The centroid in absolute geometry

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Abstract

We give a synthetic proof in absolute geometry (Birkhoff axioms without the parallel postulate) that given any triangle, its medians are concurrent. This means that the same proof is valid in both euclidean and hyperbolic geometry. We also indicate how to generalize this result.

1 Introduction

It is true both in (real) euclidean and hyperbolic geometry that the medians of a triangle are concurrent (and this point of concurrence is called the centroid of the triangle). The proof in the euclidean case uses similarity of triangles and in the hyperbolic case one uses the Klein model inside the projective plane (see [1, pp. 30-31 and 229-230], and [5, Exercise 105]). These two geometries differ only by the parallel axiom, so we can conclude that this result is a theorem of pure absolute geometry (Birkhoff axioms without the parallel postulate; the interested reader can provide a justification of this claim).

Here we present a proof of this fact using only the axioms of pure absolute geometry. The idea is simple. We project a triangle onto an isosceles one, for which it is easy to prove the result and then go back to the original triangle. We prove the necessary results on the needed projection in the same system by proving a particular configuration of Desargues' theorem. This is done in section 2. In section 3 we prove the main result. In the conclusion we indicate how to extend the result.

2 Axioms and basic results on absolute space

The axioms adopted are a weaker form of Birkhoff's axioms (as adopted in [3] and [4]). We summarize them here.

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Axiom 1 (Incidence) For each pair of distinct points there is a unique line containing them; for each set of three distinct points, no two of which lie in the same line, there is a unique plane containing them; for each pair of distinct planes, if they intersect, then the intersection is a line; every plane contain three points, not two of which lie in the same line; there are four points, no three of which in the same plane.

The following axiom is a weaker version of the ruler axiom in which the field K of the statement is the field of real numbers.

Axiom 2 (Coordinatization) There is an ordered field K which coordinatize all the lines, that is, for each line, there is a (chosen) bijection between the set of points in the line and the elements of the field K. The element of K corresponding to the point P is called the coordinate of P.

The order or betweenness relation is derived from the field K. We say that the point B, is in between A and C and write $A - B - C$ if they belong to the same line and the coordinate of B is in between the coordinates of A and C in the order of K. Given distinct points A and B, the unique line which contains them is denoted by $\overleftrightarrow{AB}$ and the ray with vertex A and points in the direction of B (that is, P such that $P \in \overleftrightarrow{AB}$ and not $P - A - B$) is denoted by $\overrightarrow{AB}$ and the segment $\overline{AB}$ is the set of points in the line $\overleftrightarrow{AB}$ containing A and B and the points P such that $A - P - B$.

Also the segment congruence is derived from K and $\equiv$ denotes the segment congruence relation.

Axiom 3 (Pasch's axiom) Given any plane π and points A, B and C in π such that A, B and C are not in the same line, and a line ℓ in π such that ℓ intersects the segment $\overline{BC}$, then ℓ intersects either the segment $\overline{AB}$ or $\overline{AC}$.

From the previous axioms we can deduce the plane and the space separation axioms of [4, chapter 4], as the readers can verify by themselves.

An angle is the set of points $\angle AOB = \overrightarrow{OA} \cup \overrightarrow{OB}$, where A, B and O are not in the same line. The triangle $\triangle ABC$ is the set $\overline{AB} \cap \overline{AC} \cap \overline{BC}$ and in this case we assume that A, B and C are not in the same line.

Axiom 4 (Angle congruence) Given the angle $\angle AOB$, a plane π, a ray $\overrightarrow{PQ} \subset \pi$ the there are points R and S in oposite sides of the line $\overleftrightarrow{PQ}$ in π such that $\angle QPR \equiv \angle AOB \equiv \angle QPS$ and the rays $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are unique with respect to this property; $\equiv$ is an equivalence relation in the set of angles.

We denote the congruence of triangles by $\equiv$ and when we write $\triangle ABC \equiv \triangle DEF$, the congruence is given by the correspondence $A \mapsto D$, $B \mapsto E$ and $C \mapsto F$.

Axiom 5 (Side-Angle-Side: SAS) Given the triangles $\triangle ABC$ and $\triangle DEF$, if $\overline{AB} \equiv \overline{DE}$, $\overline{AC} \equiv \overline{DF}$ and $\angle BAC \equiv \angle EDF$, then $\triangle ABC \equiv \triangle DEF$.

We recall that we can deduce all the other congruence criteria (namely ASA, SSS and SAA, where S stands for side and A for angle; see [3, theorems 6.2.1, 6.2.3 and 6.3.5] or [4, section 6.2]).

We say that a line ℓ is perpendicular to the plane π if $\ell \cap \pi$ contains only a point P and all lines in π containing P are perpendicular to ℓ.

The following result capture the absolute content of Euclid's Elements [2, Book XI, propositions 4, 5, 6 and 8].

Lemma 1 *Let π_1 and π_2 be two distinct planes meeting in the line ℓ. If there is a line $\ell' \subset \pi_1$ perpendicular to π_2, then all lines in π_1 perpendicular to the line ℓ are perpendicular to π_2.*

Proof: We need only to show that if $\ell_1 \subset \pi_1$ is a line perpendicular to ℓ then the line $\ell_2 \subset \pi_2$ perpendicular to ℓ at the point Q in $\ell \cap \ell_1$ is perpendicular to ℓ_1. We refer to figure 1 for a diagram of this proof.

We may suppose that $\ell' \neq \ell_1$. Let P be the meeting point of ℓ and ℓ'. Choose points $A \in \ell'$ and $B \in \ell_1$ in the same side of π_1 with respect to ℓ and such that $\overline{PA} \equiv \overline{QB}$. Let $\ell_2, \ell_3 \subset \pi_2$ be perpendicular to ℓ and such that $P \in \ell_2$ and $Q \in \ell_3$. Then $\ell' \perp \ell_3$ because we have assumed that $\ell' \perp \pi_2$. Choose points $C \in \ell_2$ and $D \in \ell_3$ in the same side of π_2 with respect to the line ℓ and such that $\overline{CP} \equiv \overline{DQ} \equiv \overline{PA}$. By Pasch's axiom (or rather by separation in each plane), the segments $\overline{PB}$ and $\overline{AQ}$ meet at a point E and the segments $\overline{PD}$ and $\overline{QC}$ meet at a point F. By SAS, $\triangle APQ \equiv \triangle BQP \equiv \triangle CPQ \equiv \triangle DQP$. Therefore, again by SAS, $\triangle PEQ \equiv \triangle QEP \equiv \triangle PFQ \equiv \triangle QFP$. Consequently, by SSS, $\triangle PEF \equiv \triangle QEF$. So, by SAS, $\triangle QAC \equiv \triangle PBD$. Finally, by SSS, $\triangle PAC \equiv \triangle QBD$. This means that $\angle QBD$ is a right angle, that is, $\ell_2 \perp \ell_1$, as required. □

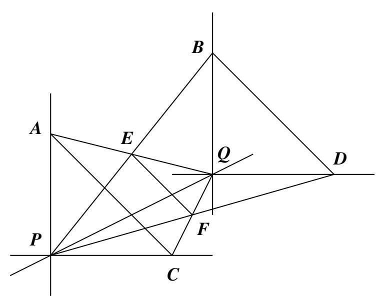

Figure 1: Diagram for the proof of Lemma 1.

In the case of this lemma we say that the planes π_1 and π_2 are perpendicular and denote $\pi_1 \perp \pi_2$.

3 The centroid

The next result is proved by using the trick of making a controlled central projection from an arbitrary triangle to an isosceles one in which it is easy to prove the desired result and then transferring it back to the original triangle.

Theorem 3.1 *The medians of a triangle $\triangle ABC$ are colinear.*

Proof: Let $L \in \overline{AB}$, $M \in \overline{AC}$ and $N \in \overline{BC}$ be the midpoints of these intervals. We refer to figure 2 for a diagram of this proof.

Firstly, following [5, exercise 105], we show that the perpendicular bissector of $\overline{BC}$ is perpendicular to the line $\overleftrightarrow{LM}$. For this, let $P, Q, R \in \overleftrightarrow{LM}$ be the feet of the perpendiculars from A, B and C, that is, $\overleftrightarrow{AP}, \overleftrightarrow{BQ}, \overleftrightarrow{CR} \perp \overleftrightarrow{LM}$. Then, using the SAA congruence theorem [3, theorem 6.2.1], we have $\triangle APL \equiv \triangle BQL$ and $\triangle BQM \equiv \triangle CRM$. From this we conclude that $\overline{AP} \equiv \overline{BQ} \equiv \overline{CR}$. Now, by the congruence of convenient triangles in the quadrilateral $\square PACR$, the line joining the midpoints of $\overline{BC}$ and $\overline{PR}$ is a common perpendicular to both segments.

Now let π_1 be the plane containing the triangle $\triangle ABC$, and π_2 be another plane intersecting π_1 in the line $\overleftrightarrow{BC}$. Let $A' \in \pi_2$ be such that $\overleftrightarrow{NA'}$ is the perpendicular bissector of $\overline{BC}$ in π_2 and let O be any point in $\overleftrightarrow{AA'}$ such that $O - A - A'$. Let $L' \in \overline{BA'}$ and $M' \in \overline{CA'}$ be such that $O - L - L'$ and $O - M - M'$. These points exist because of the Pasch axiom. The planes π_1 and π_2 contain the line $\overleftrightarrow{BC}$ which is perpendicular to the plane π_3 determined by the points A, N and A'. Let π_4 be the plane determined by O, L and M. Then, by the Lemma 1, $\pi_1 \perp \pi_4 \perp \pi_2$. This means that the line $\overleftrightarrow{NA'}$ is perpendicular to both $\overleftrightarrow{BC}$ and $\overleftrightarrow{L'M'}$. By congruence of the relevant triangles, we conclude that the segments $\overline{BM'}$ and $\overline{CL'}$ meet in a point $T' \in \overline{NA'}$. But the segment $\overline{CL'}$ is in the plane π_5 determined by the points O, C and L and the segment $\overline{BM'}$ is in the plane π_6 determined by the points O, B and M. Let π_7 be the plane determined by O, A and N. Then $A' \in \pi_7$. Therefore, the line $\overleftrightarrow{OT'}$ is in π_5, π_6 and π_7. So, if T is the point of $\overleftrightarrow{OT'}$ in π_1, then T is the meeting point of the medians of $\triangle ABC$, as required. $\square$

4 Conclusion

The trick of projecting the triangle $\triangle ABC$ onto an isosceles triangle $\triangle DEF$, proving the needed (incidence) result for $\triangle DEF$ and then pulling back the constructions to $\triangle ABC$ can be useful in the proof of various results in absolute geometry. For instance, the centroid theorem we proved here can be generalized to the appropriate versions of Ceva's and Menelau's Theorems. Actually these theorems are a particular case of the construction of harmonic conjugates in projective geometry. See [1, pp. 30-31] on how to do this. We can use these translations to make the needed constructions, prove that they hold for more convenient triangles and then pull them back in a controlled way to the original triangle.

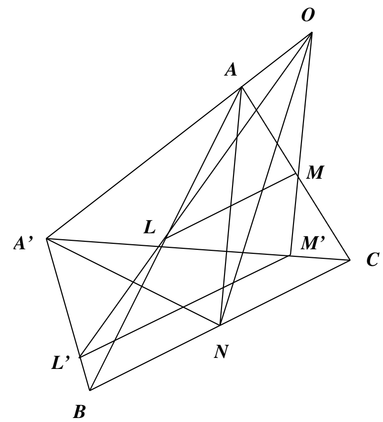

Figure 2: Diagram for the proof of Theorem 3.1.

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