Notes on measuring inconsistency in probabilistic logic

Glauber De Bona* and Marcelo Finger†

Department of Computer Science, Institute of Mathematics and Statistics
University of São Paulo, Brazil
{debona,mfinger}@ime.usp.br

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Abstract

Inconsistency measures have recently been put forward to manage inconsistent knowledge bases in the AI community. For conditional probabilistic logics, rationality postulates and computational complexity have driven the formulation of inconsistency measures. Independently, investigations in formal epistemology have used the betting concept of Dutch book to measure an agent's degree of incoherence. In this paper, we argue for the unsatisfiability of the proposed postulates and put forward alternative ones. Problematic desirable properties are weakened by analyzing the underlying consolidation process. Inconsistency measures suggested in the literature and computable with linear programs are shown to satisfy the postulates. Additionally, it is given a gambling interpretation for these practicable measures, showing they correspond to incoherence measures via Dutch books. Finally, we propose a general linear programming framework, allowing for confidence factors and encompassing measures from both communities that satisfy the reconciled postulates.

keywords: Probabilistic reasoning, Probabilistic logic, Inconsistency measures

1 Introduction

Representing real world knowledge and performing inference usually demand formalisms that cope with uncertainty. Probabilistic logics combine the deductive power of logical systems with the well-founded Theory of Probability to attend to this need. Typically, to perform inference in probabilistic logic, it

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is required the consistency of the set of premises, but many are the possible sources of inconsistency in a probabilistic knowledge base: it may contain statistical data from different samples, it could have been formed by the opinion of different experts, or even a single expert could lack the resources to check his own consistency while building the base, etc. To restore consistency in such cases, the inconsistency may be analyzed, which calls for a way to measure it. This work investigates measures of inconsistency for knowledge bases over probabilistic logic.

The problem of measuring inconsistency in knowledge bases over logical languages has increasingly received attention during the last years. Knight proposed a way to measure inconsistency in classical logic by attaching probabilities to formulas [24]. Hunter and Konieczny combined measures based on how many formulas are required to produce a contradiction with measures based on the proportion of the language affected by it [17]. The probabilistic version of measuring inconsistency has more recently been tackled by Thimm [38], Muino [26] and Potyka [29]. All three authors developed measures based on distance minimization, tailored to the probabilistic case. Potyka focused on computational aspects, looking for efficiently computable measures [29]. Muino was driven by the real knowledge base CADIAG-2, showing its infinitesimal inconsistency degree, based on a different semantics [26]. Thimm [38] adapted Hunter and Konieczny's [17] desirable properties for inconsistency measures, searching for measures that satisfy a set of postulates.

Instead of the probabilistic conditional logic [32] used by Thimm [38], we adopt a probabilistic logic with imprecise probabilities (like [10] and [25]). Knowledge bases are defined as sets whose elements have the form $(\varphi|\psi)[\underline{q}, \bar{q}]$, with the intended meaning: "the probability of φ being true given that ψ is true is between $\underline{q}$ and $\bar{q}$". An inconsistency measure is a function taking knowledge bases to non-negative numbers, which must obey some postulates. The first one, introduced by Hunter [17] for classical logic, is *consistency*, which claims that an inconsistency measure is zero if, and only if, the corresponding base is consistent. Another desirable property suggested by Hunter and Konieczny [17] is *independence*, stating that the withdrawal of a free formula of the base — *i.e.*, a formula that does not belong to any minimal inconsistent set — should not change the inconsistency measure. Thimm [38] adopts these postulates, among others, adding *continuity* to the list, which intuitively says small changes in probabilities lead to small changes at the value of the inconsistency measure. We argue that consistency, independence and continuity cannot hold together, and some of these postulates must be abandoned or exchanged for other ones that restore joint satisfiability.

While computer scientists are investigating the problem of measuring inconsistency in probabilistic knowledge bases, philosophers have been interested in degrees of incoherence for formal agents that assign probability to events. Nau was concerned with reconciling incoherent probabilities and studied methods similar to Potyka's [27]. Schervish, Seidenfeld and Kadane measure incoherence of agents through the operational interpretation of probabilities as relative prices for gambles, quantifying the agent's incoherence as the maximum sure loss she would be exposed to through a Dutch book [34]. Staffel also uses Dutch books to measure incoherence, with a different restriction on the gambling setting [37]. To the best of our knowledge, these proposals for measuring incoherence of Bayesian agents have been ignored within the Artificial Intelligence community, even though they correspond to measures on probabilistic knowledge bases.

This work has two main contributions: firstly, we identify and fix the possible problem with the postulates proposed by Thimm [38] for inconsistency measures in probabilistic logic; in a second moment, we show how practicable inconsistency measures from the Computer Science community [29] correspond to justified measures from formal epistemology research [34, 37]. We put forward a way to weaken the postulate of independence in order to reach compatibility with consistency and continuity. However, more than one measure can fulfil these compatible postulates, so computational aspects are taken into account. We review the work of Potyka, presenting two inconsistency measures that can be computed with linear programs. We show a gambling interpretation for such measures, analyzing incoherence quantification for agents via Dutch books. Furthermore, the betting interpretation not only justifies these two practicable measures but also suggests different ways of quantifying inconsistency, which are shown to be encompassed by a linear programming generalization of Potyka's framework.

The rest of the paper is organized as follows. In Section 2, probabilistic knowledge bases are defined after fixing notation on propositional classical logic. Postulates suggested in the literature for inconsistency measures are reviewed in Section 3, and it is argued against the compatibility of such desirable properties. We propose a way to circumvent this seeming incompatibility in Section 4, by refining concepts central to the postulates. Inconsistency measures computable via linear programming are presented in Section 5, where they are shown to correspond to measures based on Dutch books.

2 Preliminaries

A propositional logical language is a set of formulas formed by atomic propositions combined with logical connectives, possibly with punctuation elements. We assume a finite set of symbols $X_n = \{x_1, x_2, x_3, \ldots, x_n\}$ corresponding to *atomic propositions (atoms)*. Formulas are constructed inductively with connectives $(\neg, \wedge, \vee, \rightarrow)$ and atomic propositions as usual. The set of all these well-formed formulas is the propositional language over X_n, denoted by $\mathcal{L}_{X_n}$. Additionally, $\top$ denotes $x_i \vee \neg x_i$ for some $x_i \in X_n$, and $\bot$ denotes $\neg\top$.

Given a signature X_n, a *possible world* w is a conjunction of $|X_n| = n$ atoms containing either x_i or $\neg x_i$ for each $x_i \in X_n$. We denote by $W_{X_n} = \{w_1, \ldots, w_{2^n}\}$ the set of all possible worlds over X_n and say a $w \in W_{X_n}$ *entails* a $x_i \in X_n$ ($w \models x_i$) iff x_i is not negated in w. This entailment relation can be extended to all $\varphi \in \mathcal{L}_{X_n}$ as usual.

A *probabilistic conditional* (or simply *conditional*) is a statement of the form $(\varphi|\psi)[\underline{q}, \bar{q}]$, with the underlying meaning "the probability that φ is true given that ψ is true lies within the interval $[\underline{q}, \bar{q}]$", where $\varphi, \psi \in \mathcal{L}_{X_n}$ are propositional formulas and $\underline{q}, \bar{q} \in [0,1]$ are real numbers. Note that we do not assume $\underline{q} \leq \bar{q}$, since we are going to measure inconsistency. If ψ is a tautology, a conditional like $(\varphi|\psi)[\underline{q}, \bar{q}]$ is called an *unconditional probabilistic assessment*, usually denoted by $(\varphi)[\underline{q}, \bar{q}]$. We say a conditional in the format $(.)[q, q]$ is *precise* and denote it by $(.)[q]$.

A *probabilistic interpretation* $\pi : W_{X_n} \to [0,1]$, with $\sum_j \pi(w_j) = 1$, is a probability mass over the set of possible worlds, which induces a probability measure $P_\pi : \mathcal{L}_{X_n} \to [0,1]$ by means of $P_\pi(\varphi) = \sum\{\pi(w_j)|w_j \models \varphi\}$. A conditional $(\varphi|\psi)[\underline{q}, \bar{q}]$ is *satisfied* by π iff $P_\pi(\varphi \wedge \psi) \geq \underline{q}P_\pi(\psi)$ and $P_\pi(\varphi \wedge \psi) \leq \bar{q}P_\pi(\psi)$. Note that when $P_\pi(\psi) > 0$, a probabilistic conditional $(\varphi|\psi)[\underline{q}, \bar{q}]$ is constraining the conditional probability of φ given ψ; but any π with $P_\pi(\psi) = 0$ trivially satisfies the conditional $(\varphi|\psi)[\underline{q}, \bar{q}]$ (this semantics is adopted by Halpern [13], Frisch and Haddawy [10] and Lukasiewicz [25], for instance). A *knowledge base* is a finite set Γ of probabilistic conditionals such that, if $(\varphi|\psi)[\underline{q}, \bar{q}], (\varphi|\psi)[\underline{q}', \bar{q}'] \in \Gamma$, then $[\underline{q}, \bar{q}] = [\underline{q}', \bar{q}']$. That is, for each pair φ, ψ, only one probability interval can be assigned to $(\varphi|\psi)$ in a knowledge base[1]. A knowledge base Γ is *consistent* (or *satisfiable*) if there is a probability mass satisfying all conditionals $(\varphi|\psi)[\underline{q}, \bar{q}] \in \Gamma$. It is precise if all intervals are singletons.

The problem of verifying the consistency of a knowledge base is called *probabilistic satisfiability* (or *PSAT*) [11]. Probabilistic satisfiability has been rediscovered several times, and an analytical and unconditional version was actually proposed by Boole [2]. Hailperin [12], Bruno and Gilio [3], and Nilsson [28] suggested solutions via linear programs. This linear programming approach can be easily extended to handle conditional probabilities under the semantics we are using [14]. Recent advances in algorithms for PSAT solving can be found in [15, 8, 23].

If any probability mass π satisfying $(\varphi|\psi)[\underline{q}, \bar{q}]$ implied $P_\pi(\psi) > 0$, in an alternative semantics, the latter restriction could be added to the program, although losing the linear program standard format; this is the semantics adopted by Muino [26], for instance. De Finetti proposed an alternative setting in which the conditional probability is fundamental [7] and the satisfaction of probabilistic conditionals does not trivialize when the conditioning event has null probability. In such scenario, the consistency is called *coherence*, and its checking demands solving a sequence of linear programs [5].

When all interval bounds are rational numbers, PSAT is an NP-complete problem [11]; if there is a solution, there is a solution with only $m+1$ possible worlds receiving positive probability mass, where m is the knowledge base size. Nevertheless, column generation methods can handle large problems [22, 21], and several approaches have recently appeared [23, 8, 15, 6]. Note that this linear programming approach can be applied to other probabilistic logics (see, for instance, [1] and [20]).

[1]Note that this requirement is not too restrictive. Since nothing was said about logically equivalent propositions, a knowledge base may contain different probability intervals assigned to φ and $\varphi \wedge \top$, for instance.

3 Basic Desirable Properties for Inconsistency Measures

Approaches to measuring inconsistency in probabilistic knowledge bases have been put forward by Muino [26], Thimm [38] and Potyka [29], with different semantics for the conditionals. We follow the one adopted by Thimm and Potyka, in which a conditional is also satisfied by any measure assigning null probability to the conditioning formula. Thimm has done a groundlaying work [38], extending Hunter's postulates for inconsistency measures to the probabilistic case, which is our starting point. Potyka suggests practicable measures [29] we will review in Section 5.1, after investigating carefully the postulates. In this section, we begin with some desirable properties proposed by Thimm and then argue against their joint satisfiability.

3.1 Postulates

Let $\mathbb{K}$ ($\mathbb{K}_{prec}$) be the set of all (precise) knowledge bases. An inconsistency measure for knowledge bases is a function $\mathcal{I} : \mathbb{K} \to [0, \infty)$. Thimm's investigation is restricted to measures $\mathcal{I} : \mathbb{K}_{prec} \to [0, \infty)$ over knowledge bases with precise probabilities, to what we narrow our focus in this section. The author proposes some desirable properties such a function should satisfy, following Hunter and Konieczny's work for classical logic [17]. Although Thimm investigates a total of ten postulates, we describe in this section only four of these properties that we consider problematic. The first one claims that an inconsistency measure must at least discriminate between consistent and inconsistent bases:

Postulate 3.1 (**Consistency**). $\mathcal{I}(\Gamma) = 0$ *iff* Γ *is consistent.*

A second desirable property has to do with probabilistic conditionals one can ignore while measuring inconsistency, since they are not involved with the unsatisfiability, in some sense. Some notation is needed to formalize it.

Definition 3.2. A set Γ of probabilistic conditionals is a *minimal inconsistent set (MIS)* if Γ is inconsistent and every set $\Gamma' \subsetneq \Gamma$ is consistent.

Minimal inconsistent sets can be considered the purest form of inconsistency [18], capturing its causes. The focus on MISes is derived from the seminal work of Reiter [31] on the diagnosis problem. Reiter investigated how formulas from a base could be ruled out in order to restore consistency, by choosing at least one element from each MIS, computing thusly a hitting set of their collection.

Let $MIS(\Gamma)$ denote the collection of all MISes in Γ. Now we can define the central concept of free probabilistic conditional, following Thimm [38]:

Definition 3.3. A *free probabilistic conditional* of Γ is a probabilistic conditional $\alpha \in \Gamma$ such that, for all $\Delta \in MIS(\Gamma)$, $\alpha \notin \Delta$.

Analogously, a free probabilistic conditional of Γ is in all its maximal consistent subsets. The postulate of independence then claims that ruling out a free probabilistic conditional from a knowledge base should not change its inconsistency degree [38].

Postulate 3.4 (**Independence**). *If α is a free probabilistic conditional of Γ, then $\mathcal{I}(\Gamma) = \mathcal{I}(\Gamma \setminus \{\alpha\})$.*

A stronger condition, also introduced by Hunter and Konieczny and adopted by Thimm, deals with a sort of decomposability of the inconsistency measure through its minimal inconsistent sets. We call it a property, saving the name "postulate" to the most basic properties required from every measure. The version we present is tailored from Hunter and Konieczy's work [17], where Thimm claims to have adapted his postulate from [38].

Property 3.5 (**MIS-Separability**). *If $\Gamma = \Delta \cup \Psi$, $\Delta \cap \Psi = \varnothing$ and $MIS(\Gamma) = MIS(\Delta) \cup MIS(\Psi)$, then $\mathcal{I}(\Gamma) = \mathcal{I}(\Delta) + \mathcal{I}(\Psi)$.*

The idea behind this property is that the inconsistency of the whole knowledge base should be the sum of the inconsistency of its parts, whenever the partition does not break any minimal inconsistent set. For instance, consider $\Delta = \{(x_1)[0.5], (\neg x_1)[0.6]\}$, $\Psi = \{(x_2)[0.7], (x_2 \wedge x_3)[0.8]\}$ and $\Gamma = \Delta \cup \Psi$. It is clear that Δ and Ψ are the only minimal inconsistent sets in Γ. MIS-separability posits that the measure of inconsistency of Γ is obtained by summing the measures of Δ and Ψ; formally, $\mathcal{I}(\Gamma) = \mathcal{I}(\Delta) + \mathcal{I}(\Psi)$. MIS-separability is stronger than independence [38]:

Proposition 3.6. *If $\mathcal{I}$ satisfies MIS-separability, then $\mathcal{I}$ satisfies independence.*

These properties can be found in Hunter and Konieczny's work [18], in the definition of a "MinInc" separable basic inconsistency measure for knowledge bases over classical propositional logic. The measures they introduce are shown to fit such desiderata. Thimm revises the adaptation of these classical inconsistency measures to the probabilistic case and convincingly argues that they are not suitable to the *quantitative* nature of probabilities, since classical logic is *qualitative*.

To motivate the search for new inconsistency measures for probabilistic knowledge bases, while dispensing with measures from classical logic, Thimm puts forward the postulate of continuity. Intuitively, one expects that small changes in the probabilities of a knowledge base yield small changes in its degree of inconsistency. To formalize the continuity concept in precise knowledge bases, we introduce some notation, following Thimm [38].

That work studies precise knowledge bases of the form $\Gamma = \{(\varphi_i|\psi_i)[q_i] | 1 \leq i \leq m\}$. For each precise knowledge base Γ, there is a *characteristic function* $\Lambda_\Gamma : [0,1]^{|\Gamma|} \to \mathbb{K}_{prec}$ that, roughly speaking, changes the probabilities q_i in the base; *i.e.*, $\Lambda_\Gamma(\langle q'_1, q'_2, \ldots, q'_m \rangle) = \{(\varphi_i|\psi_i)[q'_i] | 1 \leq i \leq m\}$. To handle the (consistent) empty knowledge base, we define $\Lambda_\varnothing : \{\varnothing\} \to \{\varnothing\}$. Thimm imposes some order on the set Γ, building a sequence, for the function Λ_Γ be unique and well-defined. For simplicity, we just suppose there is some order (say, lexicographic) over the probabilistic conditionals used to uniquely specify

Λ_Γ[2]. Now the continuity postulate can be enunciated, with $\circ$ denoting function composition:

Postulate 3.7 (**Continuity** (for precise probabilities))**.** *For all* $\Gamma \in \mathbb{K}_{prec}$*, the function* $\mathcal{I} \circ \Lambda_\Gamma : [0,1]^{|\Gamma|} \to [0,\infty)$ *is continuous.*

To find inconsistency measures holding the desirable properties, including continuity, Thimm introduces a family of measures based on distance minimization, taking in account the numerical value of the probabilities. The basic idea is to quantify the inconsistency through the minimum changes, according to some distance, one has to apply on the probabilities to make the base consistency. The compatibility of consistency, independence and continuity is implicitly stated when it is proved that this whole family of inconsistency measures based on distance minimization satisfies them; and another family is proved to hold MIS-separability as well [38].

3.2 The Postulates' Incompatibility

The work done by Thimm [38] has carefully analyzed the problem of measuring inconsistency in knowledge bases over probabilistic logic. Desirable properties were borrowed from classical logic [17], and the crucial postulate of continuity was added. To attend theses properties, measures based on distance minimization were introduced and some important results were proved. However, under a closer examination, the proposed postulates seem to be incompatible.

Theorem 3.8. *There is no inconsistency measure* $\mathcal{I} : \mathbb{K}_{prec} \to [0,\infty)$ *that satisfies consistency, independence and continuity.*

Proof. To prove by contradiction, suppose there is a measure $\mathcal{I}$ satisfying consistency, independence and continuity. Consider the following knowledge bases:

$$\Gamma = \{(x_1 \wedge x_2)[0.5+\varepsilon], (x_1 \wedge \neg x_2)[0.5]\} \text{ for some } 0 < \varepsilon \leq 0.1 \tag{1}$$

$$\Delta = \Gamma \cup \{\alpha\}, \alpha = (x_1)[0.8] \tag{2}$$

We are going to use $\mathcal{I}$ to measure the inconsistency of Δ when $\varepsilon \to 0$. To apply independence, we are going to show that α free in Δ; we prove that Γ is the only MIS in Δ. To note that $\{(x_1 \wedge x_2)[0.5+\varepsilon], (x_1)[0.8]\}$ is consistent for any $\varepsilon \in (0, 0.1]$, see that such set is satisfied by the probability measure induced by the following probability mass: $\pi_1(x_1 \wedge x_2) = 0.5+\varepsilon, \pi_1(x_1 \wedge \neg x_2) = 0.3 - \varepsilon, \pi_1(\neg x_1 \wedge x_2) = \pi_1(\neg x_1 \wedge \neg x_2) = 0.1$. To prove that $\{(x_1 \wedge \neg x_2)[0.5], (x_1)[0.8]\}$ is consistent, consider the following probability mass: $\pi_2(x_1 \wedge x_2) = 0.3, \pi_2(x_1 \wedge \neg x_2) = 0.5, \pi_2(\neg x_1 \wedge x_2) = \pi_2(\neg x_1 \wedge \neg x_2) = 0.1$. Hence, all MISes of Δ must contain $\Gamma = \{(x_1 \wedge x_2)[0.5+\varepsilon], (x_1 \wedge \neg x_2)[0.5]\}$, for other subsets are all consistent. Furthermore, note that Γ is inconsistent and minimal, so it is a MIS.

[2]Technically, we could use the lexicographic order over the pairs $(\varphi_i|\psi_i)$ to construct a function Lex taking each set Γ to the corresponding sequence $\Psi = Lex(\Gamma)$, uniquely specifying a function Λ'_Ψ that changes the probabilities of the sequence Ψ. Then it could be defined $\Lambda_\Gamma(q) = Lex^{-1}(\Lambda'_\Psi(q))$.

We can conclude that Γ is the only MIS in Δ, for any value of $0 < \varepsilon \leq 0.1$. As α is a free probabilistic conditional of Δ, we can apply independence:

$$\mathcal{I}(\Delta) = \mathcal{I}(\Gamma)\,,$$

for any $0 < \varepsilon \leq 0.1$.

To exploit the continuity of $\mathcal{I}$, we need the characteristic function of Δ, $\Lambda_\Delta : [0,1]^3 \to \mathbb{K}_{prec}$, to be well-defined; so, we need an order over the probabilistic conditionals. Suppose that Γ and Δ are ordered as they were defined in (1) and (2). Let q^* be the vector $\langle 0.5, 0.5, 0.8\rangle$. It follows that $\Lambda_\Delta(q^*)$ differs from Δ only in its first conditional, which becomes $(x_1 \wedge x_2)[0.5]$. Now we prove that $\Lambda_\Delta(q^*)$ is inconsistent. For any probability measure P_π, $P_\pi(x_1 \wedge x_2) = P_\pi(x_1 \wedge \neg x_2) = 0.5$ implies $P_\pi(x_1) = 1$, contradicting $\alpha = \{(x_1)[0.8]\}$. As $\mathcal{I}$ satisfies consistency,

$$\mathcal{I} \circ \Lambda_\Delta(q^*) > 0. \tag{3}$$

By the continuity of $\mathcal{I}$, the function $\mathcal{I} \circ \Lambda_\Delta : [0,1]^3 \to [0,\infty)$ must be continuous, so there must be a limit at the point q^*, and such limit must be unique for any path approaching q^*:

$$\lim_{q \to q^*} \mathcal{I} \circ \Lambda_\Delta(q) = \lim_{\varepsilon \to 0^+} \mathcal{I} \circ \Lambda_\Delta(\langle 0.5 + \varepsilon, 0.5, 0.8\rangle) = \lim_{\varepsilon \to 0^+} \mathcal{I}(\Delta)\,.$$

By independence, we also have:

$$\lim_{\varepsilon \to 0^+} \mathcal{I}(\Delta) = \lim_{\varepsilon \to 0^+} \mathcal{I}(\Gamma)\,.$$

As $\mathcal{I}$ satisfies continuity and $\{(x_1 \wedge x_2)[0.5], (x_1 \wedge \neg x_2)[0.5]\}$ is satisfiable, the consistency of $\mathcal{I}$ implies

$$\lim_{\varepsilon \to 0^+} \mathcal{I}(\Gamma) = \mathcal{I}(\{(x_1 \wedge x_2)[0.5], (x_1 \wedge \neg x_2)[0.5]\}) = 0 = \lim_{q \to q^*} \mathcal{I} \circ \Lambda_\Delta(q)\,. \tag{4}$$

The continuity of $\mathcal{I}$ requires that $\mathcal{I} \circ \Lambda_\Gamma(q^*) = \lim_{q \to q^*} \mathcal{I} \circ \Lambda_\Gamma(q)$, which by (3) and (4) is a contradiction, finishing the proof. □

Corollary 3.9. *There is no inconsistency measure $\mathcal{I} : \mathbb{K}_{prec} \to [0,\infty)$ that satisfies consistency, MIS-separability and continuity.*

Looking at the counterexample given in the proof of Theorem 3.8 may shed some light on what is the cause of such conflict among the desirable properties. The only minimal inconsistent set in Δ is Γ, and so independence forces the degree of inconsistency of Δ to be the same as that of Γ, but this is not generally the case when inconsistency is measured via probability changing. This happens due to the fact that changing the probabilities in Γ to some consistent setting does not in general imply that Δ becomes consistent. Although Γ is the only minimal inconsistent set of Δ, there is another way to prove the contradiction. Note that Γ implies $(x_1)[1.2]$, which contradicts a probability axiom, but also contradicts $\alpha = (x_1)[0.8]$. While $\varepsilon = 0$ consolidates Γ, consolidating Δ requires a bigger change in probabilities, which is ignored by independence. By demanding $\mathcal{I}(\Delta) = \mathcal{I}(\Gamma)$ for $\varepsilon > 0$, the postulate of consistency forces a descontinuity on $\varepsilon = 0$. When $\varepsilon \to 0$, the inconsistency degree of Γ tends to zero (by continuity), and independence requires the same from Δ. But this contradicts continuity, given consistency, for $\{(x_1 \wedge x_2)[0.5], (x_1 \wedge \neg x_2)[0.5]\}$ would still contradict $(x_1)[0.8]$, and Δ would be inconsistent.

4 Reconciling the Postulates

The findings from the previous section suggest that in order to drive the rational choice of an inconsistency measure for knowledge bases, we must abandon at least one postulate among consistency, independence and continuity. We claim that a weakening of the desired properties can restore their compatibility, and in this section we investigate paths to achieve that goal. After reconciling the problematic postulates, we review other proposed properties for inconsistency measures and extend them to the general case of knowledge bases with imprecise probabilities, showing some measures to satisfy them.

The consistency postulate seems to be indisputable, since the least one can expect from an inconsistency measure is that it separates inconsistent from consistent cases, or some inconsistency from none. The answer to the question of which property we should relax to restore compatibility is thus reduced to either independence or continuity. Hunter and Konieczny have already noted problems with independence in knowledge bases over classical logic, proposing to relax it [19]. Intuition shall be inclined towards keeping continuity, for it reflects the particular quantitative nature of probabilistic reasoning. A pragmatic reason to give up independence (and so MIS-separability) is simply to keep continuity, given consistency, to save inconsistency measures based on distance minimization. In the sequel, the withdrawal of independence within probabilistic logic is argued for in a more compelling way.

The notion of free conditional and the postulate of independence are strongly related to the idea that minimal inconsistent sets are the causes of inconsistencies, as suggested by Hunter and Konieczny [17]. Thimm says that free conditionals are "harmless", in some sense, to the consistency of a knowledge base [38]. What is behind these notions is the classical way of handling inconsistency through ruling out formulas, as Reiter proposed in his diagnosis problem [31] and as the standard AGM paradigm of belief revision defines base contraction (see [16] for a general view of the AGM paradigm). Reiter's hitting sets technique views a repair of some inconsistency set of formulas as giving up of at least one element from each minimal inconsistent set. For such repair to be minimal, no free formula should be discarded. In the AGM paradigm, the *consolidation process* of a belief base can be interpreted as the contraction of $\perp$, the contradiction. The inclusion postulate claims that the result of a contraction is a subset of the belief base in question, and the relevance postulate forces the contraction of $\perp$ to contain all free formulas of the base.

When we move from classical to probabilistic logic, there is a natural way to relax formulas without completely losing their information. Note that ruling out a probabilistic conditional $(\varphi|\psi)[\underline{q}, \bar{q}]$ is semantically equivalent to changing it to $(\varphi|\psi)[0, 1]$, so it is a particular (and extreme) case of widening the probability interval. If we need to give up the belief on $(\varphi|\psi)[\underline{q}, \bar{q}]$ to restore consistency, perhaps there are some $\underline{q}' \leq \underline{q}$ and $\bar{q}' \geq \bar{q}$ such that $(\varphi|\psi)[\underline{q}', \bar{q}']$ can still be consistently believed. When inconsistency is measured continuously, through changes in probabilities, it is this more general kind of consolidation process that is being suggested. As it is indicated in the proof of Theorem 3.8, consolidating all minimal inconsistent sets (Γ) through probability changing does not imply consolidating the whole base (Δ). We can conclude that the concepts of free conditional and minimal inconsistent set are not suitable to analyze continuous inconsistency measures based on distance minimization.

Furthermore, it seems that the definition of free conditional, and so independence, can be refined to be suitable for analyzing continuous measures, while continuity is a harder definition to be contrived to be compatible with independence. Hence, we can try to weaken independence, and perhaps MIS-separability, by modifying the notion of free conditional, instead of fully forgetting this postulate.

As both independence and MIS-separability are defined via minimal inconsistent sets, in order to weaken these properties to reach compatibility with consistency and continuity, it seems reasonable to replace MIS by an alternative concept that could reconcile the desirable properties altogether. However, to do it in a principled way, we first analyze the concept of free probabilistic conditional as to the corresponding consolidation procedure and then modify it to save independence. Afterwards, a related notion of conflict that also fixes MIS-separability is introduced.

4.1 Refining the Free Probabilistic Conditional Concept

A weaker form of independence has already been suggested in the literature. Thimm [38] defines a *safe* conditional as one whose atomic propositions are disjoint from those in the rest of the base. We also demand that the conditional be satisfiable in order to be safe[3]. The *weak independence* postulate then posits that ruling a (satisfiable) safe conditional out should not change the inconsistency measure of a base. Hunter and Konieczny have suggested the same weakening for independence, in the classical setting, when they acknowledge that independence may be too strong a property to require [19]. Weak independence is compatible with consistency and continuity, since Potyka's measures satisfy them [29]. Although safe conditionals are easily recognizable, we expect that they be rare in practice, due to the natural logical dependencies among propositions within a base. We are looking for a stronger, more useful notion of independence, between the safe-based and the free-based ones, hence we look for a concept between safe and free.

[3]Thimm [38] only considers conditionals $(\varphi|\psi)[\underline{q}, \bar{q}]$ such that $\varphi \wedge \psi$ and $\neg\varphi \wedge \psi$ are (classically) satisfiable, so the conditional is also satisfiable.

Besides defining free probabilistic conditional through minimal inconsistent sets, one could equivalently do it via the notion of consolidation as giving up conditionals to restore consistency. Let us formalize this concept.

Definition 4.1. Let Γ be a knowledge base in $\mathbb{K}$. An *abrupt repair* of Γ is any set $\Delta \subseteq \Gamma$ such that $\Gamma' = \Gamma \setminus \Delta$ is consistent — we call Γ' an *abrupt consolidation*. If an abrupt repair Δ is such that, for every $\Psi \subsetneq \Delta$, $\Gamma \setminus \Psi$ is inconsistent, Δ is a *minimal abrupt repair* — and $\Gamma' = \Gamma \setminus \Delta$ is a *maximal abrupt consolidation.*

We can now prove[4] a result that states different ways to define a free probabilistic conditional, as being part of no minimal abrupt repairs (of all maximal consistent sets) or being consistent with any abrupt repair. We say a conditional α is *consistent with* a knowledge base Γ if there is a probability mass π that satisfies α and Γ.

Theorem 4.2. *Consider a knowledge base $\Gamma \in \mathbb{K}$ and a probabilistic conditional $\alpha \in \Gamma$. The following statements are equivalent:*

1. *There is no minimal abrupt repair Δ of Γ such that $\alpha \in \Delta$.*
2. *For all maximal abrupt consolidation Γ' of Γ, $\alpha \in \Gamma'$.*
3. *If $\Gamma' = \Gamma \setminus \Delta$ is an abrupt consolidation of Γ (equivalently, Δ is an abrupt repair of Γ), then α is consistent with Γ'.*
4. *There is no minimal inconsistent set $\Delta \subseteq \Gamma$ such that $\alpha \in \Delta$.*

Note that the fourth statement above is the definition of free probabilistic conditional given in Section 3.1. The first and the second statements are clearly dual to each other, so we have presented two new ways of equivalently defining a free probabilistic conditional without mentioning minimal inconsistent sets, but using abrupt repair and abrupt consolidation. As it is suggested in the previous section, ruling a conditional out is equivalent to widening the corresponding interval to $[0, 1]$ — that is why we call it an *abrupt* repair. However, a probabilistic logic allows for a more general notion of consolidation, formalized below. To save notation, we write $(\varphi|\psi)[\underline{q}, \bar{q}] \subseteq (\varphi|\psi)[\underline{q}', \bar{q}']$ if $\underline{q}' \leq \underline{q}$ and $\bar{q}' \geq \bar{q}$; and $\subsetneq$ is defined from $\subseteq$ as usual.

Definition 4.3. Let Γ be a knowledge base in $\mathbb{K}$. $\Gamma' \in \mathbb{K}$ is a *widening* of Γ if there is a bijection $f : \Gamma \to \Gamma'$ such that $\alpha \subseteq f(\alpha)$ for all $\alpha \in \Gamma$; furthermore, if a widening Γ' is consistent, we say it is a *consolidation* of Γ.

In other words, a consolidation of Γ is the result of widening the probability intervals of its conditionals to a consistent setting. Analogously to the maximal abrupt consolidation, related to a minimal abrupt repair, we can define a sort of consolidation with minimal changes, we call dominant.

Definition 4.4. A consolidation Γ' of Γ is a *dominant consolidation* (or simply a *d-consolidation*) of Γ if, for all consolidations Ψ of Γ, if Γ' is a widening of Ψ, then $\Gamma' = \Psi$.

[4]For the remaining technical results, proofs are given in a separate Appendix.

A d-consolidation Γ' of Γ is such that if some probability interval of Γ were less widened, fixing the others, the resulting base would not be consistent. In other words, it is not possible to give up strictly less information than a d-consolidation while restoring consistency; for a interval to be less widened, another must be more enlarged. In these sense, the changes in the probability bounds are minimal, and the consolidation is maximal.

From these concepts, two new definitions for free probabilistic conditional could be derived: a conditional is free if it is in any d-consolidation; or a conditional is free if it is consistent with any consolidation. We can prove these definitions are actually equivalent:

Lemma 4.5. *Consider a knowledge base $\Gamma \in \mathcal{K}$ and a probabilistic conditional $\alpha \in \Gamma$. The following statements are equivalent:*

1. *For all d-consolidation Γ' of Γ, $\alpha \in \Gamma'$.*
2. *If Γ' is a consolidation of Γ, then α is consistent with Γ'.*

A modification of the free probabilistic conditional concept is suggested by the comparison of Lemma 4.5 with Theorem 4.2, which would yield a different postulate of independence. To not overload the concept of free conditional, we say these probabilistic conditionals are *innocuous*, for they are consistent with any consolidation of the knowledge base.

Definition 4.6. An *innocuous probabilistic conditional* of Γ is a probabilistic conditional $\alpha \in \Gamma$ such that, for every dominant consolidation Γ' of Γ, $\alpha \in \Gamma'$.

The difference between free and innocuous conditionals can be seen in the knowledge base from the proof of Theorem 3.8, as the following example shows.

Example 4.7. Consider the following knowledge base:

$$\Delta = \{(x_1 \wedge x_2)[0.6], (x_1 \wedge \neg x_2)[0.5], (x_1)[0.8]\} \ .$$

As it was claimed in the proof of Theorem 3.8, $\{(x_1 \wedge x_2)[0.6], (x_1 \wedge \neg x_2)[0.5]\}$ is the only minimal inconsistent set of Δ; so $\alpha = (x_1)[0.8]$ is a free probabilistic conditional. Nonetheless, Δ has no innocuous probabilistic conditional. This can be noted through the following dominant consolidation of Δ:

$$\Delta' = \{(x_1 \wedge x_2)[0.55, 0.6], (x_1 \wedge \neg x_2)[0.45, 0.5], (x_1)[0.8, 1]\} \ .$$

Δ' is consistent and any consolidation $\Psi \neq \Delta'$ has at least one wider probability interval; so Δ' is dominant. But no original conditional of Δ is in Δ', so none is innocuous. Equivalently, any $\beta \in \Delta$ is inconsistent with Δ'. An example of innocuous conditional can be given in the knowledge base $\Psi = \Delta \cup \{(x_2)[0.3, 0.8]\}$, since $(x_2)[0.3, 0.8]$ would be consistent with any consolidation of Ψ.

An innocuous probabilistic conditional of Γ is consistent with any abrupt consolidation of Γ, since it is semantically equivalent to a consolidation with $[0, 1]$ probability intervals; furthermore, a safe conditional of Γ is clearly consistent with any consolidation of Γ:

Proposition 4.8. *Consider a probabilistic conditional $\alpha \in \Gamma$. If α is safe, it is innocuous; if α is innocuous, it is free.*

As to the independence postulate, we modify it in a corresponding way:

Postulate 4.9 (*i*-**Independence**). *If α is an innocuous probabilistic conditional of Γ, then $\mathcal{I}(\Gamma) = \mathcal{I}(\Gamma \setminus \{\alpha\})$.*

From Proposition 4.8 follows the relation among weak independence, *i*-independence and independence:

Corollary 4.10. *If $\mathcal{I}$ satisfies independence, then $\mathcal{I}$ satisfies i-independence. If $\mathcal{I}$ satisfies i-independence, then $\mathcal{I}$ satisfies weak independence.*

4.2 Refining the Minimal Conflict Concept

To redefine MIS-separability, we need a new notion of minimal conflict, related to the consolidation we introduced. Note that the union of minimal inconsistent sets is equal to the union of minimal abrupt repairs of a knowledge base, so that it forms the complement of the set of free probabilistic conditionals. To be consistent, we should provide a definition of conflicting sets such that their union is complementary to the set of innocuous conditionals. A set with all probabilistic conditionals that are not innocuous would be inconsistent when not empty, but would not have the minimality we are looking for. Such a set would be analogous to the union of all minimal inconsistent sets, but we search for a more fundamental notion of conflict, that can be derived by analyzing the consolidation properties of minimal inconsistent sets.

A minimal inconsistent set is minimal regarding set inclusion, and this is related to the abrupt consolidation:

Proposition 4.11. *A knowledge base Γ is a minimal inconsistent set iff Γ is inconsistent and there are no $\Delta_1, \ldots, \Delta_k \subsetneq \Gamma$, with $k \geq 1$, such that:*

1. $\bigcup_{i=1}^{k} \Delta_i = \Gamma$*;*
2. *For every $\Gamma' \subseteq \Gamma$, if $\Gamma' \cap \Delta_i$ is an abrupt consolidation of Δ_i for all $1 \leq i \leq k$, then Γ' is an abrupt consolidation of Γ.*

Intuitively, a minimal inconsistent set Γ is a conflict that cannot be analyzed in smaller subsets such that abruptly consolidating them implies abruptly consolidating Γ. Starting with a single inconsistent base Γ, we can find smaller subsets Δ_i satisfying both items of 4.11. We can do this recursively on the inconsistent sets Δ_i until we reach unanalyzable conflicts, which happens to be minimal inconsistent sets. So, abruptly consolidating these sets is abruptly consolidating Γ. Substituting consolidation for abrupt consolidation, we have an analogous definition of conflict:

Definition 4.12. A knowledge base Γ is an *inescapable conflict* if Γ is inconsistent and there are no $\Delta_1, \ldots, \Delta_k \subsetneq \Gamma$, with $k \geq 1$, such that:

1. $\bigcup_{i=1}^{k} \Delta_i = \Gamma$;
2. If Δ'_i is a consolidation of Δ_i for all $1 \leq i \leq k$ and $\bigcup_{i=1}^{k} \Delta'_i$ is a widening of Γ, then $\bigcup_{i=1}^{k} \Delta'_i$ is a consolidation of Γ.

The extra condition in the second item of Definition 4.12 forces consolidations of different knowledge bases $\Delta_i, \Delta_j \subsetneq \Gamma$ with some probabilistic conditional in common to agree in that probability interval; otherwise, $\bigcup_{i=1}^{k} \Delta'_i$ would not be a knowledge base. In other words, the second item says that if we widen the probability intervals of Γ making each Δ_i consistent, then Γ becomes consistent. As it happens with abrupt consolidation and MISes, to consolidate Γ, one only needs to widen its probability intervals in such a way that each inescapable conflict is solved.

Lemma 4.13. *Consider two knowledge bases $\Gamma, \Gamma' \in \mathbb{K}$ such that Γ' is a widening of Γ. If for every inescapable conflict $\Delta \subseteq \Gamma$ the base $\{\beta \in \Gamma' \mid \alpha \in \Delta$ and $\alpha \subseteq \beta\}$ is consistent, then Γ' is a consolidation of Γ.*

As all abrupt consolidations can be viewed as consolidations, an inescapable conflict is something weaker than a minimal inconsistent set:

Proposition 4.14. *If Δ is a minimal inconsistent set, then Δ is an inescapable conflict.*

Example 4.15. Consider again the knowledge base from Example 4.7:

$$\Delta = \{(x_1 \wedge x_2)[0.6], (x_1 \wedge \neg x_2)[0.5], (x_1)[0.8]\} \ .$$

As it was already shown, $\{(x_1 \wedge x_2)[0.6], (x_1 \wedge \neg x_2)[0.5]\}$ is the only minimal inconsistent set of Δ — and, by Proposition 4.14, it is an inescapable conflict. Nevertheless, it can be proved that the whole Δ is an inescapable conflict as well.

Suppose, by contradiction, there are $\Delta_1, \dots, \Delta_k \subsetneq \Delta$ such that $\bigcup_{i=1}^{k} \Delta_i = \Delta$ and, if Δ'_i is a consolidation of Δ_i for all $1 \leq i \leq k$ and $\bigcup_{i=1}^{k} \Delta'_i$ is a widening of Δ, then $\bigcup_{i=1}^{k} \Delta'_i$ is a consolidation of Δ. To build $\bigcup_{i=1}^{k} \Delta'_i$, we pick a consolidation Δ'_i for each $\Delta_i \subsetneq \Delta$. There are two cases: (a) $(x_1 \wedge x_2)[0.6] \in \Delta_i$; and (b) $(x_1 \wedge x_2)[0.6] \notin \Delta_i$. In case (a), we construct Δ'_i by widening the probability interval of the conditional $(x_1 \wedge x_2)[0.6]$ to $(x_1 \wedge x_2)[0.5, 0.6]$; formally, $\Delta'_i = (\Delta_i \setminus \{(x_1 \wedge x_2)[0.6]\}) \cup \{(x_1 \wedge x_2)[0.5, 0.6]\}$. In case (b), we choose the trivial consolidation $\Delta'_i = \Delta_i$. Even though the proof is omitted, we claim that each Δ'_i is consistent. Consider then the following knowledge base:

$$\Delta' = \bigcup_{i=1}^{k} \Delta'_i = \{(x_1 \wedge x_2)[0.5], (x_1 \wedge \neg x_2)[0.5], (x_1)[0.8]\} \ .$$

By the premises, Δ' is a consolidation of Δ, but it is inconsistent, since $\Delta' \setminus \{(x_1)[0.8]\}$ implies $(x_1)[1]$ (as shown in Section 3.2). Finally, there cannot exist such $\Delta_1, \dots, \Delta_k \subsetneq \Delta$, and Δ is an inescapable conflict.

We can now change MIS-separability to respect inescapable conflicts (IC) instead of minimal inconsistent sets. Let $IC(\Gamma)$ denote the collection of all inescapable conflicts of Γ.

Property 4.16 (IC-Separability). *If $\Gamma = \Delta \cup \Psi$, $\Delta \cap \Psi = \varnothing$ and $IC(\Gamma) = IC(\Delta) \cup IC(\Psi)$, then $\mathcal{I}(\Gamma) = \mathcal{I}(\Delta) + \mathcal{I}(\Psi)$.*

As inescapable conflict is a weaker concept than MIS, MIS-separability is stronger than IC-separability.

Corollary 4.17. *If $\mathcal{I}$ satisfies MIS-separability, then $\mathcal{I}$ satisfies IC-separability.*

Recall that a free probabilistic conditional is defined in the standard way as not belonging to any minimal inconsistent set. We prove the analogous result for innocuous conditionals and inescapable conflicts, linking all concepts introduced in this section.

Theorem 4.18. *The following statements are equivalent:*

1. *For all d-consolidation Γ' of Γ, $\alpha \in \Gamma'$.*
2. *If Γ' is a consolidation of Γ, then α is consistent with Γ'.*
3. *There is no inescapable conflict Δ in Γ such that $\alpha \in \Delta$.*
4. *α is an innocuous probabilistic conditional in Γ.*

A result analogous to Proposition 3.6 follows:

Corollary 4.19. *If $\mathcal{I}$ satisfies IC-separability, then $\mathcal{I}$ satisfies i-independence.*

As already mentioned, inescapable conflicts are to consolidations as minimal inconsistent sets are to abrupt consolidations. If consolidation via conditionals withdrawal, as in Reiter's and AGM approaches, can focus on the collection of minimal inconsistent sets (ignoring free conditionals), consolidation through widening probability intervals can be done by watching only for the inescapable conflicts (ignoring innocuous conditionals). All these relations among free and innocuous probabilistic conditionals, minimal inconsistent sets and inescapable conflicts argue in favor of the new proposed postulates, whose compatibility with consistency and continuity we will prove.

4.3 Compatible Postulates for Imprecise Probabilities

To replace the postulate of independence and the property of MIS-separability, we propose the weaker pair of i-independence and IC-separability towards building a compatible package together with consistency and continuity. Before proving such compatibility, the postulates have to be generalized to imprecise knowledge bases. To generalize consistency, i-independence and IC-separability is straightforward, we just enlarge their intended scope from knowledge bases in $\mathbb{K}_{prec}$ to bases in $\mathbb{K}$, but the continuity postulates demands some notation.

Let $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i]|1 \leq i \leq m\}$ be a knowledge base. The characteristic function of Γ can be generalized as a function $\Lambda_\Gamma : [0,1]^{2m} \to \mathbb{K}$ that changes both upper and lower bounds of each probabilistic conditional in Γ; formally, $\Lambda_\Gamma(\langle \underline{q}'_1, \bar{q}'_1, \ldots, \underline{q}'_m, \bar{q}'_m\rangle) = \{(\varphi_i|\psi_i)[\underline{q}'_i, \bar{q}'_i]\,|1 \leq i \leq m\}$. Now the continuity postulate can be generalized, with $\circ$ denoting function composition:

Postulate 4.20 (Continuity). *For all $\Gamma \in \mathbb{K}$, the function $\mathcal{I} \circ \Lambda_\Gamma : [0,1]^{2|\Gamma|} \to [0,\infty)$ is continuous.*

Note that the postulate above implies Postulate 3.7, which defines continuity for precise probabilities. Given a base Γ of size m, Postulate 3.7 considers a function $f : [0,1]^m \to \mathbb{K}_{prec}$ (the characteristic function when probabilities are precise) such that $f(\langle q'_1, q'_2, \ldots, q'_m\rangle) = \Lambda_\Gamma(\langle q'_1, q'_1, q'_2, q'_2, \ldots, q'_m, q'_m\rangle)$ and requires that $\mathcal{I} \circ f$ be continuous. But note that, if $\mathcal{I} \circ \Lambda_\Gamma$ is continuous, so is $\mathcal{I} \circ f$. Therefore, Theorem 3.8 and Corollary 3.9 also hold within the imprecise probability framework.

Hunter and Konieczny proposed another basic postulate for inconsistency measures [17] that was also adopted by Thimm [38].

Postulate 4.21 (Monotonicity). *For any knowledge bases $\Gamma, (\Gamma \cup \{\alpha\}) \in \mathbb{K}$, $\mathcal{I}(\Gamma \cup \{\alpha\}) \geq \mathcal{I}(\Gamma)$.*

Thimm actually suggests a stronger principle, super-additivity, which implies monotonicity. Since super-additivity is incompatible with normalization [38] — as also is IC-separability —, we state them as properties, and not postulates.

Property 4.22 (Super-additivity). *For any knowledge base $\Gamma \cup \Delta \in \mathbb{K}$, if $\Gamma \cap \Delta = \varnothing$, then $\mathcal{I}(\Gamma \cup \Delta) \geq \mathcal{I}(\Gamma) + \mathcal{I}(\Delta)$.*

Property 4.23 (Normalization). *For any knowledge base $\Gamma \in \mathbb{K}$, $\mathcal{I}(\Gamma) \in [0,1]$.*

To attend the desirable properties, we generalize the inconsistency measures based on distance minimization proposed by Thimm [38] to the case of imprecise probabilities. Muino introduced similar ideas under a different semantics for conditional probabilities [26]. Firstly, we define a family of p-norms.

Definition 4.24. Consider a (positive) $m \in \mathbb{N}_{>0}$ and a $p \in \mathbb{N}_{>0} \cup \{\infty\}$. Given a vector $q = \langle q_1, q_2, \ldots, q_m\rangle$ over the real numbers, the p-norm of q is $\|q\|_p = \sqrt[p]{\sum_{i=1}^{m} |q_i|^p}$ if p is finite; otherwise it is $\|q\|_\infty = \max_i |q_i|$.

Thimm defines a family $\mathcal{I}_p$ of inconsistency measures based on the p-norms, which we modify to also consider $p = \infty$ and handle the empty base.

Definition 4.25. Consider a $p \in \mathbb{N}_{>0} \cup \{\infty\}$ and a $\Gamma \in \mathbb{K}$. The function $\mathcal{I}_p : \mathbb{K} \to [0,\infty)$ is the *d^p-inconsistency measure*, defined as

$$\mathcal{I}_p(\Gamma) = \min\{\|q - q'\|_p \,|\, \Lambda_\Gamma(q) = \Gamma \text{ and } \Lambda_\Gamma(q') \text{ is consistent}\},$$

for any non-empty Γ, and $\mathcal{I}_p(\varnothing) = 0$.

Finally, we are in a position to show inconsistency measures satisfying the wanted properties. We extend Thimm's results to prove that all d^p-inconsistency measures satisfy the reconciled postulates and that some of them hold additional properties. Muino have similar results, though under a different semantics [26].

Theorem 4.26. *For any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}_p$ is well-defined and satisfies the postulates of consistency, continuity, i-independence and monotonicity.*

The compatibility of IC-separability and super-additivity with consistency, continuity, monotonicity and i-independence is confirmed by the $\mathcal{I}_1$ measure:

Lemma 4.27. *$\mathcal{I}_p$ satisfies super-additivity and IC-separability iff $p = 1$.*

And, if normalization is required, we can use the following Muino's result [26]:

Lemma 4.28. *$\mathcal{I}_p$ satisfies normalization iff $p = \infty$.*

5 Practicable Principled Inconsistency Measures

Although we have compatible postulates to drive the rational choice of inconsistency measures, these desirable properties are satisfied by a myriad of functions. We may use other arguments to pick some particular inconsistency measures among those attending to the postulates. This section investigates computational aspects of measuring inconsistency through distance minimization, reviewing and generalizing measures proposed by Potyka [29] that can be handled via linear programming. In a second moment, we show how the introduced measures can be justified by Dutch books, giving the maximum guaranteed loss an agent would be exposed to, if stakes are limited somehow. We also show that Dutch books offer other interesting measures.

5.1 Measuring Inconsistency with Linear Programs

To check the consistency of a knowledge base, one can use the well-known formulation of PSAT as a linear program [14]. Consider a knowledge base $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i]|1 \leq i \leq m\}$. Under the semantics adopted, each assessment $(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i]$ is equivalent to the pair $P_\pi(\varphi_i \wedge \psi_i) - \underline{q}_i P_\pi(\psi_i) \geq 0$ and $P_\pi(\varphi_i \wedge \psi_i) - \bar{q}_i P_\pi(\psi_i) \leq 0$ of restrictions on P_π. The knowledge base is consistent iff these $2m$ restrictions can be jointly satisfied by a probability measure P_π induced by a probability mass π. Consider two $(m \times 2^n)$-matrices, $A = [a_{ij}]$ and $B = [b_{ij}]$, with $a_{ij} = I_{w_j}(\varphi_i \wedge \psi_i) - \underline{q}_i I_{w_j}(\psi_i)$ and $b_{ij} = I_{w_j}(\varphi_i \wedge \psi_i) - \bar{q}_i I_{w_j}(\psi_i)$, in which $I_{w_j} : \mathcal{L}_{X_n} \to \{0, 1\}$ is the indicator function of the set $\{\varphi \in \mathcal{L}_{X_n} | w_j \models \varphi\}$ — I_{w_j} is the valuation relative to the possible world w_j. The knowledge base Γ

is satisfiable iff there is a $(2^n \times 1)$-vector π satisfying the system:

$$\begin{aligned} A\pi &\geq 0 \quad (5)\\ B\pi &\leq 0 \quad (6)\\ \sum \pi &= 1 \quad (7)\\ \pi &\geq 0\,. \quad (8) \end{aligned}$$

Restrictions in (5) correspond to $P_\pi(\varphi_i|\psi_i) \geq \underline{q}_i$, and those in (6) codify $P_\pi(\varphi_i|\psi_i) \leq \bar{q}_i$; Constraints (7) and (8) force π to be a probability mass over the possible worlds $w_1, w_2, \ldots, w_{2^n}$. As all constraints are linear, this system can be solved by linear programming techniques as Simplex. Despite the exponential number of columns, column generation methods can be used to handle them implicitly [22, 21], keeping the computation efficient enough to solve large knowledge bases (thousands of probabilities in [15, 9]).

To measure inconsistency using distance minimization with $\mathcal{I}_p$, we can add to the system variables $\underline{\varepsilon}_i \geq 0$ ($\bar{\varepsilon}_i \geq 0$) corresponding to decrements (increments) in lower (upper) bounds of each probability interval. Any conditional $(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i]$ yields a pair of restrictions $P_\pi(\varphi_i \wedge \psi_i) - \underline{q}_i P_\pi(\psi_i) \geq -\underline{\varepsilon}_i P_\pi(\psi_i)$ and $P_\pi(\varphi_i \wedge \psi_i) - \bar{q}_i P_\pi(\psi_i) \leq \bar{\varepsilon}_i P_\pi(\psi_i)$. Computing the $\mathcal{I}_p$ measure is so reduced to minimizing the p-norm of the vector $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle$.[5] Nonetheless, the constraints contain non-linear terms (from $\underline{\varepsilon}_i P_\pi(\psi_i)$ and $\bar{\varepsilon}_i P_\pi(\psi_i)$), and Potyka points out that these programs have (non-global) local optima [29], so convex optimization techniques cannot be directly applied. Thus, computing $\mathcal{I}_p$ is typically less efficient than deciding PSAT, as empirical results indicate [29].

Potyka emphasizes this impracticability and suggests a new family of inconsistency measures, the *minimal violation measures* [29], which we adapt here to the case of imprecise probabilities. In order to keep constraints linear, "error" variables $\underline{\varepsilon}_i, \bar{\varepsilon}_i \geq 0$ are inserted in the right-hand side of $P_\pi(\varphi_i \wedge \psi_i) - \underline{q}_i P_\pi(\psi_i) \geq 0$ and $P_\pi(\varphi_i \wedge \psi_i) - \bar{q}_i P_\pi(\psi_i) \leq 0$, yielding $P_\pi(\varphi_i \wedge \psi_i) - \underline{q}_i P_\pi(\psi_i) \geq -\underline{\varepsilon}_i$ and $P_\pi(\varphi_i \wedge \psi_i) - \bar{q}_i P_\pi(\psi_i) \leq \bar{\varepsilon}_i$. Potyka's minimal violation measures are obtained when the p-norm of $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle$ is minimized with such constraints. We denote by $\mathcal{I}_p^\varepsilon$ the optimal value from the following program, where $\underline{\varepsilon} = [\underline{\varepsilon}_i]$ and $\bar{\varepsilon} = [\bar{\varepsilon}_i]$ are $(m \times 1)$-vectors:

$$\begin{aligned} \min \|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle\|_p \quad & \text{subject to:} \quad (9)\\ A\pi &\geq -\underline{\varepsilon} \quad (10)\\ B\pi &\leq \bar{\varepsilon} \quad (11)\\ \sum \pi &= 1 \quad (12)\\ \pi, \underline{\varepsilon}, \bar{\varepsilon} &\geq 0\,. \quad (13) \end{aligned}$$

The restrictions are all linear, and non-linear terms may appear only within the objective function. We can ignore the monotone function $\sqrt[p]{\cdot}$ within the

[5]Note that if we allow $\underline{\varepsilon}_i < 0$ (and $\bar{\varepsilon}_i < 0$), it would represent the tightening of a bound, useless when searching for consistency, and the minimization would avoid it anyway.

p-norm definition, applying it only after the minimization stops. The degree of each term in the new objective function is p, and for $p = 1$ a linear program is recovered, since $\|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle\|_1 = \sum_{i=1}^{m} \underline{\varepsilon}_i + \bar{\varepsilon}_i$. Hence, one can apply the standard Simplex and column generation methods to compute $\mathcal{I}_1^{\varepsilon}$ with practically the same efficiency as deciding PSAT [29].

For any finite p different from 1, the system (9)–(13) has non-linear terms in its objective function, but this is not the case when we consider $p = \infty$. The ∞-norm is equivalent to take the maximum of the vector $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle$, but this is the same as considering all $\underline{\varepsilon}_i, \bar{\varepsilon}_i$ equal to a single scalar $\varepsilon \geq 0$. The measure $\mathcal{I}_\infty^{\varepsilon}$ is the solution of the following program [29], in which $\underline{\varepsilon} = \bar{\varepsilon} = [\varepsilon\ \varepsilon\ \ldots\ \varepsilon]^T$ are $(m \times 1)$-vectors:

$$
\begin{array}{rcll}
\min \varepsilon & \text{subject to:} & & (14)\\
A\pi & \geq & -\underline{\varepsilon} & (15)\\
B\pi & \leq & \bar{\varepsilon} & (16)\\
\sum \pi & = & 1 & (17)\\
\pi, \varepsilon & \geq & 0\ . & (18)
\end{array}
$$

The system (14)–(18) is also a linear program, like (9)–(13) when $p = 1$, but has a lesser number of variables. However, Potyka remarks that the variable ε in (14)–(18) is involved in $2m$ restrictions, while each variable $\underline{\varepsilon}_i, \bar{\varepsilon}_i$ appears in only one constraint in (9)–(13); therefore, the computation of $\mathcal{I}_\infty^{\varepsilon}$ may in practice be slightly less efficient than computing $\mathcal{I}_1^{\varepsilon}$ [29].

For sets of unconditional probabilistic assessments, when all conditioning events ψ_i are equivalent to $\top$, the inconsistency measures $\mathcal{I}_p$ and $\mathcal{I}_p^{\varepsilon}$ are extensionally identical for all p. The reason is that the restriction on P_π and $\underline{\varepsilon}_i, \bar{\varepsilon}_i$ corresponding to a conditional is the same when computing both measures. For instance, any constraint $P_\pi(\varphi_i \wedge \psi_i) - \bar{q}_i P_\pi(\psi_i) \leq 0$ becomes equivalent to $P_\pi(\varphi_i) - \bar{q}_i \leq 0$ when ψ_i is a tautology, and inserting an error to the probability bound, $P_\pi(\varphi_i) - (\bar{q}_i + \bar{\varepsilon}_i) \leq 0$, is the same as placing it in the right-hand side.

Potyka has proved that these measures, $\mathcal{I}_p^{\varepsilon}$ with $p \in \mathbb{N}_{>0} \cup \{\infty\}$, besides being computable via linear programming, satisfy the postulates of consistency, continuity and monotonicity for the case of precise probabilities [29]:

Proposition 5.1. *For any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}_p^{\varepsilon} : \mathbb{K}_{prec} \to [0, \infty)$ is well-defined and satisfies consistency, continuity, weak independence and monotonicity. $\mathcal{I}_1^{\varepsilon}$ also satisfies super-additivity.*

We can generalize the result above to encompass probability intervals and the new postulates we introduced:

Lemma 5.2. *For any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}_p^{\varepsilon} : \mathbb{K} \to [0, \infty)$ is well-defined and satisfies consistency, continuity, i-independence and monotonicity. $\mathcal{I}_1^{\varepsilon}$ also satisfies super-additivity and IC-separability; and $\mathcal{I}_\infty^{\varepsilon}$ satisfies normalization.*

Now we have a set of compatible postulates for inconsistency measures and two particular measures satisfying them that can be computed rather efficiently using linear programming techniques. On the one hand, $\mathcal{I}_1^\varepsilon$ also satisfies super-additivity and IC-separability; on the other hand, $\mathcal{I}_\infty^\varepsilon$ additionally satisfies normalization. Nonetheless, one can argue that these measures lack some proper justification, despite satisfying some postulates and being practicable, as Capotorti, Regoli and Vattari did [4]. They claim that distances between conditional probabilities are meaningless, being only geometrical measures. This might be the case, but it would only undermine the $\mathcal{I}_p$ family. For $\mathcal{I}_p^\varepsilon$ measures, distances between probabilities are computed weighting by the probabilities of the conditioning formulas, so to speak, allowing some operational interpretation. In the next section, we provide a rational for $\mathcal{I}_1^\varepsilon$ and $\mathcal{I}_\infty^\varepsilon$ based on Dutch books.

5.2 Inconsistency Measures and Dutch Books

In formal epistemology, there is an interest in measuring the incoherence of an agent whose beliefs are given as probabilities or lower previsions over propositions or random variables. If we have propositions from classical logic, the formalized problem at hand is exactly the one we are investigating. When the agent's degrees of belief are represented by a knowledge base, then to measure the agent incoherence is to measure the inconsistency of such knowledge base. Schervish, Kadane and Seidenfeld [33, 34] and Staffel [36, 37] have proposed ways to measure incoherence of an agent based on Dutch books.

Dutch book arguments are based on the agent's betting behavior, typically used to show the irrationality of her set of degrees of belief. Roughly speaking, consider a "conditional" gamble on $\varphi|\psi$ with stake λ and relative price q, which is a contract between the agent and a gambler such that: if ψ is not the case, the gamble is called off, causing neither profit nor loss for the involved parts; if ψ is the case, then the agent receives $(1-q)\lambda$ from the gambler if φ is also the case, otherwise the gambler receives $q\lambda$ from the agent. Now suppose that, if an agent believes that the probability of a proposition φ being true given that another proposition ψ is true lies within $[\underline{q}, \bar{q}]$, she finds acceptable gambles on $\varphi|\psi$ with $\lambda \geq 0$ and $q = \underline{q}$ and gambles with $\lambda \leq 0$ and $q = \bar{q}$. A *Dutch book* is a set of gambles that the agent sees as fair that causes her a guaranteed loss, no matter which possible world is the case. We assume Dutch books contain exactly two gambles on $(\varphi_i|\psi_i)$ per each each conditional $(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] \in \Gamma$, the base formalizing the agent's beliefs: one with stake $\underline{\lambda}_i \geq 0$ and the other with stake $-\bar{\lambda}_i \leq 0$. This is not restrictive, since gambles on the same $(\varphi_i|\psi_i)$ with the same relative price can be merged by summing the stakes. We can thus denote a Dutch book only by the absolute value of its stakes $\underline{\lambda}_1, \bar{\lambda}_1, \ldots, \underline{\lambda}_m, \bar{\lambda}_m \geq 0$, where $m = |\Gamma|$.

If the agent has an epistemic state represented by a set of probabilistic conditionals that turns out to be inconsistent, then she is exposed to a Dutch book [27]. In this way, Dutch book arguments were introduced to show that degrees of belief must obey the axioms of probability and are a standard proof of incoherence (introductions to Dutch books can be found in [35] and [7]). Thus, a natural approach to measuring an agent's degree of incoherence is through the magnitude of the sure loss she is vulnerable to. With no bound on the stakes, such loss would also be unlimited for incoherent agents, and different strategies to circumvent this in order to measure incoherence as a finite loss are found in the formal epistemology literature.

Schervish *et al.* propose three ways of limiting these stakes and measuring incoherence for upper and lower previsions on bounded random variables [34], which we simplify here to our case. Consider a gamble on $\varphi_i|\psi_i$ with stake $\underline{\lambda}_i \geq 0$ and relative price $\underline{q}_i$. The agent might lose $\underline{q}_i\underline{\lambda}_i$ with this gamble, while the gambler is exposed to a loss of $(1-\underline{q}_i)\underline{\lambda}_i$. Now consider a gamble on the same conditional with stake $-\bar{\lambda}_i \leq 0$ and relative price $\bar{q}_i$. The agent might have to pay $(1-\bar{q}_i)\bar{\lambda}_i$ to the gambler — who would "lose" $-(1-\bar{q}_i)\bar{\lambda}_i < 0$ —; whilst the gambler might lose $\bar{q}_i\bar{\lambda}_i \geq 0$ to the agent. Schervish *et al.* call these quantities the agent's and the gambler's *escrows*. In other words, the agent's (or gambler's) escrow for a gamble is how much she has to commit from her resources to cover an eventual lost. The first incoherence degree Schervish *et al.* introduce is the maximum guaranteed loss an agent is exposed to if her escrows sum up to one (she has limited resources). The second way they propose to measure incoherence is as the gambler's maximum guaranteed profit (agent's loss) when the gambler's escrows sum up to one. The third measure arises when the sum of all escrows, both agent's and gambler's, is limited to one, and the guaranteed loss is maximized.[6] This is the same as limiting the sum of the absolute values of the stakes, $\sum_i \underline{\lambda}_i + \bar{\lambda}_i = 1$. We denote by $\mathcal{I}^a_{SSK}(\Gamma)$, $\mathcal{I}^g_{SSK}(\Gamma)$ and $\mathcal{I}^t_{SSK}(\Gamma)$ the inconsistency measures corresponding to these three incoherence measures, respectively, when a knowledge base Γ codifies the agent's belief state.

Staffel argues that the measures introduced by Schervish *et al.* give no reasonable results in a number of cases and proposes to fix the stakes in each gamble instead of the agent's resources, the gambler's or their sum [37]. Her work is tailored to precise, unconditional probabilities. Firstly, it is assumed that, without loss of generality, only one gamble can be made on φ for each $(\varphi)[q]$ in the base representing the agent's belief state. Then stakes are limited to $\lambda_i \in \{-1, 0, 1\}$. What Staffel calls the maximum Dutch book measure [37] is defined as the maximum guaranteed loss the agent would be vulnerable to in such a setting. Unfortunately, this approach does not allow a Dutch book to be set in all cases of incoherence:

Proposition 5.3. *There is an inconsistent knowledge base* $\Gamma = \{(\varphi_i)[q_i] | 1 \leq i \leq m\} \in \mathbb{K}_{prec}$ *such that no Dutch book is possible against an incoherent agent whose belief state is represented by* Γ *if there is one gamble on* φ_i *per conditional* $(\varphi_i)[q_i] \in \Gamma$ *and stakes are limited to* $\lambda_i \in \{-1, 0, 1\}$.

To satisfy the consistency postulate, we can adapt Staffel's incoherence measure to be the maximum guaranteed loss an agent is exposed to when stakes

[6] Schervish *et al.* [34] actually measure the incoherence as maximum rates between the guaranteed loss and the limited resources of the agent, the gambler or their sum. Clearly, this is equivalent to maximizing the guaranteed loss when such resources are no greater than 1 — or, equivalently, equal to 1.

are limited to $\lambda_i \in [-1, 1]$ (equivalently, $|\lambda_i| \leq 1$). With this modification, we have a consistent measure for all precise knowledge bases, including those with conditional assessments. To cope with imprecise probabilities, we need to allow two gambles on $(\varphi_i|\psi_i)$ per conditional $(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] \in \Gamma$ to be in a Dutch book (with stakes $\underline{\lambda}_i \geq 0$ and $-\bar{\lambda}_i \leq 0$). The obtained incoherence measure is the the maximum sure loss the agent is exposed to when $\underline{\lambda}_i, \bar{\lambda}_i \leq 1$ for each $(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] \in \Gamma$. We denote by $\mathcal{I}_{\mathit{Staffel}}$ the corresponding inconsistency measure on the knowledge base representing the agent's belief state.

Even though incoherence measures based on Dutch books from the formal epistemology community and inconsistency measures based on distance minimization from Artificial Intelligence researchers seem unrelated at first, they are actually two sides of the same coin. The maximum guaranteed loss an agent is exposed to can generally be computed with linear programs that are technically dual to those that minimize distances to measure inconsistency. Though Potyka seems unaware of this connection, Nau has already investigated this matter, mentioning results similar to the following [27], which is proved in the Appendix:

Theorem 5.4. *For any* $\Gamma \in \mathbb{K}$, $\mathcal{I}^t_{SSK}(\Gamma) = \mathcal{I}^\varepsilon_\infty(\Gamma)$.

Recall that $\mathcal{I}^\varepsilon_\infty$ is exactly one of the two practicable measures proposed by Potyka [29]. Far from meaningless, such measure quantifies the maximum sure loss an agent is exposed to when the sum of the stakes is fixed at one. As to Potyka's other proposal, $\mathcal{I}^\varepsilon_1$, duality in linear programming provides a correspondence with the measure we have adapted from Staffel.

Lemma 5.5. *For any* $\Gamma \in \mathbb{K}$, $\mathcal{I}_{\mathit{Staffel}}(\Gamma) = \mathcal{I}^\varepsilon_1(\Gamma)$.

Lemma 5.5 states the extensional identity between $\mathcal{I}^\varepsilon_1$ and $\mathcal{I}_{\mathit{Staffel}}$. Within the unconditional probabilities scenario, this means that the Manhattan distance between the agent's probabilities and the closest consistent probabilities is equal to the maximum sure loss she is exposed to when (absolute values of) stakes are not higher than one.

Theorem 5.4 and Lemma 5.5 give an operational interpretation for the inconsistency measures $\mathcal{I}^\varepsilon_\infty$ and $\mathcal{I}^\varepsilon_1$ based on betting behavior. It was remarked in Section 5.1 that $\mathcal{I}^\varepsilon_p$ and $\mathcal{I}_p$ give the same inconsistency degrees to unconditional knowledge bases. Thus, the Dutch book with limited stakes ($\underline{\lambda}_i, \bar{\lambda}_i \leq 1$ or $\sum_i \underline{\lambda}_i + \bar{\lambda}_i \leq 1$) can be used to rationalize also $\mathcal{I}_1$ and $\mathcal{I}_\infty$ in the unconditional setting. However, when we take into account conditional probabilities, only $\mathcal{I}^\varepsilon_\infty$ and $\mathcal{I}^\varepsilon_\infty$ measure the maximum guaranteed loss an agent would be exposed to, when stakes are limited via $\underline{\lambda}_i, \bar{\lambda}_i \leq 1$ or $\sum_i \underline{\lambda}_i + \bar{\lambda}_i \leq 1$, respectively.

Different strategies for bounding stakes can lead to different inconsistency measures, but our motivation in this section was not to use Dutch books to determine which measures should be adopted — that is the reason of the postulates. The point here is that these two measures ($\mathcal{I}^\varepsilon_1$ and $\mathcal{I}^\varepsilon_\infty$), besides satisfying the postulates and being computable through linear programs, have a meaningful interpretation. In the next section, we show that other measures based on Dutch books have these qualities as well.

5.3 Other Practicable Principled Measures

The measures $\mathcal{I}^a_{SSK}$ and $\mathcal{I}^g_{SSK}$, which maximize sure loss via Dutch books when the agent's or the gambler's resources are respectively limited, can also be related to Potyka's minimal violation measures. Recall that $\mathcal{I}^{\varepsilon}_p$ minimizes the p-norm of $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \dots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle$ in the program given in (9)–(13). Generalizing it, we could associate *confidence factors* $\underline{\delta}_i, \bar{\delta}_i > 0$ to probability bounds $\underline{q}_i, \bar{q}_i$ in the conditionals, as Nau proposes [27]. The intuition says the more confident an agent is about a bound, the higher the penalty for changing it. We could use $\underline{\delta}_i, \bar{\delta}_i$ as coefficients to $\underline{\varepsilon}_i, \bar{\varepsilon}_i$ within the objective function of the program in (9)–(13), but we choose an option equivalent to dividing the errors within the constraints by the confidence factors.

For reasons that will be clearer soon, we think in terms of *volatility factors* $\underline{\gamma}_i, \bar{\gamma}_i$, which intuitively are inversely proportional to the confidence degrees associated to the probability bounds $\underline{q}_i, \bar{q}_i$ within a conditional. That is, the higher the volatility factor of a probability bound, the smaller is the penalty for changing it. We define $\underline{\gamma} \geq 0$ and $\bar{\gamma} \geq 0$ as functions taking conditionals to the volatility factors associated to their lower and upper bounds — with $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq m\}$ fixed, we denote $\underline{\gamma}((\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i])$ and $\bar{\gamma}((\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i])$ simply by $\underline{\gamma}_i$ and $\bar{\gamma}_i$. Combining the functions $\underline{\gamma}$ and $\bar{\gamma}$, the function $\gamma \geq 0$, taking knowledge bases to vectors of volatility factors, is defined as $\gamma(\Gamma) = \langle \underline{\gamma}_1, \bar{\gamma}_1, \dots, \underline{\gamma}_m, \bar{\gamma}_m \rangle$, with $m = |\Gamma|$. We rewrite the program from (9)–(13) to include these factors, where $\underline{\varepsilon}_\gamma$ ($\bar{\varepsilon}_\gamma$) is a $(m \times 1)$-vector whose elements are $\underline{\varepsilon}_i \underline{\gamma}_i$ ($\bar{\varepsilon}_i \bar{\gamma}_i$), for $1 \leq i \leq m$.

$$
\begin{aligned}
\min \| \langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \dots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle \|_p \quad & \text{subject to:} && (19)\\
A\pi \quad & \geq \quad -\underline{\varepsilon}_\gamma && (20)\\
B\pi \quad & \leq \quad \bar{\varepsilon}_\gamma && (21)\\
\sum \pi \quad & = \quad 1 && (22)\\
\pi, \underline{\varepsilon}, \bar{\varepsilon} \quad & \geq \quad 0\ . && (23)
\end{aligned}
$$

Given a knowledge base Γ, $\mathcal{I}^{\gamma}_p(\Gamma)$ denotes the solution to the program above. It works as an inconsistency measure when it is well-defined, but some extreme γ may turn the program infeasible — for instance, when $\gamma(\Gamma)$ is the null vector and Γ is inconsistent. Intuitively, the penalty for changing any probability bound in Γ would be infinite, and at least one bound should be relaxed, for Γ is inconsistent. We define $\mathcal{I}^{\gamma}_p(\Gamma) = \infty$ in such cases, relaxing in this section the definition of an inconsistency measure to a function $\mathcal{I} : \mathbb{K} \to [0, \infty) \cup \{\infty\}$, in order to analyze the properties of $\mathcal{I}^{\gamma}_p$.

When $p = 1$, (19)–(23) is a linear program. Again, any finite $p > 1$ leads to non-linear terms within the objective function, but this is not the case for $p = \infty$. Considering all $\underline{\varepsilon}_i, \bar{\varepsilon}_i$ equal to a single scalar $\varepsilon \geq 0$, we can modify the linear program (14)–(18) to allow for the volatility factors. The measure $\mathcal{I}^{\gamma}_\infty$ is the solution of the following program, in which $\underline{\varepsilon}_\gamma$ ($\bar{\varepsilon}_\gamma$) is a $(m \times 1)$-vector whose elements are $\varepsilon \underline{\gamma}_i$ ($\varepsilon \bar{\gamma}_i$), for $1 \leq i \leq m$:

$$\begin{aligned}
\min \varepsilon \quad & \text{subject to:} & & (24)\\
A\pi & \geq -\underline{\varepsilon}_\gamma & & (25)\\
B\pi & \leq \bar{\varepsilon}_\gamma & & (26)\\
\sum \pi & = 1 & & (27)\\
\pi, \varepsilon & \geq 0\ . & & (28)
\end{aligned}$$

From the $\mathcal{I}_p^\gamma$ family, Potyka's measures are recovered when $\underline{\gamma}$ and $\bar{\gamma}$ are constant functions taking any conditional to one, so $\gamma(\Gamma)$ is a vector whose all elements are equal to 1. Properties from Lemma 5.2 also hold for the generalized measures:

Theorem 5.6. *For any functions $\underline{\gamma}, \bar{\gamma} \geq 0$ and $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}_p^\gamma : \mathbb{K} \to [0, \infty) \cup \{\infty\}$ is well-defined and satisfies consistency, i-independence and monotonicity. $\mathcal{I}_1^\gamma$ also satisfies super-additivity and IC-separability.*

Conditions on which $\mathcal{I}_p^\gamma$ is finite, continuous and normalized are stated below:

Lemma 5.7. *If $\underline{\gamma}$ and $\bar{\gamma}$ are such that, for any conditional $\alpha = (\varphi|\psi)[\underline{q}, \bar{q}]$, $\underline{\gamma}(\alpha) = 0$ implies $\underline{q} = 0$ and $\bar{\gamma}(\alpha) = 0$ implies $\bar{q} = 1$, then $\mathcal{I}_p^\gamma(\Gamma)$ is finite for any $\Gamma \in \mathbb{K}$.*

Lemma 5.8. *Let $\gamma \circ \Lambda_\Gamma : [0, 1]^{2m} \to [0, 1]^{2m}$ be continuous for all $\Gamma \in \mathbb{K}$ and consider a $p \in \mathbb{N}_{>0} \cup \{\infty\}$. If $\gamma \circ \Lambda_\Gamma : [0, 1]^{2m} \to [0, 1]^{2m}$ is positive for all $\Gamma \in \mathbb{K}$, then $\mathcal{I}_p^\gamma$ satisfies continuity; if $\gamma \circ \Lambda_\Gamma : (0, 1)^{2m} \to [0, 1]^{2m}$ is positive for all $\Gamma \in \mathbb{K}$, then $\mathcal{I}_p^\gamma \circ \Lambda_\Gamma(q)$ is continuous for $q \in (0, 1)^{2m}$.*

Lemma 5.9. *If $\underline{\gamma}$ and $\bar{\gamma}$ are such that, for any conditional $\alpha = (\varphi|\psi)[\underline{q}, \bar{q}]$, $\underline{\gamma}(\alpha) \geq \underline{q}$ and $\bar{\gamma}(\alpha) \geq 1 - \bar{q}$, then $\mathcal{I}_\infty^\gamma$ satisfies normalization.*

After generalizing Potyka's measures based on distance minimization, their correspondence to the measures based on Dutch books follows, as Nau has already mentioned [27]:

Theorem 5.10. *If $\underline{\gamma}$ and $\bar{\gamma}$ are such that, for any conditional $\alpha = (\varphi|\psi)[\underline{q}, \bar{q}]$, $\underline{\gamma}(\alpha) = \underline{q}$ and $\bar{\gamma}(\alpha) = 1 - \bar{q}$, then $\mathcal{I}_{SSK}^a(\Gamma) = \mathcal{I}_\infty^\gamma(\Gamma)$ for any $\Gamma \in \mathbb{K}$ and both are finite.*

Note that $1 - \bar{q}_i$ is the lower bound the agent indirectly assigns to $\neg\varphi_i$ given ψ_i. Each conditional $(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i]$ is equivalent to a pair $(\varphi_i|\psi_i)[\underline{q}_i, 1], (\neg\varphi_i|\psi_i)[1-\bar{q}_i, 1]$ of lower bounds. When probabilities are assigned only through unconditional lower bounds, $\Gamma = \{(\varphi_i|\top)[q_i, 1] | 1 \leq i \leq m\}$, $\mathcal{I}_{SSK}^a$ minimizes $\max_i |q_i - q_i'|/q_i$ such that $\{(\varphi_i|\top)[q_i', 1] | 1 \leq i \leq m\}$ is consistent, which is to compute the least relative decrement in all lower bounds to reach consistency. If we interpret probability intervals as a pair of upper bounds, there is the dual result:

Theorem 5.11. *If $\underline{\gamma}$ and $\bar{\gamma}$ are such that, for any conditional $\alpha = (\varphi|\psi)[\underline{q}, \bar{q}]$, $\underline{\gamma}(\alpha) = 1 - \underline{q}$ and $\bar{\gamma}(\alpha) = \bar{q}$, then $\mathcal{I}_{SSK}^g(\Gamma) = \mathcal{I}_\infty^\gamma(\Gamma)$ for any $\Gamma \in \mathbb{K}$.*

For unconditional probabilities, this means that maximizing sure loss when the gambler's resources are limited to one is equivalent to minimizing the relative increment in all upper bounds one has to perform in order to consolidate the base. For instance, if Γ contains only unconditional probabilities and intervals are expressed as pairs of upper bounds, $\mathcal{I}^g_{SSK}(\Gamma) = 0.05$ means that an increase of 5% in each upper bound is needed to restore consistency.

Schervish *et al.* acknowledge the fact that $\mathcal{I}^g_{SSK}$ may be unbounded [33]. For instance, consider an agent whose belief state is given by $\Gamma = \{(\varphi)[1], (\neg\varphi)[1]\}$. The agent finds acceptable pairs of gambles on φ and $\neg\varphi$ in which the gambler has escrows equal to zero, and sure loss can be scaled arbitrarily up. In such cases, the program (19)–(23) is unfeasible for any p, being $\mathcal{I}^g_{SSK}$ and $\mathcal{I}^\gamma_\infty$ defined as ∞. That is, either both measures are finite and equal, or $\mathcal{I}^g_{SSK} = \mathcal{I}^\gamma_\infty = \infty$.

Since $\mathcal{I}^a_{SSK}$ and $\mathcal{I}^g_{SSK}$ are particular cases of $\mathcal{I}^\gamma_\infty$, as Theorems 5.10 and 5.11 claim, these measures also satisfies the postulates listed in Theorem 5.6. Both functions are continuous for probabilities within $(0, 1)$, by Lemma 5.8. By Lemma 5.9, $\mathcal{I}^a_{SSK}$ also satisfies normalization. Additionally, $\mathcal{I}^a_{SSK}$ and $\mathcal{I}^g_{SSK}$ can be computed by linear programs and have a meaningful betting interpretation as well. Thus, $\mathcal{I}^a_{SSK}$ seems a good alternative for measuring inconsistency when normalization is demanded, and each of these measures might be the most suitable in some contexts, such as the market scenarios described by Schervish *et al.* [33].

It is worth noting that $1-\mathcal{I}^a_{SSK}$, $1-\mathcal{I}^t_{SSK}$ and $1-\mathcal{I}_\infty$ can be seen as possible generalizations of Knight's measure of consistency for classical propositional logic [24]. Roughly speaking, a set $\Gamma \subsetneq \mathcal{L}_{X_n}$ of propositions is said to be η-consistent if the knowledge base $\Gamma_\eta = \{(\varphi)[\eta, 1]\,|\varphi \in \Gamma\}$ is satisfiable. Knight defines Γ as maximally η-consistent if η is the maximum value such that Γ is η-consistent. If we assign probability one to each element of Γ, building the base $\Gamma' = \{(\varphi)[1]\,|\varphi \in \Gamma\}$, then Γ is maximally η-consistent iff $\mathcal{I}^a_{SSK}(\Gamma') = \mathcal{I}^t_{SSK}(\Gamma') = \mathcal{I}^\varepsilon_\infty(\Gamma') = \mathcal{I}_\infty(\Gamma') = 1 - \eta$. Note that $\mathcal{I}^a_{SSK}(\Gamma)$ and $\mathcal{I}^t_{SSK}(\Gamma)$ are equal for all probabilities in Γ' are 1. Hence, Theorems 5.4 and 5.10 can rationally support the use of Knight's measure in the classical setting as well. Suppose an agent's belief base Γ contains φ iff she sees as fair a gamble on φ with $q = 1$, then Γ is maximally η-consistent iff the agent is exposed to a maximum sure loss of $1 - \eta$ when her resources are fixed at one (equivalently, absolute values of stakes sum up to one).

In general, $\mathcal{I}^\gamma_\infty$ measures the maximum guaranteed loss an agent is exposed to through a Dutch book when we limit to one the sum of the stakes' absolute values $(\underline{\lambda}_i, \bar{\lambda}_i)$ weighted by $\underline{\gamma}_i$ and $\bar{\gamma}_i$. A more meaningful interpretation to the volatility factors in $\mathcal{I}^\gamma_\infty$ can be given by considering the coverage ratios $\underline{c}_i = \underline{\gamma}_i/\underline{q}_i$ and $\bar{c}_i = (1 - \bar{\gamma}_i)/\bar{q}_i$. The number $\underline{c}_i$ ($\bar{c}_i$) is the ratio between how much the agent has to commit from her resources to cover a gamble on $\varphi_i|\psi_i$ with positive (negative) stake and how much she can lose in that bet. So $\mathcal{I}^\gamma_\infty$ maximizes the sure loss when her resources (the total she can commit) are fixed at 1 and her coverage ratios are $\underline{c}_i$ and $\bar{c}_i$. Analogously, if we define $\underline{c}'_i = (1 - \underline{\gamma}_i)/\underline{q}_i$ and $\bar{c}'_i = \bar{\gamma}_i/\bar{q}_i$, $\mathcal{I}^\gamma_\infty$ maximizes sure profit for the gambler when his resources are limited to 1 and his coverage ratios are $\underline{c}'_i$ and $\bar{c}'_i$. When coverage ratios $\underline{c}_i$ and

$\bar{c}_i$ ($\underline{c}'_i$ and $\bar{c}'_i$) are all equal to one, we recover the measure $\mathcal{I}^a_{SSK}$ ($\mathcal{I}^g_{SSK}$).

When $p = 1$, the generalization of Potyka's measure can be brought to the measure $\mathcal{I}_{\mathit{Staffel}}$. Consider a knowledge base $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i]|1 \leq i \leq m\}$ representing an agent's belief state and a vector $\delta = \langle \underline{\delta}_1, \bar{\delta}_1, \ldots, \underline{\delta}_m, \bar{\delta}_m \rangle > 0$. We define $\mathcal{I}^{\delta}_{\mathit{Staffel}}(\Gamma)$ as the maximum sure loss such an agent would be exposed to when gambles on $\varphi_i|\psi_i$, for each $(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] \in \Gamma$, have stakes $\underline{\lambda}_i \in [0, \underline{\delta}_i]$ and $-\bar{\lambda}_i \in [-\bar{\delta}_i, 0]$. Each element of δ can be seen again as a confidence factor, since the greater it is, the higher the amount the agent accepts to risk at a gamble on the corresponding probability bound.

Theorem 5.12. *For any* $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i]|1 \leq i \leq m\}$ *in* $\mathbb{K}$*, if* $\underline{\gamma}_i = 1/\underline{\delta}_i$ *and* $\bar{\gamma}_i = 1/\bar{\delta}_i$ *for all* $1 \leq i \leq m$*, then* $\mathcal{I}^{\delta}_{\mathit{Staffel}}(\Gamma) = \mathcal{I}^{\gamma}_1(\Gamma)$ *and both are finite.*

We could even permit elements of δ to be ∞, the corresponding elements of γ becoming zero. In such a case, the sure loss could be unbounded, and the program that compute $\mathcal{I}^{\gamma}_1(\Gamma)$, infeasible — $\mathcal{I}^{\gamma}_1(\Gamma) = \mathcal{I}^{\delta}_{\mathit{Staffel}}(\Gamma)$ would then be defined as ∞.

Properties from Theorem 5.6 are satisfied by $\mathcal{I}^{\delta}_{\mathit{Staffel}}$, including super-additivity and IC-separability. Furthermore, it is computable by means of a linear program and has a meaningful interpretation in the gambling scenario. Therefore, the measure $\mathcal{I}^{\delta}_{\mathit{Staffel}}$ might be appropriate to handle cases where super-additivity and IC-separability are desirable and confidence factors are available. Furthermore, if normalization is required, we coud follow Muino's ideas [26] to normalize $\mathcal{I}^{\delta}_{\mathit{Staffel}}$, although super-additivity and IC-separability would not hold anymore.

6 Conclusions and Future Work

In this work, we studied different ways of measuring inconsistency in probabilistic knowledge bases. Three aspects were discussed: postulates the measures should satisfy, the efficiency of the methods used to compute the measures, and possible meaningful interpretations for them. As it was argued for, the independence postulate shall be abandoned in favor of continuity. The causes of such incompatibility were analyzed, and a modification of independence was proposed to restore compatibility. Inconsistency measures that can be computed using linear programs were reviewed and proved to satisfy the postulates, and we gave them a rational by means of Dutch books. We showed how measures based on distances could be generalized to encompass other measures based on Dutch books, possibly considering confidence factors. In the end, we have two families of inconsistency measures, $\mathcal{I}^{\gamma}_1$ and $\mathcal{I}^{\gamma}_{\infty}$, that satisfy the postulates, can be computed through linear programs, have a meaningful interpretation and allow for confidence factors. In a given context, what might settle the case in favor of using a particular measure is its distinct properties: one family satisfies super-additivity and IC-separability, and the other satisfies normalization under some conditions.

The introduced concepts of innocuous conditional and inescapable conflict might have practical use in measuring inconsistency only if their instances are recognizable in a reasonable time. Nothing was said here about the complexity of the computational task of finding innocuous conditionals and inescapable conflicts within a knowledge base, but they are clearly very hard problems. Thus, future work includes investigating these problems aiming at devising algorithms to solve them. A possible path may involve proofs of contradiction one can build using as premises the conditionals in an inconsistent knowledge base. Intuitively, it seems that if a conditional is not essential to any proof of the contradiction, then it is innocuous. Equivalently, an inescapable conflict is apparently the set of essential conditionals used in such a proof. Hence, an isomorphism like Curry-Howard's tailored to a probabilistic logic might be needed to formalize a normal form for these proofs.

Analyzing inconsistency in knowledge bases is only a step towards restoring their consistency, required by standard methods of inference. As future work, it would be interesting to propose concrete procedures to consolidate knowledge bases, as done in [30] for instance. To achieve that, one could rely on the same triplet: rationality postulates, efficiency of computation and meaningful interpretation. Another intended continuation of this work is to study principled ways of inferring probabilistic conclusions from inconsistent bases, using the ideas here presented. For instance, this could be done by taking the models of a base to be those of the closest consistent bases, construed as the consolidations corresponding to some inconsistency measure here studied.

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Appendix: Proofs of Technical Results

Proposition 3.6 . *If $\mathcal{I}$ satisfies MIS-separability, then $\mathcal{I}$ satisfies independence.*

Proof. Let Γ be a knowledge base and $\alpha \in \Gamma$ a free conditional. By MIS-separability, as α is free and all MISes of Γ are in $\Gamma \setminus \{\alpha\}$, we have $\mathcal{I}(\Gamma) = \mathcal{I}(\Gamma \setminus \{\alpha\}) + \mathcal{I}(\alpha)$. $\square$

Corollary 3.9 . *There is no inconsistency measure $\mathcal{I} : \mathbb{K}_{prec} \to [0, \infty)$ that satisfies consistency, MIS-separability and continuity.*

Proof. It follows directly from Theorem 3.8 and Proposition 3.6. $\square$

Theorem 4.2 . *Consider a knowledge base $\Gamma \in \mathbb{K}$ and a probabilistic conditional $\alpha \in \Gamma$. The following statements are equivalent:*

1. *There is no minimal abrupt repair Δ of Γ such that $\alpha \in \Delta$.*
2. *For all maximal abrupt consolidation Γ' of Γ, $\alpha \in \Gamma'$.*
3. *If $\Gamma' = \Gamma \setminus \Delta$ is an abrupt consolidation of Γ (equivalently, Δ is an abrupt repair of Γ), then α is consistent with Γ'.*
4. *There is no minimal inconsistent set $\Delta \subseteq \Gamma$ such that $\alpha \in \Delta$.*

Proof. The first two items are clearly dual, and the fourth one is the definition of free conditional. Suppose α is free in Γ. Note that all abrupt consolidations Γ' of Γ are consistent with α. As Γ' is consistent, it has no MIS, and adding α cannot create a MIS, for it is free. Thus, if an abrupt consolidation does not contain α, it is not maximal. Now suppose there is a maximal abrupt consolidation Γ' such that $\alpha \notin \Gamma'$. For Γ' is maximal, α cannot be consistent with it. As Γ' is consistent, it has no MIS, and adding α creates a MIS (that contains α), which also is a MIS of Γ — hence, α cannot be free. $\square$

Lemma 4.5 . *Consider a knowledge base $\Gamma \in \mathcal{K}$ and a probabilistic conditional $\alpha \in \Gamma$. The following statements are equivalent:*

1. *For all d-consolidation Γ' of Γ, $\alpha \in \Gamma'$.*
2. *If Γ' is a consolidation, then α is consistent with Γ'.*

Proof. Suppose all d-consolidations of Γ contain α. For any consolidation Ψ, there is a d-consolidation Ψ' such that, for each $\beta' \in \Psi'$, there is a $\beta \in \Psi$ such that $\beta' \subseteq \beta$. Therefore, any probability mass π satisfying Ψ' must also satisfies Ψ, and $\alpha \in \Psi'$ implies π satisfies α as well. Now suppose there is a d-consolidation Ψ that does not contain α. As $\alpha \in \Gamma$, there is a $\beta \in \Psi$ such that $\alpha \subsetneq \beta$. For Ψ is dominant, $(\Psi \setminus \{\beta\}) \cup \{\alpha\}$ cannot be a consolidation and thus is inconsistent. Finally, α is not consistent with Ψ. $\square$

Proposition 4.8 . *Consider a probabilistic conditional $\alpha \in \Gamma$. If α is safe, it is innocuous; if α is innocuous, it is free.*

Proof. If $\Gamma = \{\alpha\}$, then α is safe, innocuous and free iff it is satisfiable, thus we focus on $\Gamma \neq \{\alpha\}$. Let Γ be built over the set of atoms $X_n = \{x_1, \ldots, x_n\}$. Suppose α is safe and, without loss of generality, the set of atoms appearing in α is $X_\alpha = \{x_1, \ldots, x_m\}$, for some $m < n$. As α is satisfiable, there is a probability mass $\pi_\alpha : W_{X_\alpha} \to [0, 1]$ satisfying it, where W_{X_α} is the set containing the 2^m possible worlds with atoms from X_α. The base $\Gamma' = \Gamma \backslash \{\alpha\}$ is built over the set of atoms $X_{\Gamma'} = X_n \setminus X_\alpha$. Any consolidation Ψ of Γ' must also be formed by atoms in $X_{\Gamma'}$. If Δ is a consolidation of Γ, there is a consolidation Ψ of Γ' such that $\Delta = \Psi \cup \{\beta\}$, for some β such that $\alpha \subseteq \beta$. Let $\pi_\Psi : W_{X_{\Gamma'}} \to [0, 1]$ be the probability mass satisfying Ψ, where $W_{X_{\Gamma'}}$ is the set containing the 2^{n-m} possible worlds with atoms from $X_{\Gamma'}$. Consider the probability mass $\pi : W_{X_n} \to [0, 1]$ such that $\pi(w_i \wedge w_j) = \pi_\alpha(w_i) \times \pi_\Psi(w_j)$ for any pair $(w_i, w_j) \in W_{X_\alpha} \times W_{X_{\Gamma'}}$. Note that π satisfies Ψ and α, thus π satisfies Ψ and β. Therefore, α is consistent with any consolidation $\Delta = \Psi \cup \{\beta\}$ of Γ and is innocuous by Lemma 4.5.

Now suppose α is innocuous. Any abrupt consolidation $\Delta \subseteq \Gamma$ is equivalent (and equisatisfiable) to a consolidation $\Delta' \in \Gamma$ such that $\Delta' = \Delta \cup \{(\varphi|\psi)[0, 1] | (\varphi|\psi)[\underline{q}, \bar{q}] \in \Gamma \setminus \Delta\}$. As α is innocuous, it is consistent with any consolidation Δ' and, consequently, any abrupt consolidation Δ. Finally, by Theorem 4.2, α is free. ◻

Corollary 4.10 . *If $\mathcal{I}$ satisfies independence, then $\mathcal{I}$ satisfies i-independence. If $\mathcal{I}$ satisfies i-independence, then $\mathcal{I}$ satisfies weak independence.*

Proof. It follows directly from the definitions and Proposition 4.8. ◻

Proposition 4.11 . *A knowledge base Γ is a minimal inconsistent set iff Γ is inconsistent and there are no $\Delta_1, \ldots, \Delta_k \subsetneq \Gamma$, with $k \geq 1$, such that:*

1. $\bigcup_{i=1}^{k} \Delta_i = \Gamma$;
2. *For every $\Gamma' \subseteq \Gamma$ if $\Gamma' \cap \Delta_i$ is an abrupt consolidation of Δ_i for all $1 \leq i \leq k$, then Γ' is an abrupt consolidation of Γ.*

Proof. ($\rightarrow$) Suppose Γ is a MIS and there are $\Delta_1, \ldots, \Delta_k \subsetneq \Gamma$ satisfying both items. For any $1 \leq i \leq k$, as $\Delta_i \subsetneq \Gamma$ is consistent, $\Gamma \cap \Delta_i$ is an abrupt consolidation of Δ_i. Thus, by the second item, Γ is an abrupt consolidation of itself, which contradicts the fact that Γ is inconsistent.

($\leftarrow$) Now suppose Γ is inconsistent but not a MIS. Let $MIS(\Gamma) = \{\Delta_1, \ldots, \Delta_m\}$ be the set of MISes in Γ, for some $m \geq 1$. Let Δ_{m+1} denote the set of free formulas in Γ. Clearly, $\bigcup_{i=1}^{m+1} \Delta_i = \Gamma$. Now consider a set $\Gamma' \subseteq \Gamma$ such that $\Gamma' \cap \Delta_i$ is consistent for any $1 \leq i \leq m + 1$. If Γ' was inconsistent, it would contain a MIS $\Delta_i \in MIS(\Gamma)$ and $\Gamma' \cap \Delta_i$ would be inconsistent — a contradiction. Thus Γ' is an abrupt consolidation of Γ. ◻

Lemma 4.13 . *Consider two knowledge bases $\Gamma, \Gamma' \in \mathbb{K}$ such that Γ' is a widening of Γ. If for every inescapable conflict $\Delta \subseteq \Gamma$ the base $\{\beta \in \Gamma' \mid \alpha \in \Delta$ and $\alpha \subseteq \beta\}$ is consistent, then Γ' is a consolidation of Γ.*

Proof. Consider a set $\mathcal{C}^0 = \{\Gamma\}$ and let $IC(\Gamma)$ be the set of inescapable conflicts of Γ. For any $\mathcal{C}^j$, define $\mathcal{C}^j_\perp = \{\Delta \in \mathcal{C}^j | \Delta$ is inconsistent and $\Delta \notin IC(\Gamma)\}$. Let s^j denote the size of the largest set $\Delta \in \mathcal{C}^j_\perp$, or $s^j = 0$ if $\mathcal{C}^j_\perp = \varnothing$. Starting with $\mathcal{C}^0$, we define a $\mathcal{C}^{j+1}$ from $\mathcal{C}^j$ for any j such that $s^j > 0$: select a $\Delta \in \mathcal{C}^j_\perp \subseteq \mathcal{C}^j$ such that $|\Delta| = s^j$; as Δ is inconsistent but not an inescapable conflict, there are $\Delta_1, \ldots, \Delta_k \subsetneq \Delta$ satisfying both items of Definition 4.12; define $\mathcal{C}^{j+1} = (\mathcal{C}^j \setminus \{\Delta\}) \cup \{\Delta_1\} \cup \cdots \cup \{\Delta_k\}$. By Definition 4.12, any $\mathcal{C}^j$ defined in this way is such that, if Γ' is a widening of Γ and for every $\Delta \subseteq \mathcal{C}^j$ the base $\{\beta \in \Gamma' \,|\, \alpha \in \Delta$ and $\alpha \subseteq \beta\}$ is consistent, then Γ' is a consolidation of Γ. There is a $\mathcal{C}^h$ such that $\mathcal{C}^h_\perp = \varnothing$ — *i.e.*, all conflicts in $\mathcal{C}^h$ are inescapable. Formally, let $\#(s^j)$ be the number of sets with size s^j in $\mathcal{C}^j_\perp$, or $\#(s^j) = 0$ if $\mathcal{C}^j_\perp = \varnothing$. Consider the partial order $R_< \subsetneq \mathbb{N} \times \mathbb{N}$ such that $\langle a, b\rangle R_< \langle c, d\rangle$ iff $a < c$ or $a = c$ and $b < d$. Note that $\langle s^j, \#(s^j)\rangle \in \mathbb{N} \times \mathbb{N}$ and $\langle s^j, \#(s^j)\rangle R_< \langle s^{j+1}, \#(s^{j+1})\rangle$ for any j, thus $\langle s^h, \#(s^h)\rangle = \langle 0, 0\rangle$ for some $h \in \mathbb{N}$. Let Γ' be a widening of Γ such that, for every inescapable conflict $\Delta \in IC(\Gamma)$, the base $\{\beta \in \Gamma' \,|\, \alpha \in \Delta$ and $\alpha \subseteq \beta\}$ is consistent. Let $\mathcal{C}^h_{IC} \subseteq IC(\Gamma)$ denote the set of (inescapable) conflicts in $\mathcal{C}^h$. Note that every $\Delta \in \mathcal{C}^h \setminus \mathcal{C}^h_{IC}$ is consistent. So, for every $\Delta \in \mathcal{C}^h$, the base $\{\beta \in \Gamma' \,|\, \alpha \in \Delta$ and $\alpha \subseteq \beta\}$ is consistent. Finally, Γ' is a consolidation of Γ. □

Proposition 4.14 . *If Δ is a minimal inconsistent set, then Δ is an inescapable conflict.*

Proof. Suppose Δ is inconsistent but not an inescapable conflict. Thus, there are $\Delta_1, \ldots, \Delta_k \subsetneq \Delta$ such that $\bigcup_{i=1}^k \Delta_i = \Delta$ and, if Δ'_i is a consolidation of Δ_i for all $1 \leq i \leq k$ and $\bigcup_{i=1}^k \Delta'_i$ is a knowledge base, then $\bigcup_{i=1}^k \Delta'_i = \Delta'$ is a consolidation of Δ. Now consider a $\Gamma \subseteq \Delta$ such that $\Gamma \cap \Delta_i$ is consistent of all $1 \leq i \leq k$. Note that $\Delta'_i = (\Gamma \cap \Delta_i) \cup \{(\varphi|\psi)[0, 1] | (\varphi|\psi)[\underline{q}, \bar{q}] \in \Delta_i \setminus \Gamma\}$ is a consolidation of Δ'_i. Hence, $\bigcup_{i=1}^k \Delta'_i = \Gamma' = \Gamma \cup \{(\varphi|\psi)[0, 1] | (\varphi|\psi)[\underline{q}, \bar{q}] \in \Delta \setminus \Gamma\}$ is a knowledge base and a consolidation of Δ. Finally, as $\Gamma \subseteq \Delta$ is equivalent to Γ', Γ is an abrupt consolidation, and Δ is not a MIS, by Theorem 4.11. □

Corollary 4.17 . *If $\mathcal{I}$ satisfies MIS-separability, then $\mathcal{I}$ satisfies IC-separability.*

Proof. If follows directly from the definitions and Proposition 4.14. □

Theorem 4.18 . *The following statements are equivalent:*

1. *For all d-consolidation Γ' of Γ, $\alpha \in \Gamma'$.*
2. *If Γ' is a consolidation of Γ, then α is consistent with Γ'.*
3. *There is no inescapable conflict Δ in Γ such that $\alpha \in \Delta$.*
4. *α is an innocuous probabilistic conditional in Γ.*

Proof. By the definition of innocuous conditionals and Lemma 4.5, the first, the second and the fourth statements are equivalent. It remains to prove that α is innocuous iff there is no inescapable conflict Δ in Γ such that $\alpha \in \Delta$.

($\rightarrow$) Let α be innocuous in Γ. Suppose there is an inescapable conflict $\Delta \subseteq \Gamma$ such that $\alpha \in \Delta$. Consider the base $\Psi = \Delta \setminus \{\alpha\}$. Let Ψ' be a consolidation of Ψ. Thus, $\Gamma' = \Psi' \cup \{(\varphi|\psi)[0,1] | (\varphi|\psi)[\underline{q}, \bar{q}] \in \Gamma \setminus \Psi\}$ is consistent and it is consolidation of Γ. Due to the fact that α is innocuous, α is consistent with Γ' (by Lemma 4.5) and, therefore, with Ψ'. Consequently, $\Psi' \cup \{\alpha\}$ is a consolidation of Δ for any consolidation Ψ' of Ψ. Furthermore, if $\{\beta\}$ is a consolidation of $\{\alpha\}$ (*i.e.*, $\alpha \subseteq \beta$), $\Psi' \cup \{\beta\}$ is a consolidation of Δ. As $\Psi, \{\alpha\} \subsetneq \Delta$ are such that $\Psi \cup \{\alpha\} = \Delta$, and any consolidations Ψ' and $\{\beta\}$ of theirs are such that $\Psi' \cup \{\beta\}$ is a consolidation of Δ, Δ is not an inescapable conflict, which is a contradiction.

($\leftarrow$) Suppose there is no inescapable conflict Δ in Γ such that $\alpha \in \Delta$. Consider the base $\Psi = \Gamma \setminus \{\alpha\}$. Every consolidation Γ' of Γ can be written as $\Gamma' = \Psi' \cup \{\beta\}$, where Ψ' is a consolidation of Ψ and $\alpha \subseteq \beta$. As all inescapable conflicts of Γ are in Ψ, by Lemma 4.13, $\Psi' \cup \{\alpha\}$ is a consistent. Hence, α is consistent with any consolidation $\Gamma' = \Psi' \cup \{\beta\}$ and α is innocuous by Lemma 4.5. □

Corollary 4.19 . *If $\mathcal{I}$ satisfies IC-separability, then $\mathcal{I}$ satisfies i-independence.*

Proof. Let Γ be a knowledge base and $\alpha \in \Gamma$ an innocuous conditional. As α is innocuous, all inescapable conflicts of Γ are in $\Gamma \setminus \{\alpha\}$ by Lemma 4.18. By IC-separability, we have $\mathcal{I}(\Gamma) = \mathcal{I}(\Gamma \setminus \{\alpha\}) + \mathcal{I}(\alpha)$. □

Theorem 4.26 . *For any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}_p$ is well-defined and satisfies the postulates of consistency, continuity, i-independence and monotonicity.*

Proof. To show that $\mathcal{I}_p$ is well-defined, we use results from the proof of Theorem 1 in [38]. For any $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq m\}$, Thimm shows that the set $Q_\Gamma = \{\langle q_1, \ldots, q_m\rangle \in \mathbb{R}^m | \Lambda_\Gamma(\langle q_1, q_1, \ldots, q_m, q_m\rangle) \text{ is consistent}\}$ is compact and closed, where $\Lambda_\Gamma : [0,1]^{2m} \rightarrow \mathbb{K}$ is the characteristic function of Γ. Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function such that $h(a,b) = \max(0, a-b)$ for any $a, b \in \mathbb{R}$. The measure $\mathcal{I}_p$ is the minimum of $\|f_{\underline{q},\bar{q}}(q)\|_p$ with $q \in Q_\Gamma$, where $f_{\underline{q},\bar{q}} : \mathbb{R}^m \rightarrow \mathbb{R}^{2m}$ is a function such that $f_{\underline{q},\bar{q}}(\langle q_1, \ldots, q_m\rangle) = \langle h(\underline{q}_1 - q_1), h(q_1 - \bar{q}_1), \ldots, h(\underline{q}_m - q_m), h(q_m - \bar{q}_m)\rangle$. Intuitively, $f_{\underline{q},\bar{q}}(q)$ measures, for each point q_i, how much the lower and the upper bounds have to change for we have $q_i \in [\underline{q}_i, \bar{q}_i]$. Finally, $\mathcal{I}_p$ is well defined, for Q_Γ is closed and compact [38].

Consistency: By definition, a p-norm is never negative, thus $\mathcal{I}_p(\Gamma) \geq 0$. Suppose $\Gamma = \Lambda_\Gamma(q)$ is consistent. A vector $q' = q$ is such that $\|q' - q\|_p = 0$ for any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, thus $\mathcal{I}_p(\Gamma) = 0$. Now suppose $\Gamma = \Lambda_\Gamma(q)$ is inconsistent. For every $q' \in Q_\Gamma$, $q' \neq q$, then $\|q' - q\|_p > 0$ and $\mathcal{I}_p(\Gamma) > 0$ for any $p \in \mathbb{N}_{>0} \cup \{\infty\}$.

Continuity: Given a base $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq m\}$, its characteristic function $\Lambda_\Gamma : [0,1]^{2m} \rightarrow \mathbb{K}$ and a fixed $q \in Q_\Gamma$, define the function $g_q : \mathbb{R}^{2m} \rightarrow \mathbb{R}$ such that $g_q(\langle \underline{q}_1, \bar{q}_1, \ldots, \underline{q}_m, \bar{q}_m\rangle) = \|f_{\underline{q},\bar{q}}(q)\|_p$. Note that $\mathcal{I}_p \circ \Lambda_\Gamma(\langle \underline{q}_1, \bar{q}_1, \ldots, \underline{q}_m, \bar{q}_m\rangle)$ is computed as the minimum of $\{g_q(\langle \underline{q}_1, \bar{q}_1, \ldots, \underline{q}_m, \bar{q}_m\rangle) | q \in Q_\Gamma\}$. Each g_q is continuous, and the minimum of continuous functions is continuous, hence $\mathcal{I}_p \circ \Lambda_\Gamma$ is continuous.

Monotonicity: Let $\Lambda_\Gamma(q')$ be a consolidation of $\Gamma = \Lambda_\Gamma(q)$ such that $\|q' - q\|_p$ is minimized, for a $p \in \mathbb{N}_{>0} \cup \{\infty\}$, and $\mathcal{I}_p(\Gamma) = \|q' - q\|_p$. To prove by contradiction, suppose $\mathcal{I}(\Gamma \cup \{\alpha\}) < \mathcal{I}_p(\Gamma)$, for some $\Psi = \Gamma \cup \{\alpha\} \in \mathbb{K}$. Hence, there is a consolidation $\Psi' = \Lambda_\Psi(r')$ of $\Psi = \Lambda_\Psi(r)$ such that $\|r' - r\|_p < \|q' - q\|_p$. Consider the base $\Gamma' = \Psi' \setminus \{\beta\}$, such that $\alpha \subseteq \beta$. As Ψ' is consistent, $\Gamma' = \Lambda_\Gamma(q'')$ also is, and it is a consolidation of Γ. Since q and q'' are projections (subsets, in sense) of r and r', $q'' - q$ is a projection of $r' - r$ and $\|q'' - q\|_p \leq \|r' - r\|_p < \|q' - q\|_p$. Finally, it would follow that $\mathcal{I}_p(\Gamma) \leq \|q'' - q\|_p < \|q' - q\|_p = \mathcal{I}_p(\Gamma)$, which is a contradiction.

i-**Independence:** Consider the bases $\Gamma = \Lambda_\Gamma(r)$ and $\Psi = \Gamma \setminus \{\alpha\}$ in $\mathbb{K}$, where $\alpha = (\varphi|\psi)[\underline{q}, \bar{q}]$ is innocuous in Γ. We are going to prove that $\mathcal{I}_p(\Gamma) \leq \mathcal{I}_p(\Psi)$, and the desired result follows from monotonicity. Let $\Lambda_\Psi(q')$ be a consolidation of $\Psi = \Lambda_\Psi(q)$ such that $\|q' - q\|_p$ is minimized, for a $p \in \mathbb{N}_{>0} \cup \{\infty\}$, and $\mathcal{I}_p(\Psi) = \|q' - q\|_p$. Note that $\Gamma'\Psi' \cup \{(\varphi|\psi)[0,1]\}$ is a consolidation of Γ. As $\alpha = (\varphi|\psi)[\underline{q}, \bar{q}]$ is innocuous, α is consistent with Γ' and Ψ'. Hence, $\Psi' \cup \{\alpha\} = \Lambda_\Gamma(r')$ is a consolidation of Γ. Note that $r' - r$ is $q' - q$ with two extra 0's (from alpha). Finally, $\mathcal{I}_p(\Gamma) \leq \|r' - r\|_p = \|q' - q\|_p = \mathcal{I}_p(\Psi)$. □

Lemma 4.27 . *$\mathcal{I}_p$ satisfies super-additivity and IC-separability iff $p = 1$.*

Proof. ($\rightarrow$) To note that super-additivity and IC-separability do not hold if $p > 1$, consider the bases $\Psi = \{(\top)[0.9]\}, \Delta = \{(\bot)[0.1]\}, \Gamma = \Psi \cup \Delta$. By the definition of d-consolidation, if $\mathcal{I}_p(\Gamma) = d$, then there is d-consolidation $\Lambda_\Gamma(q')$ of $\Gamma = \Lambda_\Gamma(q)$ such that $\|q' - q\|_p = d$. The only d-consolidations of Ψ, Δ, Γ are $\Psi' = \{(\top)[0.9, 1]\}, \Delta' = \{(\bot)[0, 0.1]\}, \Gamma' = \Psi' \cup \Delta'$, for changing the lower bound in Ψ and the upper bound in Δ is useless to reach consistency. For any finite p, $\mathcal{I}_p(\Psi) = \mathcal{I}_p(\Delta) = \sqrt[p]{0.1^p} = 0.1$, and $\mathcal{I}_p(\Gamma) = \sqrt[p]{0.1^p + 0.1^p} = 0.1\sqrt[p]{2}$. For $p = \infty$, $\mathcal{I}_p(\Psi) = \mathcal{I}_p(\Delta) = \max\langle 0.1\rangle = 0.1$ and $\mathcal{I}_p(\Gamma) = \max\langle 0.1, 0.1\rangle = 0.1$. Therefore, for any $p > 1 \in \mathbb{N} \cup \{\infty\}$, $\mathcal{I}_p(\Gamma) < 0.2 = \mathcal{I}_p(\Psi) + \mathcal{I}_p(\Delta)$, and both super-additivity and IC-separability fail.

($\leftarrow$) Now fix $p = 1$. To prove that super-additivity holds, suppose there are bases $\Psi, \Delta, \Gamma = \Psi \cup \Delta$ in $\mathbb{K}$ such that $\Psi \cap \Delta = \varnothing$. Let $\Psi' = \Lambda_\Psi(q'), \Delta' = \Lambda_\Delta(r'), \Gamma' = \Lambda_\Gamma(s')$ be d-consolidations of $\Psi = \Lambda_\Psi(q), \Delta = \Lambda_\Delta(r), \Gamma = \Lambda_\Gamma(s)$ that minimize $\|q' - q\|_1, \|r' - r\|_1, \|s' - s\|_1$, corresponding to $\mathcal{I}_1(\Psi), \mathcal{I}_1(\Delta), \mathcal{I}_1(\Gamma)$. Clearly, Γ' can be partitioned into $\Psi'' \cup \Delta''$ in such a way that $\Psi'' = \Lambda_\Psi(s'_\Psi), \Delta'' = \Lambda_\Delta(s'_\Delta)$ are consolidations of Ψ, Δ. By the construction of s'_Ψ and s'_Δ, $\|s' - s\|_1 = \|s'_\Psi - q\|_1 + \|s'_\Delta - r\|_1$. Hence, for $\mathcal{I}_1(\Psi) \leq \|s'_\Psi - q\|_1$ and $\mathcal{I}_1(\Delta) \leq \|s'_\Delta - r\|_1$, it follows that $\mathcal{I}(\Gamma) = \|s' - s\|_1 \geq \mathcal{I}_1(\Psi) + \mathcal{I}_1(\Delta)$.

To prove that IC-separability holds, suppose there are bases $\Psi, \Delta, \Gamma = \Psi \cup \Delta$ in $\mathbb{K}$ such that $\Psi \cap \Delta = \varnothing$, $IC(\Gamma) = IC(\Psi) \cup IC(\Delta)$. Let $\Psi' = \Lambda_\Psi(q'), \Delta' = \Lambda_\Delta(r')$, be consolidations of $\Psi = \Lambda_\Psi(q), \Delta = \Lambda_\Delta(r)$ that minimize $\|q' - q\|_1, \|r' - r\|_1$, corresponding to $\mathcal{I}_1(\Psi), \mathcal{I}_1(\Delta)$. As $\Gamma' = \Psi' \cup \Delta' = \Lambda_\Gamma(s')$ is a widening of $\Gamma = \Lambda_\Gamma(s)$ such that, for each $\Phi \in IC(\Gamma) = IC(\Psi) \cup IC(\Delta)$, the base $\{\beta \in \Gamma' \,|\, \alpha \in \Phi \text{ and } \alpha \subseteq \beta\}$ is consistent (all inescapable conflicts are solved), Γ' is a consolidation of Γ by Lemma 4.13. As $\|s' - s\|_1 = \|q' - q\|_1 + \|r' - r\|_1 =$

$\mathcal{I}_1(\Psi) + \mathcal{I}_1(\Delta)$, it follows that $\mathcal{I}_1(\Gamma) \leq \mathcal{I}_1(\Psi) + \mathcal{I}_1(\Delta)$. By super-additivity, $\mathcal{I}_1(\Gamma) \geq \mathcal{I}_1(\Psi) + \mathcal{I}_1(\Delta)$, thus $\mathcal{I}_1(\Gamma) = \mathcal{I}_1(\Psi) + \mathcal{I}_1(\Delta)$. □

Lemma 4.28 . *$\mathcal{I}_p$ satisfies normalization iff $p = \infty$.*

Proof. ($\rightarrow$) To note that normalization does not hold if p is finite, consider the base $\Gamma = \{(\top)[0], (\bot)[1]\}$. The only d-consolidation of Γ is $\Gamma' = \{(\top)[0,1], (\bot)[0,1]\}$, for changing the lower bound in $(\top)[0]$ and the upper bound in $(\bot)[1]$ is useless to reach consistency. For any finite p, $\mathcal{I}_p(\Gamma) = \sqrt[p]{1^p + 1^p} = \sqrt[p]{2} > 1$, and normalization fails.

($\leftarrow$) By definition, $\mathcal{I}_\infty(\Gamma)$ is the minimum of $\|q' - q\|_\infty$ subject to $\Gamma = \Lambda_\Gamma(q)$ and $\Lambda_\Gamma(q')$ being consistent. As the vectors q, q' are in $[0,1]^{2|\Gamma|}$, $\|q' - q\|_\infty \in [0,1]$, since $|q'_i - q_i| \in [0,1]$ for all elements q_i, q'_i of q, q'. □

Proposition 5.1 . *For any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}^\varepsilon_p : \mathbb{K} \rightarrow [0,\infty)$ is well-defined and satisfies consistency, continuity, weak independence and monotonicity. $\mathcal{I}^\varepsilon_1$ also satisfies super-additivity.*

Proof. See Section 4 in [29]. □

Lemma 5.2 . *For any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}^\varepsilon_p : \mathbb{K} \rightarrow [0,\infty)$ is well-defined and satisfies consistency, continuity, i-independence and monotonicity. $\mathcal{I}^\varepsilon_1$ also satisfies super-additivity and IC-separability; and $\mathcal{I}^\varepsilon_\infty$ satisfies normalization.*

Proof. As $\mathcal{I}^\varepsilon_1$ and $\mathcal{I}^\varepsilon_\infty$ are particular cases of $\mathcal{I}^\gamma_1$ and $\mathcal{I}^\gamma_\infty$, see the proof of Theorem 5.6. For continuity and normalization properties, see Lemmas 5.8 and 5.9. □

Proposition 5.3 . *There is an inconsistent knowledge base $\Gamma = \{(\varphi_i)[q_i] | 1 \leq i \leq m\} \in \mathbb{K}_{prec}$ such that no Dutch book is possible against an incoherent agent whose belief state is represented by Γ if there is one gamble on φ_i per conditional $(\varphi_i)[q_i] \in \Gamma$ and stakes are limited to $\lambda_i \in \{-1, 0, 1\}$.*

Proof. We prove by constructing such knowledge base. Let $\Gamma = \{(x_1)[0.3], (x_2)[0.3], (x_1 \wedge x_2)[0.3], (x_1 \wedge \neg x_2 \vee \neg x_1 \wedge x_2)[0.2]\}$. Let $x_1 \oplus x_2$ denote $x_1 \wedge \neg x_2 \vee \neg x_1 \wedge x_2$. For any probability mass π, it must hold $P_\pi(x_1) + P_\pi(x_2) = 2P_\pi(x_1 \wedge x_2) + P_\pi(x_1 \oplus x_2)$, thus Γ is inconsistent. As any proper subset of Γ is consistent, any Dutch book against an agent whose belief state is represented by Γ must have all gambles with non-zero stakes, $\lambda_i \in \{-1, 1\}$. We are left with 2^4 possible sets of gambles, since Γ has 4 conditionals. Let $\lambda_i \in \{-1, 1\}$ be the stake of a gamble on the i^{th} conditional of Γ, in the order we have presented it (λ_1 refers to a gamble on $(x_1)[0.3]$). For each $\lambda = \langle \lambda_1, \ldots, \lambda_4 \rangle$, we show in the table below a possible world w_j where there is no loss for the agent, in which $\bar{x}_i$ denotes $\neg x_i$.

λ	w_j	λ	w_j
$\langle -1,-1,-1,-1\rangle$	$\bar{x}_1 \wedge \bar{x}_2$	$\langle +1,-1,-1,-1\rangle$	$x_1 \wedge \bar{x}_2$
$\langle -1,-1,-1,+1\rangle$	$\bar{x}_1 \wedge \bar{x}_2$	$\langle +1,-1,-1,+1\rangle$	$x_1 \wedge \bar{x}_2$
$\langle -1,-1,+1,-1\rangle$	$\bar{x}_1 \wedge \bar{x}_2$	$\langle +1,-1,+1,-1\rangle$	$x_1 \wedge x_2$
$\langle -1,-1,+1,+1\rangle$	$\bar{x}_1 \wedge \bar{x}_2$	$\langle +1,-1,+1,+1\rangle$	$x_1 \wedge x_2$
$\langle -1,+1,-1,-1\rangle$	$\bar{x}_1 \wedge \bar{x}_2$	$\langle +1,+1,-1,-1\rangle$	$x_1 \wedge x_2$
$\langle -1,+1,-1,+1\rangle$	$\bar{x}_1 \wedge \bar{x}_2$	$\langle +1,+1,-1,+1\rangle$	$x_1 \wedge x_2$
$\langle -1,+1,+1,-1\rangle$	$x_1 \wedge x_2$	$\langle +1,+1,+1,-1\rangle$	$x_1 \wedge x_2$
$\langle -1,+1,+1,+1\rangle$	$x_1 \wedge x_2$	$\langle +1,+1,+1,+1\rangle$	$x_1 \wedge x_2$

For instance, consider the set of gambles $\langle -1,-1,+1,-1\rangle$ and the possible world $\bar{x}_1 \wedge \bar{x}_2$. In the gamble on (x_1), the agent earns 0.3, on (x_2), she earns 0.3, on $(x_1 \oplus x_2)$, she earns 0.2. The only gamble in which the agent loses is on $(x_1 \wedge x_2)$, but she loses only $0.3 < 0.3 + 0.3 + 0.2$ and has no net loss. $\square$

Theorem 5.4 . *For any* $\Gamma \in \mathbb{K}$, $\mathcal{I}^t_{SSK}(\Gamma) = \mathcal{I}^{\varepsilon}_{\infty}(\Gamma)$.

Proof. We are going to show a linear program that computes $\mathcal{I}^t_{SSK}(\Gamma)$ by maximizing the sure loss via Dutch books when the sum of stakes is limited to 1. Consider a knowledge base $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq m\}$. Let $\underline{\lambda}_i, \bar{\lambda}_i \geq 0$ denote gambles on $(\varphi_i|\psi_i)$, one with $q_i = \underline{q}_i$ and stake $\underline{\lambda}_i \geq 0$, the other one with $q_i = \bar{q}_i$ and stake $-\bar{\lambda}_i \leq 0$. A set of gambles G can then be represented by the vector $\lambda = \langle \underline{\lambda}_1, \bar{\lambda}_1, \ldots, \underline{\lambda}_m, \bar{\lambda}_m \rangle$. Given a gamble on $(\varphi_i|\psi_i)$ with stake $\underline{\lambda}_i$ $(-\bar{\lambda}_i)$, the agent's net profit in a possible word w_j is $\underline{\lambda}_i(I_{w_j}(\varphi_i \wedge \psi_i) - \underline{q}_i I_{w_j}(\psi_i))$ $(-\bar{\lambda}_i(I_{w_j}(\varphi_i \wedge \psi_i) - \bar{q}_i I_{w_j}(\psi_i)))$, in which $I_{w_j} : \mathcal{L}_{X_n} \to \{0,1\}$ is the indicator function of the set $\{\varphi \in \mathcal{L}_{X_n} | w_j \models \varphi\}$ — a valuation. Recall (from (5)–(8)) that $a_{ij} = I_{w_j}(\varphi_i \wedge \psi_i) - \underline{q}_i I_{w_j}(\psi_i)$ and $b_{ij} = I_{w_j}(\varphi_i \wedge \psi_i) - \bar{q}_i I_{w_j}(\psi_i)$. If a given possible world w_j is the case, the set of gambles $\langle \underline{\lambda}_1, \bar{\lambda}_1, \ldots, \underline{\lambda}_m, \bar{\lambda}_m \rangle$ gives the agent a profit of $\sum_{i=1}^m a_{ij}\underline{\lambda}_i + \sum_{i=1}^m -b_{ij}\bar{\lambda}_i$. Let $\ell \geq 0$ be the sure loss a set of gambles yields to the agent, no matter which possible world is the case. Thus, ℓ is such that $\sum_{i=1}^m a_{ij}\underline{\lambda}_i + \sum_{i=1}^m -b_{ij}\bar{\lambda}_i \leq -\ell$ for all possible world w_j. To find the set of gambles $\langle \underline{\lambda}_1, \bar{\lambda}_1, \ldots, \underline{\lambda}_m, \bar{\lambda}_m \rangle$ that maximizes the sure loss is to solve the following linear program:

$$\max \ell \quad \text{subject to:} \tag{29}$$

$$\begin{bmatrix} 1 & a_{11} & \ldots & a_{m1} & -b_{11} & \ldots & -b_{m1} \\ 1 & a_{12} & \ldots & a_{m2} & -b_{12} & \ldots & -b_{m2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{12^n} & \ldots & a_{m2^n} & -b_{12^n} & \ldots & -b_{m2^n} \end{bmatrix} \begin{bmatrix} \ell \\ \underline{\lambda}_1 \\ \vdots \\ \underline{\lambda}_m \\ \bar{\lambda}_1 \\ \vdots \\ \bar{\lambda}_m \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{30}$$

$$\underline{\lambda}_1, \bar{\lambda}_1, \ldots, \underline{\lambda}_m, \bar{\lambda}_m \geq 0 \ . \tag{31}$$

The program above is the dual of that in lines (5)–(8), which checks the consistency of Γ. Note that $B\pi \geq 0$ is equivalent to $-B\pi \leq 0$, $\pi = 1$ in (5)–(8)

can be inserted into A as a line of 1's, and 0 is the function being minimized in (5)–(8). By the duality theory, as the program above is feasible, it is unbounded iff (5)–(8) is unfeasible (for duality theory, see, for instance, [39]). That is, if Γ is inconsistent, sure loss via Dutch book is unlimited . If we add the restriction $\underline{\lambda}_1 + \bar{\lambda}_1 + \cdots + \underline{\lambda}_m + \bar{\lambda}_m \leq 1$ to the system above, we are maximizing sure loss when the sum of stakes is limited to one — that is, computing $\mathcal{I}^t_{SSK}(\Gamma)$. The dual of this new program would become the program (14)–(18), which computes $\mathcal{I}^\varepsilon_\infty(\Gamma)$. So, by the strong duality theorem, $\mathcal{I}^t_{SSK}(\Gamma) = \mathcal{I}^\varepsilon_\infty(\Gamma)$, for both programs are always feasible. $\square$

Lemma 5.5 . *For any* $\Gamma \in \mathbb{K}$*,* $\mathcal{I}_{Staffel}(\Gamma) = \mathcal{I}^\varepsilon_1(\Gamma)$.

Proof. See the proof of Theorem 5.12, since $\mathcal{I}_{Staffel}$ is a particular case of $\mathcal{I}^\delta_{Staffel}$. $\square$

Theorem 5.6 . *For any functions* $\underline{\gamma}, \bar{\gamma} \geq 0$ *and* $p \in \mathbb{N}_{>0} \cup \{\infty\}$*,* $\mathcal{I}^\gamma_p : \mathbb{K} \to [0,\infty) \cup \{\infty\}$ *is well-defined and satisfies consistency, i-independence and monotonicity.* $\mathcal{I}^\gamma_1$ *also satisfies super-additivity and IC-separability.*

Proof. Note that the linear restrictions in the program (19)–(23), when it is feasible, define a convex, closed region of feasible points (a simplex). The p-norm is a continuous function, so the minimum of the objective function in (19) is well-defined for any $p \in \mathbb{N}_{>0} \cup \{\infty\}$. If the program (19)–(23) is infeasible for some $\Gamma \in \mathbb{K}$, $\mathcal{I}^\gamma_p(\Gamma)$ is defined as ∞.

Consistency: Note that a p-norm is never negative. The base Γ is consistent iff the program (5)–(8) is feasible; and such program is feasible iff the program (19)–(23) has a feasible solution with $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle = \langle 0, 0, \ldots, 0 \rangle$; which is the case iff $\|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle\|_p = 0$ is the minimum of the objective function in (19).

Monotonicity: Consider the program $\mathcal{P}$ from lines (19)–(23), corresponding to the computation of $\mathcal{I}^\gamma_p(\Gamma)$, for some $\Gamma \in \mathbb{K}$. Let $\Psi = \Gamma \cup \{\alpha\}$ be a knowledge base. For any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, the program (19)–(23) whose solution gives $\mathcal{I}^\gamma_p(\Psi)$ has two extra constraints in comparison with $\mathcal{P}$. Thus, the program that computes $\mathcal{I}^\gamma_p(\Psi)$ cannot reach a smaller value for $\|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle\|_p$, the objective function being minimized by $\mathcal{P}$. Furthermore, $\|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_{m+1}, \bar{\varepsilon}_{m+1} \rangle\|_p \geq \|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle\|_p$, for any $p \in \mathbb{N}_{>0} \cup \{\infty\}$. Hence, $\mathcal{I}^\gamma_p(\Gamma \cup \{\alpha\}) \geq \mathcal{I}^\gamma_p(\Gamma)$, for any $p \in \mathbb{N}_{>0} \cup \{\infty\}$.

***i*-independence:** Let $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq m\}$ be a knowledge base in $\mathbb{K}$ and $\alpha = (\varphi_m|\psi_m)[\underline{q}_m, \bar{q}_m]$ be an innocuous conditional in Γ, and define $\Psi = \Gamma \setminus \{\alpha\}$. Suppose $\mathcal{I}^\gamma_p(\Psi)$ is finite. The solution on $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_{m-1}, \bar{\varepsilon}_{m-1} \rangle$ to the program (19)–(23) that computes $\mathcal{I}^\gamma_p(\Psi)$ corresponds to a consolidation of Ψ given by $\Psi' = \{(\varphi_i|\psi_i)[\underline{q}_i - \underline{\gamma}_i \underline{\varepsilon}_i, \bar{q}_i + \bar{\gamma}_i \bar{\varepsilon}_i] | 1 \leq i \leq m-1\}$. For α is innocuous in Γ, it is consistent with $\Psi' \cup (\varphi_m|\psi_m)[0,1]$ (a consolidation of Γ) and $\Psi' \cup \{\alpha\}$ is a consolidation of Γ. Hence, $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_{m-1}, \bar{\varepsilon}_{m-1}, 0, 0 \rangle$ correponds to a feasible solution to the program (19)–(23) computing $\mathcal{I}^\gamma_p(\Gamma)$. As $\|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_{m-1}, \bar{\varepsilon}_{m-1} \rangle\|_p$ is equal to $\|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_{m-1}, \bar{\varepsilon}_{m-1}, 0, 0 \rangle\|_p$ for any $p \in \mathbb{N}_{>0} \cup \{\infty\}$, $\mathcal{I}^\gamma_p(\Gamma) \leq \mathcal{I}^\gamma_p(\Psi)$. By monotonicity, $\mathcal{I}^\gamma_p(\Gamma) = \mathcal{I}^\gamma_p(\Psi)$.

Now suppose $\mathcal{I}_p^\gamma(\Psi)$ is infinite. Thus, the program (19)–(23) that computes $\mathcal{I}_p^\gamma(\Psi)$ is infeasible. Constraints in such program are inherited by the program that computes $\mathcal{I}_p^\gamma(\Gamma) = \mathcal{I}_p^\gamma(\Psi \cup \{\alpha\})$ together with the infeasibility, hence $\mathcal{I}_p^\gamma(\Gamma) = \infty$ by definition.

Super-additivity: Suppose there are bases $\Psi, \Delta, \Gamma = \Psi \cup \Delta$ in $\mathbb{K}$ such that $\Psi \cap \Delta = \varnothing$. Without loss of generality, let $\Psi = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq k\}$, $\Delta = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | k+1 \leq i \leq m\}$ and $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq m\}$. If $\mathcal{I}_1^\gamma(\Gamma) = \infty$, super-additivity trivially holds, then consider $\mathcal{I}_1^\gamma(\Gamma)$ is finite. Let $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle$ be part of a solution (that includes π) to the program (19)–(23) that computes $\mathcal{I}_1^\gamma(\Gamma)$, minimizing the objective function. As $\Gamma' = \{(\varphi_i|\psi_i)[\underline{q}_i - \underline{\varepsilon}_i\underline{\gamma}_i, \bar{q}_i + \bar{\varepsilon}_i\bar{\gamma}_i] | 1 \leq i \leq m\}$ is consistent, so are $\Psi' = \{(\varphi_i|\psi_i)[\underline{q}_i - \underline{\varepsilon}_i\underline{\gamma}_i, \bar{q}_i + \bar{\varepsilon}_i\bar{\gamma}_i] | 1 \leq i \leq k\}$ and $\Delta' = \{(\varphi_i|\psi_i)[\underline{q}_i - \underline{\varepsilon}_i\underline{\gamma}_i, \bar{q}_i + \bar{\varepsilon}_i\bar{\gamma}_i] | k+1 \leq i \leq m\}$, which are consolidations of Ψ and Δ. Thus, $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_k, \bar{\varepsilon}_k \rangle$ and $\langle \underline{\varepsilon}_{k+1}, \bar{\varepsilon}_{k+1}, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle$ correspond to feasible solutions to the programs that compute $\mathcal{I}_1^\gamma(\Psi)$ and $\mathcal{I}_1^\gamma(\Delta)$, respectively. It follows that $\mathcal{I}_1^\gamma(\Psi) \leq \|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_k, \bar{\varepsilon}_k \rangle\|_1$ and $\mathcal{I}_1^\gamma(\Delta) \leq \|\langle \underline{\varepsilon}_{k+1}, \bar{\varepsilon}_{k+1}, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle\|_1$. Finally, $\mathcal{I}_1^\gamma(\Delta) + \mathcal{I}_1^\gamma(\Psi) \leq (\sum_{i=1}^k \underline{\varepsilon}_i + \bar{\varepsilon}_i) + (\sum_{i=k+1}^m \underline{\varepsilon}_i + \bar{\varepsilon}_i) = \sum_{i=1}^m \underline{\varepsilon}_i + \bar{\varepsilon}_i = \mathcal{I}_1^\gamma(\Gamma)$.

IC-separability: To prove that IC-separability holds, suppose there are bases $\Psi, \Delta, \Gamma = \Psi \cup \Delta$ in $\mathbb{K}$ such that $\Psi \cap \Delta = \varnothing$, $IC(\Gamma) = IC(\Psi) \cup IC(\Delta)$. Without loss of generality, let $\Psi = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq k\}$, $\Delta = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | k+1 \leq i \leq m\}$ and $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq m\}$. If $\mathcal{I}_1^\gamma(\Psi) = \infty$ or $\mathcal{I}_1^\gamma(\Delta) = \infty$, then $\mathcal{I}_1^\gamma(\Gamma) = \infty$ by monotonicity, and IC-separability holds, considering that ∞ plus any non-negative number yields ∞; thus, we assume $\mathcal{I}_1^\gamma(\Psi), \mathcal{I}_1^\gamma(\Delta) < \infty$. Let $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_k, \bar{\varepsilon}_k \rangle$ and $\langle \underline{\varepsilon}_{k+1}, \bar{\varepsilon}_{k+1}, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle$ be solutions (on $\underline{\varepsilon}, \bar{\epsilon}$) to the programs in the form (19)–(23) that compute $\mathcal{I}_1^\gamma(\Psi)$ and $\mathcal{I}_1^\gamma(\Delta)$, respectively, minimizing their objective functions. As all inescapable conflicts of Γ are either in Ψ or in Δ, the union of consolidations of Ψ and Δ is a consolidation of Γ, by Lemma 4.13. Hence, $\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle$ correspond to a feasible solution to the program in the form (19)–(23) that computes $\mathcal{I}_1^\gamma(\Gamma)$ and $\mathcal{I}_1^\gamma(\Gamma) \leq \|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle\|_1 = (\sum_{i=1}^k \underline{\varepsilon}_i + \bar{\varepsilon}_i) + (\sum_{i=k+1}^m \underline{\varepsilon}_i + \bar{\varepsilon}_i) = \mathcal{I}_1^\gamma(\Psi) + \mathcal{I}_1^\gamma(\Delta)$. By super-additivity, $\mathcal{I}_1^\gamma(\Gamma) = \mathcal{I}_1^\gamma(\Psi) + \mathcal{I}_1^\gamma(\Delta)$ □

Lemma 5.7 . *If $\underline{\gamma}$ and $\bar{\gamma}$ are such that, for any conditional $\alpha = (\varphi|\psi)[\underline{q}, \bar{q}]$, $\underline{\gamma}(\alpha) = 0$ implies $\underline{q} = 0$ and $\bar{\gamma}(\alpha) = 0$ implies $\bar{q} = 1$, then $\mathcal{I}_p^\gamma(\Gamma)$ is finite for any $\Gamma \in \mathbb{K}$.*

Proof. Consider the knowledge base $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq m\}$ and the vector $q = \langle \underline{q}_1, \bar{q}_1, \ldots, \underline{q}_m, \bar{q}_m \rangle$. Suppose $\gamma(\Gamma) > 0$. Consider the restrictions $P_\pi(\varphi_i \wedge \psi_i) - \underline{q}_i P_\pi(\psi_i) \geq -\underline{\varepsilon}_i\underline{\gamma}_i$ and $P_\pi(\varphi_i \wedge \psi_i) - \bar{q}_i P_\pi(\psi_i) \leq \bar{\varepsilon}_i\bar{\gamma}_i$ in the linear program (19)–(23) that computes $\mathcal{I}_p^\gamma(\Gamma)$. As both left-hand sides have values in $[-1, 1]$ and $\underline{\gamma}_i, \bar{\gamma}_i > 0$, taking $\underline{\varepsilon}_i > 1/\underline{\gamma}_i$ and $\underline{\varepsilon}_i > \bar{\gamma}_i$ for $1 \leq i \leq m$ relaxes all such constraints. Thus, with any probability mass π we have a feasible solution with the (finite) value for the objective function $\|\langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle\|_p$, and $\mathcal{I}_p^\gamma(\Gamma)$ is finite for any $p \in \mathbb{N}_{>0} \cup \{\infty\}$.

Now suppose $\gamma(\Gamma)$ contains zeros. For any conditional $\alpha = (\varphi|\psi)[\underline{q}, \bar{q}] \in \Gamma$, $\underline{\gamma}(\alpha) = 0$ implies $\underline{q} = 0$ and $\bar{\gamma}(\alpha) = 0$ implies $\bar{q} = 1$. Hence, the linear inequalities corresponding to the probability bounds $\underline{q}_i = 0, \bar{q}_j = 1$ with $\underline{\gamma}_i = \bar{\gamma}_j = 0$ are trivially satisfied, and can be ignored. The remaining constraints can be relaxed as explained above, linear program becomes feasible, and $\mathcal{I}^{\gamma}_{p}(\Gamma)$ is finite. □

Lemma 5.8 . *Let $\gamma \circ \Lambda_\Gamma : [0,1]^{2m} \to [0,1]^{2m}$ be continuous for all $\Gamma \in \mathbb{K}$ and consider a $p \in \mathbb{N}_{>0} \cup \{\infty\}$. If $\gamma \circ \Lambda_\Gamma : [0,1]^{2m} \to [0,1]^{2m}$ is positive for all $\Gamma \in \mathbb{K}$, then $\mathcal{I}^{\gamma}_{p}$ satisfies continuity; if $\gamma \circ \Lambda_\Gamma : (0,1)^{2m} \to [0,1]^{2m}$ is positive for all $\Gamma \in \mathbb{K}$, then $\mathcal{I}^{\gamma}_{p} \circ \Lambda_\Gamma(q)$ is continuous for $q \in (0,1)^{2m}$.*

Proof. Consider the knowledge base $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}'_i, \bar{q}'_i]|1 \leq i \leq m\}$ and a vector $q = \langle \underline{q}_1, \bar{q}_1, \ldots, \underline{q}_m, \bar{q}_m \rangle$. Every probability mass $\pi : W_{X_n} \to [0,1]$ defines a vector $\varepsilon_\pi(q) = \langle \underline{\varepsilon}_1, \bar{\varepsilon}_1, \ldots, \underline{\varepsilon}_m, \bar{\varepsilon}_m \rangle$ for each q in the following way: $\underline{\varepsilon}_i = -\min\{0, (1/\underline{\gamma}_i)(P_\pi(\varphi_i \wedge \psi_i) - \underline{q}_i P_\pi(\psi_i))\}$ and $\bar{\varepsilon}_i = \max\{0, (1/\bar{\gamma}_i)(P_\pi(\varphi_i \wedge \psi_i) - \underline{q}_i P_\pi(\psi_i))\}$ for every $1 \leq i \leq m$ and $q \in [0,1]^{2m}$ (or $q \in (0,1)^{2m}$). Note that the pair $\pi, \varepsilon_\pi(q)$ is a feasible solution to the program (19)–(23) that computes $\mathcal{I}^{\gamma}_{p}(\Lambda_\Gamma(q))$ for any $q \in [0,1]^{2m}$ (or $q \in (0,1)^{2m}$), since $P_\pi(\varphi_i \wedge \psi_i) - \underline{q}_i P_\pi(\psi_i) \geq -\underline{\varepsilon}_i \underline{\gamma}_i$ and $P_\pi(\varphi_i \wedge \psi_i) - \bar{q}_i P_\pi(\psi_i) \leq \bar{\varepsilon}_i \bar{\gamma}_i$ for all $1 \leq i \leq m$. Thus, every π yields a value for the objective function $h_\pi(q) = \|\varepsilon_\pi(q)\|$ of the program (19)–(23), for any $q \in [0,1]^{2m}$ (or $q \in (0,1)^{2m}$). As $\gamma \circ \Lambda_\Gamma : [0,1]^{2m} \to [0,1]^{2m}$ ($\gamma \circ \Lambda_\Gamma : (0,1)^{2m} \to [0,1]^{2m}$) is continuous and positive, $\varepsilon_\pi(q)$ is also continuous on $q \in [0,1]^{2m}$ (or $q \in [0,1]^{2m}$), and as any p-norm is a continuous function, $h_\pi : [0,1]^{2m} \to [0,\infty)$ ($h_\pi : (0,1)^{2m} \to [0,\infty)$) also is for any π. To compute $\mathcal{I}^{\gamma}_{p}(\Lambda_\Gamma(q))$ for a particular q, one needs to take the minimum in π of $\{h_\pi(q)|\pi : W_{X_n} \to [0,1]$ is a probability mass$\}$. As the minimum of continuous functions is continuous, $\mathcal{I}^{\gamma}_{p} \circ \Lambda_\Gamma : [0,1]^{2m} \to [0,\infty) \cup \{\infty\}$ (or $\mathcal{I}^{\gamma}_{p} \circ \Lambda_\Gamma : (0,1)^{2m} \to [0,\infty) \cup \{\infty\}$) is continuous for any $p \in \mathbb{N}_{>0} \cup \{\infty\}$.

□

Lemma 5.9 . *If $\underline{\gamma}$ and $\bar{\gamma}$ are such that, for any conditional $\alpha = (\varphi|\psi)[\underline{q}, \bar{q}]$, $\underline{\gamma}(\alpha) \geq \underline{q}$ and $\bar{\gamma}(\alpha) \geq 1 - \bar{q}$, then $\mathcal{I}^{\gamma}_{\infty}$ satisfies normalization.*

Proof. Note that when $\underline{\gamma}(\alpha) = \underline{q}$ and $\bar{\gamma}(\alpha) = 1 - \bar{q}$, we are limiting the agent's escrows to one, computing $\mathcal{I}^{a}_{SSK}$ by Theorem 5.10. As the agent cannot lose more her total escrow in a Dutch book, $\mathcal{I}^{a}_{SSK}$ is normalized. But when $\underline{\gamma}(\alpha) \geq \underline{q}$ and $\bar{\gamma}(\alpha) \geq 1 - \bar{q}$, we are strengthening the restriction over the stakes in the program that maximizes sure loss (see Theorem 5.10), and it cannot have a higher maximum. Formally, when $\underline{\gamma}(\alpha) \geq \underline{q}$ and $\bar{\gamma}(\alpha) \geq 1 - \bar{q}$, $\sum_{i=1}^{m} \underline{\gamma}_i \underline{\lambda}_i + \bar{\gamma}_i \bar{\lambda}_i \leq 1$ implies $\sum_{i=1}^{m} \underline{q}_i \underline{\lambda}_i + (1 - \bar{q}_i)\bar{\lambda}_i \leq 1$, thus $\mathcal{I}^{\gamma}_{\infty} \leq \mathcal{I}^{a}_{SSK}$ in this case, and both are normalized. □

Theorem 5.10 . *If $\underline{\gamma}$ and $\bar{\gamma}$ are such that, for any conditional $\alpha = (\varphi|\psi)[\underline{q}, \bar{q}]$, $\underline{\gamma}(\alpha) = \underline{q}$ and $\bar{\gamma}(\alpha) = 1 - \bar{q}$, then $\mathcal{I}^{a}_{SSK}(\Gamma) = \mathcal{I}^{\gamma}_{\infty}(\Gamma)$ for any $\Gamma \in \mathbb{K}$ and both are finite.*

Proof. Consider a knowledge base $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i]|1 \leq i \leq m\}$. To compute $\mathcal{I}^{a}_{SSK}(\Gamma)$, we need to maximize sure loss with the agent's resources limited up to one. Remember that the agent's escrow is how much she can lose in a gamble. For gambles with non-negative stake, represented by $\underline{\lambda}_i$, the agent's escrow is $\underline{q}_i\underline{\lambda}_i$; for gambles with negative stake $-\bar{\lambda}_i < 0$, her escrow is $(1-\bar{q}_i)\bar{\lambda}_i$. Consider the linear program (29)–(31) that maximizes sure loss via Dutch books when the agent's beliefs are represented by Γ. Instead of adding the constraint $\underline{\lambda}_1 + \bar{\lambda}_1 + \cdots + \underline{\lambda}_m + \bar{\lambda}_m \leq 1$ to limit the stakes sum, we limit the agent's total escrow with the restriction $\underline{q}_1\underline{\lambda}_1 + (1-\bar{q}_1)\bar{\lambda}_1 + \cdots + \underline{q}_m\underline{\lambda}_m + (1-\bar{q}_m)\bar{\lambda}_m \leq 1$. Now, as $\underline{\gamma}_i = \underline{q}_i$ and $\bar{\gamma}_i = 1 - \bar{q}_i$ for all $1 \leq i \leq m$, such constraint is equal to $\sum_{i=1}^m \underline{\gamma}_i\underline{\lambda}_i + \bar{\gamma}_i\bar{\lambda}_i \leq 1$. Taking the dual of this linear program we recover exactly the program (24)–(28). Note that the sure loss cannot be greater than the agent's total escrow, thus it is always finite in this setting. Hence, by the strong duality theorem, $\mathcal{I}^a_{SSK}(\Gamma) = \mathcal{I}^\gamma_\infty(\Gamma)$ and both are finite. $\square$

Theorem 5.11 . *If $\underline{\gamma}$ and $\bar{\gamma}$ are such that, for any conditional $\alpha = (\varphi|\psi)[\underline{q}, \bar{q}]$, $\underline{\gamma}(\alpha) = 1 - \underline{q}$ and $\bar{\gamma}(\alpha) = \bar{q}$, then $\mathcal{I}^g_{SSK}(\Gamma) = \mathcal{I}^\gamma_\infty(\Gamma)$ for any $\Gamma \in \mathbb{K}$.*

Proof. Consider a knowledge base $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq m\}$. We need to add a restriction to the program (29)–(31) (which maximizes sure loss) in order to limit the gambler's total escrow up to one. For gambles with non-negative stake, represented by $\underline{\lambda}_i$, the gambler's escrow is $(1-\underline{q}_i)\underline{\lambda}_i$; for gambles with negative stake $-\bar{\lambda}_i$, his escrow is $\bar{q}_i\bar{\lambda}_i$. With $\underline{\gamma}_i = \underline{q}_i$ and $\bar{\gamma}_i = 1 - \bar{q}_i$ for all $1 \leq i \leq m$, such constraint is equal to $\sum_{i=1}^m \underline{\gamma}_i\underline{\lambda}_i + \bar{\gamma}_i\bar{\lambda}_i \leq 1$. Once again, the dual of this linear program is the program (24)–(28). By the strong duality theorem, $\mathcal{I}^g_{SSK}(\Gamma) = \mathcal{I}^\gamma_\infty(\Gamma)$ if both are finite. When $\mathcal{I}^g_{SSK}(\Gamma)$ is unbounded, the program (24)–(28) is infeasible, and, by our definition, $\mathcal{I}^\gamma_\infty(\Gamma) = \infty = \mathcal{I}^g_{SSK}(\Gamma)$. $\square$

Theorem 5.12 . *For any $\Gamma = \{(\varphi_i|\psi_i)[\underline{q}_i, \bar{q}_i] | 1 \leq i \leq m\}$ in $\mathbb{K}$, if $\underline{\gamma}_i = 1/\underline{\delta}_i$ and $\bar{\gamma}_i = 1/\bar{\delta}_i$ for all $1 \leq i \leq m$, then $\mathcal{I}^\delta_{Staffel}(\Gamma) = \mathcal{I}^\gamma_1(\Gamma)$ and both are finite.*

Proof. Consider the linear program (29)–(31) that maximizes sure loss via Dutch books when the agent's beliefs are represented by Γ. We can limit the stakes $\underline{\lambda}_1, \bar{\lambda}_1, \ldots, \underline{\lambda}_m, \bar{\lambda}_m$ through restrictions $\underline{\lambda}_i/\underline{\delta}_i \leq 1$ and $\bar{\lambda}_i/\bar{\delta}_i \leq 1$ in order to compute $\mathcal{I}^\delta_{Staffel}$. But then the resulting program would be exactly the dual of that in lines (19)–(23) when $\underline{\gamma}_i = 1/\underline{\delta}_i$ and $\bar{\gamma}_i = 1/\bar{\delta}_i$ for all $1 \leq i \leq m$ and $p = 1$. By the strong duality theorem, $\mathcal{I}^\delta_{Staffel}(\Gamma) = \mathcal{I}^\gamma_1(\Gamma) < \infty$ in such case, as both programs are feasible. $\square$