be a basis of S^1_{n-1} extending $\mathcal{B}_1(n)$.

In general, assume that, for $1 < j \le n$ an increasing sequence $\mathcal{B}_1(n) \subseteq \ldots \subseteq \mathcal{B}_1(j)$ of independent subsets of L_1 has been chosen so that $\mathcal{B}_1(\ell)$ is a basis of the AOS-fan S^1_ℓ $(j \le \ell \le n)$. As above, $\mathcal{B}_1(j)$ is an independent subset of the fan $S^1_{j-1} = S^1_j \cup C^1_{j-1}$. Let $\mathcal{B}_1(j-1)$ be a basis of S^1_{j-1} extending $\mathcal{B}_1(j)$. Set $\mathcal{B}_1 = \mathcal{B}_1(1)$; by construction, $\mathcal{B}_1 \cap S^1_j = \mathcal{B}_1(j)$ is a basis of S^1_j.

<u>Induction step.</u> Given an integer k, $1 \le k < n$, assume there exists a basis $\mathcal{B}_k$ of L_k satisfying property $(*)$; thus, for $k \le j \le n$, $\mathcal{B}_k(j) = \mathcal{B}_k \cap S^k_j$ is a basis of S^k_j. Further, since $S^k_n \subseteq \ldots \subseteq S^k_k = L_k$, we have $\mathcal{B}_k(n) \subseteq \ldots \subseteq \mathcal{B}_k(k) = \mathcal{B}_k$. Using Proposition VI.8.3 we define a subset $\mathcal{B}_{k+1}$ of L_{k+1} as follows:

— Firstly, fix an element $h_0 \in \mathcal{B}_k(n)$ (this set is non-empty because $n = \ell(X)$). Pick a basis $\mathcal{B}_{k+1}(n, h_0)$ of the AOS-fan $\{g \in S^{k+1}_n \mid g \rightsquigarrow h_0\}$.

— Next, for each $h \in (\mathcal{B}_k \cap S^k_{k+1}) \setminus \{h_0\}$ there is a maximal index j, $k+1 \le j \le n$, so that $h \in \mathcal{B}_k \cap S^k_j = \mathcal{B}_k(j)$; clearly, $h \notin S^k_{j+1}$, whence $h \in C^k_j = S^k_j \setminus S^k_{j+1}$ (if $j = n$, then $h \in S^k_n = C^k_n$). Since $j \ge k+1$, we have $\{g \in C^{k+1}_j \mid g \rightsquigarrow h\} \neq \emptyset$. Choose an element $g_h \in C^{k+1}_j$ so that $g_h \rightsquigarrow h$.

— Finally, set $\mathcal{B}_{k+1} = \mathcal{B}_{k+1}(n, h_0) \cup \{g_h \,|\, h \in (\mathcal{B}_k \cap S^k_{k+1}) \setminus \{h_0\}\}$.

Now, given an index j such that $k+1 \le j \le n$, we apply Proposition VI.8.3 with $\mathcal{G} = S^{k+1}_j$ —whence $\mathcal{F} = S^k_j$— and $\mathcal{B}_k \cap S^k_j$ as the basis $\mathcal{B}$ of $\mathcal{F}$. Proposition VI.7.11(a) shows that the cardinality assumption

$$\operatorname{card}(\{g \in S^{k+1}_j \mid g \rightsquigarrow h\}) = \operatorname{card}(\{g \in S^{k+1}_j \mid g \rightsquigarrow h'\}), \quad (h, h' \in S^k_j)$$

of VI.8.3 holds, and we conclude that $\mathcal{B}_{k+1} \cap S^{k+1}_j = \mathcal{B}_{k+1}(n, h_0) \cup \{g_h \,|\, h \in (\mathcal{B}_k \cap S^k_j) \setminus \{h_0\}\}$ is a basis of S^{k+1}_j, as required. □

Remarks VI.8.5 (a) In general, there are many different standard generating systems for a finite ARS-fan (X, F). The construction in VI.8.4 allows for several choices of the bases $\mathcal{B}_1(j)$ $(1 \le j \le n)$ and, at each successive step, k, for many choices of elements $h_0 \in \mathcal{B}_k(n)$, of bases $\mathcal{B}_{k+1}(n, h_0)$, and of elements $g_h \in C^{k+1}_n$ under each $h \in (\mathcal{B}_k \cap S^k_{k+1}) \setminus \{h_0\}$. In spite of this lack of uniqueness, we shall prove below that any standard generating system determines the isomorphism type of a finite ARS-fan.

(b) Some of the sets $C^k_j = S^k_j \setminus S^k_{j+1}$ may be empty. However, if $C^k_j \neq \emptyset$, then, necessarily, $\mathcal{B}_k \cap C^k_j \neq \emptyset$. Indeed, if $j = n$, then $C^k_n \neq \emptyset$ (as $n = \ell(X)$) and $C^k_n = S^k_n$ is an AOS-fan; since $\mathcal{B}_k \cap C^k_n$ is a basis of C^k_n, it must contain at least one element. If $j < n$, since S^k_{j+1} is a fan, it is a closed set; as it is disjoint from C^k_j, then no element of C^k_j is dependent on S^k_{j+1}. Hence, any basis of $S^k_j = S^k_{j+1} \cup C^k_j$ must contain an element of C^k_j. □

Any standard generating system for a finite ARS-fan has the following property:

Corollary VI.8.6 *Let $\mathcal{B}$ be a standard generating system for a finite ARS-fan (X, F); let $n = \ell(X)$, and $1 \leq k \leq m \leq n$. Then, for every $g \in \mathcal{B}_m = \mathcal{B} \cap L_m(X)$, the unique depth-$k$ successor of g in X belongs to $\mathcal{B}$ (hence to $\mathcal{B}_k = \mathcal{B} \cap L_k(X)$).*

Proof. Obvious, by construction, for $m = k + 1$. Then iterate. □

For the proof of the Isomorphism Theorem VI.8.9 below we shall need some consequences of the Small Representation Theorem III.2.15. Recall that a map $F : (X, G) \longrightarrow (Y, H)$ is an ARS-morphism iff for all $a \in H$ there is $b \in G$ so that $\widehat{a} \circ F = \widehat{b}$ (cf. [ARS-mor], proof of I.5.1).

Corollary VI.8.7 *A map $F : (X_1, F_1) \longrightarrow (X_2, F_2)$ between ARS-fans is an ARS-morphism iff F is continuous for the constructible topology (of both source and target) and preserves 3-products in X_1 (cf. III.2.14).*

Proof. ($\Leftarrow$) If F has the stated properties and $a \in F_2$, then $\widehat{a} \circ F : X_1 \longrightarrow \mathbf{3}$ also has those properties, and, by Proposition III.2.15, is represented by an element of F_1; hence, F is an ARS-morphism.

($\Rightarrow$) Assume F is an ARS-morphism. For continuity it suffices to show that $F^{-1}[V]$ is open constructible in X_1 whenever V is basic open for the constructible topology of X_2, i.e., of the form $V = U(a_1, \ldots, a_n) \cap Z(a)$ with $a, a_1, \ldots, a_n \in F_2$ (see [M], p. 111). By the assumption on F, there are $b, b_1, \ldots, b_n \in F_1$ such that $\widehat{a} \circ F = \widehat{b}$ and $\widehat{a_i} \circ F = \widehat{b_i}$ for $i = 1, \ldots, n$. These functional identities imply $F^{-1}[V] = U(b_1, \ldots, b_n) \cap Z(b)$, as required.

The preservation of products by F follows easily from the same property for $\widehat{a}$ and $\widehat{b}$ using the functional identity $\widehat{a} \circ F = \widehat{b}$. □

We shall also need:

Lemma VI.8.8 *Let (X_1, F_1), (X_2, F_2) be ARS-fans.*

(1) *For a map $F : X_1 \longrightarrow X_2$ the following are equivalent:*

(*i*) *F preserves 3-products in X_1.*

(*ii*) *a) F preserves 3-products of elements <u>of the same level</u>: for all $I \in \mathrm{Spec}(F)$ and all $h_1, h_2, h_3 \in L_I(X_1)$, $F(h_1 h_2 h_3) = F(h_1)F(h_2)F(h_3)$.*

b) F is monotone for the specialization order: for all $g, h \in X_1$, $g \underset{1}{\rightsquigarrow} h \Rightarrow F(g) \underset{2}{\rightsquigarrow} F(h)$,

($\underset{i}{\rightsquigarrow}$ denotes specialization in X_i).

(2) *If (X_1, F_1) is finite, any map verifying one of the equivalent conditions (i) or (ii) in (1) is a morphism of ARSs.*

(3) *If both (X_1, F_1), (X_2, F_2) are finite, any bijection $F : X_1 \longrightarrow X_2$ verifiying one of the equivalent conditions in (1) is an isomorphism of ARSs.*

Proof. (1). (i) $\Rightarrow$ (ii). (ii.a) is a particular case of (i).

(ii.b) $g \underset{1}{\rightsquigarrow} h \Leftrightarrow h = h^2 g$ (Lemma I.1.18). By (i), $F(h) = F(h)^2 F(g)$, and this equality (in X_2) gives $F(g) \underset{2}{\rightsquigarrow} F(h)$.

(ii) $\Rightarrow$ (i). Let h_1, h_2, h_3 be any three elements in X_1; say $Z(h_3) \subseteq Z(h_2) \subseteq Z(h_1)$. Let $I = Z(h_1)$ and for $i = 2, 3$ let h'_i be the unique successor of h_i in $L_I(X_1)$; Fact VI.7.3 shows that $h_1 h_2 h_3 = h_1 h'_2 h'_3$; then the assumption (ii.a) gives

$$F(h_1h_2h_3) = F(h_1)F(h'_2)F(h'_3).$$

By (ii.b) we have $F(h_i) \underset{2}{\rightsquigarrow} F(h'_i)$, $(i = 2,3)$. Using again VI.7.3, from $Z(F(h'_i)) \subseteq Z(F(h_1))$ we conclude

$$F(h_1)F(h'_2)F(h'_3) = F(h_1)F(h_2)F(h_3),$$

as required.

(2) follows at once from Corollary VI.8.7, since the continuity requirement is automatically fulfilled in this case: the constructible topology in X_1 is discrete.

(3) By (2) it only remains to prove that $F^{-1} : X_2 \longrightarrow X_1$ preserves 3-products in X_2. Let $g_1, g_2, g_3 \in X_2$ and let $h_i = F^{-1}(g_i)$, $i = 1,2,3$. From (1.i) we have $F(h_1h_2h_3) = g_1g_2g_3$. Composing both sides of this equality with F^{-1} gives the desired conclusion:

$$F^{-1}(g_1g_2g_3) = F^{-1}(F(h_1h_2h_3)) = h_1h_2h_3 = F^{-1}(g_1)F^{-1}(g_2)F^{-1}(g_3). \qquad \square$$

Remark. Note that any isomorphism of ARSs between fans preserves depth.

Theorem VI.8.9 (The isomorphism theorem for finite ARS-fans.) *Let (X_1, F_1), (X_2, F_2) be finite ARS-fans and let $\underset{1}{\rightsquigarrow}$, $\underset{2}{\rightsquigarrow}$ denote their respective specialization orders. If $(X_1, \underset{1}{\rightsquigarrow})$ and $(X_2, \underset{2}{\rightsquigarrow})$ are order-isomorphic, then X_1 and X_2 are isomorphic as ARSs.*

Proof. The order-isomorphism assumption implies:

(1) $\ell(X_1) = \ell(X_2)$ ($= n$, say, fixed throughout the proof).

(2) For $1 \leq k \leq j \leq n$, $\mathrm{card}(C^k_j(X_1)) = \mathrm{card}(C^k_j(X_2))$.

The proof of (2) is an easy exercise. Since $C^k_\ell \cap C^k_{\ell'} = \emptyset$ for $k \leq \ell \neq \ell' \leq n$ and $S^k_j = \bigcup_{\ell=j}^n C^k_\ell$, we get:

(3) For $1 \leq k \leq j \leq n$, $\mathrm{card}(S^k_j(X_1)) = \mathrm{card}(S^k_j(X_2))$.

(4) For $1 \leq k < n$ and all $h \in S^k_n(X_1)$, $h' \in S^k_n(X_2)$, we have:

$$\mathrm{card}(\{g \in S^{k+1}_n(X_1) \mid g \underset{1}{\rightsquigarrow} h\}) = \mathrm{card}(\{g' \in S^{k+1}_n(X_2) \mid g' \underset{2}{\rightsquigarrow} h'\}).$$

<u>Proof of (4)</u>. Consider the two-variable formula in the language $\{\leq\}$ of order:

$$\varphi(x,y) \ := \ x \in S^{k+1}_n \wedge x \leq y.$$

(It is left as an exercise for the reader to write a first-order formula in $\{\leq\}$ expressing the notion $x \in S^{k+1}_n$; cf. VI.7.1 and VI.9.1.)

If σ denotes the order isomorphism between $(X_1, \underset{1}{\rightsquigarrow})$ and $(X_2, \underset{2}{\rightsquigarrow})$, for $g, h \in X_1$ we have:

$$(X_1, \underset{1}{\rightsquigarrow}) \models \varphi[g,h] \ \Leftrightarrow \ (X_2, \underset{2}{\rightsquigarrow}) \models \varphi[\sigma(g), \sigma(h)].$$

It follows that σ maps $\{g \in S^{k+1}_n(X_1) \mid g \underset{1}{\rightsquigarrow} h\}$ bijectively onto $\{g' \in S^{k+1}_n(X_2) \mid g' \underset{2}{\rightsquigarrow} \sigma(h)\}$. Now, if $h \in S^k_n(X_1)$, then $\sigma(h) \in S^k_n(X_2)$. If $h' \in S^k_n(X_2)$, apply Proposition VI.7.11 with $h_1 = h'$ and $h_2 = \sigma(h)$, to conclude. $\square$

Since the sets in item (4) are AOS-fans (Corollary VI.7.6), they have the same dimension, i.e., any bases of each of them have the same cardinality. If $\mathcal{B}^1, \mathcal{B}^2$ are standard generating systems for X_1, X_2, respectively, then $\mathcal{B}^i \cap S^k_j(X_i)$ is a basis of the fan $S^k_j(X_i)$, for $1 \leq k \leq j \leq n$ and $i = 1,2$; from (3) we get:

(5) For $1 \leq k \leq j \leq n$, $\operatorname{card}(\mathcal{B}^1 \cap S^k_j(X_1)) = \operatorname{card}(\mathcal{B}^2 \cap S^k_j(X_2))$.

In particular, for $S^k_k = L_k$ we obtain:

(6) If $1 \leq k \leq n$, then $\operatorname{card}(\mathcal{B}^1_k) = \operatorname{card}(\mathcal{B}^2_k)$ (where $\mathcal{B}^i_k = \mathcal{B}^i \cap L_k(X_i)$).

Next, we choose an arbitrary standard generating system $\mathcal{B}^1$ for X_1. By induction on k, $1 \leq k \leq n$, we construct a standard generating system $\mathcal{B}^2$ of X_2 ($\mathcal{B}^2 = \bigcup_{k=1}^n \mathcal{B}^2_k$) and a map $f_k : \mathcal{B}^1_k \longrightarrow \mathcal{B}^2_k$ so that:

(7) i) For $k \leq j \leq n$, $f_k[\mathcal{B}^1 \cap S^k_j(X_1))] = \mathcal{B}^2 \cap S^k_j(X_2))$.

ii) If $k < n$, $g \in \mathcal{B}^1_{k+1}$, $h \in \mathcal{B}^1_k$ and $g \underset{1}{\rightsquigarrow} h$, then $f_{k+1}(g) \underset{2}{\rightsquigarrow} f_k(h)$.

<u>Construction of $\mathcal{B}^2$ and the maps f_k.</u>

<u>Level 1</u>. $\mathcal{B}^2_1$ is built as in the level 1 step in VI.8.4; with notation therein, $f_1 : \mathcal{B}^1_1 \longrightarrow \mathcal{B}^2_1$ is any bijection mapping $\mathcal{B}_1(j)$ onto $\mathcal{B}_2(j)$, for $1 \leq j \leq n$. Such a bijection exists by (5) above ($k = 1$).

<u>Induction step</u>. Assume $\mathcal{B}^2_1, \ldots, \mathcal{B}^2_k$ and $f_1, \ldots, f_k$ already constructed, so that:

— For $1 \leq j \leq k$ and $j \leq \ell \leq n$, $\mathcal{B}^2_j \cap S^j_\ell(X_2)$ is a basis of the AOS-fan $S^j_\ell(X_2)$ and $f_j[\mathcal{B}^1_j \cap S^j_\ell(X_1)] = \mathcal{B}^2_j \cap S^j_\ell(X_2)$.

— Condition (7.ii) holds for all j such that $1 \leq j < k$.

The basis $\mathcal{B}^2_{k+1}$, and along with it the map f_{k+1}, are defined by performing the construction of the inductive step in VI.8.4, with the following choice of parameters:

— If $h_0 \in \mathcal{B}^1_k \cap S^k_n(X_1)$, and $\mathcal{B}^1_{k+1}(n, h_0)$ is a basis of the fan $\{g \in S^{k+1}_n(X_1) \mid g \underset{1}{\rightsquigarrow} h_0\}$, then take $\mathcal{B}^2_{k+1}(n, f_k(h_0))$ to be a basis of the fan $\{g' \in S^{k+1}_n(X_2) \mid g' \underset{2}{\rightsquigarrow} f_k(h_0)\}$. This is possible since $f_k(h_0)) \in \mathcal{B}^2_k \cap S^k_n(X_2)$, by (7.i). Using item (4), we let $f_{k+1} \lceil \mathcal{B}^1_{k+1}(n, h_0)$ be a bijection between $\mathcal{B}^1_{k+1}(n, h_0)$ and $\mathcal{B}^2_{k+1}(n, f_k(h_0))$.

— If $g \in \mathcal{B}^1_{k+1} \cap C^{k+1}_j(X_1)$ with $k+1 \leq j \leq n$, but $g \notin \mathcal{B}^1_{k+1}(n, h_0)$, then, by the construction performed in the inductive step of VI.8.4, if h is the unique depth-k successor of g, we have $h \in \mathcal{B}^1_k \cap C^k_j(X_1)$, $h \neq h_0$ and $g = g_h$. In this case choose any element $g' \underset{2}{\rightsquigarrow} f_k(h)$ such that $g' \in C^{k+1}_j(X_2)$, and set $f_{k+1}(g) = g'$. This is possible since $f_k(h) \in \mathcal{B}^2_k \cap C^k_j(X_2)$ (which follows easily from (7.i)).

Clearly, this construction guarantees that (7.i) and (7.ii) hold for $k + 1$.

Note that (7.ii) implies, by iteration, its own generalization:

(7) iii) If $1 \leq k < m \leq n$, $g \in \mathcal{B}^1_m$, $h \in \mathcal{B}^1_k$ and $g \underset{1}{\rightsquigarrow} h$, then $f_m(g) \underset{2}{\rightsquigarrow} f_k(h)$.

Since $\mathcal{B}^i_k = \mathcal{B}^i \cap L_k(X_i)$ is a basis of the AOS-fan $L_k(X_i)$, $i = 1, 2$, we get:

(8) The bijection f_k extends (uniquely) to an AOS-isomorphism $\tilde{f}_k : L_k(X_1) \longrightarrow L_k(X_2)$ mapping $S^k_j(X_1)$ onto $S^k_j(X_2)$, for all j such that $k \leq j \leq n$.

Now set $F : X_1 \longrightarrow X_2$ to be $F = \bigcup_{k=1}^n \tilde{f}_k$. We prove:

Claim. F is an isomorphism of ARSs.

Proof of Claim. Since $X_i = \bigcup_{k=1}^{n} L_k(X_i)$ (disjoint union) for $i = 1, 2$, and $\tilde{f}_k$ maps $L_k(X_1)$ bijectively onto $L_k(X_2)$, we have:

(a) F is well-defined and bijective.

(b) For all k, $1 \leq k \leq n$, F preserves 3-products $\underline{\text{in } L_k}$.

This is clear: by (8) $F\lceil L_k(X_1) = \tilde{f}_k : L_k(X_1) \longrightarrow L_k(X_2)$ is an isomorphism of AOS-fans.

(c) F is monotone for the specialization order.

Let $g, h \in X_1$ be such that $g \underset{1}{\rightsquigarrow} h$; say $d(g) = m \geq d(h) = k$. We must prove $F(g) \underset{2}{\rightsquigarrow} F(h)$. Since $\mathcal{B}^1_m$ generates $L_m(X_1)$, then $g = g_1 \cdot \ \ldots \ \cdot g_r$ with $g_1, \ldots, g_r \in \mathcal{B}^1_m$ and r necessarily odd (possibly $= 1$). By Corollary VI.8.6, if h_i is the unique depth-k successor of g_i, then $h_i \in \mathcal{B}^1_k$. Also, $g_i \underset{1}{\rightsquigarrow} h_i$ $(i = 1, \ldots, r)$ implies $g = g_1 \cdot \ \ldots \ \cdot g_r \underset{1}{\rightsquigarrow} h_1 \cdot \ \ldots \ \cdot h_r$ (VI.7.4(a)). Since both h and $h_1 \cdot \ \ldots \ \cdot h_r$ are successors of g of the same level k, we get $h = h_1 \cdot \ \ldots \ \cdot h_r$. As F preserves products of any odd number of elements of the same level, we have:

$$F(g) = F(g_1) \cdot \ \ldots \ \cdot F(g_r) \qquad \text{and} \qquad F(h) = F(h_1) \cdot \ \ldots \ \cdot F(h_r).$$

Since $g_i \underset{1}{\rightsquigarrow} h_i$, $g_i \in \mathcal{B}^1_m$ and $h_i \in \mathcal{B}^1_k$, item (7.iii) yields $F(g_i) = f_m(g_i) \underset{2}{\rightsquigarrow} f_k(h_i) = F((h_i))$ $(i = 1, \ldots, r)$. Then, by VI.7.4(a) again,

$$F(g) = F(g_1) \cdot \ \ldots \ \cdot F(g_r) \underset{2}{\rightsquigarrow} F(h_1) \cdot \ \ldots \ \cdot F(h_r) = F(h),$$

which proves (c). The Claim follows from (a)–(c) using Lemma VI.8.8(3). This completes the proof of Theorem VI.8.9. □

VI.9 Systems of invariants for isomorphism

From the results of §§ VI.7, VI.8 we obtain:

(I) First-order axiomatizations for the root-systems of finite ARS-fans under the order of specialization.

These axioms are formulated in the language of order, and depend on two parameters: the length n of the root-system and an upper bound c on its cardinality.

(II) A system of numerical invariants for order-isomorphism of such root-systems.

By the isomorphism theorem VI.8.9, these invariants also determine the isomorphism types of finite fans in the category **ARS**. In § VI.10 we shall prove that these systems of invariants are $\underline{\text{complete}}$. □

As our results are somewhat more general, we shall proceed in three steps:

(1) Firstly, we introduce an axiom system FRS(n,c) (FRS for "finite root-system") in the first-order language for order, $\leq$, depending on fixed integers $n, c \geq 1$. It follows from Proposition VI.7.11(a) that the root-systems of finite ARS-fans under specialization are models of FRS(n,c) for suitable values of n and c. However, FRS(n,c) has, in general, models other than those arising from finite ARS-fans.

(2) Next, we introduce the systems of numerical invariants alluded to in (II) —consisting of a finite set of finite sequences of integers—, and prove that these are, in fact, invariants for the

isomorphism of models of FRS(n,c), whether or not they arise from finite ARS-fans (Theorem VI.9.5).

(3) Finally, we impose additional axioms on FRS(n,c) and additional requirements on the integers occurring in the invariant systems of (2); these additional axioms are verified by the root-systems of finite ARS-fans (and, as shown in §VI.10, do characterize them).

By enlarging the language with a binary product operation and suitable axioms for it, we also obtain first-order axiomatizations for the finite ARS-fans.

VI.9.1 The axiom-systems FRS. For fixed positive integers n, c, we introduce the following axioms in the language for order, consisting of a single binary relation symbol $\leq$. index[sub]formula@**FRS**!axiom-systems—(index[sub]axiom-systems!**FRS**—(

[FRS.1] "$\leq$ determines a root system", i.e.,

— "$\leq$ is a partial order", and

— "The sets of successors of any element is totally ordered":

$$\forall\, x\, y\, z\, (x \leq y \wedge x \leq z \longrightarrow y \leq z \vee z \leq y).$$

For the next axiom we introduce the following auxiliary predicates (implicitly used already in §§VI.7, VI.8; cf. proof of item (4) in Theorem VI.8.9); k is an integer ≥ 1:

— "x has depth $\geq k$": $d(x) \geq k \leftrightarrow \exists\, x_1 \ldots x_{k-1} \left(\bigwedge_{j=1}^{k-1} x < x_j \wedge \bigwedge_{1 \leq j < \ell \leq k-1} x_j < x_\ell \right).$

— "x has depth $\leq k$": $d(x) \leq k \leftrightarrow \exists\, x_1 \ldots x_{k-1} \Big(\bigwedge_{j=1}^{k-1} x < x_j \wedge \bigwedge_{1 \leq j < \ell \leq k-1} x_j < x_\ell) \wedge$
$\wedge \forall\, y\, (x < y \rightarrow \bigvee_{j=1}^{k-1} (y = x_j)) \Big).$

— "x has depth k": $d(x) = k \leftrightarrow d(x) \geq k \wedge d(x) \leq k$.

[FRS.2] "The root-system has length n": $\forall\, x (d(x) \leq n) \wedge \exists\, x (d(x) = n)$.

[FRS.3] "The root-system has cardinality $\leq c$."

To state the last axiom we introduce unary predicates $S_j^k(x)$ and $C_j^k(x)$, $1 \leq k \leq j \leq n$, as follows:

— $S_j^k(x) \leftrightarrow d(x) = k \wedge \exists\, y\, (y \leq x \wedge d(y) \geq j)$.

— $C_j^k(x) \leftrightarrow d(x) = k \wedge \exists\, y\, (y \leq x \wedge d(y) = j) \wedge \neg \exists\, z (z \leq x \wedge d(z) \geq j+1)$.

The following properties are easily verified (cf. VI.7.2):

Fact VI.9.2 *Axioms* [FRS.1] – [FRS.2] *imply:*

(a) $C_n^k(x) \leftrightarrow S_n^k(x)$.

(b) $\forall\, x\, (S_j^k(x) \leftrightarrow \bigvee_{\ell=j}^{n} C_\ell^k(x))$ $(k \leq j)$.

(c) For $1 \leq k < j < j' \leq n$, $\neg \exists\, x\, (C_j^k(x) \wedge C_{j'}^k(x)) \wedge \forall\, x\, (S_{j'}^k(x) \rightarrow S_j^k(x)))$.

(d) $\forall\, x\, (S_k^k(x) \leftrightarrow d(x) = k)$. $\square$

(As in VI.7.2(i) we call S_k^k the <u>k-th level</u>, and denote it by L_k.)

Recall that, given a first-order formula $\varphi(x, y_1, \ldots, y_n) = \varphi(x, \overline{y})$ on $n+1$ variables of a language L, and a (finite) bound on the cardinalities of a family of L-structures, the statement "the sets $\{\overline{y} \,|\, \varphi(x_0, \overline{y})\}$ and $\{\overline{y} \,|\, \varphi(x_1, \overline{y})\}$ have same (finite) cardinality" is expressed by a first-order L-formula, that we will denote by $\sharp\{\overline{y} \,|\, \varphi(x_0, \overline{y})\} = \sharp\{\overline{y} \,|\, \varphi(x_1, \overline{y})\}$. Details are left to the reader.

With this terminology, our last axiom is a restatement of Proposition VI.7.11(a).

[FRS.4] $\bigwedge_{1 \le k \le j \le n} \forall x x' [S^k_j(x) \wedge S^k_j(x') \rightarrow \bigwedge_{k \le t \le p \le n} (\sharp\{y \in S^t_p \,|\, y \le x\} = \sharp\{y \in S^t_p \,|\, y \le x'\})]$.

FRS(n,c) denotes the axiom-system [FRS.1] – [FRS.4]. It should be clear that the root-system $(X, \rightsquigarrow)$ of a finite ARS-fan (X, F) is a model of these axioms, for suitable integers n, c. □
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VI.9.3 Numerical invariants for isomorphism of models of FRS. For models $(X, \le)$ of FRS(n,c), we shall consider finite sets of integers, as follows:

— $n =$ the length of the root-system.

— For each integer k, $1 \le k \le n$, a decreasing sequence of integers, $s^k_k \ge \cdots \ge s^k_{n-1} \ge s^k_n \ge 1$, of length $n-k+1$, where s^k_j is interpreted as the cardinality of the set $S^k_j(X)$ $(1 \le k \le j \le n)$.

Remarks VI.9.4 (a) Since the sets C^k_j and S^k_j are interdefinable, namely:

— $C^k_j = S^k_j \setminus S^k_{j+1}$ for $1 \le k \le j < n$, and $C^k_n = S^k_n$;

— $S^k_j = \bigcup_{\ell=j}^n C^k_\ell$ (disjoint union), for $1 \le k \le j \le n$,

an equivalent system of invariants consists of a sequence $\langle c^k_n, c^k_{n-1}, \ldots, c^k_k \rangle$ of length $n-k+1$, of integers $c^k_j \ge 0$, for each k $(1 \le k \le n)$. It suffices to pose:

— $c^k_j = s^k_j - s^k_{j+1}$ for $1 \le k \le j < n$, and $c^k_n = s^k_n$; and,

— $s^k_j = \sum_{\ell=j}^n c^k_\ell$ for $1 \le k \le j \le n$.

(b) The cardinality of a model $(X, \le)$ of FRS(n,c) is determined by these systems of invariants: $\mathrm{card}(X) = \sum_{k=1}^n s^k_k = \sum_{k=1}^n \sum_{j=k}^n c^k_j$. In particular, axiom [FRS.3] implies $\sum_{k=1}^n s^k_k \le c$. □

Now we prove:

Theorem VI.9.5 *Let $(X_i, \le_i)$ $(i = 1, 2)$ be models of* FRS(n, c). *Assume that sequences of integers as in VI.9.3 are given, and that* $\mathrm{card}(S^k_j(X_1)) = \mathrm{card}(S^k_j(X_2)) = s^k_j$ *for* $1 \le k \le j < n$. *Then, $(X_1, \le_1)$ and $(X_2, \le_2)$ are isomorphic.*

Proof. By induction on k, $1 \le k \le n$, we define maps $f_k : L_k(X_1) \longrightarrow L_k(X_2)$ with the following properties:

(1) f_k is a bijection. (2) For $k \le j \le n$, $f_k[S^k_j(X_1)] = S^k_j(X_2)$.

(3) For $1 \le k < n$, $h \in L_k(X_1)$ and $g \in L_{k+1}(X_1)$: $g \le_1 h \Leftrightarrow f_{k+1}(g) \le_2 f_k(h)$.

These conditions are not independent, and imply several others that will be used in the proof; namely:

(A) Condition (3) implies its own generalization:

(4) If $1 \le k < m \le n$, $h \in L_k(X_1)$ and $g \in L_m(X_1)$ then $g \le_1 h \Leftrightarrow f_m(g) \le_2 f_k(h)$.

(Induction on m.)

(B) (2) $\Rightarrow$ (1).

Proof of (B). Since $\mathrm{card}(S^k_j(X_1)) = \mathrm{card}(S^k_j(X_2)) = s^k_j$ and these cardinals are finite, condition (2) entails that $f_k \lceil S^k_j(X_1)$ is a bijection between the $S^k_j(X_i)$'s, for $k \le j \le n$. Since $S^k_k(X_i) = L_k(X_i)$ $(i = 1, 2)$, item (1) follows.

(C) Condition (2) is equivalent to:

(2′) For $1 \le k \le j \le n$, $f_k[C^k_j(X_1)] = C^k_j(X_2)$.

(Routine checking, using VI.9.4(a) and item (1).)

(D) The implication ($\Rightarrow$) in (3) entails its own converse.

Proof of (D). Assume ($\Leftarrow$) false, i.e., there are $h \in L_k(X_1)$ and $g \in L_{k+1}(X_1)$ so that $g \not\le_1 h$ and $f_{k+1}(g) \le_2 f_k(h)$. Let $h' \in L_k(X_1)$ be the unique depth-k successor of g. Then $g \le_1 h'$ and ($\Rightarrow$) gives $f_{k+1}(g) \le_2 f_k(h')$. Thus, both $f_k(h)$ and $f_k(h')$ are depth-k successors of $f_{k+1}(g)$, whence $f_k(h) = f_k(h')$ as $(X_2, \le_2)$ is a root-system. Since $g \le_1 h'$ and $g \not\le_1 h$, we have $h \ne h'$, contradicting injectivity of f_k.

Assuming that maps $f_1, \ldots, f_n$ with properties (2) and (3) —hence also (1), (2′) and (4)— have been constructed, we define a map $F : X_1 \longrightarrow X_2$, by setting $F = \bigcup_{k=1}^n f_k$. Item (1) shows that F is well-defined and bijective, and the implication ($\Rightarrow$) (resp., ($\Leftarrow$)) in item (4) yields that F (resp., F^{-1}) is order-preserving; hence F is the order-isomorphism required to prove the Theorem.

Next, we proceed with the inductive construction of the maps f_k. It suffices to prove:

Fact. *For $1 \le k+1 \le j \le n$, $h \in S^k_j(X_1)$ and $k+1 \le t \le j$,*

$$\mathrm{card}(\{g \in S^{k+1}_t(X_1) \mid g \le_1 h\}) = \mathrm{card}(\{g' \in S^{k+1}_t(X_2) \mid g' \le_2 f_k(h)\}).$$

Assuming this Fact proved, the definition of f_{k+1} proceeds as follows:

(I) Fix j and h so that $k+1 \le j \le n$ and $h \in S^k_j(X_1)$. Clearly, $\{g \in S^{k+1}_t(X_1) \mid g \le_1 h\} \subseteq \{g \in S^{k+1}_{t'}(X_1) \mid g \le_1 h\})$ for $k+1 \le t \le t' \le j$, and similarly for the corresponding subsets of X_2. We start by choosing a bijection $f^h_{k+1,j}$ between the sets in the statement of the Fact, for $t = j$. Then, by decreasing induction on t $(k+1 \le t \le j-1)$, we pick a bijection

$$f^h_{k+1,\,t} : \{g \in S^{k+1}_t(X_1) \mid g \le_1 h\} \longrightarrow \{g' \in S^{k+1}_t(X_2) \mid g' \le_2 f_k(h)\}$$

extending the previous bijection $f^h_{k+1,\,t+1}$. Set $f^h_{k+1} = f^h_{k+1,\,k+1}$. Since $S^{k+1}_{k+1}(X_i) = L_{k+1}(X_i)$ $(i = 1, 2)$, f^h_{k+1} is a bijection from $\{g \in L_{k+1}(X_1) \mid g \le_1 h\}$ onto $\{g' \in L_{k+1}(X_2) \mid g' \le_2 f_k(h)\}$ having the following properties: for $k+1 \le t \le j$,

(I.a) $f^h_{k+1} \lceil \{g \in S^{k+1}_t(X_1) \mid g \le_1 h\} = f^h_{k+1,\,t}$.

(I.b) $f^h_{k+1}[\{g \in S^{k+1}_t(X_1) \mid g \le_1 h\}] = \{g' \in S^{k+1}_t(X_2) \mid g' \le_2 f_k(h)\}$.

(I.c) If $g \in L_{k+1}(X_1)$ then, $g \leq_1 h \Rightarrow f^h_{k+1}(g) \leq_2 f_k(h)$.

Checking these is routine, and is left to the reader. [For (I.c) note that $g \in L_{k+1}(X_1) \wedge h \in S^k_j(X_1) \wedge g \leq_1 h \Rightarrow$ there is t, $k+1 \leq t \leq j$, such that $g \in S^{k+1}_t(X_1)$.]

(II) Now, we define $f_{k+1} : L_{k+1}(X_1) \longrightarrow L_{k+1}(X_2)$ by setting, for $g \in L_{k+1}(X_1)$,

$$f_{k+1}(g) = f^h_{k+1}(g), \text{ where } h \text{ is the unique depth-}k \text{ successor of } g \text{ in } X_1.$$

We claim that f_{k+1} verifies conditions (2) and (3; $\Rightarrow$) above (this being sufficient by (D)). The latter follows at once from (I.c) and the definition of f_{k+1}. As for item (2), given $g \in S^{k+1}_j(X_1)$, $k+1 \leq j \leq n$, then $h \in S^k_j(X_1)$, $g \leq_1 h$ and (I.b), imply at once $f_{k+1}(g) = f^h_{k+1}(g) \in S^{k+1}_j(X_2)$. Conversely, given $g' \in S^{k+1}_j(X_2)$, let $h' \in L_k(X_2)$ be the immediate depth-k successor of g'; then, $h' \in S^k_j(X_2)$. By induction hypothesis f_k verifies condition (2); hence, there is $h \in S^k_j(X_1)$ so that $f_k(h) = h'$. Since f^h_{k+1} maps $\{g \in S^{k+1}_j(X_1) \mid g \leq_1 h\}$ onto $\{u \in S^{k+1}_j(X_2) \mid u \leq_2 f_k(h) = h'\}$ (see (I.b)), there is $g \in S^{k+1}_j(X_1)$ such that $g \leq_1 h$ and $f_{k+1}(g) = f^h_{k+1}(g) = g'$. This shows that $S^{k+1}_j(X_2) \subseteq f_{k+1}[S^{k+1}_j(X_1)]$, completing the proof of item (2).

It only remains the

Proof of Fact. Towards computing explicitly the cardinality of $\{g \in S^{k+1}_t \mid g \leq h\}$, we first observe:

(*) $$S^{k+1}_t = \bigcup_{h \in S^k_t} \{g \in S^{k+1}_t \mid g \leq h\}.$$

For the non-trivial inclusion $\subseteq$, given $g \in S^{k+1}_t$ and taking h to be its unique depth-k successor shows that g is in the right-hand side of (*).

Axiom [FRS.4] says that the sets in the right-hand side of (*) have same cardinality —say $\gamma(k,t)$—, for all $h \in S^k_t$. Then, (*) gives $\mathrm{card}(S^{k+1}_t) = \mathrm{card}(S^k_t) \cdot \gamma(k,t)$,

and, since $S^k_t \neq \emptyset$: $$\gamma(k,t) = \frac{\mathrm{card}\,(S^{k+1}_t)}{\mathrm{card}\,(S^k_t)}.$$

By assumption, both root-systems, $(X_i, \leq_i)$ have same cardinal invariants $\mathrm{card}(S^\ell_t(X_i)) = s^\ell_t$ $(i = 1,2)$; hence, $\gamma(k,t,X_1) = \gamma(k,t,X_2)$ for $1 \leq k \leq t \leq n$. This equality proves the Fact, upon observing that $h \in S^k_t(X_1)$ and condition (2) imply $f_k(h) \in S^k_t(X_2)$. The proof of Theorem VI.9.5 is now complete. □

VI.9.6 Binary regular root-systems. The axioms considered so far are not sufficient to characterize the root-systems of finite ARS-fans under specialization. The missing information is supplied by the following axiom:

[FRS.5] For $1 \leq k \leq j \leq n$, the cardinality of the set $S^k_j(X)$ is a power of 2 (possibly 1).

The validity of this axiom for finite ARS-fans stems from the fact —proved in Corollary VI.7.6— that the sets S^k_j have a structure of AOS-fan.

The argument in §VI.10 below shows that a particular instance of axiom [FRS.4] suffices to prove the completeness of the axiom-systems FRS. Thus, we introduce:

Definition VI.9.7 A finite root-system $(X, \leq)$ (of length n, say) is called a **binary regu-**

lar root-system (BRRS) if it verifies axiom [FRS.5] and the following instance of axiom [FRS.4]: index[sub]binary regular root-system (BRRS)index[sub]root-system!binary regular (BRRS)index[sym]BRRS

For all $1 \le k < j \le n$ and $x_1, x_2 \in S_j^k(X)$, $\mathrm{card}(\{y \in S_j^{k+1}(X) \mid y \le x_1\}) =$
$= \mathrm{card}(\{y \in S_j^{k+1}(X) \mid y \le x_2\})$. □

Remark. A first-order axiomatization of the finite ARS-fans can now be obtained by enlarging the language with a binary operation "·", and adding to FRS the following axioms:

(FF.1) Each of the sets S_j^k $(1 \le k \le j \le n)$ is 3-closed.

(FF.2) $\forall\, x\, y\, (x \le y \leftrightarrow y = y^2 x)$.

(Cf. Lemma I.1.18.) □

VI.10 Finite fans with prescribed root-systems

In this section we show that the numerical invariants introduced in section VI.9 for the order structure of finite ARS-fans under specialization, form a complete system of invariants. □
Our main result is:

Theorem VI.10.1 *For any BRRS, $(X, \le)$, there is a finite RS-fan, G, whose dual ARS, $(X_G, \rightsquigarrow)$, is order-isomorphic to $(X, \le)$.*

We start with a general construction:

VI.10.2 Construction (First step). Fix integers $p \ge 0$, $n \ge 1$, and a group H of exponent 2 and cardinality 2^n. Fix an element of $H \setminus \{1\}$ and call it -1. Let $\{-1, x_2, \ldots, x_n\}$ be a basis of H as a $\mathbb{F}_2$-vector space. We construct a ternary semigroup, G, by adding to H a new set of generators

$$\{y_1^1, \ldots, y_{\tau_1}^1;\ y_1^2, \ldots, y_{\tau_2}^2;\ \ldots\ldots;\ y_1^p, \ldots, y_{\tau_p}^p\},$$

split into packages of cardinalities $\tau_1, \ldots, \tau_p \ge 1$, respectively (as usual, if $p = 0$ this set is empty). These generators are required to verify the following relations:

(A) $(y_1^1)^2 = \cdots = (y_{\tau_1}^1)^2;\ \ldots\ldots;(y_1^p)^2 = \cdots = (y_{\tau_p}^p)^2$.

(B) $(y_1^j)^2 \cdot (y_1^k)^2 = (y_1^k)^2$, for $1 \le j \le k \le p$.

(C) $(y_1^j)^2 \ne (y_1^k)^2$, if $1 \le j \ne k \le p$.

(D) $(y_1^1)^2 \ne 1$.

Remark. To keep matters straight, recall that, in order to get a ternary semigroup, the following requirements must also be fulfilled:

i) G contains an absorbent element 0 (i.e., $a \cdot 0 = 0$ for all $a \in G$).

ii) The product operation is commutative, associative and 1 is its neutral element.

iii) The generators y_ℓ^i $(= y$, say) verify the identity $y^3 = y$.

iv) For all $a \in G \setminus \{0\}$, $-a = (-1) \cdot a \ne a$.

Then, G is the set of all finite formal products of elements of H and generators y_ℓ^i verifying these requirements, conditions (A) – (D) above, as well as other constraints to be specified later. The value of the parameters p, n, and the cardinalities $\tau_1, \ldots, \tau_p$ will be fixed later. □

Before proceeding further with our construction we derive some consequences of the relations just introduced.

VI.10.3 Remarks and Notation. (a) To ease notation, in this section we shall write $\langle\, x\,\rangle$ for the ideal $I_x = x \cdot G$ of G generated by x. Since $\langle\, x\,\rangle = \langle\, x^2\,\rangle$ holds in any ternary semigroup, the identities VI.10.2(A) imply $\langle\, y_1^k\,\rangle = \langle\, y_r^k\,\rangle$ for $1 \le k \le p$ and $1 \le r \le \tau_k$.

(b) For notational uniformity we set $y_1^{p+1} = 0$ (and $\tau_{p+1} = 1$). □

Proposition VI.10.4 *With notation as in VI.10.2 and VI.10.3, we have:*

(1) *The following relations hold in G for* $1 \le j \le k \le p$ *and all* $1 \le \ell \le \tau_j$, $1 \le r, s \le \tau_k$*:*

(B') $(y_\ell^j)^2 \cdot (y_r^k)^2 = (y_s^k)^2$. $\quad$ (C') $(y_\ell^j)^2 \neq (y_r^k)^2$, *whenever* $j \neq k$. $\quad$ (D') $(y_\ell^j)^2 \neq 1$.

In particular,

(2) *None of the generators* y_ℓ^j *is invertible; hence,* H *is the set of invertible elements of* G.

(3) $\langle\, y_1^k\,\rangle \subseteq \langle\, y_1^j\,\rangle$ *if* $1 \le j \le k \le p$*, and the inclusion is proper if* $j < k$.

(4) *For* $a \in G \setminus \{0\}$[3] *and* $1 \le k \le p$*, the following are equivalent:*

i) $a \in \langle\, y_1^k\,\rangle \setminus \langle\, y_1^{k+1}\,\rangle$.

ii) k *is the largest integer* $i \in \{1, \ldots, p\}$ *such that some generator* y_r^i *occurs in any representation of* a *as a product of elements of* H *and generators* y_ℓ^j.

(5) *For* $a \in G \setminus \{0\}$ *and* $1 \le k \le p$,

$$Z(a) = \begin{cases} \emptyset & \textit{if } \ a \in H \\ Z(y_1^k) & \textit{if } \ a \in \langle\, y_1^k\,\rangle \setminus \langle\, y_1^{k+1}\,\rangle. \end{cases}$$

[*Recall that* $Z(a) = \{f \in \mathrm{Hom}_{\mathrm{TS}}(G, \mathbf{3}) \,|\, f(a) = 0\,\}$.]

(6) *The set* $\{Z(a)\,|\, a \in G\,\}$ *is totally ordered under inclusion.*

(7) *Every non-zero proper ideal of* G *is of the form* $\langle\, y_1^k\,\rangle$ *for some* k, $1 \le k \le p$.

Remark. Item (6) means that G verifies the assumption [Z] in Theorem VI.2.1. Hence, with representation (and transversal representation) defined therein, G is a RS-fan.

Proof. (1) The relations (B′), (C′) and (D′) follow at once from (B), (C) and (D), respectively, using the identities (A) in VI.10.2; cf. VI.10.3(a).

(2) is clear from (D′) (recall that an element x of a ternary semigroup is invertible iff $x^2 = 1$).

(3) follows from VI.10.2 (B),(C). Indeed, (B) clearly implies $\langle\, y_1^k\,\rangle \subseteq \langle\, y_1^j\,\rangle$ for $1 \le j \le k \le p$, whence $y_1^k \in \langle\, y_1^j\,\rangle$, i.e., $y_1^k = z\, y_1^j$ for some $z \in G$. Let $j < k$. If equality holds, then $y_1^j \in \langle\, y_1^k\,\rangle$, i.e., $y_1^j = x\, y_1^k$ for some $x \in G$. By VI.10.2(C), $(y_1^j)^2 \neq (y_1^k)^2$. Now, we invoke the separation theorem for ternary semigroups I.1.12 to get a TS-character f so that $f((y_1^j)^2) \neq f((y_1^k)^2)$. Since $f(x^2) \in \{0, 1\}$ for any $x \in G$, one of these values is 0 and the other is 1. However, if $f((y_1^j)^2) = 0$, then $y_1^k = z\, y_1^j$ entails $f((y_1^k)^2) = 0$; and if $f((y_1^k)^2) = 0$, then $y_1^j = x\, y_1^k$ yields $f((y_1^j)^2) = 0$, a contradiction.

[3]Note that every element of $G \setminus (H \cup \{0\})$ contains at least one generator y_r^j in its expression as a product.

(4) (i) $\Rightarrow$ (ii). From $a \in \langle y_1^k \rangle$, we get $a = y_1^k b$ with $b \in G \setminus \{0\}$; this is a representation of a (as a product of elements in H and generators y_r^j) in which y_1^k occurs. If some representation of a contained a factor y_r^j with $k < j$, then $a \in \langle y_r^j \rangle = \langle y_1^j \rangle$, and by (3) we get $\langle y_1^j \rangle \subseteq \langle y_1^{k+1} \rangle$, whence $a \in \langle y_1^{k+1} \rangle$, contrary to assumption (i).

(ii) $\Rightarrow$ (i). The existence of a representation of the form $a = y_r^k b$ with $b \in G \setminus \{0\}$ and $1 \le r \le \tau_k$, obviously gives $a \in \langle y_r^k \rangle = \langle y_1^k \rangle$. However, if $a \in \langle y_1^{k+1} \rangle$, the argument proving (i) $\Rightarrow$ (ii) shows that y_1^{k+1} would occur as a factor in some representation of a, contrary to (ii).

(5) Since $a^2 = 1$ for any $a \in H$, we clearly have $Z(a) = \emptyset$. Also, $a \in \langle y_1^k \rangle$ obviously implies $Z(y_1^k) \subseteq Z(a)$. Assuming, in addition, that $a \notin \langle y_1^{k+1} \rangle$ we prove the other inclusion. By (4.ii) and the remark VI.10.3(a), we may write a in the form

$$(*) \qquad a = (y_1^k)^{\varepsilon_k} \cdot z^{k-1} \cdot \ \ldots \ \cdot z^1 \cdot b,$$

with $\varepsilon_k \in \{1, 2\}$, $b \in H$, and each z^j $(1 \le j < k)$ a product of (some) generators of the form y_r^j, i.e., $z^j = \prod_{\ell=1}^{\tau_j} (y_\ell^j)^{\eta_{j,\ell}}$, with $\eta_{j,\ell} \in \{0, 1, 2\}$ (by convention, $(y_\ell^j)^0 = 1$). Assuming $h \notin Z(y_1^k) = Z((y_1^k)^2)$, from the equality (B$'$) in (1) we get $Z(y_\ell^j) \subseteq Z(y_1^k)$, whence $h((y_\ell^j)^{\eta_{j,\ell}}) \neq 0$ for $1 \le \ell \le \tau_j$ and $1 \le j < k$, and therefore $h(z^j) \neq 0$. Since $h(b) \neq 0$ for $b \in H$, we conclude from (*) that $h(a) \neq 0$; this proves $Z(a) \subseteq Z(y_1^k)$, as required.

(6) Item (3) implies $Z(y_1^j) \subseteq Z(y_1^k)$ for $1 \le j \le k \le p$. Clearly, (6) follows from this inclusion and (5).

(7) Since G is a ternary semigroup, item (6) and Proposition VI.1.2 imply that the set of ideals of G is totally ordered under inclusion, and that every ideal is prime.

First, note that $\langle y_1^1 \rangle$ is the maximal ideal of G. Indeed, if $a \neq 0$ is a non-invertible element of G, then $a \notin H$, and some generator y_ℓ^j must occur in any representation of a; then, by item (3), $a \in \langle y_\ell^j \rangle = \langle y_1^j \rangle \subseteq \langle y_1^1 \rangle$.

Next, let $I \neq \{0\}$ be an ideal of G. By the preceding paragraph, $I \subseteq \langle y_1^1 \rangle$ (the maximal ideal is unique). Let k be the largest index in $\{1, \ldots, p\}$ such that $I \subseteq \langle y_1^k \rangle$. Since the ideals are totally ordered by inclusion and $I \neq \{0\}$, we have $\langle y_1^{k+1} \rangle \subset I$. We show that $y_1^k \in I$, thus proving $I = \langle y_1^k \rangle$. Let $a \in I \setminus \langle y_1^{k+1} \rangle$; then $a \in \langle y_1^k \rangle \setminus \langle y_1^{k+1} \rangle$. By (4.ii), all generators y_r^i occurring in any representation of a verify $i \le k$. Then, we can write $a = b \cdot c$ with $b \in H$ and c a product of generators of the form y_r^i with $1 \le i \le k$. Since $a \in I$ and b is invertible, then $c \in I$. Since I is prime, some generator y_r^i, $1 \le i \le k$, must be in I. It follows that $\langle y_r^i \rangle = \langle y_1^i \rangle \subseteq I$. From VI.10.2 (B) we get $\langle y_1^k \rangle \subseteq \langle y_1^i \rangle \subseteq I$, whence $y_1^k \in I$, as required. □

Corollary VI.10.5 *With notation as in VI.6.15, the ARS X_G dual to the fan G constructed in VI.10.2 has the following properties:*

(1) $\ell(X_G) = p + 1$.

(2) *For* $1 \le k \le p$, $L_k(X_G) = \{h \in X_G \mid Z(h) = \langle y_1^k \rangle\}$.

(3) $L_{p+1}(X_G) = \{h \in X_G \mid Z(h) = \{0\}\}$.

(4) $\mathrm{card}(L_1(X_G)) = 2^{n-1}$.

Proof. (1) G has $p + 1$ ideals (VI.10.4 (7)), ordered as in VI.10.4 (3).

(2) and (3) are obvious from VI.10.4 (7) and the definition of the level sets L_I given in VI.6.6 (b); cf. also VI.6.15.

(4) By (2), a TS-character $h \in X_G$ is in $L_1(X_G)$ if and only if $h(y^1_1) = 0$ (and hence $h(y^i_r) = 0$ for all $i \in \{1, \dots, p\}$ and all $r \in \{1, \dots, \tau_i\}$), but otherwise can take arbitrary ± 1 values on the generators $x_2, \dots, x_n$ of H. $\square$

VI.10.2. Construction (Second step). In order to prove Theorem VI.10.1 we must impose further relations between the generators of G, beyond those set in VI.10.2 above. These are of two types:

Type I. Identities of the form $(y^i_1)^2 = y^i_1$ for some $i \in \{1, \dots, p\}$. Whenever an identity of this type is required, we set $\tau_i = 1$ (i.e., a single generator of level i suffices). We denote by I the set of indices $i \in \{1, \dots, p\}$ for which an identity of this type is imposed.

Type II. Identities of the form $y^i_t \cdot y^j_1 = y^j_1$ for some indices $0 \le i < j \le p$ and $t \in \{1, \dots, \tau_i\}$.

For notational uniformity we adopt the convention that $y^0_{\ell-1} = x_\ell$ for $2 \le \ell \le n$, where the x_ℓ are the elements of the basis of H, and set $\tau_0 = n - 1$. Note that, since $x^2_\ell = 1 \neq x_\ell$ (and $y^0_{\ell-1} = x_\ell$), no equation of type (I) holds at level 0.

The number and specific form of these relations will depend on the cardinal invariants $\mathrm{card}(S^k_j(X))$ of a given BRRS $(X, \le)$, and will be fixed later, as needed. $\square$

Lemma VI.10.6 *The following are consequences of any identity* $y^i_t \cdot y^j_1 = y^j_1$ $(0 \le i < j \le p$ *and* $t \in \{1, \dots, \tau_i\})$ *of type* (II)*:*

(a) $y^i_t \cdot y^j_m = y^j_m$ *for all* $m \in \{1, \dots, \tau_j\}$.

(b) $y^i_t \cdot y^k_1 = y^k_1$ *for all* k, $j \le k \le p$.

Proof. (a) Scaling the given identity by y^j_1 we have $y^i_t \cdot (y^j_1)^2 = (y^j_1)^2$. Since $(y^j_1)^2 = (y^j_m)^2$ (VI.10.2(A)), we get $y^i_t \cdot (y^j_m)^2 = (y^j_m)^2$, which yields (a) upon scaling by y^j_m.

(b) Scaling the given identity by $(y^j_1)^2 \cdot (y^k_1)^2$ we get $y^i_t \cdot (y^j_1)^2 \cdot (y^k_1)^2 = (y^j_1)^2 \cdot (y^k_1)^2$. Since $(y^j_1)^2 \cdot (y^k_1)^2 = (y^k_1)^2$ (VI.10.2(B)), we obtain $y^i_t \cdot (y^k_1)^2 = (y^k_1)^2$; scaling by y^k_1 gives (b). $\square$

The following Lemma clarifies the role of the identities of type (I) in our construction:

Lemma VI.10.7 *Let* $1 \le i \le p$, *and assume* $\tau_i = 1$. *The following are equivalent:*

(a) The identity $(y^i_1)^2 = y^i_1$ *holds in* G.

(b) Every $h \in S^i_{i+1}(X_G)$ *has a unique* $\rightsquigarrow$*-predecessor of level* $i + 1$, *i.e.,* $\mathrm{card}(S^i_{i+1}(X_G)) = \mathrm{card}(L_{i+1}(X_G))$.

Proof. The values of any $h \in L_i(X_G)$ on generators are as follows:

— $h(y^i_1) = 0$; hence, by VI.10.4(3), $h(y^j_r) = 0$ for all $i \le j \le p$ and $r \in \{1, \dots, \tau_j\}$;

— $h(y^j_r) \neq 0$ for $0 \le j < i$ and $r \in \{1, \dots, \tau_j\}$.

If $g \in L_{i+1}(X_G)$ and $g \rightsquigarrow h$, its values on generators are:

— $g(y^j_r) = 0$ for all $i + 1 \le j \le p$ and $r \in \{1, \dots, \tau_j\}$;

— $g(y^j_r) \neq 0$ for $0 \leq j \leq i$ and $r \in \{1, \ldots, \tau_j\}$.

— $g(y^j_r) = h(y^j_r)$, whenever $h(y^j_r) \neq 0$, that is, for $0 \leq j < i$ and $r \in \{1, \ldots, \tau_j\}$ (cf. Lemma I.1.18).

In other words, given h and g as above, the values of g are determined on all generators except on y^i_1, where it must be $\neq 0$; thus we have:

(a) $\Rightarrow$ (b). If the equation $(y^i_1)^2 = y^i_1$ holds in G, then $g(y^i_1) \in \{0, 1\}$, and hence the character determined by $g(y^i_1) = 1$ is the only $\rightsquigarrow$-predecessor of $h \in S^i_{i+1}(X_G)$ of level $i + 1$.

(b) $\Rightarrow$ (a). If the equation $(y^i_1)^2 = y^i_1$ does not hold in G, any $h \in S^i_{i+1}(X_G)$ has two $\rightsquigarrow$-predecessors of level $i + 1$, by setting $g^+(y^i_1) = 1$ and $g^-(y^i_1) = -1$. $\square$

The next step is to compute the cardinalities of the sets $S^k_j(X_G)$ in terms of the number of generators and the number of equations of types (I) and (II). To do so we shall need:

VI.10.2. Construction (Third step). To each pair of indices i, j, such that $0 \leq i < j \leq p$ and $i \notin I$ (i.e., there is no equation of type (I) at level i) we associate a set T^j_i subject to the following requirements:

i) $T^j_i \subseteq \{1, \ldots, \tau_i\}$. ii) If $k, j > i$, $k \neq j$, then $T^j_i \cap T^k_i = \emptyset$.

(Some of these sets may be empty.) We write $t_{i,j} = \mathrm{card}\,(T^j_i)$. The sets T^j_i (or, rather, their cardinalities) determine which equations of type (II) the generators will verify; precisely:

— If $p \geq j > i \geq 0$ and $i \notin I$, then we include an equation $y^i_t \cdot y^j_1 = y^j_1$ if and only if $t \in T^j_i$. (Remark that, since $0 \notin I$, we do include the equations $x_t \cdot y^j_1 (= y^0_{t-1} \cdot y^j_1) = y^j_1$ for $t - 1 \in T^j_0$, $j > 0$.) $\square$

Now we are ready to prove:

Proposition VI.10.8 *For* $1 \leq m < k \leq p + 1$,

$$\mathrm{card}(S^m_k(X_G)) = \prod_{\substack{i=0 \\ i \notin I}}^{m-1} 2^{\tau_i - \sum_{j=i+1}^{k-1} t_{i,j}} .$$

Proof. Throughout this proof we omit X_G from the notation. The gist of the proof consists in finding necessary and sufficient conditions for a TS-character to be in S^m_k, in terms of its values on generators.

Assume first $h \in S^m_k$; then $h \in L_m$, and there is $h' \in L_k$ so that $h' \rightsquigarrow h$. By Corollary VI.10.5(2), Proposition VI.10.4(3) and Lemma I.1.18, these conditions amount to:

(i) $h(y^m_1) = 0$; hence, $h(y^j_r) = 0$ for $m \leq j \leq p$ and $r \in \{1, \ldots, \tau_j\}$;

(ii) $h(y^j_r) \neq 0$ for $0 \leq j \leq m - 1$ and $r \in \{1, \ldots, \tau_j\}$;

(i′) If $k \leq p$, $h'(y^k_1) = 0$; hence, $h'(y^\ell_s) = 0$ for $\ell \geq k$ and $s \in \{1, \ldots, \tau_\ell\}$;

(ii′) $h'(y^\ell_s) \neq 0$ for $0 \leq \ell \leq k - 1 \leq p$ and $s \in \{1, \ldots, \tau_\ell\}$;

(iii) $h(z) \neq 0 \Rightarrow h(z) = h'(z)$, for all $z \in G$; hence, $h(y^j_r) = h'(y^j_r) \neq 0$ for $0 \leq j \leq m - 1$ and $r \in \{1, \ldots, \tau_j\}$.

Next, if $i \in I$ —i.e., there is an equation $(y^i_1)^2 = y^i_1$—, either:

(iv) $k \leq i \leq p$ and $h(y^i_1) = h'(y^i_1) = 0$; or,

(v) $m \leq i \leq k-1$ and $h(y^i_1) = 0$, $h'(y^i_1) \neq 0$; the equation $(y^i_1)^2 = y^i_1$ yields $h'(y^i_1) = 1$; or,

(vi) $1 \leq i \leq m-1$; in this case, (iii) and the equation imply $h(y^i_1) = h'(y^i_1) = 1$.

Assume that $i \in \{0, \ldots, p\} \setminus I$. According to the third step of construction VI.10.2, we impose an equation $y^i_t \cdot y^j_1 = y^j_1$ of type (II) with $i < j$ whenever $t \in T^j_i$. If $i < j \leq k-1$, by (ii′) we have $h'(y^j_1) \neq 0$, and this equation yields:

(vi′) $h'(y^i_t) = 1$ for $t \in T^{i+1}_i \cup \ldots \cup T^{k-1}_i$ and $0 \leq i < k-1$.

From (iii) we infer:

(vii) $h(y^i_t) = 1$ for $t \in T^{i+1}_i \cup \ldots \cup T^{k-1}_i$ and $0 \leq i \leq m-1$.

These are the sole constraints on h; hence, (ii) gives:

(viii) For $i \in \{0, \ldots, m-1\} \setminus I$, $h(y^i_t)$ may take on arbitrary ± 1 values for $t \in \{1, \ldots, \tau_i\} \setminus \bigcup_{j=i+1}^{k-1} T^j_i$.

Conversely, we check that any TS-character h verifying conditions (i) and (ii) above is in S^m_k. Obviously, these conditions imply $h \in L_m$ (VI.10.5 (3)). In order to show that $h \in S^m_k$ we have to manufacture a character $h' \in L_k$ so that $h' \rightsquigarrow h$. We define h' on generators according to the constraints imposed by clauses (i′), (ii′) and (iii) above:

$$(*) \qquad \begin{cases} h'(y^j_r) = 0 & \text{for } j \geq k \text{ and } r \in \{1, \ldots, \tau_j\} & \text{(see (i′))} \\ h'(y^j_r) = h(y^j_r)\,(\neq 0) & \text{for } 0 \leq j \leq m-1 \text{ and } r \in \{1, \ldots, \tau_j\} & \text{(see (ii), (iii))} \\ h'(y^j_r) = 1 & \text{for } m \leq j \leq k-1 \text{ and } r \in \{1, \ldots, \tau_j\} & \text{(see (ii′))}\,. \end{cases}$$

Obviously, this definition guarantees that $h' \in L_k$. By Lemma I.1.18, in order to establish that $h' \rightsquigarrow h$, it suffices to check clause (iii) for all generators z of G. By the second requirement in (*), this is the case for $z = y^j_r$, with $0 \leq j \leq m-1$ and $r \in \{1, \ldots, \tau_j\}$. For $m \leq j \leq p$ and $r \in \{1, \ldots, \tau_j\}$, clause (i) ensures that $h(y^j_r) = 0$, and hence (iii) holds vacuously. This proves $h \in S^m_k$, as required.

Since conditions (i), (ii), (vi) and (vii) fix the values of any $h \in S^m_k$ on the corresponding generators, the cardinality of S^m_k is determined by clause (viii), and clearly is as in the statement of the Proposition. □

A similar argument proves:

Proposition VI.10.9 *For* $2 \leq k \leq p+1$,

$$\operatorname{card}(L_k(X_G)) = \begin{cases} \displaystyle\prod_{\substack{i=0 \\ i \notin I}}^{m-1} 2^{\tau_i - \sum_{j=i+1}^{k-1} t_{i,j}} & \text{if } k-1 \in I \\ \displaystyle\prod_{\substack{i=0 \\ i \notin I}}^{m-1} 2^{\tau_i - \sum_{j=i+1}^{k-1} t_{i,j}} \cdot 2^{\tau_{k-1}} & \text{if } k-1 \notin I\,. \end{cases}$$

Proof. The argument differs from that proving Proposition VI.10.8 at only one point.

We know that a TS-character h of G is in L_k if and only if:

(i) $h(y^j_r) = 0$ for $k \leq j \leq p$ and $r \in \{1, \ldots, \tau_j\}$; and

(ii) $h(y^j_r) \neq 0$ for $0 \leq j \leq k-1$ and $r \in \{1, \ldots, \tau_j\}$.

(Condition (i) holds vacuously if $k = p+1$; cf. VI.10.3(b).) We consider next the effect of the equations of types (I) and (II) on the values of h on generators.

If $i \in I$, then the equation $(y^i_1)^2 = y^i_1$ forces $h(y^i_1) \in \{0,1\}$; if, in addition, $0 \leq i \leq k-1$, then (ii) gives $h(y^i_1) = 1$.

If $i \in \{0, \ldots, p\} \setminus I$, there is an equation $y^i_t \cdot y^j_1 = y^j_1$ of type (II) for each j such that $i < j \leq p$ and each $t \in T^j_i$, i.e., for $t \in \bigcup_{j=i+1}^{p} T^j_i$. By (i) and (ii) these equations determine the values $h(y^i_t)$ only for $i < j \leq k-1$; for these values of j, $h(y^j_1) \neq 0$, and the corresponding equations yield:

(iii) $h(y^i_t) = 1$ for $t \in \bigcup_{j=i+1}^{p} T^j_i$ and $0 \leq i < k-1$.

From (ii) we get:

(iv) For $0 \leq i < k-1$, $h(y^i_t)$ takes on arbitrary ± 1 values for $t \in \{1, \ldots, \tau_i\} \setminus \bigcup_{j=i+1}^{k-1} T^j_i$.

Next, we consider the case $i = k-1$. If $k-1 \in I$, then, as above, $h(y^{k-1}_t) = 1$. If $k-1 \notin I$, then the corresponding equation of type (II), $y^i_t \cdot y^j_1 = y^j_1$, holds only for $j \geq k$; since $h(y^j_1) = 0$ for these j's, item (ii) gives:

(v) $h(y^{k-1}_t)$ takes on arbitrary ± 1 values for all $t \in \{1, \ldots, \tau_{k-1}\}$.

Items (iv) and (v) together yield the values of $\mathrm{card}\,(L_k(X_G))$ asserted in the statement. $\square$

Remark. The formula for $\mathrm{card}(L_k(X_G))$ just proved coincides with that obtained by setting $m = k$ in Proposition VI.10.8, provided we set $\sum_{j=k}^{k-1} t_{k,j} = 0$. This is in agreement with the fact that $L_k = S^k_k$. $\square$

From Propositions VI.10.8 and VI.10.9 we get:

Corollary VI.10.10 *With notation as in VI.10.8 and VI.10.9, we have:*

(a) For $2 \leq k \leq p+1$,

$$\mathrm{card}(L_k(X_G)) = \begin{cases} \mathrm{card}(S^{k-1}_k(X_G)) & \textit{if } k-1 \in I \\ \mathrm{card}(S^{k-1}_k(X_G)) \cdot 2^{\tau_{k-1}} & \textit{if } k-1 \notin I\,. \end{cases}$$

(b) For $2 \leq m < k \leq p+1$,

$$\mathrm{card}(S^m_k(X_G)) = \begin{cases} \mathrm{card}(S^{m-1}_k(X_G)) & \textit{if } m-1 \in I \\ \mathrm{card}(S^{m-1}_k(X_G)) \cdot 2^{\tau_{m-1} - \sum_{j=m}^{k-1} t_{m-1,j}} & \textit{if } m-1 \notin I\,. \end{cases}$$

Proof. Staightforward checking from VI.10.8 and VI.10.9. $\square$

Remark. The factor $2^{\tau_{k-1}}$ in (a) (1 in the first equality) is the number of $\rightsquigarrow$-predecessors of any element of S^{k-1}_k in L_k; this number is the same for any two elements of S^{k-1}_k; cf. VI.9.7. A similar remark applies to item (b). $\square$

For the proof of Theorem VI.10.1 we shall also need:

Lemma VI.10.11 *Let* $(X, \leq)$ *be a BRRS and* $2 \leq m \leq k \leq \ell(X)$. *Let* $s^m_k = \mathrm{card}\,(S^m_k(X))$. *Then,*

$$s_k^{m-1} \cdot s_{k+1}^m \le s_k^m \cdot s_{k+1}^{m-1}\,.$$

Proof. We write S_k^j for $S_k^j(X)$ $(j \le k)$. Given $x \in S_k^m$, let x^* denote the unique $\le$-successor of x of level $m-1$; hence, $x^* \in S_k^{m-1}$. The definition of a BRRS (cf. VI.9.7) guarantees the existence, for $x_1, x_2 \in S_k^{m-1}$, of a bijection

$$F_{x_1,x_2} : \{y \in S_k^m \mid y \le x_1\} \longrightarrow \{y \in S_k^m \mid y \le x_2\}.$$

Define a map $\beta : S_k^{m-1} \times S_{k+1}^m \longrightarrow S_k^m \times S_{k+1}^{m-1}$, as follows: for $(x_1, x_2) \in S_k^{m-1} \times S_{k+1}^m$, we set $\beta(x_1, x_2) = (F_{x_2^*, x_1}(x_2)\,, x_2^*))$.

We show that β is injective. Let $(x_1, x_2), (y_1, y_2) \in S_k^{m-1} \times S_{k+1}^m$ be such that $\beta(x_1, x_2) = \beta(y_1, y_2)$, i.e., $F_{x_2^*, x_1}(x_2) = F_{y_2^*, y_1}(y_2)$ $(= z$, say), and $x_2^* = y_2^*$. The definition of F gives $z \in S_k^m \subseteq L_m$, $z \le x_1$ and $z \le y_1$. Since $(X, \le)$ is a root-system and $x_1, y_1 \in L_{m-1}$, we get $x_1 = y_1$. Thus, $F_{x_2^*, x_1}(x_2) = F_{x_2^*, x_1}(y_2)$ and, since F is injective, $x_2 = y_2$, as required. □

Proof of Theorem VI.10.1. Given a BRRS, $(X, \le)$, we shall define the parameters n, p, $\tau_1, \dots, \tau_p \ge 1$, occuring in the construction of the fan G (VI.10.2) —as well as the set $I \subseteq \{1, \dots, p\}$ and the numbers $t_{i,j}$ $(0 \le i < j \le p)$ occuring in VI.10.8 and VI.10.9— in such a way that $\ell(X) = \ell(X_G)$ and $\mathrm{card}(S_k^m(X)) = \mathrm{card}(S_k^m(X_G))$ for all $1 \le m \le k \le \ell(X)$. Theorem VI.9.5 then guarantees that $(X_G, \rightsquigarrow)$ is order-isomorphic to $(X, \le)$.

In this proof *log* stands for base-2 logarithms, s_k^m for $\mathrm{card}(S_k^m(X))$, and $\ell_k = s_k^k = \mathrm{card}(L_k(X))$ $(1 \le m \le k \le p)$. The *log* of these numbers are integers by the first requirement (axiom [FRS.5]) in VI.9.7.

Since $\ell(X_G) = p + 1$ (VI.10.5(1)) and $\mathrm{card}(L_1(X_G)) = 2^{n-1}$ (VI.10.5(4)), we set $p = \ell(X) - 1$ and $n = 1 + \log(\ell_1)$. We also know that $\tau_0 = n - 1$ and, $\tau_i = 1$ for $i \in I = \{k \in \{1, \dots, p\} \mid$ There is an equation of type (I) at level $k\}$. The set I is presently defined in terms of X as follows:

$$\begin{aligned} I &= \{k \in \{1, \dots, p\} \mid s_{k+1}^k = \ell_{k+1}\} = \\ &= \{k \mid \text{Every element of } S_{k+1}^k(X) \text{ has a unique } \le\text{-predecessor in } L_{k+1}(X)\}. \end{aligned}$$

Let $k \in \{1, \dots, p\} \setminus I$; invoking the second clause in Corollary VI.10.10(a) (for the value $k+1$), we define:

$$\tau_k = \log(\ell_{k+1}) - \log(s_{k+1}^k).$$

Since $k \notin I$ implies that each element of $S_{k+1}^k(X)$ has at least two predecessors in $L_{k+1}(X)$, it follows that $\tau_k \ge 1$.

Next we compute the numbers $t_{m-1,k}$ $(1 \le m \le k \le p)$ that, according to Proposition VI.10.8, determine the cardinalities of the sets $S_k^m(X_G)$. First, let $m = 1$. Since $0 \notin I$, Proposition VI.10.8 gives

$$\mathrm{card}(S_k^1(X_G)) = 2^{\tau_0 - \sum_{j=1}^{k-1} t_{0,j}} \quad \text{and} \quad \mathrm{card}(S_{k+1}^1(X_G)) = 2^{\tau_0 - \sum_{j=1}^{k} t_{0,j}}.$$

Hence, $\mathrm{card}(S_{k+1}^1(X_G)) = \mathrm{card}(S_k^1(X_G)) \cdot 2^{-t_{0,k}}$. Then, we set:

$$t_{0,k} = \log(s_k^1) - \log(s_{k+1}^1), \quad \text{for } 2 \le k \le p\,.$$

Since $S_{k+1}^1(X) \subseteq S_k^1(X)$, we have $s_{k+1}^1 \le s_k^1$, and hence $t_{0,k} \ge 0$. For $k = 1$ we have $\mathrm{card}(S_2^1(X_G)) = 2^{n-1-t_{0,1}} = \mathrm{card}(L_1(X_G)) \cdot 2^{-t_{0,1}}$; so, it suffices to set

$$t_{0,1} = \log(\ell_1) - \log(s_2^1)\,,$$

which is non-negative since $s_2^1 \le \ell_1$.

Finally, if $2 \le m \le k \le p$, Corollary VI.10.10(b) gives

$$\operatorname{card}(S_{k+1}^m(X_G)) = \operatorname{card}(S_{k+1}^{m-1}(X_G)) \cdot 2^{\tau_{m-1} - \sum_{j=m}^{k-1} t_{m-1,j}} \cdot 2^{-t_{m-1,k}}\,, \qquad \text{and}$$

$$2^{\tau_{m-1} - \sum_{j=m}^{k-1} t_{m-1,j}} = \frac{\operatorname{card}(S_k^m(X_G))}{\operatorname{card}(S_k^{m-1}(X_G))}\,.$$

Thus,

$$\operatorname{card}(S_{k+1}^m(X_G)) = \operatorname{card}(S_{k+1}^{m-1}(X_G)) \cdot \frac{\operatorname{card}(S_k^m(X_G))}{\operatorname{card}(S_k^{m-1}(X_G))} \cdot 2^{-t_{m-1,k}}\,.$$

It suffices to set:

$$t_{m-1,k} = (\log(s_{k+1}^{m-1}) + \log(s_k^m)) - (\log(s_k^{m-1}) + \log(s_{k+1}^m))\,.$$

Lemma VI.10.11 shows that $t_{m-1,k} \ge 0$. This completes the proof of Theorem VI.10.1. □

VI.11 Quotients of fans

We shall now study the structure of congruences of RS-fans, giving a complete and explicit description of them. We shall prove that quotients of fans are always fans, and are transversally 2-regular. □

Proposition VI.11.1 *Let F be a RS-fan and let $\mathcal{H}$ be a proconstructible subset of X_F which is 3-closed (i.e., stable under product of any three of its elements). Then $\equiv_{\mathcal{H}}$ is a RS-congruence, the quotient $F/\mathcal{H}$ is a RS-fan, and the spectral spaces $X_{F/\mathcal{H}}$ and $\mathcal{H}$ are homeomorphic; in particular, the (Boolean) spaces $(X_{F/\mathcal{H}})_{\mathrm{con}}$ and $\mathcal{H}_{\mathrm{con}}$ are homeomorphic.*

Proof. Follows closely the proof of Theorem I.1.27; we shall use notation therein.

The quotient structure $F/\mathcal{H}$ is a ternary semigroup and $X_{F/\mathcal{H}} = \mathrm{Hom}_{\mathrm{TS}}(F/\mathcal{H}, \mathbf{3})$ is its set of (TS-) characters. The proof of item (3) in Theorem I.1.27 shows that, under our hypotheses on $\mathcal{H}$, the map $\theta : X_{F/\mathcal{H}} \longrightarrow X_F$ given by $\theta(g) = g \circ \pi$ $(g \in X_{F/\mathcal{H}})$ is a homeomorphism between the spectral spaces $X_{F/\mathcal{H}}$ and $\mathcal{H}$, as asserted.

According to equality (***) in the proof of Theorem I.1.27 (with U replaced by Z) we have $\theta^{-1}[Z(a) \cap \mathcal{H}] = Z(\pi(a))$ for $a \in F$. Since F verifies condition [Z] in VI.2.1, this equality implies that the zero-sets of elements of $F/\mathcal{H}$ are also totally ordered by inclusion; Corollary VI.2.3 implies, then, that $(X_{F/\mathcal{H}}, F/\mathcal{H})$ is a fan. □

Observe that *all* RS-congruences of a fan are obtained in the way given by the preceding Proposition:

Corollary VI.11.2 *Let F be a RS-fan and let $\equiv$ be a RS-congruence of F. Then:*

(a) $\equiv\; = \;\equiv_{\mathcal{H}}$ for some proconstructible and 3-closed set $\mathcal{H} \subseteq X_F$. Hence,

(b) $F/\!\equiv$ is a RS-fan.

(c) The correspondence $\mathcal{H} \longmapsto \equiv_{\mathcal{H}}$ establishes an inclusion-reversing bijection between proconstructible and 3-closed subsets of X_F and the set $Con(F)$ of RS-congruences of F.

Proof. (a) The set $\mathcal{H} = \mathcal{H}_{\equiv}$ is given by Proposition **??**. Items (b) and (c) follow, respectively, from Proposition VI.11.1 and Theorem I.1.27. $\square$

Since $F/\equiv$ is a fan, its representation relations are explicitly given by Theorem VI.2.1. We shall now suffer a bit more and, using this description, prove:

Theorem VI.11.3 *Let F be a RS-fan and let $\equiv$ be a RS-congruence of F. The quotient $F/\equiv$ is transversally 2-regular.*

Proof. With notation as in the preceding Corollary, $\mathcal{H}$ $(= \mathcal{H}_{\equiv})$ stands for the set of characters of F that determine the congruence $\equiv$ (whence, $F/\mathcal{H} = F/\equiv$). Given $a, b, c, d, x \in F$, we assume:

(*) $\qquad \pi(x) \in D^t_{F/\mathcal{H}}(\pi(a), \pi(b)) \cap D^t_{F/\mathcal{H}}(\pi(c), \pi(d)),$

and search for elements $a', b', c', d' \in F$ so that $a \equiv a', b \equiv b', c \equiv c', d \equiv d'$ (i.e., $\pi(a) = \pi(a'), \dots$) such that $D^t_F(a', b') \cap D^t_F(c', d') \neq \emptyset$.

Since $\{Z(y) \,|\, y \in F\}$ is totally ordered by inclusion, we assume without loss of generality throughout this proof that $Z(a) \subseteq Z(b)$ and $Z(c) \subseteq Z(d)$; in particular, $Z(\pi(a)) \subseteq Z(\pi(b))$ and $Z(\pi(c)) \subseteq Z(\pi(d))$, see proof of VI.11.1 or (***) in the proof of I.1.27. According to the values of $D^t_{F/\mathcal{H}}$ given by Theorem VI.2.1, assumption (*) gives rise to the following cases:

(1.a) $\quad Z(\pi(a)) \subset Z(\pi(b))$ and $\pi(x) = \pi(a)$, or

(1.b) $\quad Z(\pi(a)) = Z(\pi(b))$, $\pi(a)) \neq -\pi(b)$ and $\pi(x) \in \{\pi(a), \pi(b)\}$, or

(1.c) $\quad \pi(a)) = -\pi(b)$ and $\pi(x) = \pi(a)^2\, \pi(x)$ $(= \pi(b)^2\, \pi(x))$,

<u>and</u>

(2.a) $\quad Z(\pi(c)) \subset Z(\pi(d))$ and $\pi(x) = \pi(c)$, or

(2.b) $\quad Z(\pi(c)) = Z(\pi(d))$, $\pi(c)) \neq -\pi(d)$ and $\pi(x) \in \{\pi(c), \pi(d)\}$, or

(2.c) $\quad \pi(c)) = -\pi(d)$ and $\pi(x) = \pi(c)^2\, \pi(x)$ $(= \pi(d)^2\, \pi(x))$.

These relations lead to the examination of nine cases. Remark that

(†) $\;Z(\pi(v)) \subset Z(\pi(w)) \;\Rightarrow\; Z(v) \subset Z(w)$,

(††) $Z(\pi(v)) = Z(\pi(w)) \;\Rightarrow\; vw^2 \equiv v$.

[Proof of (††): Let $h \in \mathcal{H}$. If $h(v) = 0$, then $h(vw^2) = h(v) = 0$. The assumed equality of zero-sets entails $Z(v) \cap \mathcal{H} = Z(w) \cap \mathcal{H}$ (see (***), proof of I.1.27). Hence, if $h(v) \neq 0$, we get $h(w) \neq 0$, i.e., $h(w^2) = 1$, and $h(vw^2) = h(v)h(w^2) = h(v)$.]

I. (1.a) + (2.a). In this case we have $\pi(x) = \pi(a) = \pi(c)$ (i.e., $x \equiv a \equiv c$) and, by remark (†), $Z(a) \subset Z(b)$ and $Z(c) \subset Z(b)$. Since $Z(\pi(x)) = Z(\pi(a)) \subset Z(\pi(b))$, the same remark gives $Z(x) \subset Z(b)$; similarly, $Z(x) \subset Z(d)$. Then, $x \in D^t_F(x, b) \cap D^t_F(x, d)$. Setting $a' = x\,(\equiv a), b' = b, c' = x\,(\equiv c), d' = d$, we get $D^t_F(a', b') \cap D^t_F(c', d') \neq \emptyset$, as required.

II. (1.a) + (2.b). Here we consider 3 subcases.

(i) $Z(c) \subset Z(d)$. If $\pi(x) = \pi(c)$, we get $x \equiv a \equiv c$, and we conclude as in Case I. If $\pi(x) = \pi(d)$, we have $x \equiv a \equiv d$. Note that $\pi(x) = \pi(a) = \pi(d)$ gives $Z(\pi(a)) = Z(\pi(d)) = Z(\pi(c)) = Z(\pi(x)) \subset Z(\pi(b))$, and hence $Z(c) \subset Z(d) \subset Z(b)$; from (1.a), $Z(a) \subset Z(b)$. By (††) we have $cd^2 \equiv c$. Further, $\pi(c)) \neq -\pi(d)$ means $c \not\equiv -d$, yielding $cd^2 \equiv c \not\equiv -d$ (in particular, $cd^2 \neq -d$).

Since $Z(cd^2) = Z(c) \cup Z(d) = Z(d)$, from VI.2.1 (for F) we get $d \in D^t_F(cd^2, d)$. Similarly, from $Z(d) \subset Z(b)$ we get $d \in D^t_F(b, d)$. Thus, setting $a' = d, b' = b, c' = cd^2, d' = d$ we have $a \equiv a', c \equiv c'$, and $d \in D^t_F(a', b') \cap D^t_F(c', d')$, as required.

(ii) $Z(d) \subset Z(c)$. Argument similar to that of (i), interchanging c and d.

(iii) $Z(c) = Z(d)$. Since $c \neq -d$, we have $D^t_F(c, d) = \{c, d\}$. In both the cases $\pi(x) = \pi(c)$ and $\pi(x) = \pi(d)$ we proceed as in Case I.

III. (1.a) + (2.c). Here we have $\pi(x) = \pi(a)$, $\pi(c)) = -\pi(d)$ and $\pi(x) = \pi(c)^2\pi(x)$, i.e., $x \equiv a$, $c \equiv -d$ and $x \equiv c^2x$, whence $a \equiv c^2a$. Further, from (1.a) and (†) we have $Z(a) \subset Z(b)$. Since $D^t_F(c, -c) = \{y \in F \mid y = yc^2\}$ and $(c^2a)c^2 = c^2a$, we obtain $c^2a \in D^t_F(c, -c)$.

Next, we examine under what conditions $c^2a \in D^t_F(c^2a, b)$. By Theorem VI.2.1 this is clearly the case whenever $Z(c^2a) \subseteq Z(b)$. We prove next that $Z(b) \subset Z(c^2a) = Z(c) \cup Z(a)$ is impossible. Otherwise, using that assumption (1.a) implies $Z(a) \subset Z(b)$ we get $Z(b) \subset Z(c)$, which in turn implies $Z(\pi(b)) \subset Z(\pi(c))$. But, on the other hand, $\pi(x) = \pi(a) = \pi(c)^2\pi(a)$ yields $Z(\pi(c)) = Z(\pi(c^2)) \subseteq Z(\pi(c^2)\pi(a)) = Z(\pi(a)) \subset Z(\pi(b))$, contradiction.

Thus, $c^2a \in D^t_F(c^2a, b) \cap D^t_F(c, -c)$, and it suffices to set $a' = c^2a\,(\equiv a)$, $b' = b$, $c' = c$, $d' = -c\,(\equiv d)$.

IV. (1.b) + (2.a). Similar to Case II upon interchanging the roles of the pairs (a, b) and (c, d).

V. (1.b) + (2.b). Clearly, the first and last conditions in (1.b) and (2.b) imply:

$$Z(\pi(a)) = Z(\pi(b)) = Z(\pi(c)) = Z(\pi(d))\ (= Z(\pi(x))).$$

By (††) above this implies $yz^2 = y$ for all $y, z \in \{a, b, c, d\}$.

We invoke now our standing assumption that $Z(a) \subseteq Z(b)$ and $Z(c) \subseteq Z(d)$. The following cases ought to be considered:

(i) $\pi(x) = \pi(a) = \pi(d)$; (ii) $\pi(x) = \pi(c) = \pi(b)$; (iii) $\pi(x) = \pi(a) = \pi(c)$; and

(iv) $\pi(x) = \pi(b) = \pi(d)$.

(i) By assumption, $x \equiv a \equiv d$, and from $Z(a) \subseteq Z(b)$ comes $a \in D^t_F(a, b)$. From $Z(a) \subseteq Z(a) \cup Z(c) = Z(a^2c)$ follows $a \in D^t_F(a, a^2c)$. Since $a \equiv d$ and $a^2c \equiv c$, the desired conclusion is obtained by setting $a' = a$, $b' = b\,c' = a^2c$, $d' = a$.

(ii) Since $Z(c) \subseteq Z(d)$, this case is similar to (i) upon replacing a by c and d by b.

(iii) From $Z(c) \subseteq Z(d)$ comes $c \in D^t_F(c, d)$ and from $Z(c) \subseteq Z(c) \cup Z(b) = Z(c^2b)$ comes $c \in D^t_F(c, c^2b)$. Since $c^2b \equiv b$, $c \equiv a\,(\equiv x)$, setting $a' = c$, $b' = c^2b$, $c' = c$, $d' = d$, we are done.

(iv) $Z(b) \subseteq Z(b) \cup Z(c) = Z(b^2c)$ gives $b \in D^t_F(b^2c, b)$ and $Z(b) = Z(b) \cup Z(a) = Z(b^2a)$ entails $b \in D^t_F(b^2a, b)$. Hence, $a' = b^2a\,(\equiv a)$, $b' = b$, $c' = b^2c\,(\equiv c)$ and $d' = b\,(\equiv d)$, satisfy the required conditions.

VI. (1.b) + (2.c). Suppose, e.g., that in (1.b) we have $\pi(x) = \pi(a)$, i.e., $x \equiv a$. We compare the zero-sets of c^2a and b. If $Z(c^2a) \subseteq Z(b)$, we have $c^2a \in D^t_F(c^2a, b)$; we also have $c^2a \in D^t_F(c, -c)$, and we are done upon setting $a' = c^2a\,(\equiv a)$, $b' = b$, $c' = c$, $d' = -c\,(\equiv d)$ (by (2.c), $x \equiv a$ and $x \equiv c^2x$ imply $a \equiv c^2a$).

So, it only remains to examine the case $Z(b) \subset Z(c^2a) = Z(c) \cup Z(a)$. In this case, $Z(c^2b) = Z(c) \cup Z(b) \subseteq Z(c) \cup Z(a) = Z(c^2a)$, which yields $c^2b \in D^t_F(c^2a, c^2b)$. Since we also have $c^2b \in D^t_F(c, -c)$ (as $(c^2b)c^2 = c^2b$), we would be done by setting $a' = c^2a\,(\equiv a)$, $c' = c$, $d' = -c\,(\equiv d)$ and $b' = c^2b$, provided we show $c^2b \equiv b$. This is checked by evaluating characters: let $h \in \mathcal{H}$; if $h(b) = 0$, we have $h(c^2b) = h(b) = 0$. Let $h(b) \neq 0$; from the assumption of (1.b) and (††) we get $a^2b \equiv b$, whence $h(a) \neq 0$; as observed in the preceding paragraph, $a \equiv c^2a$, which gives, $h(c^2) = 1$; thus, $h(c^2b) = h(c^2)h(b) = h(b)$.

The case $\pi(x) = \pi(b)$ is similar.

VII. (1.c) + (2.a). Symmetric to Case III.

VIII. (1.c) + (2.b). Symmetric to Case VI.

IX. (1.c) + (2.c). Since $0 \in D^t_F(a, -a) \cap D^t_F(c, -c)$ and by assumption $a \equiv -b$, $c \equiv -d$, it suffices to set $a' = a$, $b' = -a$, $c' = c$, $d' = -c$. $\square$

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