Notes on Several Methods for Combining Temporal Logic Systems

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Abstract

This paper is a continuation of the work started in [FG92] on combining temporal logics. In this work, four combination methods are described and studied with respect to the transference of logical properties from the component one-dimensional temporal logics to the resulting two-dimensional temporal logic. Three basic logical properties are analysed, namely soundness, completeness and decidability.

Each combination method is composed of three submethods that combine the languages, the inference systems and the semantics of two one-dimensional temporal logic systems, generating families of two-dimensional temporal languages with varying expressivity and varying degree of transference of logical properties. The *temporalisation method* and the *independent combination method* are shown to transfer all three basic logical properties. The method of *full interlacing* of logic systems generates a considerably more expressive language but fails to transfer completeness and decidability in several cases. So a weaker method of *restricted interlacing* is proposed and shown to transfer all three basic logical properties.

The connections of our work with more generic works on combininbg (any) logic systems are unfortunately absent from the current version of this paper but will be present in its final version.

1 Introduction

We are interested in describing systems in which two distinct temporal "points of view" coexist. Descriptions of temporal systems under a single point of view, *i.e.* one-dimensional temporal systems, abound in the literature. These one-dimensional temporal logics differ from each other in several ways. They differ on the form of the logic is presented, whether proof theoretical, model theoretical or algebraic presentation. They differ on the ontology of time adopted, whether time is to be represented as a set of points, intervals or events. They can also differ on the properties assigned to flows of time, whether or branching

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[†]comments on this paper are welcome.

time, discrete or dense, continuous or allowing for gaps. We want to take advantage of this existing literature on temporal systems to study temporal systems with two coexisting temporal references, *i.e.* two dimensions.

The idea is to systematically combine two one-dimensional temporal systems into a new logical system, which we call a *two-dimensional temporal logic*, and we study if the properties of the original systems are transferred to the combined one.

It turns out that there are several possible combination methods, in the same way that there are several levels at which two temporal "points of views" can coexist. We discuss next a few of those levels of coexistence, and show how each of them can lead to a method for combining one-dimensional temporal logic. Each of these methods will have then to be studied on its own to establish whether the properties of the original systems are transferred to their combination via this particular method. With respect to the choices of presentation of logic systems, we contemplate both proof- and model-thoeretical presentations of temporal logics on a point-based ontology. Most of the results presented assume that the flow of time is linear.

First case: External time

One temporal point of view can *external* to the other. The external point of view is describing the temporal evolution of a system $\mathcal{S}$, when system $\mathcal{S}$ is itself a temporal description. Suppose $\mathcal{S}$ is described using a temporal logic T and suppose that the external point of view is given in a possibly distinct logic $\overline{\mathsf{T}}$. For example, consider an agent A, whose temporal beliefs are expressed in $\overline{\mathsf{T}}$ trying to represent the temporal beliefs of an agent B, expressed in T, this is illustrated in Figure 1.

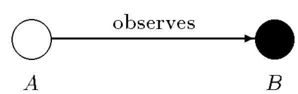

Figure 1 One agent externally observing the other

Agent A's beliefs are external to agents B's beliefs, so that $\overline{\mathsf{T}}$ is externally describing the evolution of T. Figure 2 illustrates the two coexistent temporal points of view.

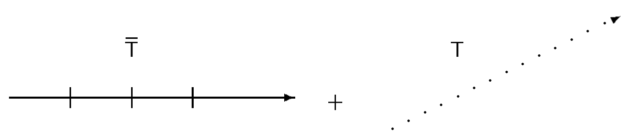

Figure 2 Coexistent temporal systems $\overline{\mathsf{T}}$ and T

The external temporal point of view $\overline{\mathsf{T}}$ is then applied to the internal system T, in a process called *temporalisation* or *adding a temporal dimension to a logic system*, [FG92]. The resulting combined logic system $\overline{\mathsf{T}}(\mathsf{T})$ is illustrated in Figure 3.

The temporalisation associates every time point in $\overline{\mathsf{T}}$with a temporal description in T, where those T-descriptions need not be all identical. Given the logical properties of T and $\overline{\mathsf{T}}$, what can be said about the logical properties of $\overline{\mathsf{T}}(\mathsf{T})$?

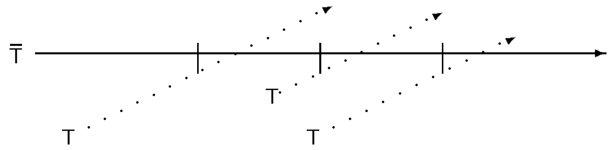

Figure 3 The combined flow of time $\bar{\mathsf{T}}(\mathsf{T})$

Second case: Independent agents

Suppose now that agent A has the ability of referring to agent B's temporal beliefs and vice versa. The agents are therefore obseving each other, as illustrated in figure 4.

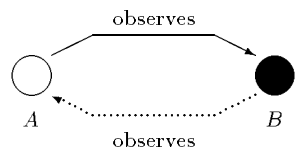

Figure 4 Independent interaction of agents

The agents' beliefs are then capable of interacting with each other through several levels of cross-reference, as in the sentence "A believes that B believes that A believes that ...". A new combination method for T and $\bar{\mathsf{T}}$ is needed in order to represent such sentence as a formula; which is called the *independent combination*, $\bar{\mathsf{T}} \oplus \mathsf{T}$. Since a formula of $\bar{\mathsf{T}} \oplus \mathsf{T}$ has a finite nature, it can be unravalled in a finite number of alternating temporalisation, as illustrated in Figure 5.

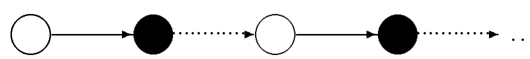

Figure 5 Unravelling $\bar{\mathsf{T}} \oplus \mathsf{T}$

Figure 5 suggests a way of analysing the properties id the independence combination method using the temporalisation method as an intermediary step. It will turn out that the independence combination method is the (infinite) union of all finite alternated temporalisations. An illustration of a possible independently combined flow of time is presented in Figure 6.

Third case: Two-dimensional plane

Yet another distinct situation can be found where we have the coexistence of two distinct temporal "points of view". This time a single agent with temporal reasoning capabilites is considered, and we want to be able to describe the evolution of his own beliefs. This is perhaps better illustrated by considering the agent as a temporal database where each piece of information is associated to a validity time (or interval). For example, consider the traditional database relation $employee(Name, Salary, Manager)$. Suppose the following is in the database at March 94

Name	*Salary*	*Dept*	*Start*	*End*
Peter	1000	R&D	Apr 93	Mar 94

where the attributes start and end represent the end points of the validity interval associated with the information. We assume that Peter's salary has not changed since Apr 93.

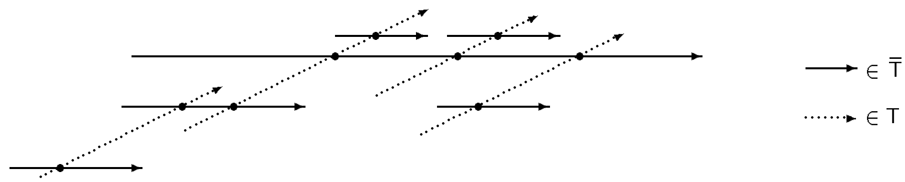

Figure 6 Independently combined flow of time

Suppose in Apr 94 Peter receives a retroactive promotion dating back to the beginning of the year, increasing his salary to 2000. The whole database evolution is illustrated at Figure 7, where only the value of Peter's salary is indicated at each point.

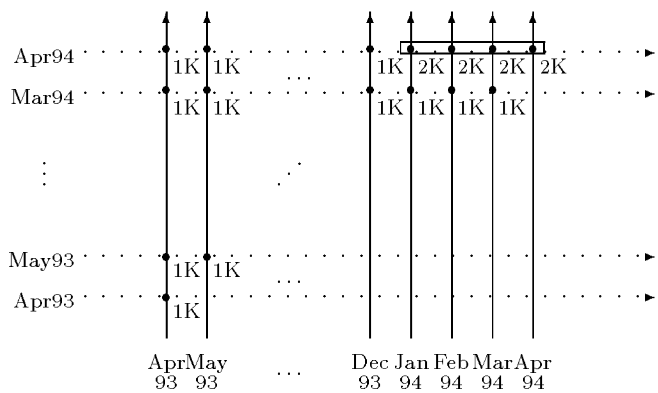

Figure 7 Two-dimensional temporal database evolution

If T represents valid-time and $\overline{\mathsf{T}}$ represents transaction-time, we have guaranteed a two-dimensional plane $\overline{\mathsf{T}} \times \mathsf{T}$ in order to represent the database evolution.

Another application of the two-dimensional plane (or its NW-semi-plane) is in the representation of intervals on a line [Ven90].

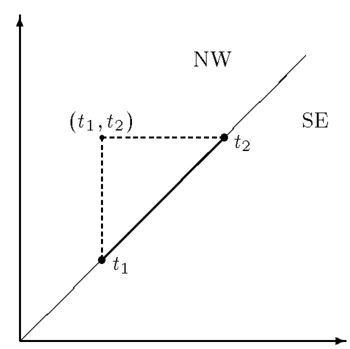

Figure 8 Two-dimensional representation of intervals

In Figure 8 we can see a line considered the diagonal of a two-dimensional plane and that a interval $[t_1, t_2]$ on that line is represented by the point (t_1, t_2) on the NW-semi-plane.

The combination between two temporal systems leading to a two-dimensional plane flow of time is stronger than the simple independent combination and will be studied on its on.

Aims

In this paper we study those three situations of coexistence of "two temporal points of view", as the result of a combination of two linear, one-dimensional temporal logics.

In this sense this paper is a continuation on the work started in [FG92] on the combination of temporal logics. There, a process for adding a temporal dimension to a logic system was described, in which a temporal logic T is *externally* applied to a generic logic system L, generating a combined logic $\mathsf{T}(\mathsf{L})$.

We now set to explore several methods for systematically combining two temporal logics, T and $\overline{\mathsf{T}}$, thus generating several new families of *two-dimensional temporal logics*.

A great number of (one-dimensional) temporal logics exist in the literature to deal with the great variety of properties one may wish to express about flows of time. When building two-dimensional temporal logics, the combination of two classes of flows of time generates an even greater number of possible systems to be studied. Furthermore, as we will see, there are several distinct classes of temporal logics, that may be considered two-dimensional, each generated by distinct combination method. It is, therefore, desirable to study if it is possible to *transfer the properties* of long known and studied (one-dimensional) temporal logic system to the two-dimensional case.

So the main goal of this paper is to study, for each combination method, the transference of logical properties from component one-dimensional temporal systems to a combined two-dimensional one.

In this work, we concentrate on the transference of three basic properties of logic systems, namely soundness, completeness and decidability. This by no means implies that those are the only properties whose transference deserve to be studied, but, as has already been noted in [FG92] for the temporal case, and in [KW91, FS91] for the monomodal case, the transference of completeness serves as a basis for the transference of several other properties of logical systems.

As for the methods for combining two temporal logics, we consider the following:

(a) The temporalisation method, ie the external application of a temporal logic to another temporal system, also known as adding a temporal dimension to a logic system;

(b) the independent combination of two temporal systems;

(c) the full interlacing of two temporal systems, where flows of time are considered over a two-dimensional plane;

(d) the restricted interlacing of two temporal system, a combination method that restricts the previous one but generates nice transference results.

We proceed as follows. Section 2 presents the basic notions of one- and two-dimensional temporal logics. Section 3 discusses combinations of logics in general terms, so that in the rest of the paper we can present special cases of combination methods. Section 4 briefly examines the transference results obtained for the temporalisation method in [FG92]. Section 5 studies the method of independent combination. Section 6 deals with the full interlacing method and Section 7 with its restricted version. Section 8 analyses the properties of a two-dimensional diagonal on the model generated by the full and restricted interlacing methods. In Section 9 we discuss the results of this work.

The current version of this paper is still missning a section that relates the combination methods described here with more general works on combining any (not only temporal) logical systems, such as [Gab92]. This connection should be presented in a later version of the paper.

2 Preliminaries

For the purposes of this paper, a logic system is composed of three elements:

(a) a language, normally given by a set of formation rules generating well found formulae over a signature and a set of logical connectives.

(b) An inference system, ie a relation, $\vdash$, between sets of formulae, normally represented by upper case Greek letters $\Delta, \Gamma, \Sigma, \Psi, \Phi$ and a single formula, normally represented by upper case letters $A, B, C, ...$; the fact that A is inferred from a set Δ is indicated by $\Delta \vdash A$. When Δ is a singleton, $\Delta = \{B\}$, the notation is abused and we write $B \vdash A$.

(c) The semantics of formulae over a class $\mathcal{K}$ of model structures. The fact that a formulae A is true of or holds at a model $\mathcal{M} \in \mathcal{K}$ is indicated by $\mathcal{M} \models A$.

In providing a method for combining two logics into a third one, it will be necessary to provide three sub-methods that combine the languages, inference systems and semantics of the component logic systems. The component systems considered in this paper will be one-dimensional linear us-temporal logics. Their language is built from a countable signature of propositional letters $\mathcal{P} = \{p_1, p_2, \ldots\}$, the Boolean connectives $\wedge$ (conjunction) and $\neg$ (negation), the two-place temporal operation U (until) and S (since), possibly renamed, and the following formation rules:

- every propositional letters is a formulae
- if A and B are formulae, so are $\neg A$ and $A \wedge B$
- if A and B are formulae, so are $U(A, B)$ (reads "until A is time in the future, B will be time") and $S(A, B)$ (reads "since A was time in the past, B has been time").
- nothing else is a formula.

The *mirror image* of a formula is another temporal formula obtained by swapping all occurrences of U and S, *e.g.* the mirror image of $U(A, S(B, C))$ is $S(A, U(B, C))$.

The other Boolean connectives $\vee$ (disjunction), $\rightarrow$ (material implication), $\leftrightarrow$ (material bi-implication) and the constants $\bot$ (false) and $\top$ (true) can be derived in the standard way. Similarly, the one-place temporal operates F ("sometime in the future"), P ("sometime in the past"), G ("always in the future") and H ("always in the past") can be defined in terms of U and S.

For the semantics of temporal formulae we have to consider a (one-dimensional) *flow of time*, $\mathcal{F} = (T <)$, where T is a set of time points and $<$ is an order over T. A temporal valuation $h : T \rightarrow 2^{\mathcal{P}}$ associates every time point with a set of propositional letters, *i.e.* $h(t)$ is the set of propositions that are true at time t[1]. A model structure $\mathcal{M} = (T, <, h)$ consists of flow of time $(T, <)$ and a temporal assignment h and, for the purposes of combination of logics, we consider a "current world" $t \in T$ as part of the model. $\mathcal{M}, t \models A$ reads "A is true at t over model $\mathcal{M}$". Classes of models are normally defined by restrictions over the order relation $<$ of the flow of time.

The semantics of temporal formulae is given by:

[1]Equivalently, and perhaps more usually, a valuation could be defined as a function $h : \mathcal{P} \rightarrow 2^T$, associating every propositional letter to a set of time points in which it holds true [Bur84, GHR94].

$\mathcal{M}, t \models p$	iff $p \in \mathcal{P}$ such that $p \in h(t)$.
$\mathcal{M}, t \models \neg A$	iff it is not the case that $\mathcal{M}, t \models A$.
$\mathcal{M}, t \models A \wedge B$	iff $\mathcal{M}, t \models A$ and $\mathcal{M}, t \models B$.
$\mathcal{M}, t \models S(A, B)$	iff there exists an $s \in T$ with $s < t$ and $\mathcal{M}, s \models A$ and for every $u \in T$, if $s < u < t$ then $\mathcal{M}, u \models B$.
$\mathcal{M}, t \models U(A, B)$	iff there exists an $s \in T$ with $t < s$ and $\mathcal{M}, s \models A$ and for every $u \in T$, if $t < u < s$ then $\mathcal{M}, u \models B$.

The following restriction will be applied throughout this presentation. Flows of time will always be considered to have the properties:

(a) irreflexivity: $\forall t \neg(t < t)$

(b) transitivity: $\forall t, s, u(t < s \wedge s < u \rightarrow t < u)$

(c) totality: $\forall t, s(t = s \vee t < s \vee s < t)$

The class of all flows respecting the restrictions above is the class $\mathcal{K}_{lin}$ of linear flows of time. We also represent the class of all models based on linear flows as $\mathcal{K}_{lin}$. Further restrictions can be applied to the nature of flows of time so that several other linear subclasses can be formed, e.g. the classes of dense ($\mathcal{K}_{dense}$), discrete ($\mathcal{K}_{dis}$), $\mathbb{Z}$-like, $\mathbb{Q}$-like and $\mathbb{R}$-like flows of time. The linearity property allows for the definition of the "at all times" temporal connective $\Box$

$$\Box A = A \wedge GA \wedge HA$$

In case of discrete flows of time, the operator "next time", $\bigcirc$, and "previous time", $\bullet$ are also defined.

$$\bigcirc A = U(A, \bot)$$
$$\bullet A = S(A, \bot)$$

The inference systems will be considered to be finite axiomatisations, i.e. a pair $(\Sigma, \mathcal{I})$ where Σ is a finite set of formulae called *axioms* and $\mathcal{I}$ is a set of inference rules. Consider the Burgess-Xu axiomatisation for $\mathcal{K}_{lin}$ [Bur82, Xu88] consisting of the following axioms:

A0 all classical tautologies

A1a $G(p \rightarrow q) \rightarrow (U(p, r) \rightarrow U(q, r))$

A2a $G(p \rightarrow q) \rightarrow (U(r, p) \rightarrow U(r, q))$

A3a $(p \wedge U(q, r)) \rightarrow U(q \wedge S(p, r), r)$

A4a $U(p, q) \rightarrow U(p, q \wedge U(p, q))$

A5a $U(q \wedge U(p, q), q) \rightarrow U(p, q)$

A6a $(U(p, q) \wedge U(r, s)) \rightarrow$
$(U(p \wedge r, q \wedge s) \vee U(p \wedge s, q \wedge s) \vee U(q \wedge r, q \wedge s))$

plus their mirror images (**b** axioms). The inference rules are:

Subst Uniform Substitution, i.e. let $A(q)$ be an axiom containing the propositional letter q and let B be any formula, then from $\vdash A(q)$ infer $\vdash A(q \backslash B)$ by substituting all appearances of q in A by B.

MP Modus Ponens: from $\vdash A$ and $\vdash A \to B$ infer $\vdash B$.

TG Temporal Generalisation: from $\vdash A$ infer $\vdash HA$ and $\vdash GA$.

A formula A is deducible from the set of formulae $\Delta, \Delta \vdash A$, if there exist a finite sequence of formulae $B_1, \ldots, B_n = A$ such that every B_i is either

(a) a formula in Δ; or

(b) an axiom; or

(c) obtained from previous formulae in the sequence through the use of an inference rule.

We write $\vdash A$ for $\emptyset \vdash A$, i.e. only items (b) and (c) above are used the deduction of A, in which case A is said to be a *theorem*. A set of formulae Δ is inconsistent if $\Delta \vdash \bot$, otherwise it is *inconsistent*. A formulae A is consistent if $\{A\}$ is consistent.

On the semantical side, a set of formulae Δ is *satisfiable* over a class of models $\mathcal{K}$ if there exist a model $\mathcal{M} \in \mathcal{K}$ (and a $t \in T$) such that, for every $B_i \in \Delta$, $\mathcal{M}, t \models B_i$. A formula A is *valid* over $\mathcal{K}$, $\mathcal{K} \models A$ if, for every model $\mathcal{M} = (T, <, h) \in \mathcal{K}$ (and $t \in T$), $\mathcal{M}, t \models A$. The expression $\Delta \models A$ represents that every model satisfying Δ also satisfies A.

An inference system is *sound* with respect to a class of models $\mathcal{K}$ iff every theorem is a valid formula, i.e. $\vdash A$ implies $\mathcal{K} \models A$. An inference system is *(weakly)* complete over $\mathcal{K}$, if every theorem $\vdash A$ is valid, $\mathcal{K} \models A$, or equivalently, if every consistent formula is satisfied over $\mathcal{K}$. Let $\mathsf{L} = \langle \mathcal{L}, \vdash, \models \rangle$ be a logic system with language L, inference system $\vdash$ and semantics $\models$. We say that L is *decidable* if there exists an algorithm (decision procedure) that determines, for every $A \in \mathcal{L}$, whether A is a theorem or not. The *validity problem* for L is to determine whether some $A \in \mathcal{L}$ is a valid formula or not.

We have the following results

Theorem 2.1 ([Bur82, Xu88]) *The Burgess-Xu axiomatisation is sound and complete over $\mathcal{K}_{lin}$.*

Theorem 2.2 ([BG85]) *The logic $\mathsf{US} = \langle \mathcal{L}_{\mathsf{US}}, \vdash_{\mathsf{US}}, \models_{US} \rangle$ is decidable over K_{lin}.*

3 Combining Logics

As we have mentioned earlier, the combination of two one-dimensional temporal logics will generate a two-dimensional temporal logic. Throughout this presentation, we refer to one of the temporal dimensions as the *horizontal dimension* and the other one as the *vertical dimension*; the symbols related to the vertical dimension are normally obtained by putting a bar on top of the corresponding horizontal ones, *e.g.* T and $\overline{\mathsf{T}}$, $F$and $\overline{F}$, $<$ and $\overline{<}$.

There are two distinct criteria for defining a modal/temporal logic system as two-dimensional:

(i) If the alphabet of the language contains two non-empty, disjoint sets of corresponding modal or temporal operators, Φ and $\overline{\Phi}$, each set associated to a distinct flow of time, $(T, <)$ and $(\overline{T}, \overline{<})$, then the system is two-dimensional.

(ii) If the truth value of a formula is evaluated with respect to two time points, then the system is two-dimensional. In this case, we even have the distinction between strong and weak interpretation of formulae that, as a consequence, generates different notions of valid formulae (a formulae is valid if it holds in all models for all pair of time points). Under the *strong interpretation*, the truth value of atoms depends on both dimensions, giving origin to *strongly valid formulae* when the evaluation of formulae is inductively extended to all connectives. In the *weak interpretation*, the truth value of atoms depends only on the one dimension, *e.g.* the horizontal dimension, giving origin to *weakly valid formulae*. Usually for this notion of two-dimensionality, both time points refer to the same flow of time, so we may also have the notion of (weak/strong) *diagonally valid* formulae by restricting validity to the case where both dimensions refer to the same point, *i.e.* A is diagonally valid iff $\mathcal{M}, t, t \models A$ for all $\mathcal{M}$ and t; see [GHR94] for more details.

Criterion (i) above will be called the *syntactic criterion* for two-dimensionality, although it is not completely syntactic, *i.e.* it depends on the semantic notion of flows of time; criterion (ii) will be called the *semantic criterion* for two-dimensionality.

Note that both cases can yield, as an extreme case, one-dimensional temporal logic. In (i), by making $T = \overline{T}$ and $\overline{<} = (<)^{-1} = (>)$, *i.e.* by taking two flows with the same set of time points such that one order is the inverse of the other; in this case, the future operators $\Phi = \{F, G, U\}$ are associated with $(T, <)$ and the past operators $\overline{\Phi} = \{P, H, S\}$ are associated with $(T, >)$. In (ii), by fixing one dimension to a single time point so that the second dimension becomes redundant.

These two distinct approaches to the two-dimensionality of a system are independent. In fact, we will see in Section 5 a system that contains two distinct sets of operators over two classes of flows of time, but its formulae are evaluated at a single point. On the other hand, there are several temporal logics in the literature satisfying (ii) but not (i), containing a single set of temporal operators in which formulae are evaluated according to two or more time points in the same flow [Aqv79, Kam71, GHR94].

A logic system that respects both the syntactic and the semantic criteria for two-dimensionality is called *broadly two-dimensional*, and this will be the kind of system we will be aiming to achieve through combination methods; we consider in this work only strong evaluation and validity; the weak interpretation generates systems with the expressivity of only monadic first-order language [GHR94], but for broadly two-dimensional systems we are interested in the expressivity of dyadic first-order language, although it is known that no set of temporal operators can be expressively complete[2] over dyadic first-order language [Ven90]. Venema's [Ven90] two-dimensional temporal logic, Segerberg's [Seg73] two-dimensional modal logic and the temporalisation of a temporal logic are all broadly two-dimensional; so are the combined logics in Sections 6 and 7.

In the study of one-dimensional temporal logics (1DTLs) several classes of flows of time are taken into account. When we move to 2DTLs, the number of such classes increases considerably, and every pair of one-dimensional classes can be seen as generating a different two-dimensional class. The study of 2DTLs would benefit much if the properties known to hold for 1DTLs could be systematically transferred to 2DTLs, avoiding the repetition of much of the work that has been published in the literature. This is a strong motivation

[2] A modal/temporal language is *expressively complete* over a class of first-order formulae if, for any first-order formula A in that class, there exists a modal/temporal formula B such that A is first-order equivalent to B^*, where B^* is the standard first-order translation of B [GHR94].

to consider methods of combination of 1DTLs into 2DTLs and studying the transference of logical properties through each method. Also in favour of such an approach is the fact that the results concerning 2DTLs are then presented in a general, compact and elegant form.

In providing a method to combine two 1DTLs $\overline{\mathsf{T}}$ and T we have to pay attention to the following points:

(a) A method for combining logics $\overline{\mathsf{T}}$ and T is composed of three sub-methods, namely a method for combining the languages of $\overline{\mathsf{T}}$ and T, a method for combining their inference systems and a method for combining their semantics.

(b) We study the combined logic system with respect to the way certain logical properties of $\overline{\mathsf{T}}$ and T are transferred to the two-dimensional combination. We focus here on the properties of soundness, completeness and decidability of the combined system given those of the component ones.

(c) The combined language should be able to express some properties of the interaction between the two-dimensions; otherwise the combination is just a partial one, and the two systems are not fully combined. For example, it is desirable to express formulae like $F\overline{F}A \leftrightarrow \overline{F}FA$ and $P\overline{F}A \leftrightarrow \overline{F}PA$ that are not in the temporalised language of $\overline{\mathsf{T}}(\mathsf{T})$.

(d) If we want to strengthen the interaction between the two systems, some properties of the interaction between the two-dimensions are expected to be theorems of the combining system, *e.g.* the commutativity of horizontal and vertical future operators such as $F\overline{F}A \leftrightarrow \overline{F}FA$ and $P\overline{F}A \leftrightarrow \overline{F}PA$.

(e) We want the combination method to be as independent as possible from the underlying flows of time.

All methods of combination must comply with item (a). The method for combining the languages of $\overline{\mathsf{T}}$ and T includes the choice of which sublanguage of $\overline{\mathsf{T}}$ and T is going to be part of the combined two-dimensional language, as well as the way in which this combination is done; in this presentation we will work, in the most general case, with the standard languages of S and U, $\overline{S}$ and $\overline{U}$, but we also consider some sublanguages, *e.g.* the sublanguage generated by a set of derived operators, as the vertical "previous" ($\overline{\bullet}$) and "next" ($\overline{\bigcirc}$) in Section 7. In combining the inference systems of $\overline{\mathsf{T}}$ and T, we will assume that they are both an extension of classical logic and that they are presented in the form of a regular, normal axiomatic system $(\Sigma, \mathcal{I})$, where Σ is a set of axioms and $\mathcal{I}$ is a set of inference rules; one important requirement is that the combined system be a conservative extension of the two components. The conservativeness property states that if A is a formula in the language of T and T^* is a logic system extending T (*i.e.* the language of T is a sublanguage of the language of T^*) then A will be a theorem of T^* only if it is a theorem of T already; conservativeness guarantees that no new information about the original system T is present in the extended one T^*.

The combined semantics has to deal with the structure of the combined model, the evaluation of two-dimensional formulae over those structures and also with the combinations of classes of flows of time.

Items (b), (c), (d) and (e) may conflict with each other. In fact, the rest of this paper shows that this is the case, as we try to compromise between expressivity, independence of the underlying flow of time and the transference of logical properties.

4 Temporalising a Logic

The first of the combination methods, known as "adding a temporal dimension to a logic system" or simply "temporalising a logic system", has been extensively discussed in [FG92].

Temporalisation is a methodology whereby an arbitrary logic system L can be enriched with temporal features to create a new system $\mathsf{T(L)}$. The new system is constructed by combining L with a pure propositional temporal logic T (such as linear temporal logic with "Since" and "Until") in a special way.

Although we are only interested here in temporalising an already temporal system, so as to generate a 2DTL, the original method is more general and is applicable to a generic logic L; L is to actually constrained to be an extension of classical logic, i.e. all propositional tautologies must be valid in it, but such a constraint does not affect us, for we are assuming that both temporal systems T and L are extensions of $US/\mathcal{K}_{lin}$ The language of a temporalised system is based on the US language and is a subset of the language of L, $\mathcal{L}_L$. The set $\mathcal{L}_\mathsf{L}$ is partitioned in two sets, BC_L and ML_L. A formula $A \in \mathcal{L}_\mathsf{L}$ belongs to the set of *boolean combinations*, BC_L, iff it is built up from other formulae by the use of one of the boolean connectives $\neg$ or $\wedge$ or any other connective defined only in terms of those; it belongs to the set of *monolithic formula* ML_L otherwise.

The result of temporalising over $\mathcal{K}$ the logic system L is the logic system $\mathsf{T(L)}/\mathcal{K}$. The alphabet of the temporalised language uses the alphabet of L plus the two-place operators S and U, if they are not part of the alphabet of L; otherwise, we use $\overline{S}$ and $\overline{U}$ or any other proper renaming.

Definition 4.1 Temporalised formulae The set $\mathcal{L}_{\mathsf{T(L)}}$ of formulae of the logic system L is the smallest set such that:

1. If $A \in ML_\mathsf{L}$, then $A \in \mathcal{L}_{\mathsf{T(L)}}$;
2. If $A, B \in \mathcal{L}_{\mathsf{T(L)}}$ then $\neg A \in \mathcal{L}_{\mathsf{T(L)}}$ and $(A \wedge B) \in \mathcal{L}_{\mathsf{T(L)}}$;
3. If $A, B \in \mathcal{L}_{\mathsf{T(L)}}$ then $S(A, B) \in \mathcal{L}_{\mathsf{T(L)}}$ and $U(A, B) \in \mathcal{L}_{\mathsf{T(L)}}$.

□

Note that, for instance, if $\Box$ is an operator of the alphabet of L and A and B are two formulae in $\mathcal{L}_\mathsf{L}$, the formula $\Box U(A, B)$ is *not* in $\mathcal{L}_{\mathsf{T(L)}}$. The language of $\mathsf{T(L)}$ is independent of the underlying flow of time, but not its semantics and inference system, so we must fix a class $\mathcal{K}$ of flows of time over which the temporalisation is defined; if $\mathcal{M}_\mathsf{L}$ is a model in the class of models of L, $\mathcal{K}_\mathsf{L}$, for every formula $A \in \mathcal{L}_\mathsf{L}$ we must have either $\mathcal{M}_\mathsf{L} \models A$ or $\mathcal{M}_\mathsf{L} \models \neg A$. In the case that L is a temporal logic we must consider a "current time" o as part of its model to achieve that condition.

Definition 4.2 Semantics of the temporalised logic Let $(T, <) \in \mathcal{K}$ be a flow of time and let $g : T \to \mathcal{K}_\mathsf{L}$ be a function mapping every time point in T to a model in the class of models of L. A model of $\mathsf{T(L)}$ is a triple $\mathcal{M}_{\mathsf{T(L)}} = (T, <, g)$ and the fact that A is true in $\mathcal{M}_{\mathsf{T(L)}}$ at time t is written as $\mathcal{M}_{\mathsf{T(L)}}, t \models A$ and defined as:

$\mathcal{M}_{\mathsf{T(L)}}, t \models A, A \in ML_{\mathsf{L}}$	iff $g(t) = \mathcal{M}_{\mathsf{L}}$ and $\mathcal{M}_{\mathsf{L}} \models A$.
$\mathcal{M}_{\mathsf{T(L)}}, t \models \neg A$	iff it is not the case that $\mathcal{M}_{\mathsf{T(L)}}, t \models A$.
$\mathcal{M}_{\mathsf{T(L)}}, t \models (A \wedge B)$	iff $\mathcal{M}_{\mathsf{T(L)}}, t \models A$ and $\mathcal{M}_{\mathsf{T(L)}}, t \models B$.
$\mathcal{M}_{\mathsf{T(L)}}, t \models S(A, B)$	iff there exists $s \in T$ such that $s < t$ and $\mathcal{M}_{\mathsf{T(L)}}, s \models A$ and for every $u \in T$, if $s < u < t$ then $\mathcal{M}_{\mathsf{T(L)}}, u \models B$.
$\mathcal{M}_{\mathsf{T(L)}}, t \models U(A, B)$	iff there exists $s \in T$ such that $t < s$ and $\mathcal{M}_{\mathsf{T(L)}}, s \models A$ and for every $u \in T$, if $t < u < s$ then $\mathcal{M}_{\mathsf{T(L)}}, u \models B$.

□

Figure 3 illustrates a temporalised model. The inference system of $\mathsf{T(L)}/\mathcal{K}$ is given by the following:

Definition 4.3 Axiomatisation for $\mathsf{T(L)}$ An axiomatisation for the temporalised logic $\mathsf{T(L)}$ is composed of:

- The axioms of $\mathsf{T}/\mathcal{K}$;
- The inference rules of $\mathsf{T}/\mathcal{K}$;
- For every formula A in $\mathcal{L}_{\mathsf{L}}$, if $\vdash_{\mathsf{L}} A$ then $\vdash_{\mathsf{T(L)}} A$, *i.e.* all theorems of L are theorems of $\mathsf{T(L)}$. This inference rule is called **Persist**.

□

Example 4.1 Temporalising propositional logic Consider classical propositional logic $\mathsf{PL} = \langle \mathcal{L}_{\mathsf{PL}}, \vdash_{\mathsf{PL}}, \models_{\mathsf{PL}} \rangle$. Its temporalisation generates the logic system $\mathsf{T(PL)} = \langle \mathcal{L}_{\mathsf{T(PL)}}, \vdash_{\mathsf{T(PL)}}, \models_{\mathsf{T(PL)}} \rangle$.

It is not difficult to see that $\mathcal{L}_{\mathsf{T(PL)}} = \mathcal{L}_{\mathsf{US}}$ and $\vdash_{\mathsf{T(PL)}} = \vdash_{\mathsf{US}}$, i.e. the temporalised version of PL over any $\mathcal{K}$ is actually the temporal logic $\mathsf{T} = \mathsf{US}/\mathcal{K}$. With respect to $\mathcal{M}_{\mathsf{T(L)}}$, the function g actually assigns, for every time point, a PL model. □

Example 4.2 Temporalising US-temporal logic If we temporalise over $\mathcal{K}$ the one-dimensional logic system $\mathsf{US}/\mathcal{K}$ we obtain the two-dimensional logic system $\mathsf{T(US)} = \langle \mathcal{L}_{\mathsf{T(US)}}, \vdash_{\mathsf{T(US)}}, \models_{\mathsf{T(US)}} \rangle = \mathsf{T}^2(\mathsf{PL})/\mathcal{K}$. In this case we have to rename the two-place operators S and U of the temporalised alphabet to, say, $\overline{S}$ and $\overline{U}$.

In order to obtain a model for $\mathsf{T(US)}$, we must fix a "current time", o, in $\mathcal{M}_{\mathsf{US}} = (T_1, <_1, g_1)$, so that we can construct the model $\mathcal{M}_{\mathsf{T(US)}} = (T_2, <_2, g_2)$ as previously described. Note that, in this case, the flows of time $(T_1, <_1)$ and $(T_2, <_2)$ need not to be the same. $(T_2, <_2)$ is the flow of time of the upper-level temporal system whereas $(T_1, <_1)$ is the flow of time of the underlying logic which, in this case, happens to be a temporal logic.

The logic system we obtain by temporalising US-temporal logic is the two-dimensional temporal logic described in [Fin92]. □

Example 4.3 N-dimensional temporal logic If we repeat the process started in the last two examples, we can construct an n-dimensional temporal logic $\mathsf{T}^{\mathsf{n}}(\mathsf{PL})/\mathcal{K}$ (its alphabet including S_n and U_n) by temporalising a $(n-1)$-dimensional temporal logic.

Every time we add a temporal dimension, we are able to describe changes in the underlying system. Temporalising the system L once, we are creating a way of describing the history of L; temporalising for the second time, we are describing how the history of L is viewed in different moments of time. We can go on indefinitely, although it is not clear what is the purpose of doing so. □

From now on we restrict the logic systems to $\mathsf{L} = \mathsf{US}/\mathcal{K}$ and $\mathsf{T} = \bar{\mathsf{U}}\bar{\mathsf{S}}/\overline{\mathcal{K}}$, where $\mathcal{K}, \overline{\mathcal{K}} \subseteq \mathcal{K}_{lin}$. We write $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$ instead of $\mathsf{T}(\mathsf{L})$ and the generated class of models is referred to as $\overline{\mathcal{K}}(\mathcal{K})$. For this system, we enumerate a series of results that are proved in [FG92]. Those results will be useful for the discussion of the independent combination method.

Theorem 4.1 (Transference via temporalisation) *Consider the logic systems* $\bar{\mathsf{U}}\bar{\mathsf{S}}/\overline{\mathcal{K}}$ *and* $\mathsf{US}/\mathcal{K}$, $\overline{\mathcal{K}}, \mathcal{K} \subseteq \mathcal{K}_{lin}$.

- *(a) If* $\bar{\mathsf{U}}\bar{\mathsf{S}}$ *is sound with respect to* $\overline{\mathcal{K}}$ *and* US *is sound with respect to* $\mathcal{K}$, *then* $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$ *is sound w.r.t.* $\overline{\mathcal{K}}(\mathcal{K})$.
- *(b) If* $\bar{\mathsf{U}}\bar{\mathsf{S}}$ *is complete w.r.t.* $\overline{\mathcal{K}}$ *and* US *is complete w.r.t.* $\mathcal{K}$ *then* $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$ *is complete w.r.t.* $\overline{\mathcal{K}}(\mathcal{K})$.
- *(c) If* $\bar{\mathsf{U}}\bar{\mathsf{S}}$ *is complete w.r.t.* $\mathcal{K}$, *then* $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$ *is a conservative extension of both* $\bar{\mathsf{U}}\bar{\mathsf{S}}$ *and* US.
- *(d) If* $\bar{\mathsf{U}}\bar{\mathsf{S}}$ *is complete and is decidable over* $\overline{\mathcal{K}}$ *and* US *is complete and decidable over* $\mathcal{K}$ *then* $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$ *is decidable over* $\overline{\mathcal{K}}(\mathcal{K})$.

5 Independent Combination

We have seen in the previous Section how to add a temporal dimension to a logic system. In particular, if a temporal logic is itself temporalised we obtain a two-dimensional temporal logic. Such a logic system is, however, very weakly expressive; if US is the internal (horizontal) temporal logic in the temporalisation process (F is derived in US), and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ is the external (vertical) one ($\overline{F}$ is defined in $\bar{\mathsf{U}}\bar{\mathsf{S}}$), we cannot express that vertical and horizontal future operators commute,

$$F\overline{F}A \leftrightarrow \overline{F}FA.$$

In fact, the subformula $F\overline{F}A$ is not even in the temporalised language of $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$, nor is the whole formula. In other words, the interplay between the two-dimensions is not expressible in the language of the temporalised $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$.

The idea is then to define a new method of combination of logic systems that puts together all the expressivity of the two component logic systems in an independent way; for that we assume that the language of a system is given by a set of formation rules.

Definition 5.1 Let $Op(\mathsf{L})$ be the set of non-boolean operators of a generic logic L. Let $\bar{\mathsf{T}}$ and T be logic systems such that $Op(\mathsf{T}) \cap Op(\bar{\mathsf{T}}) = \emptyset$. The *fully combined language* of logic systems $\bar{\mathsf{T}}$ and T over the set of atomic propositions $\mathcal{P}$, is obtained by the union of the respective set of connectives and the union of the formation rules of the languages of both logic systems. □

Let the operators U and S be in the language of US and $\overline{U}$ and $\overline{S}$ be in that of $\bar{\mathsf{U}}\bar{\mathsf{S}}$. Note that the renaming of the temporal operator is done prior to the combination, so that the combined systems contains the set of boolean operators $\{\neg, \wedge\}$ coming from both components, plus the set of temporal operators $\{U, S, \overline{U}, \overline{S}\}$. Their fully combined language over a set of atomic propositions $\mathcal{P}$ is given by

- every atomic proposition is in it;
- if A, B are in it, so are $\neg A$ and $A \wedge B$;
- if A, B are in it, so are $U(A, B)$ and $S(A, B)$.
- if A, B are in it, so are $\overline{U}(A, B)$ and $\overline{S}(A, B)$.

In general, we do not want any non-boolean operator to be shared between the two languages, for this may cause problems when combining their axiomatisations. For example[3], if a generic operator $\Box$ belongs to both temporal logic system such that T contains axiom $q \leftrightarrow \Box q$ and system $\overline{\mathsf{T}}$ contains axiom $\neg q \leftrightarrow \Box q$, the union of their axiomatisations will result in an inconsistent systems even though each system might have been itself consistent. To avoid such a behaviour the restriction $Op(\mathsf{T}) \cap Op(\overline{\mathsf{T}}) = \emptyset$ was imposed on the fully combined language of $\overline{\mathsf{T}}$ and T. Not only are the two languages taken to be independent of each other, but the set of axioms of the two systems are supposed to be disjoint; so we call the following combination method the *independent combination* of two temporal logics.

This new method of combination is called *independent* because it takes the independent union of the axiomatisation of its two component systems, and it is based on their fully combined language.

Definition 5.2 Let US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ be two US-temporal logic systems defined over the same set $\mathcal{P}$ of propositional atoms such that their languages are independent. The *independent combination* $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ is given by the following:

- The fully combined language of US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$.
- If $(\Sigma, \mathcal{I})$ is an axiomatisation for US and $(\overline{\Sigma}, \overline{\mathcal{I}})$ is an axiomatisation for $\bar{\mathsf{U}}\bar{\mathsf{S}}$, then $(\Sigma \cup \overline{\Sigma}, \mathcal{I} \cup \overline{\mathcal{I}})$ is an axiomatisation for $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$. Note that, apart from the classical tautologies, the set of axioms Σ and $\overline{\Sigma}$ are supposed to be disjoint, but not the inference rules.
- The class of independently combined flows of time is $\mathcal{K} \oplus \overline{\mathcal{K}}$ composed of biordered flows of the form $(\tilde{T}, <, \overline{<})$ where the connected components of $(\tilde{T}, <)$ are in $\mathcal{K}$ and the connected components of $(\tilde{T}, \overline{<})$ are in $\overline{\mathcal{K}}$, and $\tilde{T}$ is the (not necessarily disjoint) union of the sets of time points T and $\overline{T}$ that constitute each connected component; such a biordered flow of time has been discussed in [KW91] for the case of the independent combination of two mono-modal systems.

  A model structure for $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ over $\mathcal{K} \oplus \overline{\mathcal{K}}$ is a 4-tuple $(\tilde{T}, <, \overline{<}, g)$, where $(\tilde{T}, <, \overline{<}) \in \mathcal{K} \oplus \overline{\mathcal{K}}$ and g is an assignment function $g : \tilde{T} \rightarrow 2^{\mathcal{P}}$. An independently combined model is illustrated in Figure 6.

  The semantics of a formula A in a model $\mathcal{M} = (\tilde{T}, <, \overline{<}, g)$ is defined as the union of the rules defining the semantics of $\mathsf{US}/\mathcal{K}$ and $\bar{\mathsf{U}}\bar{\mathsf{S}}/\overline{\mathcal{K}}$. The expression $\mathcal{M}, t \models A$ reads that the formula A is true in the (combined) model $\mathcal{M}$ at the point $t \in \tilde{T}$. The semantics of formulae is given by induction in the standard way:
  $\mathcal{M}, t \models p$ iff $p \in g(t)$ and $p \in \mathcal{P}$.
  $\mathcal{M}, t \models \neg A$ iff it is not the case that $\mathcal{M}, t \models A$.
  $\mathcal{M}, t \models A \wedge B$ iff $\mathcal{M}, t \models A$ and $\mathcal{M}, t \models B$.

[3] this example is due to Ian Hodkinson

$\mathcal{M}, t \models S(A,B)$ iff there exists an $s \in \tilde{T}$ with $s < t$ and $\mathcal{M}, s \models A$ and for every $u \in \tilde{T}$, if $s < u < t$ then $\mathcal{M}, u \models B$.

$\mathcal{M}, t \models U(A,B)$ iff there exists an $s \in \tilde{T}$ with $t < s$ and $\mathcal{M}, s \models A$ and for every $u \in \tilde{T}$, if $t < u < s$ then $\mathcal{M}, u \models B$.

$\mathcal{M}, t \models \overline{S}(A,B)$ iff there exists an $s \in \tilde{T}$ with $s \overline{<} t$ and $\mathcal{M}, s \models A$ and for every $u \in \tilde{T}$, if $s \overline{<} u \overline{<} t$ then $\mathcal{M}, u \models B$.

$\mathcal{M}, t \models \overline{U}(A,B)$ iff there exists an $s \in \tilde{T}$ with $t \overline{<} s$ and $\mathcal{M}, s \models A$ and for every $u \in \tilde{T}$, if $t \overline{<} u \overline{<} s$ then $\mathcal{M}, u \models B$.

□

Note that, despite the combination of two flows of time, formulae are evaluated according to a single point. The independent combination generates a system that is two-dimensional according to the first criterion but fails the second one, so it is not broadly two-dimensional.

The following result is due to [Tho80] and is more general than the independent combination of two US-logics.

Proposition 5.1 *With respect to the validity of formulae, the independent combination of two modal logics is a conservative extension of the original ones.*

Note that we have previously defined conservative extension in proof theoretical terms; completeness for the independently combined case will lead to the conservativeness with respect to derivable theorems.

As usual, we will assume that $\mathcal{K}, \overline{\mathcal{K}} \subseteq \mathcal{K}_{lin}$, so $<$ and $\overline{<}$ are transitive, irreflexive and total orders; similarly, we assume that the axiomatisations are extensions of $\mathsf{US}/\mathcal{K}_{lin}$.

The temporalisation process will be used as an inductive step to prove the transference of soundness, completeness and decidability for $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ over $\mathcal{K} \oplus \overline{\mathcal{K}}$. Let us first consider the *degree of alternation* of a $(\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}})$-formula A for US, $dg(A)$, and $\bar{\mathsf{U}}\bar{\mathsf{S}}$, $\overline{dg}(A)$.

$dg(p) = 0$	$\overline{dg}(p) = 0$
$dg(\neg A) = dg(A)$	$\overline{dg}(\neg A) = \overline{dg}(A)$
$dg(A \wedge B) = max\{dg(A), dg(B)\}$	$\overline{dg}(A \wedge B) = max\{\overline{dg}(A), \overline{dg}(B)\}$
$dg(S(A,B)) = max\{dg(A), dg(B)\}$	$\overline{dg}(\overline{S}(A,B)) = max\{\overline{dg}(A), \overline{dg}(B)\}$
$dg(U(A,B)) = max\{dg(A), dg(B)\}$	$\overline{dg}(\overline{U}(A,B)) = max\{\overline{dg}(A), \overline{dg}(B)\}$
$dg(\overline{S}(A,B)) = 1 + max\{\overline{dg}(A), \overline{dg}(B)\}$	$\overline{dg}(S(A,B)) = 1 + max\{dg(A), dg(B)\}$
$dg(\overline{U}(A,B)) = 1 + max\{\overline{dg}(A), \overline{dg}(B)\}$	$\overline{dg}(U(A,B)) = 1 + max\{dg(A), dg(B)\}$

Any formula A of $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ can be seen as a formula of some finite number of alternating temporalisations of the form $\mathsf{US}(\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US}(\ldots)))$; more precisely, A can be seen as a formula of $\mathsf{US}(\mathsf{L}_n)$, where $dg(A) = n$, $\mathsf{US}(\mathsf{L}_0) = \mathsf{US}$, $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{L}_0) = \bar{\mathsf{U}}\bar{\mathsf{S}}$, and $\mathsf{L}_{n-2i} = \bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{L}_{n-2i-1})$, $\mathsf{L}_{n-2i-1} = \mathsf{US}(\mathsf{L}_{n-2i-2})$, for $i = 0, 1, \ldots, \lceil \frac{n}{2} \rceil - 1$. This fact is illustrated in Figure 5. The following Lemma actually allows us to see the independent combination as the (infinite) union of finite number of alternating temporalisations of US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$; it will also be used in the proof of transferrence of completeness and decidability (given completeness) for $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$.

Lemma 5.1 *Let* US *and* $\bar{\mathsf{U}}\bar{\mathsf{S}}$ *be two complete logic systems. Then, A is a theorem of* $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ *iff it is a theorem of* $\mathsf{US}(\mathsf{L}_n)$*, where $dg(A) = n$.*

Proof If A is a theorem of $\mathsf{US}(\mathsf{L}_n)$, all the inferences in its deduction can be repeated in $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$, so it is a theorem of $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$.

Suppose A is a theorem of $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$; let $B_1, \ldots, B_m = A$ be a deduction of A in $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ and let $n' = max\{dg(B_i)\}$, $n' \geq n$. We claim that each B_i is a theorem of $\mathsf{US}(\mathsf{L}_{n'})$. In fact, by induction on m, if B_i is obtained in the deduction by substituting into an axiom, the same substitution can be done in $\mathsf{US}(\mathsf{L}_{n'})$; if B_i is obtained by Temporal Generalisation from B_j, $j < i$, then by the induction hypothesis, B_j is a theorem of $\mathsf{US}(\mathsf{L}_{n'})$ and so is B_i; if B_i is obtained by Modus Ponens from B_j and B_k, $j, k < i$, then by the induction hypothesis, B_j and B_k are theorems of $\mathsf{US}(\mathsf{L}_{n'})$ and so is B_i.

So A is a theorem of $\mathsf{US}(\mathsf{L}_{n'})$ and, since US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ are two complete logic systems, by Theorem 4.1, each of the alternating temporalisations in $\mathsf{US}(\mathsf{L}_{n'})$ is a conservative extension of the underlying logic; it follows that A is a theorem of $\mathsf{US}(\mathsf{L}_n)$, as desired. $\square$

The transference of soundness, completeness and decidability follows directly from this result.

Theorem 5.1 (Independent Combination) *Let* US *and* $\bar{\mathsf{U}}\bar{\mathsf{S}}$ *be two sound and complete logic systems over the classes* $\mathcal{K}$ *and* $\overline{\mathcal{K}}$*, respectively. Then their independent combination* $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ *is sound and complete over the class* $\mathcal{K} \oplus \overline{\mathcal{K}}$*. If* US *and* $\bar{\mathsf{U}}\bar{\mathsf{S}}$ *are complete and decidable, so is* $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$*.*

Proof Soundness follows immediately from the validity of axioms and inference rules. For completeness, suppose that A is a consistent formula in $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$; by Lemma 5.1, A is consistent in $\mathsf{US}(\mathsf{L}_n)$, so we construct a temporalised model for it, and we obtain a model $(\tilde{T}_1, <_1, g_1, o_1)$ over $\mathcal{K}(\overline{\mathcal{K}}(\mathcal{K}(\ldots)))$, where o_1 is the "current time" necessary for the successive temporalisations. We show now how it can be transformed into a model over $\mathcal{K} \oplus \overline{\mathcal{K}}$.

Without loss of generality, suppose that US is the outermost logic system in the multi-layered temporalised system $\mathsf{US}(\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US}(\ \ldots\ )))$, and let n be the number of alternations. The construction is recursive, starting with the outermost logic. Let $i \leq n$ denote the step of the construction; if i is odd, it is a US-temporalisation, otherwise it is a $\bar{\mathsf{U}}\bar{\mathsf{S}}$-temporalisation. At every step i we construct the sets $\tilde{T}_{i+1}$, $<_{i+1}$ and $\overline{<}_{i+1}$ and the function g_{i+1}.

We start the construction of the model at step $i = 0$ with the temporalised model $(\tilde{T}_1, <_1, g_1, o_1)$ such that $(\tilde{T}_1, <_1) \in \mathcal{K}$, and we take $\overline{<}_1 = \emptyset$. At step $i < n$, consider the current set of time points $\tilde{T}_i$; according to the construction, each $t \in \tilde{T}_i$ is associated to:

- a temporalised model $g_i(t) = (\tilde{T}^t_{i+1}, <^t_{i+1}, g^t_{i+1}, o^t_{i+1}) \in \mathcal{K}$ and take $\overline{<}^t_{i+1} = \emptyset$, if i is even; or
- a temporalised model $g_i(t) = (\tilde{T}^t_{i+1}, \overline{<}^t_{i+1}, g_{i+1}, o^t_{i+1}) \in \overline{\mathcal{K}}$ and take $<^t_{i+1} = \emptyset$, if i is odd.

The point t is made identical to $o^t_{i+1} \in \tilde{T}^t_{i+1}$, so as to add the new model to the current structure; note that this preserves the satisfiability of all formulae at t. Let $\tilde{T}_{i+1}$ be the (possibly infinite) union of all $\tilde{T}^t_{i+1}$ for $t \in \tilde{T}_i$; similarly, $<_{i+1}$ and $\overline{<}_{i+1}$ are generated. And finally, for every $t \in \tilde{T}_{i+1}$, the function g_{i+1} is constructed as the union of all g^t_{i+1} for $t \in \tilde{T}_i$.

Repeating this construction n times, we obtain a combined model over $\mathcal{K} \oplus \overline{\mathcal{K}}$, $\mathcal{M} = (\tilde{T}_n, <_n, \overline{<}_n, g_n)$, such that for all $t \in \tilde{T}_n$, $g_n(t) \subseteq \mathcal{P}$. Since satisfiability of formulae is preserved at each step, it follows that $\mathcal{M}$ is a model for A, and completeness is proved.

For decidability, again by Lemma 5.1, we can recursively apply the decision procedure of $\mathsf{US}(\mathsf{L}_n)$ and $\bar{\mathsf{U}}\bar{\mathsf{S}}(L_{n-1})$, starting with $n = dg(A)$, thus obtaining a decision procedure for $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$. □

6 Full Interlacing

With respect to the generation of two-dimensional systems, the method of independent combination has two main drawbacks. First, it generates logic systems whose formulae are evaluated at one single time point, not generating a broadly two-dimensional logic. Second, since the method independently combines the two component logic systems, no interaction between the dimension is provided by it. As a consequence, although a formula like $F\overline{F}A \leftrightarrow \overline{F}FA$ is expressible in its language, it will not be valid, as can easily be verified, for it expresses an interplay between the dimensions. We therefore introduce the notion of a *two-dimensional plane model*.

Definition 6.1 Let $\mathcal{K}$ and $\overline{\mathcal{K}}$ be two classes of flow of time. A *two-dimensional plane model* over the *fully combined class* $\mathcal{K} \times \overline{\mathcal{K}}$ is a 5-tuple $\mathcal{M} = (T, <, \overline{T}, \overline{<}, g)$, where $(T, <) \in \mathcal{K}$, $(\overline{T}, \overline{<}) \in \overline{\mathcal{K}}$ and $g : T \times \overline{T} \to 2^{\mathcal{P}}$ is a two-dimensional assignment. The semantics of the horizontal and vertical operators are independent of each other.

$\mathcal{M}, t, x \models S(A, B)$	iff	there exists $s < t$ such that $\mathcal{M}, s, x \models A$ and for all u, $s < u < t$, $\mathcal{M}, u, x \models B$.
$\mathcal{M}, t, x \models \overline{S}(A, B)$	iff	there exists $y \overline{<} x$ such that $\mathcal{M}, t, y \models A$ and for all z, $y \overline{<} z \overline{<} x$, $\mathcal{M}, t, z \models B$.

Similarly for U and $\overline{U}$, the semantics of atoms and boolean connectives remaining the standard one. A formula A is (strongly) valid over $\mathcal{K} \times \overline{\mathcal{K}}$ if for all models $\mathcal{M} = (T, <, \overline{T}, \overline{<}, g)$, for all $t \in T$ and $x \in \overline{T}$ we have $\mathcal{M}, t, x \models A$. □

With respect to the expressivity of fully combined two-dimensional languages, Venema [Ven90] has shown that no finite set of two-dimensional temporal operators is expressively complete over the class of linear flows with respect to dyadic first-order logic — despite the fact that US-temporal logic is expressively complete with respect to monadic first-order logic over $\mathbb{N}$ and over $\mathbb{R}$, and that, with additional operators (the Stavi operators), we can get expressive completeness over $\mathbb{Q}$ and $\mathcal{K}_{lin}$ [Gab81b]. So expressive completeness is not transferred by full interlacing.

It is easy to verify that the following formulae expressing the commutativity of future and past operators between the two dimensions are valid formulae in two-dimensional plane models.

I1 $F\overline{F}A \leftrightarrow \overline{F}FA$
I2 $F\overline{P}A \leftrightarrow \overline{P}FA$
I3 $P\overline{F}A \leftrightarrow \overline{F}PA$
I4 $P\overline{P}A \leftrightarrow \overline{P}PA$

Therefore, if we want to satisfy both the syntactic and the semantic criteria for two-dimensionality, we may define the method of *full interlacing* containing the fully combined language of US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ and their fully combined class of models. The question is whether there is a method for combining their axiomatisations so as to generate a *fully interlaced* *axiomatisation* that transfers the properties of soundness, completeness and decidability. The answer, however, is no, not in general. In some cases we can obtain the transference of completeness, in some other cases the transference fails. To illustrate that, we consider completeness results over classes of the form $\mathcal{K}\times\mathcal{K}$.

We start by defining some useful abbreviations. Let p be a propositional atom; define:

$$hor(p) = \Box(p\wedge\overline{H}\neg p\wedge\overline{G}\neg p)$$
$$ver(p) = \overline{\Box}(p\wedge H\neg p\wedge G\neg p)$$

It is clear that $hor(p)$ makes p true along the horizontal line and false elsewhere; similarly for $ver(p)$ with respect to the vertical.

The axiomatisation of $\mathsf{US}\times\bar{\mathsf{U}}\bar{\mathsf{S}}$ over $\mathcal{K}_{lin}\times\mathcal{K}_{lin}$ extends that of $\mathsf{US}\oplus\bar{\mathsf{U}}\bar{\mathsf{S}}$ over $\mathcal{K}_{lin}\oplus\mathcal{K}_{lin}$ by including the interlacing axioms **I1**–**I4** and the following inference rules:

IR1 if $\vdash hor(p)\rightarrow A$ and p does not occur in A, then $\vdash A$
IR2 if $\vdash ver(p)\rightarrow A$ and p does not occur in A, then $\vdash A$

IR1 and **IR2** are two-dimensional extensions of the irreflexivity inferrence rule (IRR) defined in [Gab81a] for the one-dimensional case: if $\vdash p\wedge H\neg p\rightarrow A$ and p does not occur in A, then $\vdash A$.

Theorem 6.1 (2D-completeness) *There is a sound and complete axiomatisation over the class of full two-dimensional temporal models over $\mathcal{K}_{lin}\times\mathcal{K}_{lin}$.*

A proof can be found in [Fin94] showing that the axiomatisation above is sound and complete over $\mathcal{K}_{lin}\times\mathcal{K}_{lin}$. If $\mathcal{K}_{dis}$ is the class of all linear and discrete flows, [Fin94] also shows completeness results for the classes $\mathcal{K}_{dis}\times\mathcal{K}_{dis}$, $\mathbb{Q}\times\mathbb{Q}$, $\mathcal{K}_{lin}\times\mathcal{K}_{dis}$, $\mathcal{K}_{lin}\times\mathbb{Q}$ and $\mathbb{Q}\times\mathcal{K}_{dis}$.

The negative result is the following.

Proposition 6.1 (2D-unaxiomatisability) *There are no finite axiomatisations for the (strongly) valid two-dimensional formulae over the classes $\mathbb{Z}\times\mathbb{Z}$, $\mathbb{N}\times\mathbb{N}$ and $\mathbb{R}\times\mathbb{R}$.*

This proposition follows directly from Venema's proof that the valid formulae over the upper half two-dimensional plane are not enumerable for $\mathbb{Z}\times\mathbb{Z}$, $\mathbb{N}\times\mathbb{N}$ and $\mathbb{R}\times\mathbb{R}$, which in its turn was based on [HS86]. Since there are sound, complete and decidable US-temporal logics over $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ [Rey92], the general conclusion on full interlacing is the following.

Theorem 6.2 (Full Interlacing) *Completeness and decidability do not transfer in general through full interlacing.*

It has to be noted that two-dimensional temporal logics seem to behave like modal logics in the following sense. We can see the result of the independent combination of US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ as generating a "minimal" combination of the logics, *i.e.* one without any interference between the dimensions. The addition of extra axioms, inference rules or an extra condition on its models has to be studied on its own, just as adding a new axiom to a modal logic or imposing a new property on its accessibility relation has to be analysed on its own.

The full interlacing method illustrates the conflict between the generality of a method and its ability to achieve the transference of logical properties. We next restrict the interlacing method so as to recover the transference of logical properties.

7 Restricted Interlacing

The fact that the transference of logical properties fails for the interlacing of two US-temporal logics does not mean that the interlacing of any two temporal logic systems fails to achieve this transference. We restrict the vertical logic system to a temporal logic $\bar{\mathsf{N}}\bar{\mathsf{P}}$ with operators $\bar{\bigcirc}$ for Next time and $\bar{\bullet}$ for Previous time; the formation rules for the formulae of $\bar{\mathsf{N}}\bar{\mathsf{P}}$ are the standard ones. This is a restriction of the $\bar{\mathsf{U}}\bar{\mathsf{S}}$-language for $\bar{\bigcirc}$ and $\bar{\bullet}$ can be defined in terms of $\bar{U}$ and $\bar{S}$, namely by

$$\bar{\bigcirc}A =_{def} \bar{U}(A, \bot)$$
$$\bar{\bullet}A =_{def} \bar{S}(A, \bot)$$

Not only is the expressivity of the language reduced this way, but also the underlying flow of time is now restricted to a discrete one; in fact, we concentrate our attention on integer-like flows of time.

Let $h : \mathbb{Z} \to \mathcal{P}$ be a temporal assignment over the integers so that the semantics of $\bar{\mathsf{N}}\bar{\mathsf{P}}$ over the integers is the usual for atoms and boolean operators and

$$(\mathbb{Z}, <, h), t \models \bar{\bigcirc}A \quad \text{iff} \quad (\mathbb{Z}, <, h), t+1 \models A$$
$$(\mathbb{Z}, <, h), t \models \bar{\bullet}A \quad \text{iff} \quad (\mathbb{Z}, <, h), t-1 \models A$$

An axiomatisation for $\mathsf{NP}/\mathbb{Z}$ is given by the classical tautologies plus

NP1 $\bar{\bigcirc}\bar{\bullet}p \to p$
NP2 $\bar{\bigcirc}\neg p \leftrightarrow \neg\bar{\bigcirc}p$
NP3 $\bar{\bigcirc}(p \wedge q) \to \bar{\bigcirc}p \wedge \bar{\bigcirc}q$
NP4 The mirror image of **NP1–3** obtained by interchanging $\bar{\bigcirc}$ with $\bar{\bullet}$

The rules of inference are the usual Substitution, Modus Ponens and Temporal Generalisation (from A infer $\bar{\bigcirc}A$ and $\bar{\bullet}A$).

The converse of each axiom can be straightforwardly derived, so the formulae on both sides of the $\to$-connective are actually equivalent. It follows that every $\bar{\mathsf{N}}\bar{\mathsf{P}}$-formula can be transformed into an equivalent one by "pushing in" the temporal operators, e.g. by following the arrows the axioms, and by "cancelling" the occurrences of $\bar{\bigcirc}$ and $\bar{\bullet}$ in a string of temporal operators, e.g. $\bar{\bigcirc}\bar{\bullet}\bar{\bullet}\bar{\bigcirc}\bar{\bullet}p$ is equivalent to $\bar{\bullet}p$; the resulting $\bar{\mathsf{N}}\bar{\mathsf{P}}$-*normal form* formula is a boolean combination of formulae of the form $\bar{\bigcirc}^k p$ and $\bar{\bullet}^l q$, where p and q are atoms, $k, l \in \mathbb{N}$ and $\bar{\bigcirc}^k$ is a sequence of $\bar{\bigcirc}$-symbols of size k, similarly for $\bar{\bullet}^l$; it is useful sometimes to consider k negative or 0, so we define $\bar{\bigcirc}^{-k}A = \bar{\bullet}^k A$ and $\bar{\bigcirc}^0 A = A$. As an example, the formula $\bar{\bigcirc}\bar{\bigcirc}(\bar{\bullet}\bar{\bullet}\bar{\bullet}(p \wedge q) \vee p)$ has normal form $(\bar{\bullet}p \wedge \bar{\bullet}q) \vee \bar{\bigcirc}\bar{\bigcirc}p$. The existence of such normal form gives us very simple proofs for completeness and decidability of $\bar{\mathsf{N}}\bar{\mathsf{P}}/\mathbb{Z}$ that we outline next.

For completeness, let Σ be a possibly infinite consistent set of $\bar{\mathsf{N}}\bar{\mathsf{P}}$-formulae and assume all formulae in the set is in the normal form. Σ can be seen as a consistent set of propositional formulae where each maximal subformulae of the form $\bar{\bigcirc}^k p$ is understood as a new propositional atom, so let h_0 be a propositional valuation assigning every extended atom into $\{true, false\}$. For $n \in \mathbb{Z}$, let $h(n) = \{p \in \mathcal{P} \mid h_0(\bar{\bigcirc}^n p) = true\}$. Clearly $(\mathbb{Z}, <, h)$ is a model for the original set.

For decidability, let A be a formula of $\bar{\mathsf{N}}\bar{\mathsf{P}}$ and let A^* be its normal form; clearly there exists an algorithm to transform A into A^*. By considering subformulae of the form $\bar{\bigcirc}^k p$ as new atoms, k possibly negative, we apply any decision procedure for propositional logic to A^*. A is a $\bar{\mathsf{N}}\bar{\mathsf{P}}$-valid formula iff A^* is a propositional tautology.

Definition 7.1 The *restricted interlacing* of temporal logic systems $\mathsf{US}/\mathcal{K}$ and $\bar{\mathsf{N}}\bar{\mathsf{P}}/\mathbb{Z}$ is the two-dimensional temporal logic system $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$ given by:

- the fully combined language of US and $\bar{\mathsf{N}}\bar{\mathsf{P}}$;
- the two-dimensional plane model over $\mathcal{K} \times \mathbb{Z}$, equipped with the broadly two-dimensional semantics;
- the union of the axioms of $\mathsf{US}/\mathcal{K}$ and $\bar{\mathsf{N}}\bar{\mathsf{P}}/\mathbb{Z}$ plus the interlacing axioms

  $$\overline{\bigcirc} U(p,q) \rightarrow U(\overline{\bigcirc} p, \overline{\bigcirc} q)$$
  $$\overline{\bigcirc} S(p,q) \rightarrow S(\overline{\bigcirc} p, \overline{\bigcirc} q)$$

  plus their duals obtained by swapping $\overline{\bigcirc}$ with $\overline{\bullet}$; the inference rules are just the union of the inference rules of both component systems. □

The following gives us a normal form for $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$.

Lemma 7.1 *Let A be a formula of $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$. There exists a normal form formula A^* equivalent to A, such that all the occurrences of $\overline{\bigcirc}$ and $\overline{\bullet}$ in it are in the form $\overline{\bigcirc}^k p$ and $\overline{\bullet}^l q$, where p and q are atoms.*

Proof First we show that converse of the interlacing axioms are theorem too. For that, note that U and S respect the *congruence property*, *i.e.* if $A \leftrightarrow C$ and $B \leftrightarrow D$ then $U(A,B) \leftrightarrow U(C,D)$ and $S(A,B) \leftrightarrow S(C,D)$. Also note that

equiv $\quad (p \leftrightarrow \overline{\bigcirc}\,\overline{\bullet} p) \wedge (p \leftrightarrow \overline{\bullet}\,\overline{\bigcirc} p)$

The transitivity of $\rightarrow$ connects the steps in the proof of $U(\overline{\bigcirc} p, \overline{\bigcirc} q) \rightarrow \overline{\bigcirc} U(p,q)$ below:

$$\begin{array}{rll} U(\overline{\bigcirc} p, \overline{\bigcirc} q) & \rightarrow \overline{\bigcirc}\,\overline{\bullet} U(\overline{\bigcirc} p, \overline{\bigcirc} q) & \text{by } \mathbf{equiv} \\ & \rightarrow \overline{\bigcirc} U(\overline{\bullet}\,\overline{\bigcirc} p, \overline{\bullet}\,\overline{\bigcirc} q) & \text{by interlacing axiom} \\ & \rightarrow \overline{\bigcirc} U(p,q) & \text{by } \mathbf{equiv} \text{ and congruence} \end{array}$$

It follows that $U(\overline{\bigcirc} p, \overline{\bigcirc} q) \leftrightarrow \overline{\bigcirc} U(p,q)$. It is completely analogous to show the converse of other interlacing axioms, so we omit the details.

Given A in the language of $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$, the equivalence between both sides of the interlacing axioms allows for "pushing in" the vertical operators $\overline{\bigcirc}$ and $\overline{\bullet}$, so a simple induction on the number of nested temporal operators in A shows an algorithmic way to generate an equivalent formula A^* in the desired normal form. □

Theorem 7.1 (Completeness via restricted interlacing) *Let US be a logic system complete over the class $\mathcal{K} \subseteq \mathcal{K}_{lin}$. Then the two-dimensional system $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$ is complete over $\mathcal{K} \times \mathbb{Z}$.*

Proof Consider a $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$-consistent formula A and assume it is in the normal form. So we can see A as a US-formulae over the extended set of atoms $\overline{\bigcirc}^k$, k possibly negative or 0. From the completeness of $\mathsf{US}/\mathcal{K}$ there exist a one-dimensional model $(T, <, h_{US})$ for A at a point $o \in T$, where $(T, <) \in \mathcal{K}$. Define the two-dimensional assignment

$$h(k,t) = \{p \in \mathcal{P} \mid \overline{\bigcirc}^k p \in h_{US}(t)\}.$$

Clearly, $(T, <, \mathbb{Z}, <_{\mathbb{Z}}, h)$ is a two-dimensional plane $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$-model for A at $(o, 0)$. □

Corollary 7.1 *If* $\mathsf{US}/\mathcal{K}$ *is strongly complete, so is* $\mathsf{US}\times\bar{\mathsf{N}}\bar{\mathsf{P}}/\,\mathcal{K}\times\mathbb{Z}$.

Theorem 7.2 (Decidability via restricted interlacing) *If the logic system* US *is decidable over* $\mathcal{K}$, *so is* $\mathsf{US}\times\bar{\mathsf{N}}\bar{\mathsf{P}}$ *over* $\mathcal{K}\times\mathbb{Z}$.

Proof The argument of the proof is the same as that of the decidability of NP, all we have to do is note that there exists an algorithmic way to convert a combined two-dimensional formula into its normal form, so it can be seen as a US-formula and we can apply the US-decision procedure to it. □

So by restricting the expressivity and the underlying class of flows of time, we can obtain the transference of the basic logical properties via restricted interlacing. It should not be difficult to extend these results to $\mathbb{N}$ instead of $\mathbb{Z}$, although we do not explore this possibility here.

It is also worth noting that the restricted interlacing method answers a conjecture posed by Venema [1990] on the distance of some expressively limited two-dimensional temporal logic over $\mathbb{Z}\times\mathbb{Z}$ that was "well behaved" in the sense of having the completeness and decidability properties.

8 The Two-dimensional Diagonal

We now study some properties of the diagonal in two-dimensional plane models. The diagonal is a privileged line in the two-dimensional model intended to represent the sequence of time points we call "now", *i.e.* the time points on which an historical observer is expected to be traverse . The observer is, therefore, on the diagonal when he or she poses a query (*i.e.* evaluates the truth value of a formula) on a two-dimensional model. The diagonal is illustrated in Figure 9.

So let δ be a special atom and consider the formulae:

D1 $\Diamond\delta\wedge\overline{\Diamond}\delta$
D2 $\delta\rightarrow(G\neg\delta\wedge H\neg\delta\wedge\overline{G}\neg\delta\wedge\overline{H}\neg\delta)$
D3 $\delta\rightarrow(\overline{H}G\neg\delta\wedge\overline{G}H\neg\delta)$

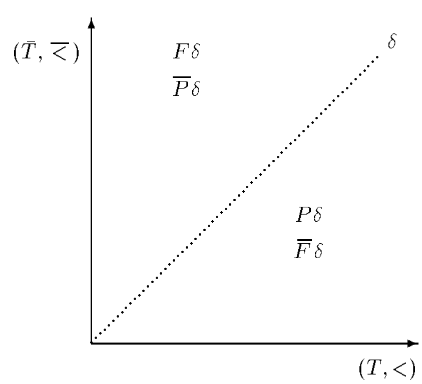

Figure 9 The two-dimensional diagonal

Let $Diag = \Box\overline{\Box}(\mathbf{D1}\wedge\mathbf{D2}\wedge\mathbf{D3})$. The intuition behind $Diag$ is the following. **D1** implies that the two-dimensional diagonal can always be reached in both vertical and horizontal directions; **D2** implies that there are no two diagonal points on the same horizontal line and on the same vertical line and **D3** implies that the diagonal goes in the direction SW–NE. We say that *Diag* characterises a two-dimensional diagonal in the following sense.

Lemma 8.1 *Let $\mathcal{M} = (T, <, \overline{T}, \overline{<}, g)$ be a full two-dimensional model over $\mathcal{K}\times\overline{\mathcal{K}}$, $\mathcal{K}, \overline{\mathcal{K}} \subseteq \mathcal{K}_{lin}$, and let δ be a propositional letter. Then the following are equivalent.*

(a) $\mathcal{M}, t, x \models Diag$, *for some* $t \in T$ *and* $x \in \overline{T}$.

(b) $\mathcal{M}, t, x \models Diag$, *for all* $t \in T$ *and* $x \in \overline{T}$.

(c) There exists an isomorphism $i : T \to \overline{T}$ *such that* $\mathcal{M}, t, x \models \delta$ *iff* $x = i(t)$.

Proof It is straightforward to show that (a) $\Longleftrightarrow$ (b) and (c) $\Longrightarrow$ (a); we show only (b) $\Longrightarrow$ (c). So assume that $\mathcal{M}, t, x \models Diag$, for all $t \in T$ and $x \in \overline{T}$. Define

$$i = \{(t, x) \in T\times\overline{T} \mid \mathcal{M}, t, x \models \delta\}.$$

All we have to show is that i is an isomorphism.

- i, i^{-1} are functions such that $dom(i) = T$ and $dom(i^{-1}) = \overline{T}$. Suppose that both (t, x_1) and (t, x_2) are in i; then $\mathcal{M}, t, x_1 \models \delta$ and $\mathcal{M}, t, x_2 \models \delta$. By linearity of $\overline{T}$, $x_1 = x_2$, $x_1 \overline{<} x_2$ or $x_2 \overline{<} x_1$, but **D2** eliminates the latter two; **D1** gives us that $dom(i) = T$. Similarly, the linearity of T and **D2** gives us that i^{-1} is a function and **D1** gives us that $dom(i^{-1}) = \overline{T}$.

- $i(t) = x$ iff $i^{-1}(x) = t$ follows directly from the definition. So i is a bijection.

- i preserves ordering. Suppose $t_1 < t_2$; by the linearity of $\overline{T}$ we have three possibilities:

  - $i(t_1) = i(t_2)$ contradicts i is a bijection.
  - $i(t_2) \overline{<} i(t_1)$ contradicts **D3**.
  - $i(t_1) \overline{<} i(t_2)$ is the only possible option.

Therefore i is an isomorphism, which proves the result. □

This result shows that by adding **D1**–**D3** to the axiomatisation over $\mathcal{K}_{lin}\times\mathcal{K}_{lin}$ of Section 6 gives us completeness over the class of models of the form $(T, <, T, <, g)$, where $(T, <) \in \mathcal{K}_{lin}$. It follows from [HS86], however, that such logic system is undecidable.

The diagonal is interpreted as the sequence of time points we call "now". The diagonal divides the two-dimensional plane in two semi-planes. The semi-plane that is to the (horizontal) left of the diagonal is "the past", and the formula $F\delta$ holds over all points of this semi- plane. Similarly, the semi-plane that is to the (horizontal) right of the diagonal is "the future", and the formula $P\delta$ holds over all points of this semi-plane. Figure 9 puts this fact in evidence. If we assume that *Diag* holds over $\mathcal{M}$ such that i is the isomorphism defined in Lemma 8.1, $t < s$ iff $i(t) \overline{<} i(s)$, then

$$\begin{aligned}\mathcal{M}, t, x \models F\delta &\text{ iff exists } s > t \text{ such that } \mathcal{M}, s, x \models \delta \text{ and } i(s) = x\\ &\text{ iff exists } y = i(t) \overline{<} x \text{ such that } \mathcal{M}, t, y \models \delta\\ &\text{ iff } \mathcal{M}, t, x \models \overline{P}\delta.\end{aligned}$$

Similarly, it can be shown that:

$$\mathcal{M}, t, x \models P\delta \text{ iff } \mathcal{M}, t, x \models \overline{F}\delta.$$

It follows that the following formula is valid for $\mathsf{US}\times\bar{\mathsf{U}}\bar{\mathsf{S}}$ over $\mathcal{K}_{lin}\times\mathcal{K}_{lin}$:

$$Diag \rightarrow (\ (F\delta \leftrightarrow \overline{P}\delta) \ \wedge \ (P\delta \leftrightarrow \overline{F}\delta) \ ).$$

As a consequence, $\overline{P}\delta$ holds over all points of the "past" semi-plane and $\overline{F}\delta$ holds over all points of the "future" semi-plane, as is indicated in Figure 9.

The formula $Diag$ is in the language of $\mathsf{US} \times \bar{\mathsf{U}}\bar{\mathsf{S}}$but not in the language of $\mathsf{US}\times\bar{\mathsf{N}}\bar{\mathsf{P}}$, for $Diag$ contains the vertical temporal operators $\overline{G}$, $\overline{H}$, $\overline{\Box}$ and $\overline{\Diamond}$. To characterise a two-dimensional diagonal in $\mathsf{US}\times\bar{\mathsf{N}}\bar{\mathsf{P}}$ we do the following. We say that a formula A *holds over* or *is valid over* a two-dimensional model $\mathcal{M}$ if for every $t \in T$ and every $x \in \overline{T}$, it is the case that $\mathcal{M}, t, x \models A$. Consider the formulae

d1 $\overline{\Diamond}\delta$
d2 $\delta \rightarrow (G\neg\delta \wedge H\neg\delta)$
d3 $\delta \leftrightarrow \overline{\bigcirc}\bigcirc\delta$

where δ is a proposition. Those formulae are all in the language of $\mathsf{US}\times\bar{\mathsf{N}}\bar{\mathsf{P}}$, for $Diag$ (so also in the language of $\mathsf{US} \times \bar{\mathsf{U}}\bar{\mathsf{S}}$ and they can characterize the two-dimensional diagonal due to the following property.

Proposition 8.1 *Let $\mathcal{M}$ be a two-dimensional plane model over $\mathbb{Z}\times\mathbb{Z}$. Then the formula* **D1**$\wedge$**D2**$\wedge$**D3** *holds over $\mathcal{M}$ iff* **d1**$\wedge$**d2**$\wedge$**d3** *holds over $\mathcal{M}$.*

Proof By Lemma 8.1 we know that **D1**$\wedge$**D2**$\wedge$**D3** holds over $\mathcal{M}$ iff the relation i defined as below

$$i = \{(t, x) \in \mathbb{Z}\times\mathbb{Z} \mid \mathcal{M}, t, x \models \delta\}.$$

is an isomorphism in $\mathbb{Z}$. So all we have to do is to prove that i as defined above is an isomorphism iff **d1**$\wedge$**d2**$\wedge$**d3** holds over $\mathcal{M}$. The *only if* is a straightforward verification that for all x and t in $\mathbb{Z}$, $\mathcal{M}, t, x \models$ **d1**$\wedge$**d2**$\wedge$**d3**.

Assume **d1**$\wedge$**d2**$\wedge$**d3** holds over $\mathcal{M}$. Then:

1. **d1** gives us that for every x there exists a t such that $\mathcal{M}, t, x \models \delta$;

2. **d2** gives us that for every x, t, t', $t \neq t'$, $\mathcal{M}, t, x \models \delta$ implies $\mathcal{M}, t', x \not\models \delta$;

3. **d3** give us that for every x, t, $\mathcal{M}, t, x \models \delta$ iff $\mathcal{M}, t+1, x+1 \models \delta$ iff for every $n \in \mathbb{Z}$, $\mathcal{M}, t+n, x+n \models \delta$

The first two items give us that $i^{-1} : \mathbb{Z} \rightarrow \mathbb{Z}$ is a function. To show that i is also a function, suppose that $(t, x_1), (t, x_2) \in i$. By linearity of $\mathbb{Z}$, it follows that either $x_1 < x_2$ or $x_2 < x_1$ or $x_1 = x_2$. Let $x_1 - x_2 =$ m; then, by the third item above, $(t+m, x_2+m = x_1) \in i$, so $t = (t+m)$ and $m = 0$. It follows that $x_1 = x_2$, so $i : \mathbb{Z} \rightarrow \mathbb{Z}$ is a function. Directly by the definition of i, it follows that i is a bijection.

Again by the third item above, if $i(t_1) = x_1$ and $i(t_2) = x_2$, then $t_1 - t_2 = x_1 - x_2$. It follows that i is order preserving and hence an isomorphism, which finishes the proof. $\Box$

It would be desirable to generalise the idea of a diagonal as the sequence of "now" moments to any pair of flows of time that are not necessarily isomorphic. For that, we would have to create an order between the points of the two flows, *i.e.* we would have to merge the flows.

So let $(T, <)$ and $(\overline{T}, \overline{<})$ be two flows of time such that T and $\overline{T}$ are disjoint. Then there always exists a flow $(T', <')$ and a mapping $f : T \cup \overline{T} \to T'$ such that f is one-to-one and order preserving. The *f-merge of* $(T, <)$ *and* $(\overline{T}, \overline{<})$ is the flow of time consisting of the image of f ordered by the restriction of $<'$ to the image of f. An example of an f-merge is shown in Figure 10, where $f(y)$ is made equal, via merge, to $f(\bar{x})$ and on the merged flow the order is preserved, *i.e.* originally $x < y$ and $\bar{x} \overline{<} \bar{y}$ and on the f-merged flow $f(x) <' f(y) = f(\bar{x}) <' f(\bar{y})$.

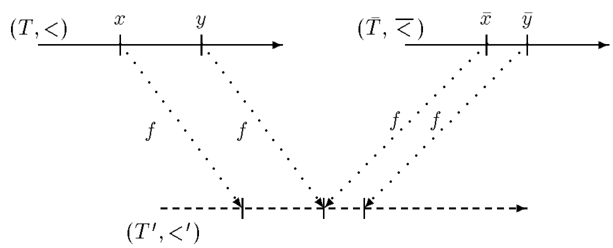

Figure 10 The f-merge

We can then construct a two dimensional model with two copies of the f-merge, in which we can define a diagonal over $(T', <') \times (T', <')$ as shown in Figure 11.

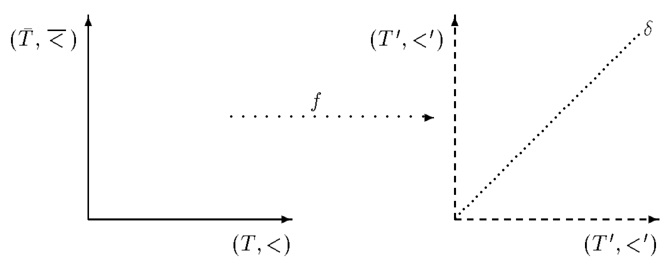

Figure 11 The diagonal of two distinct flows

This construction motivates a method of combining two one-dimensional temporal logics into another one-dimensional logic, namely that over the class of all f-merges of its two-component flows of time. We could then study the transference of logical properties in the same way as we have done in this and the previous section, but we do not investigate those matters here.

9 Conclusion

This paper dealt with the combination of two logic systems in order to obtain a new logic system. The issues were:

- Several methods of combination of two logic systems were presented. Each combination involved at least one temporal logic system. Each method had a particular discipline for combining the language, the semantics and the inference system of two logic systems. Each combination generated a single logic system.

- The study of transference of logical properties from the component systems into their combined form has been the major point in the analysis of combination methods. The basic logical properties whose transference was analysed were soundness, completeness and decidability; for some combination methods, the transference of other properties was also investigated such as conservativeness and the compactness property (in the form of strong completeness).

- The investigation of four basic methods has been accomplished. The temporalisation method and the independent combination method were shown to transfer all basic properties, although they do not generate an expressive enough system to be called fully two-dimensional. The full interlacing method does generate a fully two-dimensional temporal system, but in many cases it failed to transfer even the completeness property. As a compromise, it was shown that a restricted interlacing method, although generating two-dimensional temporal logic systems that were not as expressive and generic as the fully interlaced one, accomplishes the transference of all basic logical properties.

Another contribution of our analysis was to answer a question raised by Venema [Ven90] on the existence of a fragment of the two-dimensional plane temporal logic that, in his own words, was 'better behaved' than the two-dimensional plane system with respect to completeness and decidability properties. We have shown that the two-dimensional temporal logic systems obtained by restricted interlacing are an example of such fragments.

Another question raised by Venema in that same work remains open, namely, whether it is possible to have a complete axiomatisation over the two-dimensional model using only canonical inference rules, *i.e.* without using the special inference rules **IR1** and **IR2**. This problem seems to be a very hard one. Nevertheless we succeeded in extending Venema's completeness result, that originally holds for only two-dimensional flows built from two identical one-dimensional flows, to any two-dimensional flow built from any flow in the classes $\mathcal{K}_{lin}$, $\mathcal{K}_{dis}$, K_{dense} and $\mathbb{Q}$.

Comparisons, Extensions and Further Work

With respect to combination of logics, the works in the literature that most closely approximate ours in spirit and aims, are those of Kracht and Wolter [KW91] and of Fine and Schurz [FS91]. Both works concentrated on monomodal logics, and investigated the transference of logical properties for only the method we called here independent combination. However, their work investigated several paths that suggest that further work may be done in our studies. First, they analysed the transference of many other properties from two logic systems to its combined form, *e.g.* finite model property and interpolation. Second, both works did not concentrate only in linear systems and they were able to extend their results to any class of underlying Kripke frames. Third, Fine and Schurz's work generalised the independent combination method to more than two monomodal logics.

Those two papers cited above therefore suggest several extensions to our work. Note, however, that the temporalisation method was easily shown to be extensible to many temporal logic systems in Example 2.4. The focus on linear flows of time was due to database applications of two-dimensional temporal logics as in [FG92, Fin94], but we believe that this restriction may be lifted without damaging the transference results of the temporalisation and independent combination methods. These have to be further investigated and the transference of any other logical property has to be analysed on its own.

The generalisation of combination methods other than the independent combination method to modal logics is another area for further work. As noted in [FG92], the temporalisation process is directly extensible to monomodal logics. It may even be the case that, for monomodal logics, the full interlacing method achieves transference of completeness over several classes of fully two-dimensional Kripke frames using only canonical inference rules, as it is suggested by the results in [Seg73].

The complexity class of the decision problem for the combined logic is another interesting subject for study. For the independent combination of monomodal logics, such a study was done by Spaan [Spa93] and the conclusion was that the satisfiability problem of an independently combined logic is either reducible to that of one of the component logics, or it is PSPACE-hard or it is in NP. We believe a similar result can be obtained for the temporalisation and the independent combination of temporal logics, although the details have not yet been worked out. The complexity of the full and restricted interlacing methods still have to be studied.

All the systems dealt with in this paper were extensions of classical logic. It is possible that the temporalisation process preserves its transference properties even in the case the underlying system is not an extension of classical logic. What if the external temporal logic is non-classical itself? The same question applies to other combination methods. Do they achieve transference of logical properties when one or both of the combined temporal of modal logics is not classical? Gabbay [Gab92] has recently posed that question in a very generic framework involving Labelled Deductive Systems (LDS) and found that in order to obtain the transference of completeness we do not need the full power of classical logic but only some weaker form of monotonicity. He has also developed other methods of combination called *fibring* that depends on the choice of a fibring function. A fibring function maps the truth value of atoms in one logic's semantics with the semantics of formulae in other logic's semantics. Gabbay's *dovetailing* process, obtained with a certain class of fibring functions, is similar to the independent combination method extended to logics respecting those weaker conditions of monotonicity. More work on this area is needed to clarify exactly how fibring is related to existing combination methods.

There are also other possible types of combinations of one-dimensional temporal logics that may be explored. As pointed out in Section 8, two linear flows of time can be merged into another one; the question is then how to combine two one-dimensional temporal logics into another one-dimensional temporal logic over the merged flow.

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