Advanced Tense Logics

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1 The Expressive Power of Temporal Connectives

The expressivity of a language is always measured with respect to some other language. That is, when talking about expressivity, we are always comparing two or more languages. When measuring the expressivity of a large number of languages, it is usually more convenient to have a single language with respect to which all other languages can be compared, if such a language is known to exist.

In the case of propositional one-dimensional temporal languages defined by the presence of a fixed number of *temporal connectives* (also called *temporal modalities*), the expressivity of those languages can be all measured against a fragment of first-order logic, namely the *monadic first-order language*. This is the fragment that contains a binary $<$ (to represent the underlying temporal order), $=$ (which we assume is always in the language) and a set of unary predicates $Q_1(x), Q_2(x), \ldots$ (which account for the interpretation of the propositional letters, that are interpreted as a subset of the temporal domain T). Indeed, any one-dimensional temporal connective can be defined as a well-formed formula in such a fragment, known as the connective's *truth table*; one-dimensionality forces such truth tables to have a single free variable.

In the case of comparing the expressivity of temporal connectives, another parameter must be taken into account, namely the underlying flow of time. Two temporal languages may have the same expressivity over one flow of time (say, the integers) but may differ in expressivity over another (e.g. the rationals); see the discussion on the expressivity of the US connectives below.

Let us exemplify what we mean by those terms. Consider the connectives *since*(S), *until*(U), *future*(F), and *past*(P). Given a flow of time $(T, <, h)$, the truth value of each of the above connectives at a point $t \in T$ is determined as follows:

$$
\begin{array}{lll}
(T,<,h),t \models Fp & \text{iff} & (\exists s > t)(T,<,h),s \models p, \\
(T,<,h),t \models Pp & \text{iff} & (\exists s < t)(T,<,h),s \models p, \\
(T,<,h),t \models U(p,q) & \text{iff} & (\exists s > t)((T,<,h),s \models p \wedge \\
 & & \forall y(t < y < s \to (T,<,h),y \models q)), \\
(T,<,h),t \models S(p,q) & \text{iff} & (\exists s < t)((T,<,h),s \models p \wedge \\
 & & \forall y(s < y < t \to (T,<,h),y \models q))
\end{array}
$$

If we assume that $h(p)$ represents a first-order unary predicate that is interpreted as $h(p) \subseteq T$, then these truth values above can be expressed as first-order formulas. Thus:

(a) $(T,<,h),t \models Fq$ iff $\chi_F(t,h(q))$ holds in $(T,<)$,

(b) $(T,<,h),t \models Pq$ iff $\chi_P(t,h(q))$ holds in $(T,<)$,

(c) $(T,<,h),t \models U(q_1,q_2)$ iff $\chi_U(t,h(q_1),h(q_2))$ holds in $(T,<)$, and

(d) $(T,<,h),t \models S(q_1,q_2)$ iff $\chi_S(t,h(q_1),h(q_2))$ holds in $(T,<)$.

where

(a) $\chi_F(t,Q) = (\exists s > t)Q(s)$,

(b) $\chi_P(t,Q) = (\exists s < t)Q(s)$,

(c) $\chi_U(t,Q_1,Q_2) = (\exists s > t)(Q_1(s) \wedge \forall y(t < y < s \rightarrow Q_2(y)))$,

(d) $\chi_S(t,Q_1,Q_2) = (\exists s < t)(Q_1(s) \wedge \forall y(s < y < t \rightarrow Q_2(y)))$.

$\chi_{\#}(t,Q_1,\ldots,Q_n)$ is called the *truth table* for the connective $\#$. The number n of parameters in the truth table will be the number of places in the connective, e.g. F and P are one place connective, and their truth tables have a single parameter; S and U are two-place connectives, with truth tables having two parameters.

It is clear that in such a way, we start defining any number of connectives. For example consider $\chi(t,Q) = \exists xy(t < x < y \wedge \forall s(x < s < y \rightarrow Q(s)))$; then $\chi(t,Q)$ means 'There is an interval in the future of t inside which P is true.' This is a table for a connective F_{int}: $(T,<,h),t \models F_{\text{int}}(p)$ iff $\chi(t,h(p))$ holds in $(T,<)$.

We are in condition of presenting a general definition of what a temporal connective is:

Definition 1.1

1. Any formula $\chi(t,Q_1,...,Q_m)$ with one free variable t, in the monadic first-order language with predicate variable symbols Q_i, is called an m-place truth table (in one dimension).

2. Given a syntactic symbol $\#$ for an m-place connective, we say it has a truth table $\chi(t,Q_1,...,Q_m)$ iff for any T,h and t, $(*)$ holds:

$$(*): \ (T,<,h),t \models \#(q_1,...,q_m) \text{ iff } (T,<) \models \chi(t,h(q_1),...,h(q_m)).$$

This way we can define as many connectives as we want. Usually, some connectives are definable using other connectives. For example, it is well known that F is definable using U as $Fp \equiv U(p, \top)$. As another example, consider a connective that states the existence of a "next" time point: $\delta \equiv U(\top, \bot)$.

The connective δ is a nice example on how the definability of a connective by others depends on the class of flows of time being considered. For example, in a dense flow of time, δ can be defined in terms of F and P — actually, since there are no "next" time points anywhere, $\delta \equiv \bot$. Similarly, in an integer-like flow of time, δ is equivalent to $\top$.

On the other hand, consider the flow $(T, <)$ of time with a single point without a "next time": $T = \{... -2, -1, 0, 1, 2, ...\} \cup \{(1/n) \mid n = 1, 2, 3...\}$, with $<$ being the usual order; then δ is not definable using P and F. To see that, suppose for contradiction that δ is equivalent to A where A is written with P and F and, maybe, atoms. Replace all appearances of atoms by $\bot$ to obtain A'. Since $\delta \leftrightarrow A$ holds in the structure $(T, <, h')$ with all atoms always false, in this structure $\delta \leftrightarrow A'$ holds. As neither δ nor A' contain atoms, $\delta \leftrightarrow A'$ holds in all other $(T, <, h)$ as well. Now A' contains only P and F, $\top$, and $\bot$ and the classical connectives. Since $F\top \equiv P\top \equiv \top$ and $F\bot \equiv P\bot \equiv \bot$, at every point, A' must be equivalent (in $(T, <)$) to either $\top$ or $\bot$ and so cannot equal δ which is true at 1 and false at 0. As a consequence, δ is not definable using P and F over linear time.

In general, given a family of connectives, e.g. $\{F, P\}$ or $\{U, S\}$, we can build new connectives using the given ones. That these new connectives are connectives in the sense of Definition 1 follows from the following.

Lemma 1.2 *Let $\#_1(q_1, ..., q_{m_1}), ..., \#_n(q_1, ..., q_{m_n})$ be n temporal connectives with tables $\chi_1, ..., \chi_n$. Let A be any formula built up from atoms $q_1, ..., q_m$, the classical connectives, and these connectives. Then there exists a monadic $\psi_A(t, Q_1, ..., Q_m)$ such that for all T and h,*

$$(T, <, h), t \models A \text{ iff } (T, <) \models \psi_A(t, h(q_1), ..., h(q_m)).$$

Proof. We construct ψ_A by induction on A. The simple cases are: $\psi_{q_j} = Q_j(t)$, $\psi_{\neg A} = \neg\psi_A$ and $\psi_{A \wedge B} = \psi_A \wedge \psi_B$.

For the temporal connective case, we construct the formula $\psi_{\#_i(A_1, ..., A_{m_i})} = \chi_i(t, \psi_{A_1}, ..., \psi_{A_{m_i}})$; the right-hand side is a notation for the formula obtained by substituting $\psi_{A_j}(x)$ in χ_i wherever $Q_j(x)$ appears, with the appropriate renaming of bound variables to avoid clashes. The induction hypothesis is applied over $\psi_{A_1}, ..., \psi_{A_{m_i}}$ and the result is simply obtained by truth table of the connective $\#_i$. ■

The formula ψ_A built above is called the *first-order translation* of a temporal formula A. An m-palce connective $\#$ with truth table $\chi(t, Q_1, ..., Q_m)$ is said to be *definable* from connectives $\#_1, \ldots, \#_n$ in a flow of time $(T, <)$ if there exists a temporal formula A built from those connectives whose first order translation is ψ_A such that

$$(T, <) \models \psi_A \leftrightarrow \chi.$$

The *expressive power* of a family of connectives over a flow of time is measured by how many connectives it can express over the flow of time. If it can express any conceivable connective (given by a monadic formula), then that family of connectives is expressively complete.

Definition 1.3 A temporal language with one-dimensional connectives is said to be *expressively complete* or, equivalently, *functionally complete*, in one dimension over a class $\mathcal{T}$ of partial orders iff for any monadic formula $\psi(t, Q_1, ..., Q_m)$, there exists an A of the language such that for any $(T, <)$ in $\mathcal{T}$, for any interpretation h for $q_1, ..., q_m$,

$$(T, <) \models \forall t(\psi \leftrightarrow \psi_A)(t, h(q_1), ..., h(q_m)).$$

In the cases where $\mathcal{T} = \{(T, <)\}$ we talk of expressive completeness over $(T, <)$. For example, the language of Since and Until is expressively complete over integer time and real number flow of time, as we are going to see in Section 1.2; but they are not expressively complete over rational numbers time [GPSS80].

Definition 1.4 A flow of time $(T, <)$ is said to be expressively complete (or functionally complete) (in one dimension) iff there exists a *finite* set of (one-dimensional) connectives which is expressively complete over $(T, <)$, in one dimension.

The qualification of one-dimensionality in the definitions above will be explained when we introduce the notion of H-dimension below.

These notions parallel the definability and expressive completeness of classical logic. We know that in classical logic $\{\neg, \rightarrow\}$ is sufficient to define all other connectives. Furthermore, for any n-place truth table $\psi : 2^n \rightarrow 2$ there exists an $A(q_1, ..., q_n)$ of classical logic such that for any h,

$$h(A) = \psi(h(q_1), ..., h(q_n)).$$

This is the expressive completeness of $\{\neg, \rightarrow\}$ in classical logic.

The notion of expressive completeness leads us to formulate two questions:

(a) Given a finite set of connectives and a class of flows of time, are these connectives expressively complete?

(b) In case the answer to (a) is no, we would like to ask: given a class of flows, does there exist a finite set of one-dimensional connectives that is expressively complete?

These questions occupy us to the rest of this section. We show that the notion of expressive completeness is intimately related to the *separation property*, which we investigate in Section.

The answer to question (b) is related to the notion of H-dimension, discussed in Section 1.3.

1.1 Separation and Expressive Completeness

The notion of separation involves partitioning a flow of time in disjoint parts (typically: present, past and future). A formula is separable if it is equivalent to another formula whose temporal connectives refer only to one of the partitions.

If every formula in a language is separable, that means that we have at least one connective that has enough expressivity over each of the partitions. So we might expect that that set of connectives is expressively complete over a class of flows that admits such partitioning, provided the partitioning is also expressible by the connectives.

The notion of separation was initially analysed in terms of linear flows, where the notion of present, past and future most naturally applies. So we start our discussion with separation over linear time. We later extend separation to generic flows.

1.1.1 Separation over linear time

Consider a linear flow of time $(T, <)$. Let h, h' be two assignments and $t \in T$. We say that h, h' *agree on the past of* t, $h =_{<t} h'$, iff for any atom q and any $s < t$,

$$s \in h(q) \text{ iff } s \in h'(q).$$

We define $h' =_{=t} h$ for *agreement of the present*, iff for any atom q

$$t \in h(q) \text{ iff } t \in h'(q).$$

and $h' =_{>t} h$, for *agreement on the future*, iff for any atom q and any $s > t$,

$$s \in h(q) \text{ iff } s \in h'(q).$$

Let $\mathcal{T}$ be a class of linear flows of time and A be a formula in a temporal language over $(T, <)$. We say that A is a *pure past formula over* $\mathcal{T}$, iff for all $(T, <)$ in $\mathcal{T}$, for all $t \in T$,

$$\forall h, h', (h =_{<t} h') \text{ implies that } (T, <, h), t \models A \text{ iff } (T, <, h'), t \models A.$$

Similarly, we define *pure future* and *pure present* formulas.

Such a definition of purity is a semantic one. In a temporal language containing only S and U there is also have a notion of syntactic purity as follows. A formula is a *boolean combination* of $\phi_1, \ldots, \phi_n$ if it is built from $\phi_1, \ldots, \phi_n$ using only boolean connectives. A *syntactically pure present* formula is a boolean combination of atoms only. A *syntactically pure past* formula is a boolean combination of formulas of the form $S(A, B)$ where A and B are either pure present or pure past. Similarly, a *syntactically pure future* formula is a boolean combination of formulas of the form $U(A, B)$ where A and B are either pure present or pure future.

It is clear that if A is a syntactically pure past formula, then A is a pure past formula; similarly for pure present and pure future formulas. The converse, however, is not true. For example, from the semantical definition, all temporal temporally valid formulas are pure future (and pure past, and pure present), including those involving S.

We are now in a position to define the separation property.

Definition 1.5 Let $\mathcal{T}$ be a class of linear flows of time and A be a formula in a temporal language $\mathbf{L}$. We say A is *separable* in $\mathbf{L}$ over $\mathcal{T}$ iff there exists a formula in $\mathbf{L}$ which is a boolean combination of pure past, pure future, and atomic formulas and is equivalent to A everywhere in any $(T, <)$ from $\mathcal{T}$.

A set of temporal connectives is said to have the separation property over $\mathcal{T}$ iff every formula in the temporal language of these connectives is separable in the language (over $\mathcal{T}$).

We now show that separation implies expressive completeness.

Theorem 1.6 *Let* $\mathbf{L}$ *be a temporal language built from any number (finite or infinite) of connectives in which* P *and* F *are definable over a class* $\mathcal{T}$ *of linear flows of time. If* $\mathbf{L}$ *has the separation property over* $\mathcal{T}$ *then* $\mathbf{L}$ *is expressively complete over* $\mathcal{T}$.

Proof. If $\mathcal{T}$ is empty, **L** is trivially expressively complete, so suppose not. We have to show that for any $\varphi(t,\overline{Q})$ in the monadic theory of linear order with predicate variable symbols $\overline{Q} = (Q_1,...,Q_n)$, there exists a formula $A = A(q_1,...,q_n)$ in the temporal language such that for all flows of time $(T,<)$ from $\mathcal{T}$, for all h,t, $(T,<,h),t \models A$ iff $(T,<) \models \varphi(t,h(q_1),...,h(q_n))$.

We denote this formula by $A[\varphi]$ and proceed by induction on the depth m of nested quantifiers in φ. For $m=0$, $\varphi(t)$ is quantifier free. Just replace each appearance of $t=t$ by $\top$, $t<t$ by $\bot$, and each $Q_j(t)$ by q_j to obtain $A[\varphi]$.

For $m>0$, we can assume $\varphi = \exists x\psi(t,x,\overline{Q})$ where ψ has quantifier depth $\leq m$ (the $\forall$ quantifier is treated as derived).

Assuming that we do not use t as a bound variable symbol in ψ and that we have replaced all appearances of $t=t$ by $\top$ and $t<t$ by $\bot$ then the atomic formulas in ψ which involve t have one of the following forms: $Q_i(t)$, $t<y$, $t=y$, or $y<t$, where y could be x or any other variable letter occurring in ψ.

If we regard t as fixed, the relations $t<y, t=y, t>y$ become unary and can rewritten, respectively, as $R_<(y)$, $R_=(y)$ and $R_>(y)$, where $R_<$, $R_=$ and $R_>$ are new unary predicate symbols.

Then ψ can be rewritten equivalently as

$$\psi_0^t(x,\overline{Q},R_=,R_>,R_<),$$

in which t appears only in the form $Q_i(t)$. Since t is free in ψ, we can go further and prove (by induction on the quantifier depth of ψ) that ψ_0^t can be equivalently rewritten as

$$\psi^t = \bigvee_j [\alpha_j(t) \wedge \psi_j^t(x,\overline{Q},R_=,R_>,R_<)],$$

where

- $\alpha_j(t)$ is quantifier free,
- $Q_i(t)$ appear only in $\alpha_j(t)$ and not at all in ψ_j^t,
- and each ψ_j^t has quantifier depth $\leq m$.

By the induction hypothesis, there is a formula $A_j = A_j(\overline{q},r_=,r_>,r_<)$ in the temporal language such that, for any h,x,

$$(T,<,h),x \models A_j \text{ iff } (T,<) \models \psi_j^t(x,h(q_1),...,h(q_n),h(r_=),h(r_>),h(r_<)).$$

Now let $\Diamond q$ be an abbreviation for a temporal formula equivalent (over $\mathcal{T}$) to $Pq \vee q \vee Fq$ whose existence in $\mathbf{L}$ is guaranteed by hypothesis. Then let $B(\overline{q}, r_=, r_>, r_<) = \bigvee_j (A[\alpha_j] \wedge \Diamond A_j)$. $A[\alpha_j]$ can be obtained from the quantifier free case.

In any structure $(T, <)$ from $\mathcal{T}$ for any h interpreting the atoms $\overline{q}$, $r_=, r_>$ and $r_<$, the following are straightforward equivalences

$$
\begin{gathered}
(T, <, h), t \models B \\
(T, <, h), t \models \bigvee_j (A[\alpha_j] \wedge \Diamond A_j) \\
\bigvee_j ((T, <, h), t \models A[\alpha_j] \wedge (T, <, h), t \models \Diamond A_j) \\
\bigvee_j (\alpha_j(t) \wedge \exists x((T, <, h), x \models A_j)) \\
\bigvee_j (\alpha_j(t) \wedge \exists x \psi_j^t(x, h(q_1), ..., h(q_n), h(r_=), h(r_>), h(r_<))) \\
\exists x \bigvee_j (\alpha_j(t) \wedge \psi_j^t(x, h(q_1), ..., h(q_n), h(r_=), h(r_>), h(r_<))) \\
\exists x \psi_0^t(x, h(q_1), ..., h(q_n), h(r_=), h(r_>), h(r_<)).
\end{gathered}
$$

Now provided we interpret the r atoms as the appropriate R predicates, i.e.:

- $h^*(r_=) = \{t\}$,
- $h^*(r_<) = \{s \mid t < s\}$, and
- $h^*(r_>) = \{s \mid s < t\}$,

we obtain

$$(T, <, h^*), t \models B \text{ iff } \exists x \psi(t, x, h^*(q_1), ..., h^*(q_n)) \text{ iff } \varphi(t, h^*(q_1), ..., h^*(q_n)).$$

B is almost the $A[\varphi]$ we need except for one problem. B contains, besides the q_i, also three other atoms, $r_=, r_>$, and $r_<$, and equation $(*)$ from Definition 9.1.1 above is valid for any h^* which is arbitrary on the q_i but very special on $r_=, r_>, r_<$. We are now ready to use the separation property (which we haven't used so far in the proof). We use separation to eliminate $r_=, r_>, r_<$ from B. Since we have separation B is equivalent to a boolean combination of atoms, pure past formulas, and pure future formulas.

So there is a boolean combination $\beta = \beta(\overline{p}_+, \overline{p}_-, p_0)$ such that

$$B \leftrightarrow \quad \beta(\overline{B}_+, \overline{B}_-, B_0),$$

where $B_0(\overline{q}, r_>, r_=, r_<)$ is a combination of atoms, $B_+(\overline{q}, r_>, r_=, r_<)$ are pure future, and $B_-(\overline{q}, r_>, r_=, r_<)$ are pure past formulas.

Finally, $B^* = \beta(B_+^*, B_-^*, B_0^*)$ where

- $B_0^* = B_0(\overline{q}, \bot, \top, \bot)$;
- $B_+^* = B_+(\overline{q}, \top, \bot, \bot)$;
- $B_-^* = B_-(\overline{q}, \bot, \bot, \top)$.

Then we obtain for any h^*,

$$\begin{aligned}(T, <, h^*), t \models B \quad & \text{iff} (T, <, h^*), t \models \beta(B_+, B_-, B_0) \\ & \text{iff} (T, <, h^*), t \models \beta(B_+^*, B_-^*, B_0^*) \\ & \text{iff} (T, <, h^*), t \models B^*.\end{aligned}$$

Hence

$$(T, <, h^*), t \models B^* \quad \text{iff} \quad (T, <) \models \varphi(t, h^*(\overline{q})).$$

This equation holds for any h^* arbitrary on $\overline{q}$, but restricted on $r_<, r_>, r_=$. But $r_<, r_>, r_=$ do not appear in it at all and hence we obtain that for any h, $(T, <, h), t \models B^*$ iff $(T, <) \models \varphi(t, h^*(\overline{q}))$. So make $A[\varphi] = B^*$ and we are done. ■

The converse is also true: expressive completeness implies separation over linear time. The proof involves using the first-order theory of linear time to first separate a first-order formula over linear time; a temporal formula is translated into the first-order language, where it is separated; expressive completeness is needed then to translate each separated first-order subformula into a temporal formula. Details are omitted, but can be found in [GHR94].

1.1.2 Generalized Separation

The separation property is not restricted to linear flows of time. In this section we generalize the separation property over any class of flows of time and see that Theorem 1.6 has a generalised version.

The basic idea is to have some relations that will partition every flow of time in $\mathcal{T}$, playing the role of $<$, $>$ and $=$ in the linear case.

Definition 1.7 Let $\varphi_i(x, y), i = 1, ..., n$ be n given formulas in the monadic language with $<$ and let $\mathcal{T}$ be a class of flows of time. Suppose $\varphi_i(x, y)$ partition $\mathcal{T}$, that is, for every t in each $(T, <)$ in $\mathcal{T}$ the sets $T(i, t) = \{s \in T \mid \varphi_i(s, t)\}$ for $i = 1, ..., n$ are mutually exclusive and $\bigcup_i T(i, t) = T$.

In analogy to the way that F and P represented $<$ and $>$, we assume that for each i there is a formula $\beta_i(t, x)$ such that $\varphi_i(t, x)$ and $\beta_i(t, x)$ are equivalent over $\mathcal{T}$ and β_i is a boolean combination of some $\varphi_j(x, t)$. Also assume that $<$ and $=$ can be expressed (over $\mathcal{T}$) as boolean combinations of the φ_i.

Then we have the following series of definitions:

- For any t from any $(T, <)$ in $\mathcal{T}$, for any $i = 1, ..., n$, we say that truth functions h and h' agree on $T(i, t)$ if and only if $h(q)(s) = h'(q)(s)$ for all s in $T(i, t)$ and all atoms q.

- We say that a formula A is *pure* φ_i over $\mathcal{T}$ if for any $(T, <)$ in $\mathcal{T}$, any $t \in T$ and any two truth functions h and h' which agree on $T(i, t)$, we have
$$(T, <, h), t \models A \text{ iff } (T, <, h'), t \models A.$$

- The logic $\mathbf{L}$ has the *generalized separation property* over $\mathcal{T}$ iff every formula A of $\mathbf{L}$ is equivalent over $\mathcal{T}$ to a boolean combination of pure formula.

Theorem 1.8 (generalized separation theorem) *Suppose the language* $\mathbf{L}$ *can express over* $\mathcal{T}$ *the 1-place connectives* $\#_i$*,* $i = 1, ..., n$*, defined by:*

$$\begin{array}{lll} (T, <, h), t \models \#_i(p) & \textit{iff} & \exists s \quad \varphi_i(s, t) \textit{ holds in } (T, <) \\ & & \quad \textit{and } (T, <, h), s \models p. \end{array}$$

If has the generalized separation property over a class $\mathcal{T}$ *of flows of time then* $\mathbf{L}$ *is expressively complete over* $\mathcal{T}$*.*

A proof of this result appears in [Ami85]. See also [GHR94].

The converse does not always hold in the general case, for it depends on the theory of the underlying class $\mathcal{T}$.

A simple application of the generalised separation theorem is the following. Suppose we have a first order language with the binary order predicates $<, >, =$ with their usual interpretation, and suppose it also contains a *parallel* operator $|$ defined by:

$$x|y =_{def} \neg[(x = y) \vee (x < y) \vee (y < x)].$$

Suppose we have a new temporal connective D, defined by

$$(T, <, h), t \models Dq \text{ iff } \exists x|t\text{suchthat}(T, <, h), x \models q.$$

Finally, A is said to be *pure parallel* over a class $\mathcal{T}$ of flows of time iff for all t from any $(T, <)$ from $\mathcal{T}$, for all $h =_{|t} h'$,

$$(T, <, h), t \models A\text{iff}(T, <, h'), t \models A,$$

where $h =_{|t} h'$ iff $\forall x|t \forall q(x \in h(q) \leftrightarrow x \in h'(q))$.

It is clear what separation means in the context of pure present, past, future, and parallel. It is simple to check that the $<, >, =, |$ satisfy the general separation property and other preconditions for using the generalized separation theorem. Thus that theorem gives immediately the following.

Corollary 1.9 *Let* $\mathbf{L}$ *be a language with* F, P, D *over any class of flows of time. If* $\mathbf{L}$ *has a separation then* $\mathbf{L}$ *is expressively complete.*

1.2 Expressive Completeness of Since and Until over Integer Time

As an example of the applications of separation to the expressive completeness of temporal language, we are going to sketch the proof of separation of the Since and Until-temporal logic containing over linear time. The full proof can be found in [Gab89, GHR94]. With separation and Theorem 1.6 we immediately obtain that the connectives S and U are expressively complete over the integers; the original proof of the expressive completeness of S and U over the integers is due to Kamp [Kam68].

The basic idea of the separation process is to start with a formula in which S and U may be nested inside each other and through several transformation steps we are going to systematically remove U from inside S and vice-versa. This gives us a syntactical separation which, obviously, implies separation.

As we shall see there are eight cases of nested occurrences of U within an S to worry about. It should be noted that all the results in the rest of this section have dual results for the *mirror images* of the formulas. The *mirror image* of a formula is the formula obtained by interchanging U and S; for example, the mirror image of $U(p \wedge S(q, r), u)$ is $S(p \wedge U(q, r), u)$.

We start dealing with boolean connectives inside the scope of temporal operators, with some equivalences over integer flows of time. We say that a formula A is *valid* over a flow of time $(T, <)$ if it is true at all $t \in T$; notation: $(T, <) \models A$

Lemma 1.10 *The following formulas (and their mirror images) are valid over integer time:*

- $U(A \vee B, C) \leftrightarrow U(A, C) \vee U(B, C)$;
- $U(A, B \wedge C) \leftrightarrow U(A, B) \wedge U(A, C)$;
- $\neg U(A, B) \leftrightarrow G(\neg A) \vee U(\neg A \wedge \neg B, \neg A)$;

- $\neg U(A,B) \leftrightarrow G(\neg A) \vee U(\neg A \wedge \neg B, B \wedge \neg A)$.

Proof. Simply apply the semantical definitions. ■

We now show the eight separation cases involving simple nesting and atomic formulas only.

Lemma 1.11 *Let p, q, A, and B be atoms. Then each of the formulas below is equivalent, over integer time, to another formula in which the only appearances of the until connective are as the formula $U(A,B)$ and no appearance of that formula is in the scope of an S:*

1. $S(p \wedge U(A,B), q)$,
2. $S(p \wedge \neg U(A,B), q)$,
3. $S(p, q \vee U(A,B))$,
4. $S(p, q \vee \neg U(A,B))$,
5. $S(p \wedge U(A,B), q \vee U(A,B))$,
6. $S(p \wedge \neg U(A,B), q \vee U(A,B))$,
7. $S(p \wedge U(A,B), q \vee \neg U(A,B))$, *and*
8. $S(p \wedge \neg U(A,B), q \vee \neg U(A,B))$.

Proof. We prove the first case only; omitting the others. Note that $S(p \wedge U(A,B), q)$ is equivalent to

$$\begin{array}{cc} & S(p,q) \wedge S(p,B) \wedge B \wedge U(A,B) \\ \vee & [A \wedge S(p,B) \wedge S(p,q)] \\ \vee & S(A \wedge q \wedge S(p,B) \wedge S(p,q), q). \end{array}$$

Indeed, the original formula holds at t iff there is $s < t$ and $u > s$ such that p holds at s, A at u, B everywhere between s and u, and q everywhere between s and t. The three disjuncts correspond to the cases $u > t$,$u = t$, and $u < t$ respectively. Note that we make essential use of the linearity of time. ■

We now know the basic steps in our proof of separation. We simply keep pulling out Us from under the scopes of Ss and vice versa until there are no more. Given a formula A, this process will eventually leave us with a *syntactically separated* formula, i.e. a formula B which is a boolean combination of atoms, formulas $U(E, F)$ with E and F built without using S and formulas $S(E, F)$ with E and F built without using U. Clearly, such a B is separated.

We start dealing with more than one U inside an S. In this context, we call a formula in which U and S do not appear *pure*.

Lemma 1.12 *Suppose that A and B are pure formulas and that C and D are such that any appearance of U is as $U(A, B)$ and is not nested under any Ss. Then $S(C, D)$ is equivalent to a syntactically separated formula in which U only appears as the formula $U(A, B)$.*

Proof. If $U(A, B)$ does not appear then we are done. Otherwise, by rearrangement of C and D into disjunctive and conjunctive normal form, respectively, and repeated use of Lemma 1.10 we can rewrite $S(C, D)$ equivalently as a boolean combination of formulas $S(C_1, D_1)$ with no U appearing. Then the preceding lemma shows that each such boolean constituent is equivalent to a boolean combination of separated formulas. Thus we have a separated equivalent. ■

Next let us begin the inductive process of removing Us from more than one S. We present the separation in a crescendo. Each step introduces extra complexity in the formula being separated and uses the previous case as a starting point.

Lemma 1.13 *Suppose that A, B, possibly subscripted, are pure formulas. Suppose C, D, possibly subscripted, contain no S. Then E has a syntactically separated equivalent if:*

- *the only appearance of U in E is as $U(A, B)$;*
- *the only appearances of U in E are as $U(A_i, B_i)$;*
- *the only appearances of U in E are as $U(C_i, D_i)$;*
- *E is any U, S formula.*

We omit the proof, referring to [GHR94, Chapter 10] for a detailed account. But note that since each case above uses the previous one as an induction basis, this process of separation tends to be highly exponential. Indeed, the separated version of a formula can be many times larger than the initial one. We finally have the main results.

Theorem 1.14 (separation theorem) *Over the integer flow of time, any formula in the $\{U, S\}$-language is equivalent to a separated formula.*

Proof. This follows directly from the preceding lemma because, as we have already noted, syntactic separation implies separation. ■

Theorem 1.15 *The language $\{U, S\}$ is expressively complete over integer time.*

Proof. This follows from the separation theorem and Theorem 1.6. ■

Other known separation and expressive completeness results over linear time are [GHR94]:

- The language $\{U, S\}$ is separable over real time. Indeed, it is separable over any Dedekind complete linear flow of time. As a consequence, it is also expressively complete over such flows.
- The language $\{U, S\}$ is *not* separable over the rationals; as a result, it is not separable over the class of linear flows of time, nor is it expressively complete over such flows.

The problem of $\{U, S\}$ over generic linear flows of time is that they may contain *gaps*. It is possible to define a first order formula that makes a proposition true up until a gap and false afterwards. Such formula, however, cannot be expressed in terms of $\{U, S\}$. So is there an extra set of connectives that is expressively complete over the rationals? The answer in this case is yes, and they are called the Stavi connectives. These are connectives whose truth value depends on the existence of gaps in the flow of time, and therefore are always false over integers or reals. For a detailed discussion on separation in the presence of gaps, please refer to [GHR94, Chapters 11 and 12].

We remain with the following generic question: given a flow of time, can we find a set of connectives that is expressively complete over it? This is the question that we investigate next.

1.3 H-dimension

The notion of *Henkin-* or H-dimension involves limiting the number of bound variables employed in first-order formulas. We will see that a *necessary* condition for there to exist a finite set of connectives which is expressively complete over a flow of time is that such flow of time have a finite H-dimension.

As for a *sufficient* condition for a finite expressively complete set of connectives, we will see that if *many-dimensional connectives* are allowed, than finite H-dimension implies the existence of such finite set of connectives. However, when we consider *one-dimensional connectives* such as Since and Until, finite H-dimension is no longer a sufficient condition.

In fact our approach in this discussion will be based on a *weak many-dimensional logic*. It is *many dimensional* because the truth value of a formula is evaluates at more than one time-point. It is *weak* because atomic formulas are evaluated only at a single time point (called the *evaluation point*), while all the other points are the *reference points*). Such weak many dimensionality allows us to define the truth table of many dimensional systems as formulas in the monadic first-order language, as opposed to a full m-dimensional system (in which atoms are evaluated at m time points) which would require an m-adic language.

An *m-dimensional table for an n-place connective* is a formula of the form $\chi(x_1, \ldots, x_m; R_1, \ldots, R_n)$, where χ is a formula of the first-order predicate language, written with symbols from $\{<\} \cup \{R_1, \ldots, R_n\}$, where $R_1, \ldots, R_n$ are special m-place relation symbols. Without loss of expressivity, we will further assume that each term y_j ocurring in $R_i(y_1, \ldots, y_m)$ is a always a variable.

Fix a temporal system $\mathcal{T}$ whose language contains atoms $q_1, q_2, \ldots$, the classical connectives, and the special symbols $\#_1, \ldots, \#_j$, standing for n_1-,$\ldots, n_j$-place connectives respectively. Let $\chi_1, \ldots, \chi_j$ be their m-dimensional n_1-,$\ldots, n_j$-place tables respectively.

Remark 1.16 Since there are finitely many χ_i to consider, we can further assume that there is $b \geq m$ such that each χ_i is written with variables $x_1, \ldots, x_b$ only.

The semantics of m-dimensional formulas is given by:

Definition 1.17 Let $(T, <)$ be a flow of time. Let h be an assignment into T, i.e. for any atom q, $h(q) \subseteq T$. We define the truth value of each formula A of the language of $\mathcal{T}$ at m indices $a_1, \ldots, a_{m-1}, t \in T$ under h, as follows:

1. $(T, <, h), a_1, \ldots, a_{m-1}, t \models q$ iff $t \in h(q)$, q atomic.
2. $(T, <, h), a_1, \ldots, a_{m-1}, t \models A \wedge B$ iff $(T, <, h), a_1, \ldots, a_{m-1}, t \models A$ and $(T, <, h), a_1, \ldots, a_{m-1}, t \models B$.
3. $(T, <, h), a_1, \ldots, a_{m-1}, t \models \neg A$ iff $(T, <, h), a_1, \ldots, a_{m-1}, t \not\models A$.
4. For each i $(1 \leq i \leq j)$, $(T, <, h), a_1, \ldots, a_{m-1}, t \models \#_i(A_1, \ldots, A_{n_i})$ iff

$T \models \chi_i(a_1, \ldots, a_{m-1}, t, h(A_1), \ldots, h(A_{n_i}))$, where

$$h(A_k) =_{\text{def.}} \{(t_1, \ldots, t_m) \in T^m \mid (T, <, h), t_1, \ldots, t_m \models A_k\}.$$

Let L^M denote the monadic language with $<$, first-order quantifiers over elements, and an arbitrary number of monadic predicate symbols Q_i for subsets of T. We will regard the Q_i as predicate (subset) variables, implicitly associated with the atoms q_i. We define the translation of an m-dimensional temporal formula A into a monadic formula δA:

1. If A is an atom q_i, we set $\delta A = (x_1 = x_1) \wedge \ldots \wedge (x_{m-1} = x_{m-1}) \wedge Q_i(x_m)$.
2. $\delta(A \wedge B) = \delta A \wedge \delta B$, and $\delta(\neg A) = \neg \delta A$.
3. Let $A = \#_i(A_1, \ldots, A_{n_i})$, where $\chi_i(x_1, \ldots, x_m; R_1, \ldots, R_{n_i})$ is the table of $\#_i$. Since we can always rewrite χ such that all occurrences of $R_k(y_1, \ldots, y_m)$ in χ are such that the terms y_i are variables, after a suitable variable replacement we can write δA using only the variables $x_1, \ldots, x_b$ as:

$$\delta A = \chi_i(x_1, \ldots, x_m, \delta A_1, \ldots, \delta A_{n_i}).$$

Clearly, a simple induction gives us that:

$$(T, <, h), a_1, ..., a_m \models B \text{ iff } T \models \delta B(a_1, ..., a_m, h(q_1), \ldots, h(q_k)).$$

such that $\delta B(a_1, ..., a_m, h(q_1), \ldots, h(q_k))$ uses only the variables $x_1, \ldots, x_b$.

Suppose that $\mathcal{K}$ is a class of flows of time, $\bar{x} = x_1, \ldots, x_m$ are variables, and $\bar{Q} = Q_1, \ldots, Q_r$ are monadic predicates. If $\alpha(\bar{x}, \bar{Q})$, $\beta(\bar{x}, \bar{Q})$ are formulas in L^M with free variables $\bar{x}$ and free monadic predicates $\bar{Q}$, we say that α and β are *$\mathcal{K}$-equivalent* if for all $T \in \mathcal{K}$ and all subsets $S_1, \ldots, S_r \subseteq T$,

$$T \models \forall \bar{x} \Big(\alpha(\bar{x}, S_1, \ldots S_r) \leftrightarrow \beta(\bar{x}, S_1, \ldots, S_r) \Big).$$

We say the temporal system $\mathcal{T}$ is *expressively complete over $\mathcal{K}$ in n dimensions* $(1 \leq n \leq m)$ if for any $\alpha(x_1, \ldots, x_n, \bar{Q})$ of L^M with free variables $x_1, \ldots, x_n$, there exists a temporal formula $B(\bar{q})$ of $\mathcal{T}$ built up from the atoms $\bar{q} = q_1, \ldots, q_r$, such that $\alpha \wedge \bigwedge_{n<i\leq m} x_i = x_i$ and δB are equivalent in $\mathcal{K}$. In this case, $\mathcal{K}$ is said to be *m-functionally complete* in n dimensions (symbolically, FC^m_n); $\mathcal{K}$ is *functionally complete* if it is FC^m_1 for some m.

Finally, we define the *Henkin or H-dimension* d of a class $\mathcal{K}$ of flows as the smallest d such that:

- For any monadic formula $\alpha(x_1, \ldots, x_n, Q_1, \ldots, Q_r)$ in L^M with free variables among $x_1, \ldots, x_n$ and monadic predicates $Q_1, \ldots, Q_r$ (with n, r arbitrary), there exists an L^M-formula $\alpha'(x_1, \ldots, x_n, Q_1, \ldots, Q_r)$ that is $\mathcal{K}$-equivalent to α and uses no more than d different bound variable letters.

We now show that for any class of flows, finite Henkin dimension is equivalent to functional completeness (FC_1^m for some m).

Theorem 1.18 *For any class $\mathcal{K}$ of flows of time, if $\mathcal{K}$ is functionally complete then $\mathcal{K}$ has finite H-dimension.*

Proof. Let $\sigma(\bar{Q})$ be any sentence of L^M. By functional completeness, there exists a $B(\bar{q})$ of $\mathcal{T}$ such that the formulas $x_1 = x_1 \wedge ... \wedge x_m = x_m \wedge \sigma(\bar{Q})$ and $\delta B(x_1, \ldots, x_m, \bar{Q})$ are $\mathcal{K}$-equivalent. We know that δB is written using variables $x_1, \ldots, x_b$ only. Hence the sentence $\sigma^* = \exists x_1 ... \exists x_m \delta B(x_1, \ldots, x_m, \bar{Q})$ has at most b variables, and is clearly $\mathcal{K}$-equivalent to σ. So every sentence of L^M is $\mathcal{K}$-equivalent to one with at most b variables. This means that $\mathcal{K}$ has H-dimension at most b, so it is finite. ■

We now show the converse. That is, we assume that the class $\mathcal{K}$ of flows of time has finite H-dimension m. Then we are going to construct a temporal logic that is expressively complete over $\mathcal{K}$ and that is weakly $m+1$-dimensional (and that is why such proof does not work for 1-dimensional systems: it always constructs a logic of dimension at least 2).

Let us call this logic system **d**. Besides atomic propositions $q_1, q_2, \ldots$ and the usual boolean operators, this system has a set of constants (0-place operators) $C_{i,j}^{<}$ and $C_{i,j}^{=}$ and unary temporal connectives Π_i and $\Box_i$, for $0 \leq i, j \leq m$. If h is an assignment such that $(h(q) \subseteq T$ for atomic q, the semantics of **d**-formulas is given by:

1. $(T, <, h), x_0, ..., x_m \models q$ iff $x_0 \in h(q)$ for q atomic.
2. The tables for $\neg, \wedge$ are the usual ones.
3. $(T, <, h), x_0, \ldots, x_m \models C_{i,j}^{<}$ iff $x_i < x_j$. Similarly we define the semantics of $C_{i,j}^{=}$. $C_{i,j}^{=}$ are thus called *diagonal* constants.
4. $(T, <, h), x_0, ..., x_m \models \Pi_i A$ iff $(T, <, h), x_i, \ldots, x_i \models A$. So Π_i "projects" the truth value on the i-th dimension.
5. $(T, <, h), x_0, ..., x_m \models \Box_i A$ iff $(T, <, h), x_0, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_m \models A$ for all $y \in T$. So $\Box_i$ is an "always" operator for the i-th dimension.

Lemma 1.19 *Let β be a formula of L^M written only using the variable letters $u_0, \ldots, u_m$, and having $u_{i_1}, .., u_{i_k}$ free for arbitrary $k \leq m$. Then there exists a temporal formula A of* **d** *such that for all $h, t_0, \ldots, t_m \in T$,*

$$(T, <, h), t_0, ..., t_m \models A \textit{ iff } \mathcal{K}, h \models \beta(t_{i_1}, \ldots, t_{i_k}).$$

Proof. By induction on β. Assume first that β is atomic. If β is $u_i < u_j$ let $A = C^{<}_{i,j}$ if $i \neq j$, and $\perp$ otherwise. Similarly for $u_i = u_j$. If β is $Q(u_i)$, let A be $\Pi_i(q)$.

The classical connectives present no difficulties. We turn to the case where β is $\forall u_i \alpha(u_{i_1}, .., u_{i_k})$. By induction hypothesis, let A be the formula corresponding α; then $\Box_i A$ is the formula suitable for β. ■

We are now in a position to prove the converse of Theorem 1.18.

Theorem 1.20 *For any class $\mathcal{K}$ of flows of time, if $\mathcal{K}$ has finite H-dimension then $\mathcal{K}$ is functionally complete.*

Proof. Let $\beta(u_0)$ be any formula of L^M with one free variable u_0. As $\mathcal{K}$ has H-dimension m, we can suppose that β is written with variables $u_0, ..., u_m$. By Lemma 1.19 there exists an A of $\mathcal{T}$ such that for any $T \in \mathcal{K}$, $t \in T$, and assignment h into T, $(T, <, h), t, ..., t \models A$ iff $\mathcal{K}, h \models \beta(t)$. ■

As an application of the results above, we show that the class of partial orders is not functionally complete. For consider the formula corresponding to the statement *there are at least n elements in the order*:

$$\sigma_n = \exists x_1, \ldots, x_n \bigwedge_{i \neq j} [(x_i \neq x_j) \wedge \neg(x_i < x_j)].$$

It can be shown that such formula cannot be written with less then n variables (e.g. [GHR94]). Since we are able to say that there are at least n elements in the order for any finite n, the class partial orders have infinite H-dimension and by Theorem 1.18 it is not functionally complete.

On the other hand, the reals and the integers must have finite H-dimension, for the $\{U, S\}$ temporal logic is expressively complete over both. Indeed, [GHR94] shows that it has H-dimension at most 3, and so does the theory of linear order.

2 Combining Temporal Logics

There is a profusion of logics proposed in the literature for the modelling of a variety of phenomena, and many more will surely be proposed in the future. A great part of those logics deal only with "static" aspects, and the temporal evolution is left out. But eventually, the need to deal with the temporal evolution of a model appears. What we want to avoid is the so called *reinvention of the wheel*, that is, reworking from scratch the whole logic, its language, inference system and models, and reproving all its basic properties, when the temporal dimension is added.

We therefore show here several methods for combining logic systems and we study if the properties of the component systems are *transferred* to their combination. We understand a logic system $\mathcal{L}_{\mathsf{L}}$ as composed of three elements:

(a) a language $\mathcal{L}_{\mathsf{L}}$, normally given by a set of formation rules generating well formed formulas over a signature and a set of logical connectives.

(b) An inference system, i.e. a relation, $\vdash_{\mathsf{L}}$, between sets of formulas, represented by $\Delta \vdash_{\mathsf{L}} A$. As usual, $\vdash_{\mathsf{L}} A$ indicates that $\varnothing \vdash_{\mathsf{L}} A$.

(c) The semantics of formulas over a class $\mathcal{K}$ of model structures. The fact that a formulas A is true of or holds at a model $\mathcal{M} \in \mathcal{K}$ is indicated by $\mathcal{M} \models A$.

Each method for combining logic systems proposes a way of generating the language, inference system and model structures from those of the component system.

The first method presented here adds a temporal dimension T to a logic system L, called the *temporalisation* of a logic system $\mathsf{T}(\mathsf{L})$, with an automatic way of constructing:

- the language of $\mathsf{T}(\mathsf{L})$;
- the inference system of $\mathsf{T}(\mathsf{L})$; and
- the class of temporal models of $\mathsf{T}(\mathsf{L})$.

We do that in a way that the basic properties of soundness, completeness and decidability are *transfered* from the component logics T and L to the combined system $\mathsf{T}(\mathsf{L})$.

If the temporalised logic is itself a temporal logic, we have a two dimensional temporal logic $\mathsf{T}(\mathsf{T}')$. Such a logic is too weak, however, because, by construction, the temporal logic T' cannot refer to the the logic system T.

We therefore present the *independent combination* $\mathsf{T} \oplus \mathsf{T}'$ in which two temporal logics are symmetrically combined. As before, the language, inference systems and models of $\mathsf{T} \oplus \mathsf{T}'$, and show that the properties of soundness, completeness and decidability are transferred form T and T' to $\mathsf{T} \oplus \mathsf{T}'$.

The independent combination is not the strongest way to combine logics; in particular, the independent combination of two linear temporal logic does not necessarily produce a two-dimensional grid model. So we show how to produce the *full join* of two linear temporal logics $\mathsf{T} \times \mathsf{T}'$, such that all models will be two-dimensional grids. However, in this case we cannot guarantee that the basic properties of T and T' are transferred to $\mathsf{T} \times \mathsf{T}'$. In this sense, the independent combination $\mathsf{T} \oplus \mathsf{T}'$ is a *minimal symmetrical combination* of logics that automatically transfers the basic properties. Any further interaction between the logics has to be separately investigated.

As a final way of combining logics, we present methods of combination that are motivated by the study of Labelled Deductive Systems (LDS) [Gab96].

All temporal logics considered for combination here are assumed to be linear.

2.1 Temporalising a Logic

The first of the combination methods, known as "adding a temporal dimension to a logic system" or simply "temporalising a logic system", has been initially presented in [FG92].

Temporalisation is a methodology whereby an arbitrary logic system L can be enriched with temporal features from a linear temporal logic T to create a new, *temporalised* system $\mathsf{T}(\mathsf{L})$.

We assume that the language of temporal system T is the US language and its inference system is an extensions of that of $US/\mathcal{K}_{lin}$, with its corresponding class of temporal linear models $\mathcal{K} \subseteq \mathcal{K}_{lin}$.

With respect to the logic L we assume it is an extension of classical logic, that is, all propositional tautologies are valid in it. The set $\mathcal{L}_{\mathsf{L}}$ is partitioned in two sets, BC_{L} and ML_{L}. A formula $A \in \mathcal{L}_{\mathsf{L}}$ belongs to the set of *boolean combinations*, BC_{L}, iff it is built up from other formulas by the use of one of the boolean connectives $\neg$ or $\wedge$ or any other connective defined only in terms of those; it belongs to the set of *monolithic formula* ML_{L} otherwise.

If L is not an extension of classical logic, we can simply "encapsulate" it in L' with a one-place symbol # not occurring in either L or T, such that for each formula $A \in \mathcal{L}_{\mathsf{L}}$, $\#A \in \mathcal{L}_{\mathsf{L}'}$, $\vdash_{\mathsf{L}} A$iff $\vdash_{\mathsf{L}'} \#A$ and the model structures of $\#A$ are those of A. Note that $ML_{\mathsf{L}'} = \mathcal{L}_{\mathsf{L}'}$, $BC_{\mathsf{L}'} = \varnothing$.

The alphabet of the temporalised language uses the alphabet of L plus the two-place operators S and U, if they are not part of the alphabet of L; otherwise, we use $\overline{S}$ and $\overline{U}$ or any other proper renaming.

Definition 2.1 Temporalised formulas The set $\mathcal{L}_{\mathsf{T(L)}}$ of formulas of the logic system L is the smallest set such that:

1. If $A \in ML_{\mathsf{L}}$, then $A \in \mathcal{L}_{\mathsf{T(L)}}$;
2. If $A, B \in \mathcal{L}_{\mathsf{T(L)}}$ then $\neg A \in \mathcal{L}_{\mathsf{T(L)}}$ and $(A \wedge B) \in \mathcal{L}_{\mathsf{T(L)}}$;
3. If $A, B \in \mathcal{L}_{\mathsf{T(L)}}$ then $S(A, B) \in \mathcal{L}_{\mathsf{T(L)}}$ and $U(A, B) \in \mathcal{L}_{\mathsf{T(L)}}$.

Note that, for instance, if $\Box$ is an operator of the alphabet of L and A and B are two formulas in $\mathcal{L}_{\mathsf{L}}$, the formula $\Box U(A, B)$ is *not* in $\mathcal{L}_{\mathsf{T(L)}}$. The language of $\mathsf{T(L)}$ is independent of the underlying flow of time, but not its semantics and inference system, so we must fix a class $\mathcal{K}$ of flows of time over which the temporalisation is defined; if $\mathcal{M}_{\mathsf{L}}$ is a model in the class of models of L, $\mathcal{K}_{\mathsf{L}}$, for every formula $A \in \mathcal{L}_{\mathsf{L}}$ we must have either $\mathcal{M}_{\mathsf{L}} \models A$ or $\mathcal{M}_{\mathsf{L}} \models \neg A$. In the case that L is a temporal logic we must consider a "current time" o as part of its model to achieve that condition.

Definition 2.2 *Semantics of the temporalised logic.* [1] Let $(T, <) \in \mathcal{K}$ be a flow of time and let $g : T \to \mathcal{K}_{\mathsf{L}}$ be a function mapping every time point in T to a model in the class of models of L. A model of $\mathsf{T(L)}$ is a triple $\mathcal{M}_{\mathsf{T(L)}} = (T, <, g)$ and the fact that A is true in $\mathcal{M}_{\mathsf{T(L)}}$ at time t is written as $\mathcal{M}_{\mathsf{T(L)}}, t \models A$ and defined as:

$$
\begin{array}{ll}
\mathcal{M}_{\mathsf{T(L)}}, t \models A,\ A \in ML_{\mathsf{L}} & \text{iff } g(t) = \mathcal{M}_{\mathsf{L}} \text{ and } \mathcal{M}_{\mathsf{L}} \models A. \\
\mathcal{M}_{\mathsf{T(L)}}, t \models \neg A & \text{iff it is not the case that } \mathcal{M}_{\mathsf{T(L)}}, t \models A. \\
\mathcal{M}_{\mathsf{T(L)}}, t \models (A \wedge B) & \text{iff } \mathcal{M}_{\mathsf{T(L)}}, t \models A \text{ and } \mathcal{M}_{\mathsf{T(L)}}, t \models B. \\
\mathcal{M}_{\mathsf{T(L)}}, t \models S(A, B) & \text{iff there exists } s \in T \text{ such that } s < t \text{ and} \\
& \quad \mathcal{M}_{\mathsf{T(L)}}, s \models A \text{ and for every } u \in T, \text{ if} \\
& \quad s < u < t \text{ then } \mathcal{M}_{\mathsf{T(L)}}, u \models B. \\
\mathcal{M}_{\mathsf{T(L)}}, t \models U(A, B) & \text{iff there exists } s \in T \text{ such that } t < s \text{ and} \\
& \quad \mathcal{M}_{\mathsf{T(L)}}, s \models A \text{ and for every } u \in T, \text{ if} \\
& \quad t < u < s \text{ then } \mathcal{M}_{\mathsf{T(L)}}, u \models B.
\end{array}
$$

The inference system of $\mathsf{T(L)}/\mathcal{K}$ is given by the following:

Definition 2.3 Axiomatisation for $\mathsf{T(L)}$ An axiomatisation for the temporalised logic $\mathsf{T(L)}$ is composed of:

[1] We assume that the a model of T is given by $(T, <, h)$ where h maps time points into sets of propositions (instead of the more common, but equivalent, mapping of propositions into sets of time points); such notation highlights that in the temporalised model each time point is associated to a model of L.

- The axioms of $\mathsf{T}/\mathcal{K}$;
- The inference rules of $\mathsf{T}/\mathcal{K}$;
- For every formula A in $\mathcal{L}_{\mathsf{L}}$, if $\vdash_{\mathsf{L}} A$ then $\vdash_{\mathsf{T(L)}} A$, i.e. all theorems of L are theorems of $\mathsf{T(L)}$. This inference rule is called **Persist**.

Example 2.4 Consider classical propositional logic $\mathsf{PL} = \langle \mathcal{L}_{\mathsf{PL}}, \vdash_{\mathsf{PL}}, \models_{\mathsf{PL}} \rangle$. Its temporalisation generates the logic system $\mathsf{T(PL)} = \langle \mathcal{L}_{\mathsf{T(PL)}}, \vdash_{\mathsf{T(PL)}}, \models_{\mathsf{T(PL)}} \rangle$. It is not difficult to see that the temporalised version of PL over any $\mathcal{K}$ is actually the temporal logic $\mathsf{T} = \mathsf{US}/\mathcal{K}$.

If we temporalise over $\mathcal{K}$ the one-dimensional logic system $\mathsf{US}/\mathcal{K}$ we obtain the two-dimensional logic system $\mathsf{T(US)} = \langle \mathcal{L}_{\mathsf{T(US)}}, \vdash_{\mathsf{T(US)}}, \models_{\mathsf{T(US)}} \rangle = \mathsf{T}^2(\mathsf{PL})/\mathcal{K}$. In this case we have to rename the two-place operators S and U of the temporalised alphabet to, say, $\overline{S}$ and $\overline{U}$. Note, however, how weak this logic is, for $\overline{S}$ and $\overline{U}$ cannot occur within the scope of U and S.

In order to obtain a model for $\mathsf{T(US)}$, we must fix a "current time", o_1, in $\mathcal{M}_{\mathsf{US}} = (T_1, <_1, g_1)$, so that we can construct the model $\mathcal{M}_{\mathsf{T(US)}} = (T_2, <_2, g_2)$ as previously described. Note that, in this case, the flows of time $(T_1, <_1)$ and $(T_2, <_2)$ need not to be the same. $(T_2, <_2)$ is the flow of time of the upper-level temporal system whereas $(T_1, <_1)$ is the flow of time of the underlying logic which, in this case, happens to be a temporal logic. The satisfiability of a formula in a model of $\mathsf{T(US)}$ needs two evaluation points, o_1 and o_2; therefore it is a *two-dimensional temporal logic.*

The logic system we obtain by temporalising US-temporal logic is the two-dimensional temporal logic described in [Fin92].

This temporalisation process can be repeated n times, generating an n dimensional temporal logic with connectives U_i, S_i, $1 \leq i \leq n$, such that for $i < j$ U_j, S_j cannot occur within the scope of U_i, S_i.

We analyse now the transfer of soundness, completeness and decidability from T and L to $\mathsf{T(L)}$; that is, we are asuming the logics T and L have sound, complete and decidable axiomatisations with respect to their semantics, and we will analyse how such properties transfer to the combined system $\mathsf{T(L)}$. It is a routine task to analyse that if the inference systems of T and L are sound, so is $\mathsf{T(L)}$. So we concentrate on the proof of transference of completeness.

Completeness

We prove the completeness of $\mathsf{T(L)}/\mathcal{K}$ indirectly by transforming a consistent formula A of $\mathsf{T(L)}$ into $\varepsilon(A)$ and then mapping it into a consistent formula of T. Completeness of $\mathsf{T}/\mathcal{K}$ is used to find a T-model for A^* that is used to construct a model for the original $\mathsf{T}(\mathsf{L})$ formula A.

We first define the transformation and mapping. Given a formula $A \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$, consider the following sets:

$$\begin{array}{rcl} Lit(A) & = & Mon(A) \cup \{\neg B \mid B \in Mon(A)\} \\ Inc(A) & = & \{\bigwedge \Gamma \mid \Gamma \subseteq Lit(A) \text{ and } \Gamma \vdash_{\mathsf{L}} \bot\} \end{array}$$

where $Mon(A)$ is the set of maximal monolithic subformulae of A. $Lit(A)$ is the set of literals occurring in A and $Inc(A)$ is the set of inconsistent formulas that can be built with those. We transform A into A as: $\varepsilon(A)$:

$$\varepsilon(A) \quad = \quad A \wedge \bigwedge_{B \in Inc(A)} (\neg B \ \wedge \ G\neg B \ \wedge \ H\neg B)$$

The big conjunction $in\varepsilon(A)$ is a theorem of $\mathsf{T}(\mathsf{L})$, so we have the following lemma.

Lemma 2.5 $\vdash_{\mathsf{T}(\mathsf{L})} \varepsilon(A) \leftrightarrow A$

If K is a subclass of linear flows of time, we also have the following property (this is where linearity is used in the proof).

Lemma 2.6 *Let $\mathcal{M}_{\mathsf{T}}$ be a temporal model over $\mathcal{K} \subseteq \mathcal{K}_{lin}$ such that for some $o \in T$, $\mathcal{M}_{\mathsf{T}}, o \models \sigma(\Box A)$. Then, for every $t \in T$, $\mathcal{M}_{\mathsf{T}}, t \models \sigma(\Box A)$.*

Therefore, if some subset of $Lit(A)$ is inconsistent, the transformed formula $\varepsilon(A)$ puts that fact in evidence so that, when it id mapped into T, inconsistent subformulae will be mapped into falsity.

Now we want to map a $\mathsf{T}(\mathsf{L})$-formula into a T-formula. For that, consider an enumeration $p_1, p_2, \ldots$, of elements of $\mathcal{P}$ and consider an enumeration A_1, A_2, $\ldots$, of formulae in ML_{L}. The *correspondence mapping* $\sigma : \mathcal{L}_{\mathsf{T}(\mathsf{L})} \to \mathcal{L}_{\mathsf{T}}$ is given by:

$$\begin{array}{lcl} \sigma(A_i) & = & p_i \text{ for every } A_i \in ML_{\mathsf{L}}, i = 1, 2 \ldots \\ \sigma(\neg A) & = & \neg \sigma(A) \\ \sigma(A \wedge B) & = & \sigma(A) \wedge \sigma(B) \\ \sigma(S(A, B)) & = & S(\sigma(A), \sigma(B)) \\ \sigma(U(A, B)) & = & U(\sigma(A), \sigma(B)) \end{array}$$

The following is the *correspondence lemma*.

Lemma 2.7 *The correspondence mapping is a bijection. Furthermore if A is $\mathsf{T}(\mathsf{L})$-consistent then $\sigma(A)$ is T-consistent.*

Lemma 2.8 *If A is $\mathsf{T}(\mathsf{L})$-consistent, then for every $t \in T$, $G_A(t) = \{B \in Lit(A) \mid \mathcal{M}_{\mathsf{T}}, t \models \sigma(B)\}$ is finite and L-consistent.*

Proof. Since $Lit(A)$ is finite, $G_A(t)$ is finite for every t. Suppose $G_A(t)$ is inconsistent for some t, then there exist $\{B_1, \ldots, B_n\} \subseteq G_A(t)$ such that $\vdash_{\mathsf{L}} \bigwedge B_i \to \bot$. So $\bigwedge B_i \in Inc(A)$ and $\Box\neg(\bigwedge B_i)$ is one of the conjuncts of $\varepsilon(A)$. Applying Lemma 2.6 to $\mathcal{M}_{\mathsf{T}}, o \models \sigma(\varepsilon(A))$ we get that for every $t \in T$, $\mathcal{M}_{\mathsf{T}}, t \models \neg(\bigwedge \sigma(B_i))$ but by, the definition of G_A, $\mathcal{M}_{\mathsf{T}}, t \models \bigwedge \sigma(B_i)$, which is a contradiction. ■

We are finally ready to prove the completeness of $\mathsf{T}(\mathsf{L})/\mathcal{K}$.

Theorem 2.9 (Completeness transfer for $\mathsf{T}(\mathsf{L})$) *If the logical system L is complete and T is complete over a subclass of linear flows of time $\mathcal{K} \subseteq \mathcal{K}_{lin}$, then the logical system $\mathsf{T}(\mathsf{L})$ is complete over K.*

Proof. Assume that A is $\mathsf{T}(\mathsf{L})$-consistent. By Lemma 2.8, we have $(T, <) \in \mathcal{K}$ and associated to every time point in T we have a finite and L-consistent set $G_A(t)$. By (weak) completeness of L, every $G_A(t)$ has a model, so we define the temporalised valuation function g:

$$g(t) \quad = \quad \{\mathcal{M}^t_{\mathsf{L}} \mid \mathcal{M}^t_{\mathsf{L}} \text{ is a model of } G_A(t)\}$$

Consider the model $\mathcal{M}_{\mathsf{T}(\mathsf{L})} = (T, <, g)$ over K. By structural induction over B, we show that for every B that is a subformula of A and for every time point t,

$$\mathcal{M}_{\mathsf{T}}, t \models \sigma(B) \text{ iff } \mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models B$$

We show only the basic case, $B \in Mon(A)$. Suppose $\mathcal{M}_{\mathsf{T}}, t \models \sigma(B)$; then $B \in G_A(t)$ and $\mathcal{M}^t_{\mathsf{L}} \models B$, and hence $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models B$. Suppose $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models B$ and assume $\mathcal{M}_{\mathsf{T}}, t \models \neg\sigma(B)$; then $\neg B \in G_A(t)$ and $\mathcal{M}^t_{\mathsf{L}} \models \neg B$, which contradicts $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models B$; hence $\mathcal{M}_{\mathsf{T}}, t \models \sigma(B)$. The inductive cases are straightforward and omitted.

So, $\mathcal{M}_{\mathsf{T}(\mathsf{L})}$ is a model for A over K and the proof is finished. ■

Theorem 2.9 gives us sound and complete axiomatisations for $\mathsf{T}(\mathsf{L})$ over many interesting classes of flows of time, such as the class of all linear flows of time, $\mathcal{K}_{lin}$, the integers, $\mathbb{Z}$, and the reals, $\mathbb{R}$. These classes are, in their T versions, decidable and the corresponding decidability of $\mathsf{T}(\mathsf{L})$ is dealt next.

Note that the construction above is finitistic, and therefore does not itself guarantee that compactness is transferred. However, an important corollary of the construction above is that the temporalised system is a *conservative* *extension* of both original systems, that is, no new theorem in the language of an original system is provable in the combined system. Formally, L_1 is a *conservative extension* of L_2 if it is an extension of L_2 such that if $A \in \mathcal{L}_{\mathsf{L}_2}$, then $\vdash_{\mathsf{L}_1} A$ only if $\vdash_{\mathsf{L}_2} A$.

Corollary 2.10 *Let L be a sound and complete logic system and T be sound and complete over $\mathcal{K} \subseteq \mathcal{K}_l in$. The logic system $\mathsf{T}(\mathsf{L})$ is a conservative extension of both L and T.*

Proof. Let $A \in \mathcal{L}_{\mathsf{L}}$ such that $\vdash_{\mathsf{T}(\mathsf{L})} A$. Suppose by contradiction that $\not\vdash_l$ $ogicLA$, so by completeness of L, there exists a model $\mathcal{M}_{\mathsf{L}}$ such that $\mathcal{M}_{\mathsf{L}} \models \neg A$. We construct a temporalised model $\mathcal{M}_{\mathsf{T}(\mathsf{L})} = (T, <, g)$ by making $g(t) = \mathcal{M}_{\mathsf{L}}$ for all $t \in T$. $\mathcal{M}_{\mathsf{T}(\mathsf{L})}$ clearly contradicts the soundness of $\mathsf{T}(\mathsf{L})$ and therefore that of T, so $\vdash_{\mathsf{L}} A$. This shows that $\mathsf{T}(\mathsf{L})$ is a conservative extension of L; the proof of extension of T is similar. ■

Decidability

The transfer of decidability is also done using the correspondence mapping σ and the transformation η. Such a transformation is actually computable, as the following two lemmas state.

Lemma 2.11 *For any $A \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$, if the logic system L is decidable then there exists an algorithm for constructing $\varepsilon(A)$.*

Lemma 2.12 *Over a linear flow of time, for every $A \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$,*

$$\vdash_{\mathsf{T}(\mathsf{L})} A \textit{ iff } \vdash_{\mathsf{T}} \sigma(\varepsilon(A)).$$

Decidability is a direct consequence of these two lemmas.

Theorem 2.13 *If L is a decidable logic system, and T is decidable over $\mathcal{K} \subseteq \mathcal{K}_{lin}$, then the logic system $\mathsf{T}(\mathsf{L})$ is also decidable over $\mathcal{K}$.*

Proof. Consider $A \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$. Since L is decidable, by Lemma 2.11 there is an algorithmic procedure to build $\varepsilon(A)$. Since σ is a recursive function, we have an algorithm to construct $\sigma(\varepsilon(A))$, and due to the decidability of T over $\mathcal{K}$, we have an effective procedure to decide if it is a theorem or not. Since $\mathcal{K}$ is linear, by Lemma 2.12 this is also a procedure for deciding whether A is a theorem or not. ■

2.2 Independent Combination

We now deal with the combination of two temporal logic systems. One of the will be called the *horizontal* temporal logic US, while the other will be the *vertical* temporal logic $\bar{\mathsf{U}}\bar{\mathsf{S}}$. If we temporalise the horizontal logic with the vertical logic, we obtain a very weakly expressive system; if US is the internal (horizontal) temporal logic in the temporalisation process (F is derived in US), and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ is the external (vertical) one ($\overline{F}$ is defined in $\bar{\mathsf{U}}\bar{\mathsf{S}}$), we cannot express that vertical and horizontal future operators commute,

$$F\overline{F}A \leftrightarrow \overline{F}FA.$$

In fact, the subformula $F\overline{F}A$ is not even in the temporalised language of $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$, nor is the whole formula. In other words, the interplay between the two-dimensions is not expressible in the language of the temporalised $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$.

The idea is then to define a method for combining temporal logics that is symmetrical. As usual, we combine the languages, inference systems and classes of models.

Definition 2.14 Let $Op(\mathsf{L})$ be the set of non-boolean operators of a generic logic L. Let $\bar{\mathsf{T}}$ and T be logic systems such that $Op(\mathsf{T}) \cap Op(\bar{\mathsf{T}}) = \varnothing$. The *fully combined language* of logic systems $\bar{\mathsf{T}}$ and T over the set of atomic propositions $\mathcal{P}$, is obtained by the union of the respective set of connectives and the union of the formation rules of the languages of both logic systems.

Let the operators U and S be in the language of US and $\overline{U}$ and $\overline{S}$ be in that of $\bar{\mathsf{U}}\bar{\mathsf{S}}$. Their fully combined language over a set of atomic propositions $\mathcal{P}$ is given by

- every atomic proposition is in it;
- if A, B are in it, so are $\neg A$ and $A \wedge B$;
- if A, B are in it, so are $U(A, B)$ and $S(A, B)$.
- if A, B are in it, so are $\overline{U}(A, B)$ and $\overline{S}(A, B)$.

Not only are the two languages taken to be independent of each other, but the set of axioms of the two systems are supposed to be disjoint; so we call the following combination method the *independent combination* of two temporal logics.

Definition 2.15 Let US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ be two US-temporal logic systems defined over the same set $\mathcal{P}$ of propositional atoms such that their languages are independent. The *independent combination* $US \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ is given by the following:

- The fully combined language of US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$.
- If $(\Sigma, \mathcal{I})$ is an axiomatisation for US and $(\overline{\Sigma}, \overline{\mathcal{I}})$ is an axiomatisation for $\bar{\mathsf{U}}\bar{\mathsf{S}}$, then $(\Sigma \cup \overline{\Sigma}, \mathcal{I} \cup \overline{\mathcal{I}})$ is an axiomatisation for $US \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$. Note that, apart from the classical tautologies, the set of axioms Σ and $\overline{\Sigma}$ are supposed to be disjoint, but not the inference rules.
- The class of independently combined flows of time is $\mathcal{K} \oplus \overline{\mathcal{K}}$ composed of biordered flows of the form $(\tilde{T}, <, \overline{<})$ where the connected components of $(\tilde{T}, <)$ are in $\mathcal{K}$ and the connected components of $(\tilde{T}, \overline{<})$ are in $\overline{\mathcal{K}}$, and $\tilde{T}$ is the (not necessarily disjoint) union of the sets of time points T and $\overline{T}$ that constitute each connected component.

  A model structure for $US \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ over $\mathcal{K} \oplus \overline{\mathcal{K}}$ is a 4-tuple $(\tilde{T}, <, \overline{<}, g)$, where $(\tilde{T}, <, \overline{<}) \in \mathcal{K} \oplus \overline{\mathcal{K}}$ and g is an assignment function $g : \tilde{T} \to 2^{\P}$.

  The semantics of a formula A in a model $\mathcal{M} = (\tilde{T}, <, \overline{<}, g)$ is defined as the union of the rules defining the semantics of $US/\mathcal{K}$ and $\bar{\mathsf{U}}\bar{\mathsf{S}}/\overline{\mathcal{K}}$. The expression $\mathcal{M}, t \models A$ reads that the formula A is true in the (combined) model $\mathcal{M}$ at the point $t \in \tilde{T}$. The semantics of formulas is given by induction in the standard way:

$$
\begin{array}{ll}
\mathcal{M}, t \models p & \text{iff } p \in g(t) \text{ and } p \in \mathcal{P}. \\
\mathcal{M}, t \models \neg A & \text{iff it is not the case that } \mathcal{M}, t \models A. \\
\mathcal{M}, t \models A \wedge B & \text{iff } \mathcal{M}, t \models A \text{ and } \mathcal{M}, t \models B. \\
\mathcal{M}, t \models S(A, B) & \text{iff there exists an } s \in \tilde{T} \text{ with } s < t \text{ and } \mathcal{M}, s \models A \\
& \quad \text{and for every } u \in \tilde{T}, \text{ if } s < u < t \text{ then } \mathcal{M}, u \models B. \\
\mathcal{M}, t \models U(A, B) & \text{iff there exists an } s \in \tilde{T} \text{ with } t < s \text{ and } \mathcal{M}, s \models A \\
& \quad \text{and for every } u \in \tilde{T}, \text{ if } t < u < s \text{ then } \mathcal{M}, u \models B. \\
\mathcal{M}, t \models \overline{S}(A, B) & \text{iff there exists an } s \in \tilde{T} \text{ with } s \overline{<} t \text{ and } \mathcal{M}, s \models A \\
& \quad \text{and for every } u \in \tilde{T}, \text{ if } s \overline{<} u \overline{<} t \text{ then } \mathcal{M}, u \models B. \\
\mathcal{M}, t \models \overline{U}(A, B) & \text{iff there exists an } s \in \tilde{T} \text{ with } t \overline{<} s \text{ and } \mathcal{M}, s \models A \\
& \quad \text{and for every } u \in \tilde{T}, \text{ if } t \overline{<} u \overline{<} s \text{ then } \mathcal{M}, u \models B.
\end{array}
$$

The also independent combination of two logics appears in the literature under the names of *fusion* or *join*.

As usual, we will assume that $\mathcal{K}, \overline{\mathcal{K}} \subseteq \mathcal{K}_{lin}$, so $<$ and $\overline{<}$ are transitive, irreflexive and total orders; similarly, we assume that the axiomatisations are extensions of $US/\mathcal{K}_{lin}$.

The temporalisation process will be used as an inductive step to prove the transference of soundness, completeness and decidability for $US \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ over $\mathcal{K} \oplus \overline{\mathcal{K}}$. We define the *degree alternation* of a $(US \oplus \bar{\mathsf{U}}\bar{\mathsf{S}})$-formula A for US, $dg(A)$:

$$
\begin{aligned}
&dg(p) = 0\\
&dg(\neg A) = dg(A)\\
&dg(A \wedge B) = dg(S(A,B)) = dg(U(A,B)) = max\{dg(A), dg(B)\}\\
&dg(\overline{U}(A,B)) = dg(\overline{S}(A,B)) = 1 + max\{\overline{dg}(A), \overline{dg}(B)\}
\end{aligned}
$$

and similarly define $\overline{dg}(A)$ for $\bar{\mathsf{U}}\bar{\mathsf{S}}$.

Any formula of the fully combined language can be seen as a formula of some finite number of alternating temporalisations of the form $\mathsf{US}(\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US}(\ldots)))$; more precisely, A can be seen as a formula of $\mathsf{US}(\mathsf{L_n})$, where $dg(A) = n$, $\mathsf{US}(\mathsf{L_0}) = \mathsf{US}$, $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{L_0}) = \bar{\mathsf{U}}\bar{\mathsf{S}}$, and $\mathsf{L}_{n-2i} = \bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{L}_{n-2i-1})$, $\mathsf{L}_{n-2i-1} = US(\mathsf{L}_{n-2i-2})$, for $i = 0, 1, \ldots, \lceil \frac{n}{2} \rceil - 1$.

Indeed, not only the language of $US \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ is decomposable in a finite number of temporalisation, but also its inferences, as the following important lemma indicates.

Lemma 2.16 *Let US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ be two complete logic systems. Then, A is a theorem of $US \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ iff it is a theorem of $\mathsf{US}(\mathsf{L_n})$, where $dg(A) = n$.*

Proof. If A is a theorem of $\mathsf{US}(\mathsf{L_n})$, all the inferences in its deduction can be repeated in $US \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$, so it is a theorem of $US \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$.

Suppose A is a theorem of $US \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$; let $B_1, \ldots, B_m = A$ be a deduction of A in $US \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ and let $n' = max\{dg(B_i)\}$, $n' \geq n$. We claim that each B_i is a theorem of $\mathsf{US}(\mathsf{L_{n'}})$. In fact, by induction on m, if B_i is obtained in the deduction by substituting into an axiom, the same substitution can be done in $\mathsf{US}(\mathsf{L_{n'}})$; if B_i is obtained by Temporal Generalisation from B_j, $j < i$, then by the induction hypothesis, B_j is a theorem of $\mathsf{US}(\mathsf{L_{n'}})$ and so is B_i; if B_i is obtained by Modus Ponens from B_j and B_k, $j, k < i$, then by the induction hypothesis, B_j and B_k are theorems of $\mathsf{US}(\mathsf{L_{n'}})$ and so is B_i.

So A is a theorem of $\mathsf{US}(\mathsf{L_{n'}})$ and, since US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ are two complete logic systems, by Theorem 2.9, each of the alternating temporalisations in $\mathsf{US}(\mathsf{L_{n'}})$ is a conservative extension of the underlying logic; it follows that A is a theorem of $\mathsf{US}(\mathsf{L_n})$, as desired. ■

Note that the proof above gives conservativeness as a corollary. The transference of soundness, completeness and decidability also follows directly from this result.

Theorem 2.17 (Independent Combination) *Let US and $\mathsf{\bar{U}\bar{S}}$ be two sound and complete logic systems over the classes $\mathcal{K}$ and $\overline{\mathcal{K}}$, respectively. Then their independent combination $US \oplus \mathsf{\bar{U}\bar{S}}$ is sound and complete over the class $\mathcal{K} \oplus \overline{\mathcal{K}}$. If US and $\mathsf{\bar{U}\bar{S}}$ are complete and decidable, so is $US \oplus \mathsf{\bar{U}\bar{S}}$.*

Proof. Soundness follows immediately from the validity of axioms and inference rules.

We only sketch the proof of completess here. Given a $US \oplus \mathsf{\bar{U}\bar{S}}$-consistent formula A, Lemma 2.16 is used to see that it is also consistent in $\mathsf{US(L_n)}$, so a temporalised $\mathsf{US(L_n)}$-model is built for it. Then, by induction on the degree of alternation of A, this $\mathsf{US(L_n)}$ is used to construct a $US \oplus \mathsf{\bar{U}\bar{S}}$-model; each step of such construction preserves the satisfatibility of formulas of a limited degree of alternation, so in the final model, A, is satisfiable; and completeness is proved. For details, see [FG96].

For decidability, suppose we want to decide whether a formula $A \in US \oplus \mathsf{\bar{U}\bar{S}}$ is a theorem. By Lemma 2.16, this is equivalent to deciding whether $A \in \mathsf{US(L_n)}$ is a theorem, where $n = dg(A)$. Since $US/\mathcal{K}$ and $\mathsf{\bar{U}\bar{S}}/\overline{\mathcal{K}}$ are both complete and decidable, by successive applications of Theorems 2.9 and 2.13, it follows that the following logics are decidable: $\mathsf{US(\bar{U}\bar{S})}$, $\mathsf{\bar{U}\bar{S}(US(\bar{U}\bar{S}))} = \mathsf{\bar{U}\bar{S}(L_2)}, \ldots, \mathsf{\bar{U}\bar{S}(L_{n-1})} = \mathsf{L_n}$; so a the last application of Theorems 2.9 and 2.13 yields that $\mathsf{US(L_n)}$ is decidable. ■

2.2.1 The minimality of the independent combination

The logic $US \oplus \mathsf{\bar{U}\bar{S}}$ is the minimal logic that conservatively extends both US and $\mathsf{\bar{U}\bar{S}}$. This result was first shown for the independent combination of monomodal logics independently by [KW91] and [FS91].

Indeed, suppose there is another logic $\mathsf{T_1}$ that conservatively extends both US and $\mathsf{\bar{U}\bar{S}}$ but some theorem A of $US \oplus \mathsf{\bar{U}\bar{S}}$ is not a theorem of $\mathsf{T_1}$. But A can be obtained by a finite number of inferences $A_1, \ldots, A_n = A$ using only the axioms of US and $\mathsf{\bar{U}\bar{S}}$. But any conservative extension of US and $\mathsf{\bar{U}\bar{S}}$ must be able to derive A_i, $1 \leq i \leq n$, from $A_1, \ldots, A_{i-1}$, and therefore it must be able to derive A; contradiction.

Once we have this minimal combination between two logic systems, any other interaction between the logics must be considered on its own. As an example, consider the following formulas expressing the commutativity of future and past operators between the two dimensions are not generally valid over a model of$US \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$:

I_1 $F\overline{F}A \leftrightarrow \overline{F}FA$
I_2 $F\overline{P}A \leftrightarrow \overline{P}FA$
I_3 $P\overline{F}A \leftrightarrow \overline{F}PA$
I_4 $P\overline{P}A \leftrightarrow \overline{P}PA$

Now consider the *product* of two linear temporal models, given as follows.

Definition 2.18 Let $(T, <) \in \mathcal{K}$ and $(\overline{T}, \overline{<}) \in \overline{\mathcal{K}}$ be two linear flows of time. The *product* of those flows of time is $(T \times \overline{T}, <, \overline{<})$. A *product model* over $\mathcal{K} \times \overline{\mathcal{K}}$ is a 4-tuple $\mathcal{M} = (T \times \overline{T}, <, \overline{<}, g)$, where $g : T \times \overline{T} \to 2^{\P}$ is a two-dimensional assignment. The semantics of the horizontal and vertical operators are independent of each other:

$\mathcal{M}, t, x \models S(A, B)$	iff	there exists $s < t$ such that $\mathcal{M}, s, x \models A$ and for all u, $s < u < t$, $\mathcal{M}, u, x \models B$.
$\mathcal{M}, t, x \models \overline{S}(A, B)$	iff	there exists $y \overline{<} x$ such that $\mathcal{M}, t, y \models A$ and for all z, $y \overline{<} z \overline{<} x$, $\mathcal{M}, t, z \models B$.

Similarly for U and $\overline{U}$, the semantics of atoms and boolean connectives remaining the standard one. A formula A is valid over $\mathcal{K} \times \overline{\mathcal{K}}$ if for all models $\mathcal{M} = (T, <, \overline{T}, \overline{<}, g)$, for all $t \in T$ and $x \in \overline{T}$ we have $\mathcal{M}, t, x \models A$.

It is easy to verify that the formulas I_1–I_4 are valid over product models. We wonder if such product of logics transfers the properties we have investigated for the previous logics. The answer is: it depends. We have the following results.

Proposition 2.19 *(a) There is a sound and complete axiomatisation for $US \times \bar{\mathsf{U}}\bar{\mathsf{S}}$ over the classes of product models $\mathcal{K}_{lin} \times \mathcal{K}_{lin}$, $\mathcal{K}_{dis} \times \mathcal{K}_{dis}$, $\mathbb{Q} \times \mathbb{Q}$, $\mathcal{K}_{lin} \times \mathcal{K}_{dis}$, $\mathcal{K}_{lin} \times \mathbb{Q}$ and $\mathbb{Q} \times \mathcal{K}_{dis}$ [Fin94].*

(b) There are no finite axiomatisations for the valid two-dimensional formulas over the classes $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$ and $\mathbb{R} \times \mathbb{R}$ [Ven90].

Note that the all the component one-dimensional mentioned above logic systems are complete and decidable, but their product sometimes is complete, sometimes not. Also, the logics in (a) are all decidable and those in (b) are undecidable.

This is to illustrate the following idea: given an independent combination of two temporal logics, the addition of extra axioms, inference rules or an extra condition on its models has to be studied on its own, just as adding a new axiom to a modal logic or imposing a new property on its accessibility relation has to be analysed on its own.

Combinations of logics in the literature

The work on combining temporal logics presented here has first appeared in the literature in [FG92, FG96].

General combinations of logics have been addressed in the literature in various forms. Combinations of tense and modality were discussed in [Tho84], without explicitly providing a general methodology for doing so. A methodology for constructing logics of belief based on existing deductive systems is the *deductive model* of Konolige [Kon86]; in this case, the language of the original system was the base for the construction of a new modal language, and the modal logic system thus generated had its semantics defined in terms of the inferences of the original system. This is a methodology quite different from the one adopted here, in which we separately combine language, inference systems and class of models.

Combination of two monomodal logics and the transference of properties have been studied by Kracht and Wolter [KW91] and Fine and Schurz [FS91]; the latter even considers the transference of properties through the combination of n-monomodal logics. These works differ from the combination of temporal logics in several ways: their modalities have no interaction whatsoever (unlike S and U, which actually interact with each other); they only consider one-place modalities ($\Box$); and their constructions are not a recursive application of the temporalisation (or any similar external application of one logic to another).

A stronger combination of logics have been investigated by Gabbay and Shehtman [GS98], where the starting point is the product of two Kripke frames, generating the *product* of the two monomodal logics. It shows that the transference of completeness and decidability can either succeed or fail for the product, depending on the properties of the component logics. The failure of transference of decidability for temporal products in $FP/\mathcal{K}_{lin} \times FP/\mathcal{K}_{lin}$ has been shown in [MR99], and fresh results on the products of logics can be foun din [RZar].

The transference of soundness, completeness and decidability are by no means the only properties to study. Kracht and Wolter [KW91] study the transference of interpolation between two monomodal logics. The complexity of the combination of two monomodal logics is studied in [Spa93]; the complexity of products are studied in [Marar]. Gabbay and Shehtman [GS98] report the failure of transference of the finite model property for their product of modal logics. With respect to specific temporal properties, the transference of the separation property is studied in [FG92].

2.3 Fibering

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