THE UNRESTRICTED ADDITION OF A TEMPORAL DIMENSION TO A LOGIC SYSTEM

MARCELO FINGER AND M. ANGELA WEISS

Abstract. This papers generalises and completes the work on combining temporal logics started by Finger and Gabbay [5]. We present proofs of transference of completeness and decidability for the temporalisation of logics $\mathsf{T}(\mathsf{L})$ for any flow of time, eliminating the original restriction that required linear time for the transference of those properties through logic combination.

This generalisation over generic flows of times propagates to other combinations of logics that can be interpreted in terms of temporalisations. In this way, the independent combination of temporal logics is obtained over generic flows of time.

1. Introduction

In this paper we extend the study of the *temporalisation* of logic systems introduced by Finger and Gabbay in [5]. There, the temporalisation process was restricted to linear flows of time. Here, we aim to generalize it to any flow of time. We are interested in studying the *transference of properties* from the logic system L into its temporalised version $\mathsf{T}(\mathsf{L})$.

The system $\mathsf{T}(\mathsf{L})$ combines two logics: a temporal logic T, which is applied *externally* to a given logic system L. This combination process, called *temporalisation*, involves the combination of the languages, inference systems, and model structures of T and L into a language, inference system and model structure of $\mathsf{T}(\mathsf{L})$. We show that if the logic systems T and L are sound, complete or decidable, then $\mathsf{T}(\mathsf{L})$ is also sound, complete or decidable; no constraints are imposed on the nature of the flow of time.

To show the transference of completeness via logic combination, we maintain the same general proof strategy of by Finger and Gabbay [5]. However, because here we cannot rely on the linearity of the flow of time, the underlying proof construction has to be almost fully reworked. For that, we introduce a bound associated to the number of steps in "the past" and the number of steps in "the future" one must take to evaluate a given formula ψ in a temporal model. This construction allows us to select the "relevant" time points in the evaluation of a formula. The set of "relevant" time points may be infinite, but each point can be reached in finitely many steps. This construction allows us to do without the original restriction of linearity.

Our approach naturally leads us to decision procedures. In Section 4 we show that provided that L and T are decidable, so is $\mathsf{T}(\mathsf{L})$.

Combination of logics have been previously analyzed in the literature. Combinations of tense and modality were discussed in [6], without explicitly providing a general methodology for doing so. Fine and Schurz [1] and Kracht and Wolter [4] have studied the transfer properties of systematically combining independently axiomatisable monomodal systems. The work of [1] generalizes to more than two independent modalities. Finger and Gabbay [5, 2] were the first to address the issue of

combining logics with two-place modalities, S ("since") and U ("until"), and with modalities that were not all independent, for "since" and "until" interact with each other. The results of [5, 2], however, are restricted to the case of linear flows of time and, because non-linear flows, e.g. over trees or over some other partially ordered sets, often appears in Mathematics as well as in Computer Science, our approach is needed.

It is important to note that this generalisation over generic flows of times propagates to other combinations of logics that can be interpreted in terms of temporalisations. In [2], it was shown that the *independent combination* of two US logics, $US \oplus \bar{U}\bar{S}$ can be seen as the infinite union of several temporalisation $US(\bar{U}\bar{S})$ and $US(\bar{U}\bar{S}(US)), \ldots$, and thus the results here also generalise the transference of soundness, completeness and decidability for $US \oplus \bar{U}\bar{S}$ over generic flows of time.

2. The Temporalised System $\mathsf{T}(\mathsf{L})$

In this section we describe the system $\mathsf{T}(\mathsf{L})$ introduced in [5]. By a *logic system* we mean a *triple* $S = (\mathcal{L}_S, \vdash_S, \mathcal{K}_S)$, where $\mathcal{L}_S$ is the system's language, $\vdash_S$ is an inference system and $\mathcal{K}_S$ is the system's associated class of models.

The language of $\mathsf{T}(\mathsf{L})$. The language $\mathcal{L}_{\mathsf{US}}$ of the temporal system T is built from a denumerable set of atoms $\mathcal{A}$, applying the two-place modalities U *(until) and* S*, (*since), and the Boolean connectives $\neg$ (*negation*) and $\wedge$ (*conjunction*).

Very little is required of the internal logic L, except that its language is described from a denumerable set of *atoms* and that it has the classical Boolean connectives $\neg$ and $\wedge$; but see below in case it does not have them. Apart from that, any other type of modalities or predicates are accepted in the language.

Before we define the language of the *temporalised system* $\mathsf{T}(\mathsf{L})$ we need to introduce a few definitions.

The language $\mathcal{L}_{\mathsf{L}}$ of L is partitioned into the sets BC_{L} and ML_{L}, where:

- BC_{L}, the set of *Boolean combinations* consists of the formulas built up from *any* other formulas with the use of the Boolean connectives $\neg$ or $\wedge$;
- ML_{L}, the set of *monolithic formulas* is the complementary set of BC_{L} in $\mathcal{L}_{\mathsf{L}}$.

If the external logic L does not contain the classical connectives $\neg$ and $\wedge$, we assume that $ML_{\mathsf{L}} = \mathcal{L}_{\mathsf{L}}$ and $BC_{\mathsf{L}} = \emptyset$, so every formula in L is considered monolithic.

The set of temporalised formulas, $\mathcal{L}_{\mathsf{T}(\mathsf{L})}$, is defined as the smallest set closed under the rules

1. If $A \in ML_{\mathsf{L}}$, then $A \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$;
2. If $A, B \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$, then $\neg A \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$ and $A \wedge B \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$;
3. If $A, B \in ML_{\mathsf{L}}$, then $S(A, B) \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$ and $U(A, B) \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$.

Note that the atoms of $\mathcal{L}_{\mathsf{US}}$ are not elements of $\mathcal{L}_{\mathsf{T}(\mathsf{L})}$. As an example of a temporalised language, consider the atoms $p, q \in \mathcal{L}_{\mathsf{L}}$ and $\Box$ is a modal symbol in L, then $\Box p$ and $\Box(p \wedge q)$ are monolithic formulas whereas $\neg\Box p$ and $\Box p \wedge \Box q$ are two Boolean combinations.

The *mirror image* of a given formula is given by replacing U by S and vice-versa. We will use the connectives $\vee$ and $\rightarrow$ and the constants $\top$ and $\bot$ in its usual meaning. Also, the formulas PA,

FA, GA and HA abbreviate respectively $S(A, \top)$, $U(A, \top)$, $\neg F(\neg A)$ and $\neg P(\neg A)$. The *complexity* of a formula A is the cardinality of its subformulas.

The Semantics of $\mathsf{T}(\mathsf{L})$. A *flow of time* is a pair $(T, <)$ where T is a set of time points and $<$ is a binary relation on T. By imposing restrictions on $<$ we generate *classes of flows of time*, e.g. the class $\mathcal{K}_{lin}$ of all transitive, irreflexive and linear flows of time.

Let $\mathcal{K}_L$ be the class of models of L. Let $(T, <)$ be a flow of time and let g be a mapping from T into $\mathcal{K}_L$. The mapping g is assumed to be such that for all formula $A \in \mathcal{L}_{\mathsf{L}}$ for all $t \in T$, either $g(t) \models A$ or $g(t) \models \neg A$. E.g., if L is a $S5$ system, the mapping g reaches at each t a triple $(\mathcal{W}_t, R_t, x_t)$, where $(\mathcal{W}_t, R_t)$ is a Kripke frame and $x_t \in \mathcal{W}_t$ is a possible world, so that either $\mathcal{W}_t, R_t, x_t \models A$ or $\mathcal{W}_t, R_t, x_t \models \neg A$, for every formula A of L; this would not be the case if x_t were not included.

A triple $\mathcal{M}_{\mathsf{T}(\mathsf{L})} = (T, <, g)$ is a *temporalised model* of $\mathsf{T}(\mathsf{L})$. We say that a temporal model $(T, <, g)$ belongs to a class $\mathcal{K}$ iff $(T, <) \in \mathcal{K}$.

The satisfaction relation $\models$ is defined recursively over structure of temporalised formulas:

1. $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models A$, $A \in ML_{\mathsf{L}}$, iff $g(t) = \mathcal{M}_L$ and $\mathcal{M}_L \models A$;
2. $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models \neg A$ iff $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \not\models A$;
3. $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models A \wedge B$ iff $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models A$ and $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models B$;
4. $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models S(A, B)$ iff there exists $s < t$ such that $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, s \models A$ and for all $s < r < t$, $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, r \models A$;
5. $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models U(A, B)$ iff there exists $t < s$ such that $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, s \models A$ and for all $t < r < s$, $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, r \models A$.

A formula is *valid* in a class $\mathcal{K}$ if it is verified at all times at all models over that class.

The Inference System of $\mathsf{T}(\mathsf{L})$. We assume that an *inference system* for a generic logic system is a mechanism capable of recursively enumerating the set of all provable formulas of the system, here called *theorems* of the logic system.

An inference system is *sound* with respect to a class of models $\mathcal{C}$ if all its theorems are valid over $\mathcal{C}$. Conversely, it is *complete* if all valid formulas are theorems. We assume that L's inference system is sound and complete.

We will assume that the temporal logic T's inference system is given in an axiomatic form, consisting of a set of *axioms* and a set of inference rules. For example, consider the possible axiomatizations of US over several classes of flows of time presented in [7] or in [3]. When a temporal logic T is sound and complete over the class $\mathcal{K}$ of flows, we write $\mathsf{T}/\mathcal{K}$.

Given $\mathsf{T}/\mathcal{K}$, the inference system of $\mathsf{T}(\mathsf{L})$ is denoted by $\mathsf{T}(\mathsf{L})/\mathcal{K}$ and consists of the following elements:

- The axioms of $\mathsf{T}/\mathcal{K}$;
- The inference rules of $\mathsf{T}/\mathcal{K}$;
- The inference rule *Preserve*: For every formula φ in $\mathcal{L}_{\mathsf{L}}$, if $\vdash_L \varphi$ then $\vdash_{T(L)} \varphi$.

In [5] it is shown that if $\mathsf{T}/\mathcal{K}$ and L have a sound inference system, then the inference system of $\mathsf{T}(\mathsf{L})/\mathcal{K}$ is sound; no extra restrictions are made on the nature of $\mathcal{K}$. Also, in case L has a complete

inference system and $\mathcal{K}_{lin}$ is a class of linear flows of time, then the inference system of if $\mathsf{T}(\mathsf{L})/\mathcal{K}_{lin}$ is complete. We want to eliminate this restriction on linearity.

3. Completeness of $\mathsf{T}(\mathsf{L})$

To show the transference of completeness we maintain the same proof strategy of [5], but we introduce a new technique and rework its underlying constructions. In the presence of linearity, one can write a formula that expresses the fact that a formula A "is true everywhere" in a model. This simplifies life a lot, but cannot be reproduced in a generic model. So we introduce a technique that picks up the "relevant" worlds in a model for the evaluation of a given formula, and we construct a formula that forces A to be true over all such relevant worlds.

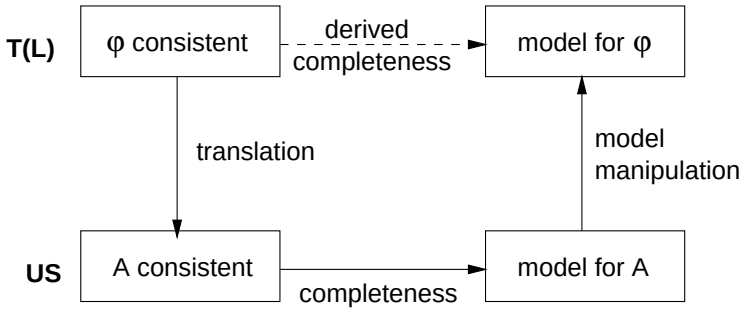

Figure 1. Completeness proof strategy

The strategy of the proof is illustrated in Figure 1. We start with a consistent $\mathcal{L}_{\mathsf{T}(\mathsf{L})}$-formula φ, translate it into a pure $\mathcal{L}_{\mathsf{US}}$-temporal logic consistent formula A; then completeness of $\mathcal{L}_{\mathsf{US}}/\mathcal{K}$ gives us a model for A; after some model manipulation (and the completeness of L) we obtain a $\mathsf{T}(\mathsf{L})/\mathcal{K}$-model for φ, thus deriving the completeness for $\mathsf{T}(\mathsf{L})/\mathcal{K}$. The more sophisticated bit of the proof is the initial translation step, which in the generic case has to deal with the nesting of temporal operators in φ instead of the simpler translation used for the linear case. Such initial elaboration allows us later to do a straightforward model manipulation to construct a model for φ.

To deal with the nesting of temporal operator in a formula, we define the *operator nesting tree* of a temporal or temporalised formula ψ, D_ψ. A tree is represented here as a set of strings of 0's and 1's, with the symbol $*$ representing concatenation of strings; the empty string is represented by ε. The tree is closed under prefix formation of its strings, that is, if $101 \in D_\psi$, then $\varepsilon, 1, 10 \in D_\psi$ as well. The 0 represents a past operator (a step to the past) and the 1 represents a future operator (or a step to the future).

Notation 3.1. In the following we will use the Greek letters φ, ψ and χ to indicate $\mathsf{T}(\mathsf{L})$ formulas, and the letters A, B and C to indicate temporal US formulas. We use the Greek letters φ, ψ and χ also to refer to either a temporal or temporalised formula.

Definition 3.2. *Given a formula $\psi \in \mathcal{L}_{\mathsf{US}} \cup \mathcal{L}_{\mathsf{T}(\mathsf{L})}$ we build its* operator nesting tree D_ψ *recursively over the structure of ψ:*

1. *If ψ is a literal or monolithic, then $D_\psi = \{\varepsilon\}$;*
2. *If $\psi = \varphi_1 \wedge \varphi_2$, then $D_\psi = D_{\varphi_1} \cup D_{\varphi_2}$;*
3. *If $\psi = \neg\varphi$, then $D_\psi = D_\varphi$;*
4. *If $\psi = S(\varphi_1, \varphi_2)$, then $D_\psi = \{\varepsilon\} \cup \{0 * s | s \in D_{\varphi_1} \cup D_{\varphi_2}\}$;*
5. *If $\psi = U(\varphi_1, \varphi_2)$, then $D_\psi = \{\varepsilon\} \cup \{1 * s | s \in D_{\varphi_1} \cup D_{\varphi_2}\}$.*

This definition implies that $\varepsilon \in D_\psi$ for any ψ and, as a consequence, the prefix of any string in D_ψ will also be a member of D_ψ. For example, consider the US formula

$$A = S(U(p, S(p,q)), S(p,p)) \wedge U(\neg U(p,q), S(p,q))$$

It's associated operator nesting tree will be:

$$\begin{aligned}
D_A = &\ D_{S(U(p,S(p,q)),S(p,p))} \cup D_{U(\neg U(p,q),S(p,q))}, \\
D_A = &\ \{\varepsilon\} \cup \{0 * s | s \in D_{U(p,S(p,q))} \cup D_{S(p,p)}\} \cup \{1 * r | r \in D_{U(p,q)} \cup D_{S(p,q)}\}, \\
D_A = &\ \{\varepsilon, 0, 1\} \cup \{01 * s' | s' \in D_p \cup D_{S(p,q)}\} \cup \{00 * s'' | s'' \in D_p\} \cup \\
&\ \{11 * r' | r' \in D_p \cup D_q\} \cup \{10 * r'' | r'' \in D_p \cup D_q\}, \\
D_A = &\ \{\varepsilon, 0, 1, 01, 00, 11, 10\} \cup \{010 * s''' | s''' \in \cup D_p \cup D_q\}\ , \\
D_A = &\ \{\varepsilon, 0, 1, 01, 00, 11, 10, 010\}.
\end{aligned}$$

Let 1^m represents a string of m 1's, and similarly for 0^m. Let 0^0 and 1^0 represent the empty string. So each string can be represented as $1^{m_1}0^{m_2}\ldots 1^{m_{n-1}}0^{m_n}$, where all $m_i > 0$, except for m_1 and m_n, that can be 0. Note that n is always an even number.

Each such string is then associated to a temporal operator over H and G. Let $H^0\psi = G^0\psi = \psi$; let $G^{n+1}\psi = G(G^n\psi)$; and $H^{n+1}\psi = H(H^n\psi)$. So each string $1^{m_1}0^{m_2}\ldots 1^{m_{n-1}}0^{m_n}$ is associated with the temporal operator $G^{m_1}H^{m_2}\ldots G^{m_{n-1}}H^{m_n}$, which we abbreviate as $\Box_{m_1,m_2,\ldots,m_{n-1},m_n}$.

As an example, $\Box_{0,2}(\Box_{0,3,1,0}\psi) \equiv \Box_{0,5,1,0}\psi$ instead of $\Box_{0,2,0,3,1,0}\psi$.

We can now start defining the translation of *consistent* formulas in $\mathsf{T}(\mathsf{L})$ into *consistent* formulas in US. The first step is the *correspondence mapping*.

Definition 3.3. Let $\{p_1, p_2, \ldots\}$ be an enumeration of the set of atoms of US, and let $\{\psi_1, \psi_2, \ldots\}$ be an enumeration of ML_L, the set of monolithic formulas of $\mathsf{T}(\mathsf{L})$. Define the *correspondence mapping* σ from $\mathcal{L}_{\mathsf{T}(\mathsf{L})}$ into $\mathcal{L}_\mathsf{US}$, inductively over a formula as:

$$\begin{aligned}
(\forall \psi_i \in ML_\mathsf{L})(\sigma(\psi_i)) &= p_i \\
\sigma(\neg\chi) &= \neg\sigma(\chi) \\
\sigma(\chi_1 \wedge \chi_2) &= \sigma(\chi_1) \wedge \sigma(\chi_2) \\
\sigma(S(\chi_1, \chi_2)) &= S(\sigma(\chi_1), \sigma(\chi_2)) \\
\sigma(U(\chi_1, \chi_2)) &= U(\sigma(\chi_1), \sigma(\chi_2))
\end{aligned}$$

The following two lemmas are shown in [5]:

Lemma 3.4 (The correspondence Lemma)**.** *The correspondence mapping σ is a bijection.* ■

Lemma 3.5. *For all $\chi \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$, if χ is $\mathsf{T}(\mathsf{L})$-consistent, then $\sigma(\chi)$ is US-consistent.* ■

The reverse of Lemma 3.5 is not true, as we can see in this example:

Example 3.6. In a modal normal logic with the modality $\Box$, for all atoms φ, ψ,

$$\chi \equiv \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

is a theorem in L. The formulas $\Box(\varphi \to \psi)$, $\Box\varphi$ and $\Box\psi$ are monolithic, so they are mapped by σ into some atoms of US, say p_1, p_2 and p_3, respectively.

Thus, $\sigma(\chi) = p_1 \to (p_2 \to p_3)$, that is not a theorem in T. ■

For the model manipulation in the final part of the proof of completeness, we will need also the converse of Lemma 3.5, that is, $\mathsf{T}(\mathsf{L})$ theorems must be mapped into US theorems. To achieve that, we define the transformation η, which makes use of the operator nesting tree D_ψ, and preserves consistency and which will guarantee that theorems in are mapped into theorems of US.

Definition 3.7. Given two formulas $\varphi, \psi \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$, define:

1. $Mon(\varphi)$ is the set of monolithic subformulas of φ;
2. $Lit(\varphi) = Mon(\varphi) \cup \{\neg\psi | \psi \in Mon(\varphi)\}$;
3. $Inc(\varphi) = \{\bigwedge F | F \subseteq Lit(\varphi) \text{ and } F \vdash_{\mathsf{L}} \bot\}$; that is $Inc(\varphi)$ is the set of L-inconsistent formulas that can be built using the monolithic subformulas of φ;
4. $\Box_\varphi\psi$ is the conjunction of all formulas of the form $\Box_{m_1,\ldots,m_n}\psi$ where $\Box_{m_1,\ldots,m_n}$ is a temporal operator associated to a string in the operator nesting tree D_φ;
5. $\eta(\varphi) = \bigwedge\{\Box_\varphi\neg\psi | \psi \in Inc(\varphi)\}$.

Example 3.8. *If $\varphi = S(p,q)$, then $D_\varphi = \{\varepsilon, 0\}$. So, for any formula ψ, $\Box_\varphi\psi = \Box_{0,0}\psi \wedge \Box_{0,1}\psi = \psi \wedge H\psi$.* ■

The terminology used in Definition 3.7 were introduced in [5]. The modification for the general case we had to make here is restricted to the definition of $\Box_\varphi\psi$ (used in the definitions of $\eta(\varphi)$).

The following Lemma is an adaptation of [5] for a the case of generic flows of time.

Lemma 3.9. $\vdash_{\mathsf{T}(\mathsf{L})} \eta(\psi)$.

Proof: Every formula φ in $Inc(\psi)$ is a contradiction, and therefore its negation is a theorem of $\mathsf{T}(\mathsf{L})$. Now, if $\neg\varphi$ is a theorem, so are $H\neg\varphi$ and $G\neg\varphi$; by induction we get that $\Box_{m_1,\ldots,m_n}\neg\varphi$ is a theorem too, for any $m_1, \ldots, m_n$. ■

Putting together Lemmas 3.5 and 3.9, we have that if ψ is $\mathsf{T}(\mathsf{L})$-consistent, then $\sigma(\psi \wedge \eta(\psi))$ is US-consistent. We can apply completeness of $\mathsf{US}/\mathcal{K}$ and obtain a US-model $\mathcal{M}_{\mathsf{US}}$ for $\sigma(\psi \wedge \eta(\psi))$ over $\mathcal{K}$. Furthermore, the theoremhood of the monolithic L-formulas in ψ is captured in $\eta(\psi)$ and will guarantee that its translation will be true in the "relevant part" of $\mathcal{M}_{\mathsf{US}}$. It is this notion of "relevant part" of a temporal model that we define next by associating subflows of time to binary trees (not very surprisingly). At this part of the proof we will be working at the US level.

Let $(T, <) \in \mathcal{K}$ be a flow of time, and let $t, s \in T$. We say that s is 1-related to t if $t < s$ (s is in the future of t); similarly, s is 0-related to t if $s < t$ (s is in the past of t). Let $t_1, \ldots, t_n \in T$ be a sequence of time points such that each pair t_i, t_{i+1} is 0- or 1-related. Such a sequence can then be associated to a string of 0's and 1's of length $n-1$, where the ith position is 1 if t_i and t_{i+1} are 1-related, and 0 otherwise; we represent it as $\mathsf{string}(t_1, \ldots, t_n)$.

The "relevant part" of a flow of time $(T, <)$, with respect to a temporal formula A at a point t, is formally defined as the *range of A at t over $(T, <)$*, $Rg(A, t)$:

$$Rg(A,t) = \{t\} \cup \{s \in T \mid \mathsf{string}(t, t_1, \ldots, t_n, s) \in D_A \text{ for some } t_1, \ldots, t_n \in T\}$$

Note that since $D_A = D_{\neg A}$, it follows that $Rg(A,t) = Rg(\neg A, t)$.

It is important to highlight that we are *not* constructing a submodel of a given model generated by $Rg(A,t)$. Our aim is to construct a model that belongs to a class $\mathcal{K}$. If we start in a model over $\mathcal{K}$ and generate a submodel based on $Rg(A,t)$, there is no way to guarantee that the generated submodel belongs to $\mathcal{K}$, and in general it does not. So $Rg(A,t)$ will be used to focus on a relevant part of the model. The satisfaction of a formula A at a point t in a temporal model depends only on the temporal valuation at points in $Rg(A,t)$, as shown below.

Lemma 3.10. *Consider a temporal model $\mathcal{M} = (T, <, g)$, a formula $A \in \mathcal{L}_{\mathsf{US}}$, and a point $t \in T$. Then for any model $\mathcal{M}' = (T, <, g')$ such that $g'(s) = g(s)$ for every $s \in Rg(A,t)$,*

$$\mathcal{M}, t \models A \text{ iff } \mathcal{M}', t \models A.$$

Proof: Initially note that, both $\mathcal{M}$ and $\mathcal{M}'$ are based on the same flow of time, so for every formula B of A and for every $s \in T$, $Rg(B,s)$ is the same set. We proceed by structural induction over A.

- If A is atomic, then $g(t) = g'(t)$.
- If $A = \neg B$, then $Rg(A,t) = Rg(B,t)$, so the induction hypothesis directly fives us the result.
- If $A = B_1 \wedge B_2$, then $Rg(A,t) = Rg(B_1,t) \cup Rg(B_2,t)$, and therefore for every $s \in Rg(B_i,t)$, $g(t) = g'(t)$ $[i = 1,2]$, so the induction hypothesis applies and gives us that $\mathcal{M}, t \models B_i$ *iff* $\mathcal{M}', t \models B_i$, from which the result follows immediately.
- If $A = S(B,C)$, then $\mathcal{M}, t \models A$ iff there exists a $t' < t$ with $\mathcal{M}, t' \models B$ and for every t'' such that $t' < t'' < t$, $\mathcal{M}, t'' \models C$. Note that both $t', t'' \in Rg(A,t)$. Furthermore, because the temporal nesting of B and C is smaller than that of A, we have $Rg(B,t') \subseteq Rg(A,t)$ and therefore $g(s) = g(s')$ for every $s \in Rg(B,t')$, so the induction hypothesis applies and yields $\mathcal{M}, t' \models B$ iff $\mathcal{M}', t' \models B$; analogously, we get that for every t'' such that $t' < t'' < t$, $\mathcal{M}, t'' \models C$ iff $\mathcal{M}', t' \models C$, and therefore the result follows.
- If $A = S(B,C)$ the reasoning is totally analogous to the previous case, finishing the proof. ■

The following lemma shows that the definition of $\eta(\psi)$ preserves ψ's truth value over that "relevant part" of a model.

Lemma 3.11. *Let $\mathcal{M}_{\mathsf{US}} = (T, <, g)$ be a temporal model over $\mathcal{K}$ and $\varphi, \psi \in \mathcal{L}_{\mathsf{T(L)}}$. Let $t \in T$ so that $\mathcal{M}_{\mathsf{US}}, t \models \sigma(\Box_\varphi \psi)$. Then for every $s \in Rg(\sigma(\varphi), t)$, $\mathcal{M}_{\mathsf{US}}, s \models \sigma(\psi)$.*

Proof: We know that

$$\Box_\varphi \psi = \bigwedge_{1^{m_1} \ldots 0^{m_n} \in D_\varphi} \Box_{m_1, \ldots, m_n} \psi.$$

A simple induction shows that $D_\varphi = D_{\sigma(\varphi)}$, and therefore

$$\sigma(\Box_\varphi \psi) = \bigwedge_{1^{m_1} \ldots 0^{m_n} \in D_{\sigma(\varphi)}} \Box_{m_1, \ldots, m_n} \sigma(\psi).$$

Consider $s \in Rg(\sigma(\varphi), t)$. Then either $s = t$ or there are $t_1, \ldots, t_n \in Rg(\sigma(\varphi), t)$ such that $\mathsf{string}(t, t_1, \ldots, t_n, s) \in D_{\sigma(\varphi)}$. If $s = t$, since $\varepsilon \in D_{\sigma(\varphi)}$, it follows that $\mathcal{M}_{\mathsf{US}}, s \models \sigma(\psi)$. In the latter case, we show the result by induction on n.

For $n = 1$, we have that either $s < t$, in which case we have that $\mathcal{M}_{\mathsf{US}}, t \models H\sigma(\psi)$ so $\mathcal{M}_{\mathsf{US}}, s \models \sigma(\psi)$, or $t < s$, in which case we have that $\mathcal{M}_{\mathsf{US}}, t \models G\sigma(\psi)$ so $\mathcal{M}_{\mathsf{US}}, s \models \sigma(\psi)$.

For the inductive case, we have that $\mathsf{string}(t, t_1, \ldots, t_n, s) \in D_{\sigma(\varphi)}$. Again we have two possibilities. If $t_n < s$ then the rightmost operator in $\Box_{m_1,\ldots,m_n}$ is a G, and the induction hypothesis gives us that $\mathcal{M}_{\mathsf{US}}, t_n \models G\sigma(\psi)$ so $\mathcal{M}_{\mathsf{US}}, s \models \sigma(\psi)$. If $s < t_n$ then the rightmost operator in $\Box_{m_1,\ldots,m_n}$ is an H, and the induction hypothesis gives us that $\mathcal{M}_{\mathsf{US}}, t_n \models H\sigma(\psi)$ so $\mathcal{M}_{\mathsf{US}}, s \models \sigma(\psi)$.

This finishes the induction and the proof. ■

We can now finally to glue the pieces of the completeness proof.

Theorem 3.12. *If the logical system* L *is complete and* US *is complete over a class of flows of time* $\mathcal{K}$*, then the logical system* $\mathsf{T}(\mathsf{L})$ *is complete over* $\mathcal{K}$*.*

Proof: Let ψ be a $\mathsf{T}(\mathsf{L})/\mathcal{K}$-consistent formula. We will construct a $\mathsf{T}(\mathsf{L})$-model for ψ over the class $\mathcal{K}$.

By Lemma 3.9, $\psi \wedge \eta(\psi)$ is also a $\mathsf{T}(\mathsf{L})$-consistent formula. So, by Lemma 3.5, $\sigma(\psi \wedge \eta(\psi))$ is a T-consistent formula. As we assume that $\mathsf{US}/\mathcal{K}$ is complete, then there exists a temporal model $\mathcal{M}_{\mathsf{US}} = (T, <, g)$ with $(T, <) \in \mathcal{K}$ such that for some $t \in T$, $\mathcal{M}_{\mathsf{US}}, t \models \sigma(\psi \wedge \eta(\psi))$. For every $s \in Rg(\psi, t)$, define:

$$G_\psi(t) = \{\varphi \in Lit(\psi) | \mathcal{M}_{\mathsf{US}}, t \models \sigma(\varphi)\}$$

Claim: For every $s \in Rg(\psi, t)$, $G_\psi(s)$ is finite and L-consistent.

Indeed, $G_\psi(t)$ is finite because $Lit(\psi)$ is finite. To prove consistency, suppose by absurd that for some $s \in T$, $G_\psi(s)$ is L-inconsistent. Then there exists a subset of $G_\psi(s)$, $\{\varphi_1, \ldots, \varphi_n\}$ such that $\vdash_{\mathsf{L}} \bigwedge_{1 \leq i \leq n} \varphi_i \rightarrow \bot$. Thus $\bigwedge_{1 \leq i \leq n} \varphi_i \in Inc(\psi)$.

Let $\xi = \Box_\psi \sigma(\bigwedge_{1 \leq i \leq n} \varphi_i)$. As $\mathcal{M}_{\mathsf{US}}, t \models \sigma(\psi \wedge \eta(\psi))$ and $\neg\xi$ is a conjunct of $\sigma(\psi \wedge \eta(\psi))$, i.e. $\sigma(\psi \wedge \eta(\psi)) = \chi \wedge \neg\xi$, for some formula χ; by Lemma 3.11 and $\mathcal{M}_{\mathsf{US}}, t \models \neg\xi$, it follows that $\mathcal{M}_{\mathsf{US}}, s \models \neg\xi$. By the definition of $G_\psi(s)$, $\mathcal{M}_{\mathsf{US}}, s \models \sigma(\bigwedge_{1 \leq i \leq n} \varphi_i)$, contradicting the inconsistency of $\bigwedge_{1 \leq i \leq n} \varphi_i$.

Therefore $G_\psi(s)$ is always L-consistent, proving the claim.

This claim is then used to build a model for ψ in the following way. By Lemma 3.11, for each $s \in Rg(\psi, t)$, $\mathcal{M}_{\mathsf{US}}, s \models \sigma(G_\psi(t))$. By hypothesis, L is complete, so for each $s \in Rg(\psi, t)$ there exists a model for the L-consistent formula $G_\psi(s)$, $\mathcal{M}^s_{\mathsf{L}}$. So, we can define a valuation h as:

$$h(s) = \mathcal{M}^s_{\mathsf{L}}$$

for every $s \in Rg(\psi, t)$; for $s \in T - Rg(\psi, t)$, $h(s)$ can be any model of L.

Consider $\mathcal{M}_{\mathsf{T}(\mathsf{L})} = (T, <, h)$. To obtain completeness, all we have to do is to prove that $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models \psi$. First, note that for every $s \in Rg(\psi, t)$, and every monolithic subformula B of ψ, $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models \varphi$ iff $\mathcal{M}_{\mathsf{US}}, t \models \sigma(\varphi)$. Then a straightforward structural induction on φ generalizes this to show that $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models \psi$ iff $\mathcal{M}_{\mathsf{US}}, t \models \sigma(\psi)$; details omitted.

But since we have that $\mathcal{M}_{\mathsf{US}}, t \models \sigma(\psi)$, it follows that $\mathcal{M}_{\mathsf{T(L)}}$ is a temporalised model for ψ over $\mathcal{K}$, finishing the proof. ■

4. Decidability and Complexity

Decidability and Complexity results are shown in [5] conditioned to completeness of a given subsystem of the linear flow of time system $\mathcal{K}$.

We extend here the decidability and complexity shown in [5]. Recall that a given system L is *decidable* if for any formula $\psi \in \mathsf{L}$, there exists a decision procedure to show if ψ is a theorem or not. So, if L is complete then L is decidable if for any formula $\psi \in \mathsf{L}$, it is possible to show if ψ is valid or not.

Let us suppose that both the temporal system T and the external system L are decidable. We assume for simplicity that both T and the external system L are sound and complete. Then, $\mathsf{T(L)}$ is also sound and complete, and it follows that decidability is obtainable if we can decide the validity of a $\mathsf{T(L)}$ formula ψ in any temporalised model.

Now, because L is decidable, the formula $\eta(\psi)$ can be constructed and is a $\mathsf{T(L)}$-theorem, so ψ is a theorem iff $\eta(\psi) \to \psi$ is. We want to show that ψ is a $\mathsf{T(L)}$-theorem iff $\sigma(\eta(\psi) \to \psi)$ is a US-theorem.

It is trivial to show that if $\sigma(\psi)$ is a US-theorem, then ψ is a $\mathsf{T(L)}$-theorem; it suffices to mimic the US-proof at the temporalised level, since all US axioms and inference rules are present at $\mathsf{T(L)}$.

For the other direction, suppose ψ is a $\mathsf{T(L)}$-theorem. I follows from Lemmas 3.10 and 3.11, that if there was a countermodel for $\sigma(\eta(\psi) \to \psi)$, we would be able to construct a countermodel for $\eta(\psi) \to \psi$, and thus also a countermodel for ψ, which contradicts completeness.

Therefore we have proved that to decide whether ψ is a theorem, we can decide if $\sigma(\eta(\psi) \to \psi)$ is a theorem. But since $\eta(\psi)$ is constructible by the decidability of L, we can apply the decidability of US as the final part of our decision procedure. We have thus proved that:

Theorem 4.1. *If* T *and* L *are sound, complete and decidable,* $\mathsf{T(L)}$ *is decidable.*

To eliminate the need to assume soundness and completeness, a proof method like that of [5] can be applied, with considerably more proof theoretical manipulation.

5. Conclusion

We have extended the original result on temporalisation of [5] to any flow of time. Recursive temporalisations were used in [2] to show the transference of completeness and decidability for the full combination of two linear US-temporal logics. Such combinations do not have the restriction that one logic is *external* to the other, and allows arbitrary nesting of one kind of modality inside the other.

What is notable is that the same original construction of [2] applies directly to the recursive temporalisation of generic (not necessarily linear) temporal logics. That is, the results here presented also generalise the transference of completeness and decidability to the full combination of US-temporal logics.

Acknowledgements

Marcelo Finger was partly supported by the Brazilian Research Council (CNPq), grant PQ 300597/95-5.

References

[1] K. Fine and G. Schurz. Transfer theorems for stratified multimodal logics. To appear in the Proceedings of Arthur Prior Memorial Conference, Christchurch, New Zealand.

[2] M. Finger and D. Gabbay. Combining Temporal Logic Systems. *Notre Dame Journal of Formal Logic*, 37(2):204–232, Spring 1996.

[3] D. M. Gabbay, I. M. Hodkinson, and M. A. Reynolds. *Temporal Logic: Mathematical Foundations and Computational Aspects*, volume 1. Oxford University Press, 1994.

[4] M. Kracht and F. Wolter. Properties of independently axiomatizable bimodal logics. *Journal of Symbolic Logic*, 56(4):1469–1485, 1991.

[5] M. Finger and D. Gabbay. Adding a temporal dimension to a logic system. *Journal of Logic Language and Information*, 1:203–233, 1992.

[6] R. H. Thomason. Combinations of Tense and Modality. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume II, pages 135–165. D. Reidel Publishing Company, 1984.

[7] M. Xu. On some U, S-Tense Logics. *Journal of Philosophical Logic*, 17:181–202, 1988.