When is a Substructural Logic Paraconsistent?

Structural conditions for paraconsistency in ternary frames*

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Abstract
In this work, we study structural, model-theoretical conditions that support paraconsistency in Substructural Logics. The idea is to follow the notion of *Correspondence Theory* from Modal Logics and apply it to *Substructural Logics*.

Several logics in the family of Substructural Logics were initially defined with goals similar to those of Paraconsistent Logic. There are several possible ways of defining paraconsistency, but this work takes a neutral way towards all such definitions. We note that the formalization of such definitions vary according to the set of connectives present in the logical language, and also according to whether we view paraconsistency as the possibility to deny the principles of Non-contradiction or Trivialization. All this yields a number of possible definitions of *paraconsistency*. We propose a method that allows us to compute which effects a given definition may have upon the model theoretical structures of a Substructural Logic that adopt one such definition.

It has been known since the work of Routley and Meyer [RM73] that binary logical connectives can be seen as modalities interpreted over Kripke frames (W, R) with a ternary accessibility relationship $R \subseteq W \times W \times W$. More recently, a correspondence theory was developed for substructural logics in analogy to the usual modal correspondence theory.

In this a setting, we derive structural restrictions over ternary frames corresponding to the violation of a *consistency condition*, that is, an axiom. Such a process is performed on a fragment consisting of the connectives $\otimes$ (*tensor product*, also called *multiplicative conjunction*), $\rightarrow$(multiplicative implication), $\neg$ (classical negation), $\sim$ (intuitionistic negation) and $\wedge$ (classical conjunction).

*Marcelo Finger was partly supported by the Brazilian Research Council (CNPq), grants PQ 300597/95-5 and APQ 468765/2000-0.

1 Introduction

In this work, we study structural, model-theoretical conditions that support paraconsistency [dC74] in Substructural Logics [Res00].. One of the initial motivations for the proposal of Relevant Logics was to avoid the classical trivialization of theories, where from a formula A and its negation one can infer a formula B, even if A has nothing in common with B [AB75]. The way that Relevant and other Substructural Logics followed to achieve that goal was to restrict the set of classical structural rules in deductions; hence the name of the family of logics. With the elimination of structural rules, classical connectives unfolded into several others, so many new fragments were created for Substructural Logics. Actually, the family was unified as such only much later [Doš93], and for several years there were just several groups of logics (Relevant, Linear, Lambek, Intuitionistic, etc). A semantics for Relevant Logic based on ternary frames was proposed by Routley and Meyer [RM73], which was later extended to the whole family of Substructural Logics [Res00].

The way Paraconsistency is treated in da Costa's approach is different [dC74], and consists of weakening the notion of classical negation. Initial tentatives to create a semantics for paraconsistent logics tried to provide set theoretical constructions to accommodate the "inconsistent elements" present in most paraconsistent systems, with partial success [CA81]. Recent approaches to a semantics of paraconsistent logics have totally avoided the manipulation of the usual set theoretical structures, preferring to give a semantics based on the *translation* of a paraconsistent logics into a set of many-valued logics, plus some mechanism for the combination/interaction of these translations [Car98].

We do not deny that there are interesting aspects in these translation-based approaches to semantics, but since we are taking the substructural point of view, we will study the model theoretical conditions present at the intersection between Substructural and Paraconsistent Logics in the light of model theoretical constructions for substructural paraconsistency.

It is important to note that we do not mean that Paraconsistent Logics *are* Substructural Logics. Quite the opposite, we simply note that some Substructural Logics display a *paraconsistent behaviour*, e.g. Relevant Logics as mentioned above. So some substructural logics do accept some paraconsistent theories, but some others do not. This does not rule out the possibility of existing other logics termed Paraconsistent that are not Substructural or vice versa.

In this way, we proceed with our study of model theoretical conditions that permit a substructural logic to accept paraconsistent theories.

1.1 Paraconsistency and Substructural Logics

Our approach here does not start with *the* definition of a Paraconsistent Logic, so that we can put forward a sound and complete semantics for it. We do not have a final definition for paraconsistency, nor do we think that one such definition is desirable.

In the literature, there are two basic notions related to paraconsistency, both involving a formula A and its negation $\mathsf{not}\,A$, both related to the violation of a logical principle:

- *Non-contradiction:* according to this principle, a theory should not derive a formula and its negation. Therefore, a *paraconsistent theory* that violates non-contradiction cannot validate an axiom of the form $\mathsf{not}(A\ \mathsf{and}\ \mathsf{not}\ A)$.

- *Trivialization:* according to this principle, a theory containing both a formula and its negation derives any formula. A *paraconsistent theory* that violates triviality must not validate an axiom of the form $(A\ \mathsf{and}\ \mathsf{not}\ A)\ \mathsf{implies}\ B$.

In this explanation above, we have used the connectives $\mathsf{not}, \mathsf{and}, \mathsf{implies}$ to remain neutral as to their definition, for in substructural logics there may exist several possible connectives for negation, conjunction and implication. The present work is also neutral towards such definitions and we analyse structural conditions for several possible definitions of these connectives.

As stated earlier, our approach is based on the semantics. We start with a pure semantical structure for substructural logics, that is, a semantical structure free from any structural pressuposition. We then study what kind of properties should be imposed on that structure for each alternative definition of paraconsistency.

1.2 Paraconsistency and Correspondence Theory

The idea is to follow the notion of *Correspondence Theory* from Modal Logics [vB84]. In modal semantics we have the notion of a basic Kripke frame, $\mathcal{F} = (W, R)$, consisting of a set W of possible worlds with a binary relation R, called the *accessibility relation*, which provides a sound and complete semantic basis for the minimal modal logic K. We know that by adding some property to the system, e.g. reflexivity, some formulas become valid in the class of all Kripke models obeying that property; e.g. the axiom T, $\Box p \rightarrow p$, is valid in all reflexive Kripke frames. Conversely, if we add an axiom to a modal axiomatization, we get completeness over some class of Kripke frames; e.g. logic K + axiom $\Box p \rightarrow p$ is complete over the class of reflexive Kripke frames [BS84, Che80].

In this way, the relationship between modal axioms and classes of Kripke frames can be studied without the need to define *the* modal logic.

We develop here a similar approach for *substructural logics* [Doš93, Res00], that is, the family of logics obtained by rejecting some of the structural rules used in classical logic deductions. The works of Roorda [Roo91] and Kurtonina [Kur94] have shown that, in the same way that monadic modalities are interpreted over binary accessibility relationships, binary connectives can be seen as modalities interpreted over Kripke frames with a ternary accessibility relationship. In particular, we may study the usual connectives (implication, conjunction, negation) as modalities.

In such a setting we can start asking what sort of properties corresponds to a given axiom, as is done in modal correspondence theory. In particular, some axiom may be taken as the definition of consistency in the system, so that we may investigate what structural properties correspond to each definition of consistency.

Note that it follows from the modal examples above that if we want to allow for the falsity of modal axiom T at some worlds, we may not have all worlds reflexive; that is, $\forall x Rxx$ must fail for some x. This is the way we are going to treat paraconsistency conditions, namely by falsifying the structural conditions imposed by *consistency axioms* on ternary frames.

1.3 Automated Methods

Recently, we have been able to find an automatic way to compute a first-order condition on ternary frames associated to an axiom [Fin00], in a manner analogous to the way that modal Sahlqvist formulas can computationally generate a restriction on traditional (binary) Kripke frames [vB84]. Such automatic computation is performed on a substructural fragment known as Categorial Grammar [Car97, Moo97], consisting of the connectives $\rightarrow$ (*right-implication*), $\leftarrow$ (*left-implication*) and $\otimes$ (*tensor product*, also called *multiplicative conjunction* or *fusion*).[1]

We claim that such techniques can be applied for the study of first-order condition on ternary frames that allows a logic to support paraconsistent theories.

The rest of the paper develops as follows. Ternary frames, and its relationship to first-order formulas are presented in Section 2, with an example on how to compute the first-order restriction associated with an axiom. Then in Section 3 we show that different definitions of what constitutes a consistency axiom lead to distinct structural constraints; in particular, we study consistency conditions based on:

- non-contradiction *vs.* trivialization principles;
- boolean *vs.* intuitionistic negation;
- boolean *vs.* multiplicative conjunction.

Finally, we analyse in Section 4, we apply those methods for relevant negation and in Section 5 we discuss several other possible negations which can be analysed by our method.

2 Ternary Frames

The idea of using ternary frame for the semantics of substructural logics goes back to [RM73], where it was used to provide a semantics for relevance logics. In a context free of structural pressuposition, that semantics has been used in, for example, [Kur94, DM97].

A ternary frame is a pair $\mathcal{F} = (W, R)$, where R is a any ternary relation on $W \times W \times W$. The set W is a set of possible worlds. We normally represent that a triple $\langle a, b, c\rangle \in R$ by writing $Rabc$. The elements of R are seen as a binary tree, with a being the root node, b its left daughter, and c its right daughter. To reinforce this point of view, $Rabc$ is sometimes written as Ra, bc.

Every model has a distinguished world $\mathbf{0} \in W$. Unlike modal Kripke models, a valid formula is not required to hold at all worlds of every model, but only at the distinguished world of every model. The distinguished $\mathbf{0}$ has the following properties:

$$Ra\mathbf{0}a \text{ and } Raa\mathbf{0}$$

The language fragment we work with in this section consists of a countable set of propositions, $\mathcal{P} = \{p_1, p_2, \ldots\}$, and the binary connectives $\rightarrow, \leftarrow, \otimes$. We use A, B, C as variables ranging over substructural formulas. The connectives $\Rightarrow$, $\wedge$ and $\neg$ are, respectively, the classical implication, conjunction and negation.

[1] These connectives also appear in the literature as $/$, $\backslash$ and $\bullet$.

A model $\mathcal{M} = (W, R, V, \mathbf{0})$ consists of a ternary frame plus a valuation $V : \mathcal{P} \to 2^W$ that maps propositional variables into a set of possible worlds. Formulas are evaluated with respect to a possible world $a \in W$, so that $\mathcal{M}, a \models A$ reads that the formula A holds at a in model $\mathcal{M}$. The semantics of the binary connectives over a ternary model is given by:

$$\begin{array}{ll}
\mathcal{M}, a \models p & \textit{iff } a \in V(p) \\
\mathcal{M}, a \models A \otimes B & \textit{iff } \exists b \exists c (Rabc \wedge \mathcal{M}, b \models A \wedge \mathcal{M}, c \models B) \\
\mathcal{M}, a \models A \to B & \textit{iff } \forall b \forall c (Rcab \wedge \mathcal{M}, b \models A \Rightarrow \mathcal{M}, c \models B) \\
\mathcal{M}, a \models B \leftarrow A & \textit{iff } \forall b \forall c (Rcba \wedge \mathcal{M}, b \models A \Rightarrow \mathcal{M}, c \models B)
\end{array}$$

A formula is *valid* if it holds at $\mathbf{0}$ in all models. It is easy to see that a formula of the form $A \to A$ or $A \leftarrow A$ is valid at ternary formulas.

A ternary model $\mathcal{M} = (W, R, V)$ can be seen as a first-order model structure over $\mathcal{M}_{FO} = (W, R, P_1, P_2, \ldots)$, where each unary predicate P_i corresponds to a propositional letter $p_i \in \mathcal{P}$. A substructural formula can thus be translated into a first-order one, with respect to a world a, in the following way:

$$\begin{array}{lcl}
FO_a(p_i) & = & P_i(a) \\
FO_a(A \otimes B) & = & \exists b \exists c (Rabc \wedge FO_b(A) \wedge FO_c(B)) \\
FO_a(A \to B) & = & \forall b \forall c (Rcab \wedge FO_b(A) \Rightarrow FO_c(B)) \\
FO_a(B \leftarrow A) & = & \forall b \forall c (Rcba \wedge FO_b(A) \Rightarrow FO_c(B))
\end{array}$$

It is straightforward to see that $\mathcal{M}, a \models A$ iff $\mathcal{M}_{FO} \models FO_a(A)$.

Like in usual modal correspondence theory, if we want to make a formula A valid over all models, this means that A should be true in all models, for all valuations; this translates into a second-order formula, obtained by the universal closure of $FO_a(A)$ over a and over all the predicate symbols occurring in it, that is:

$$\forall P_1 \ldots \forall P_n \forall a FO_a(A).$$

Such a formula provides a second-order constraint over the ternary relation R. It is particularly interesting here (as in modal logic) to know whether this second-order formula is equivalent to a first-order formula. However, it is not always possible to find such a first-order equivalent to a second-order frame constraint. We illustrate next a case where it is possible.

EXAMPLE 1 Consider the formula $A = (p \to q) \to (q \leftarrow p)$. We want to know what restrictions should be imposed on ternary frames for it to be a valid formula. For that, we compute $FO_a(A)$:

$$\begin{array}{l}
FO_a((p \to q) \to (q \leftarrow p)) = \\
\quad = \ \forall bc(Rcab \wedge FO_b(p \to q) \Rightarrow FO_c(q \leftarrow p)) \\
\quad = \ \forall bc(Rcab \wedge \forall de(Rebd \wedge P(d) \Rightarrow Q(e)) \Rightarrow \forall fg(Rgfc \wedge P(f) \Rightarrow Q(g))) \\
\quad = \ \forall bcfg \exists de(Rcab \wedge (Rebd \wedge P(d) \Rightarrow Q(e)) \wedge (Rgfc \wedge P(f) \Rightarrow Q(g)))
\end{array}$$

At this point we know that for A to be a valid formula, the ternary frame has to obey the second-order restriction $\forall P \forall Q \forall a(FO_a(A))$. To obtain a first-order equivalent to this formula, an appropriated valuation for P and Q must be provided; this is equivalent to finding a valuation for p and q in the modal context. Finding such a valuation is the crucial point of this method. Although we have a way of computing one [Fin00], if one exists, for the substructural fragment, here we just present one:

$$\begin{array}{lcl} V(p) = \{f\} & \Longrightarrow & \forall x(P(x) \Leftrightarrow x = f) \\ V(q) = W - \{g\} & \Longrightarrow & \forall x(Q(x) \Leftrightarrow x \neq g) \end{array}$$

By substituting such a valuation in $\forall a(FO_a)(A)$ we obtain:

$$\begin{array}{l} \forall abcfg\exists de(Rcab \wedge (Rebd \wedge d = f \Rightarrow e \neq g) \Rightarrow (Rgfc \wedge \top \Rightarrow \bot)) \Longleftrightarrow \\ \forall cfg(\exists abRcab \wedge \forall de(d = f \wedge e = g \Rightarrow \neg Rebd) \Rightarrow \neg Rgfc) \Longleftrightarrow \\ \forall bcfg(\exists aRcab \wedge \neg Rgbf \Rightarrow \neg Rgfc) \end{array}$$

But since we know that, $\forall cRc\mathbf{0}c$, it is always the case that, for $c = b$, $\exists aRcab$, so we end up with the first-order restriction:

$$\forall cfg(Rgfc \Rightarrow Rgcf)$$

That is, the restriction imposed on R is the commutativity of its second and third arguments. It remains to be shown that whenever we have the commutativity of R's second and third arguments, the formula A is valid; such a proof can be found in [Kur94]. It follows that $(p \rightarrow q) \rightarrow (q \leftarrow p)$ corresponds to the restriction of 2,3-commutativity over ternary frames. Note that it is well known that $(p \rightarrow q) \rightarrow (q \leftarrow p)$ is a theorem of substructural logics that allow for commutativity of premises in a sequent deduction [Doš93].

The really interesting part of the procedure above is to know whether the second-order formula generated is equivalent to a first-order one and what is the substitution that will lead to it. This is the basic task of our algorithm developed in [Fin00]; as there is no space for a full presentation of the method, we only briefly present it next.

2.1 The SLaKE-Tableaux Method

We compute a first order formula equivalent to a substructural sequent (or formula) by means of a construction of a tableau. This method is called SLaKE-tableau (<u>S</u>ubstructural <u>La</u>belled <u>KE</u>).

Each formula in a SLaKE-tableau is signed with T or F and receives a label; the signed labeled formulas $T\ A : a$ and $F\ A : b$ are called *opposites*. The original sequent $A_1, \ldots, A_n \vdash C$ is associated with an initial SLaKE-tableau:

$$\begin{array}{c} T\ A_1 : a_1 \\ \vdots \\ T\ A_n : a_n \\ F\ C : a \end{array}$$

and with a first-order formula:

$$\psi = \neg\exists a a_1 \ldots a_n [V_{a_1}(A_1) \wedge \ldots \wedge V_{a_n}(A_n) \wedge \neg V_a(C) \wedge Ra(a_1 \ldots a_{n-1})a_n \wedge \sharp_1]$$

where $V_a(A)$ is the valuation of the formula A at label a and is defined as follows:

- $V_a(A) =_{def} \top$ if A is not atomic
- $V_a(p) =_{def} (a \neq a_1) \wedge \ldots \wedge (a \neq a_n)$, where $p : a_1, \ldots, p : a_n$ occur in a branch above $p : a$ with opposite sign. If no opposite formula occurs above $p : a$, $V_a(p) =_{def} \top$.

Each of the tableau linear expansion rules is associated with an expansion of the correspondence formula of the form $\sharp_i := \psi(R, A_1, \ldots, A_n, \sharp_{i+1})$, where R is the ternary accessibility relation, $A_1, \ldots, A_n$ are the formulas generated in the expansion, and $\sharp$ is the "substitution place" for next expansion and can be read simply as *truth*. The tableau rules for SLaKE-tableaux are illustrated in Figure 1.

SLaKE Expansion	*Formula Expansion*
$\begin{array}{c} T\ B \to A : a \\ T\ B : b \\ \hline T\ A : c \text{ (new } c) \end{array}$	$\sharp_i := \forall c(Rcab \Rightarrow (V_c(A) \wedge \sharp_{i+1}))$
$\begin{array}{c} F\ B \to A : a \\ \hline T\ B : b \text{ (new } b) \\ F\ A : c \text{ (new } c) \end{array}$	$\sharp_i := \exists b \exists c(Rcab \wedge V_b(B) \wedge \neg V_c(A) \wedge \sharp_{i+1})$
$\begin{array}{c} T\ A \leftarrow B : a \\ T\ B : b \\ \hline T\ A : c \text{ (new } c) \end{array}$	$\sharp_i := \forall c(Rcba \Rightarrow (V_c(A) \wedge \sharp_{i+1}))$
$\begin{array}{c} F\ A \leftarrow B : a \\ \hline T\ B : b \text{ (new } b) \\ F\ A : c \text{ (new } c) \end{array}$	$\sharp_i := \exists b \exists c(Rcba \wedge V_b(B) \wedge \neg V_c(A) \wedge \sharp_{i+1})$
$\begin{array}{c} T\ A \bullet B : a \\ \hline T\ A : b \text{ (new } b) \\ T\ B : c \text{ (new } c) \end{array}$	$\sharp_i := \exists b \exists c(Rabc \wedge V_b(A) \wedge V_c(B) \wedge \sharp_{i+1})$
$\begin{array}{c} F\ A \bullet B : a \\ T\ A : b \\ \hline F\ B : c \text{ (new } c) \end{array}$	$\sharp_i := \forall c(Rabc \Rightarrow (\neg V_c(B) \wedge \sharp_{i+1}))$
$\begin{array}{cc} \hline T\ A : x & F\ A : x \end{array}$	$\sharp_i := \forall x((V_x(A) \wedge \sharp^1_{i+1}) \vee (\neg V_x(A) \wedge \sharp^2_{i+1}))$

Figure 1: SLaKE rules

In each linear rule in Figure 1, the formulas above the horizontal line are the *premises* of the rule, and those below it are the *conclusions* of the rule. There are one-premised and two-premised rules, but each rule has exactly one premise that is a *compound* formula, which is called the *main premise*; other premises are called *auxiliary*. Two-premised rules are $\forall$-rules and one-premised rules are $\exists$-rules. If either of the conclusions of an $\exists$-rule is present on the current branch, it is not added again with a new label. $\forall$-rules always generate a new conclusion.

The last rule in Figure 1 is the Principle of Bivalence (PB), the only branching rule. It introduces two "substitution places" in the correspondence formula, $\sharp^1_{i+1}$

and $\sharp_{i+1}^2$, one for each new branch. A branch that can still be expanded is called *active*. Each active branch in a SLaKE tableau always has exactly one substitution place.

The importance of substitution places is that they guarantee that each formula introduced in the correspondence formula will "see the correct context", that is, it will be in the scope of the correct quantifiers.

A full presentation of the method is beyond the scope of this paper. Here we repeat Example 1 using the SLaKE-tableau method.

EXAMPLE 2 Consider the sequent $q \to p \vdash p \leftarrow q$. Its associated SLaKE tableau is:

1.	$T\ q \to p : a$		
2.	$F\ p \leftarrow q : a$		$\psi = \neg\exists a(\sharp_1)$
3.	$T\ q : b$	from 2	
4.	$F\ p : c$	from 2	$\sharp_1 := \exists b \exists c(Rcba \wedge \top \wedge \top \wedge \sharp_2)$
5.	$T\ p : d$	from 1, 3	$\sharp_2 := \forall d(Rdab \Rightarrow d \neq c \wedge \sharp_3)$

By putting together all substitution places we obtain the formula:

$$\psi = \neg\exists a \exists b \exists c(Rcba \wedge \forall d(Rdab \Rightarrow d \neq c))$$

which is equivalent to $\forall a \forall b \forall c(Rcba \Rightarrow Rcab)$, the commutativity of the second and third R-positions.

A tableau as above is *deterministic*, that is, at all expansions of a branch, there is only a single expansion rule to be applied. In [Fin00] it has been shown that:

PROPOSITION 3 If the SLaKE tableau generated by a sequent is finite, saturated and deterministic, then the associated first-order formula ψ it computes is the sequent's correspondence formula.

We note that SLaKE-tableaux may be infinite, in which case no first-order formula is computed. I may also be non-deterministic, in which case we have to take the conjunction of the formulas associated to all possible SLaKE-tableaux.

2.2 Extending the Method

As the example above shows, the method is based on the semantics of the connectives. We can in this way extend the method to other connectives, such as *classical negation* ($\neg$) and *classical conjunction* ($\wedge$) given by their semantical definitions:

$$\begin{array}{ll} \mathcal{M}, a \models \neg A & \text{iff } \mathcal{M}, a \not\models A \\ \mathcal{M}, a \models A \wedge B & \text{iff } \mathcal{M}, a \models A \textit{ and } \mathcal{M}, a \models B \end{array}$$

These semantical rules translate generate the following tableau rules:

$$\frac{T\ \neg A : a}{F\ A : a} \quad \sharp_i := \neg V_a(A) \wedge \sharp_{i+1} \qquad\qquad \frac{F\ \neg A : a}{T\ A : a} \quad \sharp_i := \neg V_a(A) \wedge \sharp_{i+1}$$

The computational results in [Fin00] do not immediately apply to such extensions, so we cannot affirm that it is a decidable process. However, the method can still be applied to particular examples with success.

But the point we are going to make here is that such a method (even if not fully automated for larger fragments) can be applied to the study of structural conditions for paraconsistency.

3 Consistent and Paraconsistent Restrictions on Ternary Frames

A *consistency condition* is a formula that one wants to see valid so that the system is considered consistent. As a consequence, a system will be *paraconsistent* with respect to a consistency condition if such a formula is invalidated.

We want to apply the techniques described above to associate a constraint over ternary frames with a consistency formula. The rejection of such constraint will therefore characterize paraconsistency over ternary models.

Usually, consistency formulas have to deal with negation. So we introduce *classical negation* ($\neg$) in our language with its usual semantics:

$$\mathcal{M}, a \models \neg A \textit{ iff } \mathcal{M}, a \not\models A$$

The obvious extension of the first-order translation is: $FO_a(\neg A) = \neg FO_a(A)$. We can thus explore the constraint associated with consistency conditions related to the principle of *non-contradiction*.

Consistency Condition 1: $\neg(p \otimes \neg p)$

We start by computing the first order translation of $\neg(p \otimes \neg p)$:

1.	$F\ \neg(p \otimes \neg p) : \mathbf{0}$	$\psi := \neg\sharp_1$
2.	$T\ (p \otimes \neg p) : \mathbf{0}$	$\sharp_1 := \neg V_{\mathbf{0}}(p \otimes \neg p) \wedge \sharp_2$
3.	$T\ p : b$	
4.	$T\ \neg p : c$	$\sharp_2 := \exists bc(R\mathbf{0}bc \wedge V_b(p) \wedge V_c(\neg p) \wedge \sharp_3)$
5.	$Fp : c$	$\sharp_3 := b \neq c$

Putting everything together and doing some classical equivalences, we get the formula

$$\forall bc(R\mathbf{0}bc \rightarrow b = c)$$

That is, for the consistency condition to be valid on ternary frames, the the special world $\mathbf{0}$ is related only to pairs of identical worlds. A structural condition to paraconsistency in this case would be:

$$\exists bc(R\mathbf{0}bc \wedge b \neq c)$$

Hence for a paraconsistency that rejects the consistency condition above, it suffices that in every model there is a triple $\langle \mathbf{0}, b, c\rangle \in R$ with distinct last two arguments.

Consistency Condition 2: $\neg(p \wedge \neg p)$

Suppose now that we want to add boolean conjunction in our language so that we can study the constraint associated with the usual boolean consistency condition $\neg(p \wedge \neg p)$.

For that, first, we add the obvious semantic definition

$$\mathcal{M}, a \models A \wedge B \textit{ iff } \mathcal{M}, a \models A \textit{ and } \mathcal{M}, a \models B$$

together with its obvious first-order translation

$$FO_a(A \wedge B) = FO_a(A) \wedge FO_a(B)$$

and the tableau rules

$$\frac{T\ \ A \wedge B : a}{\begin{array}{l} T\ \ A : a \\ T\ \ B : a \end{array}} \qquad \sharp_i := V_a(A) \wedge V_a(B) \wedge \sharp_{i+1}$$

$$\frac{\begin{array}{l} F\ \ A \wedge B : a \\ T\ \ A : a \end{array}}{F\ \ B : a} \qquad \sharp_i := V_a(A) \wedge \neg V_a(B) \wedge \sharp_{i+1}$$

If we now apply our method to $\vdash \neg(p \wedge \neg p)$ we see that it is logically equivalent to $\top$; details omitted. This is not at all surprising, since we are dealing with both boolean negation and conjunction, which are enough to define all classical connectives, thus rejecting inconsistency.

Intuitionistic Negation

The main idea of intuitionistic negation (which we represent here as $\sim$) is to assert the negation of a formula in a world provided that this formula is not asserted at any other world "above" it. In our ternary models, if $Rabc$ then a is above b and c, which we write $a > b$ and $a > c$. Formally:

$$a > b \textit{ iff } \exists c(Rabc)$$

Such a definition is inspired on a similar one in [RM73][2]. We then have, for ternary frames, the usual intuitionistic definition of negation over Kripke models [Fit69]:

$$\mathcal{M}, a \models \sim A \textit{ iff } \forall b(b > a \Rightarrow \mathcal{M}, b \not\models A)$$

This definition generates a first-order translation:

$$FO_a(\sim A) = \forall b(b > a \Rightarrow \neg FO_b(A))$$

and SLaKE-tableau rules

$$\frac{T\ \ \sim A : a}{F\ \ A : b} \qquad \sharp_i := \forall b(b > a \Rightarrow \neg V_b(A) \wedge \sharp_{i+1})$$

$$\frac{F\ \ \sim A : a}{T\ \ A : b} \qquad \sharp_i := \exists b(b > a \wedge V_b(A) \wedge \sharp_{i+1})$$

We then choose as a consistency condition the formula $\sim(p \otimes \sim p)$. For space reasons we omit here the details, but when we develop the expansion we get that $\vdash \sim(p \otimes \sim p)$ corresponds to the first-order restriction:

$$\forall abc(Rabc \Rightarrow b > c)$$

[2] In fact, since we do not assume any properties of R, we could define two orders, the other one being $a >_2 c$ *iff* $\exists c(Rabc)$.

imposing the order $>$ on all R-related worlds. The paraconsistency condition here states that in every model there must exist an R-related triple $Rabc$ such that b is not above c.

Similarly, a consistency condition of the form $\sim(\sim p \otimes p)$ would generate a restriction of the form $\forall abc(Rabc \Rightarrow c > b)$, leading to a different imposition of $>$-ordering.

If both consistency conditions are required, a structural condition for paraconsistency should be that in every model there must exist an R-related triple $Rabc$ such that neither b nor c is above the other. This is expressed by the following structural condition:

$$\exists abc(Rabc \wedge \neg(b > c) \wedge \neg(c > b)).$$

Finally, we consider the consistency condition $\sim(p \wedge \sim p)$. The development of a SLaKE-tableau for $\vdash \sim(p \wedge \sim p))$ leads us to the first-order condition

$$\forall a(a > a)$$

Thus the intuitionistic consistency condition $\sim(p \otimes \sim p)$ imposes $>$-reflexivity, which is a condition normally expected in intuitionistic models. Those models support the semantic of $\wedge$ in exactly the terms defined here[3]; see e.g. [Fit69].

So a paraconsistent condition that rejects this intuitionistic view of consistency requires that every ternary model contains a $>$-irreflexive world:

$$\exists a \neg(a > a).$$

Consistency as Trivialization

Another possible way of defining a consistency condition, perhaps more in conformity with the original formulation of paraconsistency [dC74], is to state that an inconsistency trivializes implication, that is, from p and its negation we can derive any q. If we focus only on boolean conjunction, two new consistency conditions arise, namely:

1. $(p \wedge \neg p) \rightarrow q$;
2. $(p \wedge \sim p) \rightarrow q$.

By applying our method, we get their correspondent first-order restriction over ternary frames, respectively as:

1. $\top$;
2. $\forall a(a > a)$.

Item 1 implies that the consistency conditions for boolean negation based on non-contradiction and triviality lead exactly to the same restrictions over ternary frames, and hence to the same paraconsistent condition. Item 2 tells us that exactly the same fact occurs for intuitionistic negation, and the structural restriction of $>$-reflexivity is the same for both non-contradiction and triviality conditions.

[3] the transitivity of $>$ found in intuitionistic Kripke models is imposed by intuitionistic implication.

3.1 Summary and Analysis

Consistency Condition	*Structural Restriction*
$\neg(p \otimes \neg p)$	$\exists bc(R\mathbf{0}bc \wedge b \neq c)$
$(p \otimes \neg p) \to q$	$\exists abc(Rabc \wedge b \neq c)$
$\sim (p \otimes \sim p)$	$\exists abc(Rabc \wedge \forall d \neg Rbcd)$
$(p \otimes \sim p) \to q$	$\exists abc(Rabc \wedge \forall d \neg Rbcd)$
$\sim (p \wedge \sim p)$	$\exists a \forall b \neg Raab$
$(p \wedge \sim p) \to q$	$\exists a \forall b \neg Raab$
$\neg(p \wedge \neg p)$	impossible to violate
$(p \wedge \neg p) \to q$	impossible to violate

Table 1: Structural conditions for paraconsistency

Table 1 summarizes the results obtained by our method. Each consistency condition is associated to the structural restriction that violates it, and is expressed in terms of the ternary R relation.

What calls the attention in this result is that the pairs:

$$\begin{array}{cc} \sim (p \otimes \sim p) & (p \otimes \sim p) \to q \\ \sim (p \wedge \sim p) & (p \wedge \sim p) \to q \\ \neg(p \wedge \neg p) & (p \wedge \neg p) \to q \end{array}$$

generate the same structural conditions for paraconsistency. That is, non-contradiction and the corresponding trivialization condition yield the same structural condition.

The other pair examined here is

- $\neg(p \otimes \neg p)$
- $(p \otimes \neg p) \to q$

where the latter leads to a structural restriction for paraconsistency that is implied by the the structural condition of the former.

But it is widely known that there are logics for which the non-contradiction and trivialization conditions are totally independent.

The conclusion is that such logics employ a kind of negation that is neither classical (in the sense of the semantic definition: $\mathcal{M}, a \models \neg A$ iff $\mathcal{M}, a \not\models A$) nor intuitionistic, also semantically defined. In fact, the semantics of negation may take extra parameters in these logics; for example, in [Res00] we find semantics for substructural negations that depend not only on the ternary relation R but also in a partial order $\sqsubseteq$ of *information refinement* where $Rabc$ does not necessarily imply $b \sqsubseteq a$. Other kinds of semantical definitions for negation can be found in [Dun94].

In the cases where intuitionistic or classical negation is employed with its fixed semantics, trivialization and non-contradiction always yield structural conditions that are either identical or strongly connected. As a last example of such connection, we will examine the structural conditions associated with *relevant negation*.

4 Relevant Negation

There are a great range of relevant logics defined in the literature [AB75]. In several of the proposed systems, and in particular in system R, a kind of negation is used, which is represented as $\overline{A}$, meaning that it is inconsistent with the formula A.

To provide a semantics for such a negation over ternary frames, Routley and Meyer [RM73] postulated the existence of a unary function $^* : W \to W$ such that, for every $a, b, c \in W$:

1. $a^{**} = a$

2. $Rabc \Rightarrow (Ra^*bc^* \wedge Ra^*b^*c)$

With such a function, the System-R's relevant negation [AB75] is defined as:

$$\mathcal{M}, a \models \overline{A} \text{ iff } \mathcal{M}, a^* \not\models A$$

Note that in such a system, it is possible not to have neither A nor $\overline{A}$ holding at a possible world a.

With such semantics we apply our method to the following consistency conditions:

- $\overline{(A \otimes \overline{A})}$

- $(A \otimes \overline{A}) \to q$

By applying our method to it, we see that the first one imposes on the model the condition:

$$\forall bc(R\mathbf{0}^*bc \Rightarrow b = c^*)$$

whose negation leads to the paraconsistency condition:

$$\exists bc(R\mathbf{0}^*bc \wedge b \neq c^*)$$

On the other hand, by applying our method to the trivialization formula $(A \otimes \overline{A}) \to q$ we obtain the frame condition:

$$\forall abc(Rabc \Rightarrow b = c^*)$$

which is leads to the following structural restriction:

$$\exists abc(b \neq c^* \wedge Rabc)$$

Again, we see that the latter paraconsistency condition — associated with trivialization — is logically implied by the former one — associated with non-contradiction.

5 Conclusions

We have provided a method that allows us to find structural conditions on ternary Kripke frames to support paraconsistency. Our method is not biased towards any particular definition of paraconsistency. The examples developed here were based on possible definitions of consistency conditions to be refuted by a paraconsistent model.

Admittedly, the examples of consistency condition displayed here were quite simple. For the cases of consistency conditions based on the principle of non-contradiction and involving boolean conjunction and the use of boolean and intuitionistic negation, namely the formulas $\neg(p \wedge \neg q)$ and $\sim (p \wedge \sim p)$, the results obtained were the expected ones; the corresponding conditions based on the trivialization principle provided coincident conditions. This represents a validation of the method presented here.

More importantly, the examples presented show that the method, whether automated or not, is really quite flexible and may, in principle, be applicable to more daring definitions of paraconsistency than those presented here. There are several candidates for alternative negation, such as those in [Res00]:

- split negation;
- simple negation;
- De Morgan Negation;
- ortho-negation; and
- Strict De Morgan Negation;

These negations need a more refined semantics, for which the simple ternary semantics used in this paper is a limit case. We know that in such cases the formula computed by our SLaKE-tableau method is *implied* by the correspondence formula, but we do not know if the formula thus computed is *the* correspondence formula (nor do we know whether the method can decide in the generic case, as it can in the simple fragment of $\{\otimes, \rightarrow, \leftarrow\}$, whether the condition does have a first-order correspondence formula.

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