REMARKS ON THE P-LAPLACIAN ON THIN DOMAINS

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ABSTRACT. The limiting behavior of solutions of quasilinear elliptic equations on thin domains is investigated. As we will see the boundary conditions play an important role. If one considers homogeneous Dirichlet boundary conditions the sequence of solutions will converge to the null function, whereas, if one considers Neumann boundary conditions there is a non trivial equation which determines the limiting behavior.

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To Djairo G. de Figueiredo, on the occasion of his 80th birthday

1. INTRODUCTION

Over the last years partial differential equations on thin domains have received considerable attention in the literature of pure and applied mathematics. They occur in applications as in mechanics of nano structures (thin rods, plates or shells), fluids in thin channels (lubrication models, blood circulation), chemical diffusion process on membranes or narrow strips (catalytic process), homogenization of reticulated structures, as in the study of the stability (or instability) of the asymptotic dynamics of singularly perturbed parabolic equations, see for instance [1, 2, 3, 4, 8, 11, 14, 15, 17, 18, 19, 20, 21, 22, 23, 25].

In all of these problems the aim is to describe the effective behavior of perturbed elements. Generally this is made establishing a formal limit in order to compare them, since, in many cases, the limit is simpler to study than the perturbed one. Indeed, this is our goal here, establish the limiting regime for the family of solutions of the elliptic equation

$$-\Delta_p u + |u|^{p-2}u = f(u),$$

posed on thin domains of the form

$$\Omega_\epsilon = \{(x, \epsilon y) : (x, y) \in \Omega\},$$

where $\Omega \subset \mathbb{R}^{m+n}$ is a smooth bounded domain and $\epsilon > 0$ is a small parameter. $\Delta_p u := \text{div}(\|\nabla u\|^{p-2}\nabla u)$ denotes the p-Laplacian operator, $p > 1$ and $f : \mathbb{R} \to \mathbb{R}$ is a suitable nonlinearity which will be specified later. We consider (1.1) coupled by Dirichlet

$$u = 0, \quad \text{on} \ \partial \Omega^\epsilon,$$

or Neumann boundary conditions

$$\frac{\partial u}{\partial \eta^\epsilon} = 0, \quad \text{on} \ \partial \Omega^\epsilon,$$

where $\eta^\epsilon$ denotes the outward unitary normal vector field to $\partial \Omega^\epsilon$.

If Dirichlet boundary condition is considered, we shown (Theorem 2.5) that solutions of the perturbed problem (1.1)–(1.2) converge to the null function as $\epsilon \to 0$. On the other hand, assuming Neumann boundary...
conditions we obtain a nontrivial limiting equation (see (3.9)) posed in a lower dimensional domain which determines the behavior of the solutions of the problem (1.1) and (1.3) as $\epsilon \to 0$.

We notice that if one considers quasilinear parabolic equations of the form
\[ u_t - \Delta_p u + |u|^{p-2} u = f(u), \quad [0, \infty) \times \Omega^\epsilon, \]
which are relevant in a variety of physical phenomena (see [7, 10, 13, 27]), the solutions of (1.1) are the steady states of (1.4). If $E_\epsilon$ is the set of solutions of (1.1) and (1.3) and $E_0$ the solutions of (3.9), the results discussed here say respect to the upper semicontinuity of $E_\epsilon$ at $\epsilon = 0$, i.e., with some abuse of notation,
\[ \sup_{u_\epsilon \in E_\epsilon} \inf_{u_0 \in E_0} \| u_\epsilon - u_0 \|_{W^{1,p}(\Omega_\epsilon)} \to 0. \]
Such result is the first step in order to prove the stability of the asymptotic dynamics (attractors) of (1.4). For recent works on this topic we indicate [5, 6, 26].

Finally, we would like to express our gratitude to Prof. Djairo G. de Figueiredo who opened, and still opens paths through the field of PDE’s for generations of mathematicians in Brazil. Besides, the way he works as professor and scientist is an inspiration for us. Speaking on inspiration, let us mention that Djairo’s work [12] gave us the idea of write down this paper which generalizes to the $p$-Laplacian operator some results obtained by Hale and Raugel in the seminal work [14] in the case $p = 2$.

The paper is organized as follows: in Section 2 we study the problem (1.1) with Dirichlet boundary conditions and in Section 3 the case of Neumann boundary conditions. In Section 4 we present as example a particular class of thin domains defined as graphic of smooth functions.

2. DIRICHLET BOUNDARY CONDITION

Let $\Omega$ be a smooth and bounded domain in $\mathbb{R}^m \times \mathbb{R}^n$ not necessary a product domain. As usual, we identify $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$ writing $(x, y)$ for a generic point of $\mathbb{R}^{m+n}$ where $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$ and $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$. We also write $||(x, y)|| = (|x|^2 + |y|^2)^{1/2}$ for the euclidian norm in the space $\mathbb{R}^{m+n}$ where $|| \cdot ||$ denotes indistinctly the euclidian norm in $\mathbb{R}^m$ or $\mathbb{R}^n$, and we adopt standard notation for the inner product $(x, y) \cdot (w, z) = x \cdot w + y \cdot z$ for all $(x, y), (w, z) \in \mathbb{R}^{m+n}$. For a function $u \in W^{1,p}(\Omega)$, we denote by $\nabla u = (\nabla_x u, \nabla_y u) \in L^p(\Omega)^{m+n}$ the (distributional) gradient of $u$. We will endow $W^{1,p}(\Omega)$ with the equivalent norm
\[ \|u\|_{W^{1,p}(\Omega)} = \left[ \int_\Omega \left( \|\nabla_x u\|^p + \|\nabla_y u\|^p + |u|^p \right) dx dy \right]^{1/p} \]
which still preserves the uniformly convexity of $W^{1,p}(\Omega)$ (19). Along the paper $\epsilon$ will represent a small positive parameter which will converges to zero.

2.1. The thin domain problem. Considering the squeezing operator $\Phi_\epsilon : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ defined by $\Phi_\epsilon(x, y) = (x, \epsilon y)$, we set up the thin domain $\Omega^\epsilon := \Phi_\epsilon(\Omega)$ and we analyze convergence of the solutions of the family of elliptic equations
\[ -\Delta_p u + |u|^{p-2} u = h^\epsilon, \quad \text{in } \Omega^\epsilon, \]
\[ u = 0, \quad \text{on } \partial \Omega^\epsilon, \]
where
\[ \Delta_p u := \text{div}(\|\nabla u\|^{p-2}\nabla u) = \sum_{i=1}^m \frac{\partial}{\partial x_i} \left[ \left( \|\nabla_x u\|^2 + \|\nabla_y u\|^2 \right)^{(p-2)/2} \frac{\partial u}{\partial x_i} \right] + \sum_{i=1}^n \frac{\partial}{\partial y_i} \left[ \left( \|\nabla_x u\|^2 + \|\nabla_y u\|^2 \right)^{(p-2)/2} \frac{\partial u}{\partial y_i} \right] \]
denotes the $p$-Laplacian operator, $p > 1$, and $h^\epsilon \in L^{p'}(\Omega^\epsilon)$, $1/p + 1/p' = 1$.

**Definition 2.1.** Given $h^\epsilon \in L^{p'}(\Omega^\epsilon)$, we say that $u \in W^{1,p}(\Omega^\epsilon)$ is solution of the problem (2.1) if

\[
\int_{\Omega^\epsilon} \left( \|\nabla u\|^{p-2} \nabla u \cdot \nabla \varphi + |u|^{p-2} u \varphi \right) dx dy = \int_{\Omega^\epsilon} h^\epsilon \varphi dx dy,
\]

for all $\varphi \in W_0^{1,p}(\Omega^\epsilon)$.

It is well known (see e.g. [16]) that for each value of $\epsilon > 0$, the p-Laplacian operator can be seen as

\[-\Delta_p : W^{1,p}(\Omega^\epsilon) \rightarrow W^{-1,p'}(\Omega^\epsilon)\]

where $\langle \cdot, \cdot \rangle_{W^{-1,p'},W^{1,p}}$ denotes the pair of duality between $W^{-1,p'}(\Omega^\epsilon)$ and $W^{1,p}(\Omega^\epsilon)$. With $-\Delta_p$ defined above, $-\Delta_p u + |u|^{p-2} u = J_\psi u$, where $J_\psi$ is the duality mapping corresponding to the normalization function $\psi(t) = t^{p-1}$ on the space $W^{1,p}_0(\Omega^\epsilon)$. We know from [16] that $J_\psi$ is single valued and therefore, for each value of the parameter $\epsilon > 0$, the problem (2.1) has an unique solution $\bar{u}^\epsilon \in W^{1,p}_0(\Omega^\epsilon)$. Moreover, since the functional $\phi : W^{1,p}_0(\Omega^\epsilon) \rightarrow \mathbb{R}$ defined by

\[
\phi(u) = \int_0^{|u|^{1,p}(\Omega^\epsilon)} \psi(t) dt - \int_{\Omega^\epsilon} h^\epsilon u dx dy = \frac{1}{p} \|u\|_{W^{1,p}_0(\Omega^\epsilon)}^p - \int_{\Omega^\epsilon} h^\epsilon u dx dy,
\]

is Gâteaux differentiable and

\[
\langle \phi'(u), v \rangle_{W^{1,p}_0, W^{-1,p'}} = \langle -J_\psi u, v \rangle_{W^{-1,p'}, W^{1,p}} - \int_{\Omega^\epsilon} h^\epsilon v dx dy,
\]

we have that $\bar{u}^\epsilon$ is solution of the problem (2.1) if and only if $\phi'(\bar{u}^\epsilon) = 0$. Since $\phi$ is convex and $W^{1,p}_0(\Omega^\epsilon)$ is reflexive, we have the following characterization: $\bar{u}^\epsilon \in W^{1,p}_0(\Omega^\epsilon)$ is solution of (2.1) if and only if

\[
\frac{1}{p} \|\bar{u}^\epsilon\|_{W^{1,p}_0(\Omega^\epsilon)}^p - \int_{\Omega^\epsilon} h^\epsilon \bar{u}^\epsilon dx dy = \min_{\varphi \in W^{1,p}_0(\Omega^\epsilon)} \left\{ \frac{1}{p} \|\varphi\|_{W^{1,p}_0(\Omega^\epsilon)}^p - \int_{\Omega^\epsilon} h^\epsilon \varphi dx dy \right\}.
\]

Now, in order to obtain the limiting regime of the family of solutions $\{\bar{u}^\epsilon\}_{\epsilon > 0}$, we perform a dilatation of the domain $\Omega^\epsilon$ by a factor $\epsilon^{-1}$ in the $y$-direction. Introducing the operator

\[
\Delta^\epsilon_x u := \text{div} \left( |(\nabla_x u, \frac{1}{\epsilon} \nabla_y u)|^{p-2} (\nabla_x u, \frac{1}{\epsilon^2} \nabla_y u) \right),
\]

we obtain in the fixed domain $\Omega$ the equivalent equation

\[
-\Delta^\epsilon_x u + |u|^{p-2} u = f^\epsilon, \quad \text{in } \Omega,
\]

\[
\text{on } \partial \Omega,
\]

where $f^\epsilon(x,y) := h^\epsilon(x,\epsilon y)$.

The relation between the functions spaces set in $\Omega^\epsilon$ and those set in $\Omega$ is given by the isomorphism

\[
\Phi^\epsilon : L^p(\Omega^\epsilon) \rightarrow L^p(\Omega) \quad u \mapsto u \circ \Phi^\epsilon,
\]

\[
J_\psi u := \{ u^* \in W^{-1,p'}(\Omega^\epsilon) : \|u^*\|_{W^{-1,p'}(\Omega^\epsilon)} = \psi(\|u\|_{W^{1,p}_0(\Omega^\epsilon)}) \|u\|_{W^{1,p}_0(\Omega^\epsilon)} \}, \quad \langle u^*, u \rangle_{W^{-1,p'}, W^{1,p}} = \psi(\|u\|_{W^{1,p}_0(\Omega^\epsilon)})
\]
which also define an isomorphism from $W^{1,p}(\Omega^\epsilon)$ onto $W_0^{1,p}(\Omega)$, as well as from $W_0^{1,p}(\Omega^\epsilon)$ onto $W_0^{1,p}(\Omega)$. It is not difficult to see that $\overline{u^\epsilon} \in W_0^{1,p}(\Omega^\epsilon)$ is solution of \eqref{1.1} if and only if $u^\epsilon := \Phi_\epsilon(\overline{u^\epsilon}) \in W_0^{1,p}(\Omega)$ is solution of \eqref{2.5}, i.e., for all $\varphi \in W_0^{1,p}(\Omega)$, $u^\epsilon$ must satisfy
\[
\int_\Omega \left( \|\nabla_x u^\epsilon\|^p + \frac{1}{\epsilon^2} \|\nabla_y u^\epsilon\|^p + |u^\epsilon|^p \right) \, dx \, dy = \int_\Omega f^\epsilon \varphi \, dx \, dy.
\]

As we can see from the last identity, when we consider the problem \eqref{1.1} in the fixed domain $\Omega$ (which is not thin anymore), it appears a factor $\epsilon^{-1}$ on the gradient of $u^\epsilon$ in the $y$-direction. Physically this means that there is a very strong diffusion mechanism acting on the $y$-direction, and therefore one expects solutions become homogeneous in this direction, i.e., is expected that in the limit $\epsilon \to 0$, the limiting solution do not depends on the variable $y$. We formalize this in the next Proposition.

**Proposition 2.2.** Let $f^\epsilon \in L^p(\Omega)$ be an uniformly bounded (with respect to $\epsilon$) family of functions. If $u^\epsilon \in W_0^{1,p}(\Omega)$ is the solution of \eqref{2.5} there exists $u^0 \in W_0^{1,p}(\Omega)$ such that $\nabla_y u^0 = 0$ a.e. in $\Omega$ and, up to subsequence,
\[
u^\epsilon \rightharpoonup u^0, \text{ weakly in } W_0^{1,p}(\Omega) \text{ and strongly in } L^p(\Omega).
\]

**Proof.** Taking $u^\epsilon$ as test function in \eqref{2.7} we obtain
\[
\int_\Omega \left( \|\nabla_x u^\epsilon\|^p + \|\epsilon^{-1}\nabla_y u^\epsilon\|^p + |u^\epsilon|^p \right) \, dx \, dy = \int_\Omega f^\epsilon u^\epsilon \, dx \, dy.
\]
Therefore
\[
\|u^\epsilon\|_{W_0^{1,p}(\Omega)}^p \leq \int_\Omega \left( \|\nabla_x u^\epsilon\|^p + \|\epsilon^{-1}\nabla_y u^\epsilon\|^p + |u^\epsilon|^p \right) \, dx \, dy \leq \|f^\epsilon\|_{L^p(\Omega)} \|u^\epsilon\|_{L^p(\Omega)},
\]
which implies that $\|u^\epsilon\|_{W_0^{1,p}(\Omega)} = O(1)$. Since $W_0^{1,p}(\Omega)$ is reflexive and $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ compactly, up to subsequence there exists $u^0 \in W_0^{1,p}(\Omega)$ satisfying \eqref{2.8}. Also, due to \eqref{2.9}, $\|\nabla_y u^\epsilon\|_{L^p(\Omega)} = O(\epsilon)$ and then, $\nabla_y u^\epsilon \rightharpoonup 0$ in $L^p(\Omega)$ implying $\nabla_y u^0 = 0$ in $L^p(\Omega)$.

**Corollary 2.3.** Let $f^\epsilon$, $u^\epsilon$ and $u^0$ as before. Then $u^0 = 0$.

**Proof.** Since $\Omega$ is bounded, by Poincare’s inequality \cite[Lemma 5.1]{24} there exists $C = C(\Omega, p)$ such that
\[
\|u^0\|_{L^p(\Omega)} \leq C \|\nabla_y u^0\|_{L^p(\Omega)}.
\]
The result follows from Proposition 2.2.

**Corollary 2.4.** $\|u^\epsilon\|_{W_0^{1,p}(\Omega)} \to 0$.

**Proof.** It follows from Proposition 2.2 Corollary 2.3 and \eqref{2.8} that $u^\epsilon \to 0$ in $L^p(\Omega)$. Hence, we get the result by estimate \eqref{2.9}.

2.2. **Nonlinearities.** Now we consider the problem \eqref{2.7} with a nonlinearity $f$ at the right side, i.e., we consider the problem: find $u^\epsilon \in W_0^{1,p}(\Omega)$ such that
\[
\int_\Omega \left( \|\nabla_x u\| + \frac{1}{\epsilon^2} \|\nabla_y u\| \right)^{p-2} \left( \nabla_x u \cdot \nabla_x \varphi + \frac{1}{\epsilon^2} \nabla_y u \cdot \nabla_y \varphi \right) \, dx \, dy = \int_\Omega f(u) \varphi \, dx \, dy,
\]
for all $\varphi \in W_0^{1,p}(\Omega)$.

This class of equations were deeply studied by Djairo’s et al in \cite{12}. In order to simplify our discussion, let us assume that the nonlinearity $f$ satisfies
\[
|f(s)| \leq c + d \, |s|^{p-1-\alpha}, \quad \forall s \in \mathbb{R},
\]
where $c$, $d$ and $\frac{p}{p-1-\alpha}$ are such that $\frac{p}{p-1-\alpha} > 1$.
for some constants $c, d \in \mathbb{R}$, $\alpha > 0$ such that $p > 1 + \alpha$. The Nemytskii operator $L^p(\Omega) \ni u \mapsto f \circ u \in L^{p/(p-1-\alpha)}(\Omega)$ maps bounded sets into bounded sets and is not difficult to see that the family of solutions $u^\epsilon$ of (2.10) is uniformly bounded in $W^{1,p}_0(\Omega)$. Indeed, for any $\epsilon \in (0, 1)$, we have similarly to (2.9) that
\[
\|u^\epsilon\|_{W^{1,p}_0(\Omega)}^p \leq \int_\Omega \left( \|\nabla u^\epsilon\|^{p} + \|\epsilon^{-1}\nabla_y u^\epsilon\|^p + \|u^\epsilon\|^p \right) dx dy \leq \|f(u^\epsilon)\|_{L^p(\Omega)} \|u^\epsilon\|_{L^{p\alpha}(\Omega)} \leq C\|u^\epsilon\|^{p-\alpha}_{L^p(\Omega)},
\]
where $(p-1-\alpha)/p + 1/p_\alpha = 1$. Note that $L^p(\Omega) \hookrightarrow L^{p\alpha}(\Omega)$.

Hence, we can argue as in the proofs of Proposition 2.2 and Corollary 2.3 to conclude that the family of solutions of the nonlinear problem (2.10) converges to the null function obtaining the following result.

**Theorem 2.5.** Let $u^\epsilon$ be a family satisfying (2.10) with $f : \mathbb{R} \rightarrow \mathbb{R}$ continuous and satisfying (2.11). Then $\|u^\epsilon\|_{W^{1,p}_0(\Omega)} \xrightarrow{\epsilon \rightarrow 0} 0$.

### 3. Neumann Boundary Condition

Now we consider a similar problem but with Neumann boundary condition rather Dirichlet boundary condition. Given a family $h^\epsilon \in L^{p'}(\Omega^\epsilon)$, we analyze convergence properties of the solutions of the family of elliptic equations
\[
-\Delta_p u + |u|^{p-2} u = h^\epsilon, \quad \text{in } \Omega^\epsilon, \\
\frac{\partial u}{\partial \eta^\epsilon} = 0, \quad \text{on } \partial \Omega^\epsilon,
\]
where $\eta^\epsilon$ denotes the outward unitary normal vector field to the boundary of the thin domain $\Omega^\epsilon$.

**Definition 3.1.** We say that $u \in W^{1,p}(\Omega^\epsilon)$ is solution of the problem (3.1) if
\[
\int_{\Omega^\epsilon} \left( \|\nabla u\|^{p-2}\nabla u \cdot \nabla \phi + |u|^{p-2} u \phi \right) dx dy = \int_{\Omega^\epsilon} h^\epsilon \phi dx dy,
\]
for all $\phi \in W^{1,p}(\Omega^\epsilon)$.

As before, in order to study the limiting behavior of the family of solutions of equation (3.1), we stretch the domain $\Omega^\epsilon$ by the factor $\epsilon^{-1}$ in the $y$-direction. In this case the equivalent equation in $\Omega$ has the form
\[
-\Delta^\epsilon_p u + |u|^{p-2} u = f^\epsilon, \quad \text{in } \Omega, \\
\nabla_x u \cdot \eta_x + \frac{1}{\epsilon^2} \nabla_y u \cdot \eta_y = 0, \quad \text{on } \partial \Omega,
\]
where $\Delta^\epsilon_p$ is the operator introduced in (2.4). $f^\epsilon(x, y) := h^\epsilon(x, \epsilon y)$ and $(\eta_x, \eta_y)$ denotes the outward unitary normal vector field to $\partial \Omega$.

Hence, $u^\epsilon \in W^{1,p}(\Omega^\epsilon)$ is solution of (3.1) if and only if $u^\epsilon := \Phi^\epsilon(u^\epsilon) \in W^{1,p}(\Omega)$ is solution of (3.3), i.e., $u^\epsilon$ satisfies
\[
\int_{\Omega} \left( \|\nabla_x u^\epsilon\|^{p-2} \nabla_x u^\epsilon \cdot \nabla_x \phi + \frac{1}{\epsilon^2} \nabla_y u^\epsilon \cdot \nabla_y \phi \right) dx dy = \int_{\Omega} f^\epsilon \phi dx dy,
\]
for all $\phi \in W^{1,p}(\Omega)$.

Here, we will see that the family of solutions $\{u^\epsilon\}_{\epsilon > 0}$ of the problem (3.3) will converges to a limit $u^0$, which is solution of a suitable equation defined on a lower dimensional domain. In order to obtain this limiting regime, we perform a dilatation of the $(m+n)$-dimensional Lebesgue measure by a factor $1/\epsilon^n$. 

With this measure, namely $\rho^\epsilon := 1/\epsilon^n \times \text{Lebesgue measure}$, we set the Lebesgue and Sobolev spaces $L^p(\Omega^\epsilon; \rho^\epsilon)$ and $W^{1,p}(\Omega^\epsilon; \rho^\epsilon)$. Now is natural to consider in $W^{1,p}(\Omega)$ the equivalent norm
\[
\|u\|_\epsilon := \left[ \int_\Omega \left( \|\nabla u\|^p + \frac{1}{\epsilon^p} \|\nabla y u\|^p + |u|^p \right) dx dy \right]^{1/p}.
\]
It is easy to see that the isomorphism defined in (2.6)
\[\Phi^\epsilon_* : W^{1,p}(\Omega^\epsilon; \rho^\epsilon) \rightarrow W^{1,p}_\epsilon(\Omega)\]
is indeed an isometry where $W^{1,p}_\epsilon(\Omega) := (W^{1,p}(\Omega), \|\cdot\|_\epsilon)$. Similar observation can be done considering $\Phi^\epsilon_* : L^p(\Omega^\epsilon; \rho^\epsilon) \rightarrow L^p(\Omega)$.

We also denote by $\Omega_1 := \pi_1(\Omega)$ where $\pi_1 : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m : (x, y) \mapsto x$, is the projection onto the first $m$ components. For each $x \in \Omega_1$, let $\Gamma_x = \{ y \in \mathbb{R}^n : (x, y) \in \pi_1^{-1}\{x\}\}$. We define $\omega : \Omega_1 \rightarrow \mathbb{R}_+$ by
\[\omega(x) = |\Gamma_x|,\]
the $m$-dimensional Lebesgue measure of $\Gamma_x$.

Now we consider the spaces $L^p(\Omega_1; \omega)$ and $W^{1,p}(\Omega_1; \omega)$. The norm in $W^{1,p}(\Omega_1; \omega)$ will be denoted by
\[\|u\|_0 = \left[ \int_{\Omega_1} \omega(\|\nabla u\|^p + |u|^p) \, dx dy \right]^{1/p}.
\]
Due to the nature of this specific kind of perturbation, we also introduce the following operators

(Average projector)
\[M_\epsilon : L^p(\Omega) \rightarrow L^p(\Omega_1), \quad (M_\epsilon u)(x) = \frac{1}{\omega(x)} \int_{\Gamma_x} u(x, y) \, dy \quad (3.5)\]

(Extension operator)
\[E_\epsilon : L^p(\mathbb{R}^n) \rightarrow L^p(\Omega^\epsilon), \quad (E_\epsilon u)(x, y) = u(x) \quad (3.6)\]

Notice that $M_\epsilon \circ E_\epsilon = I$, the identity operator in $L^p(\Omega_1)$. Furthermore the extension operator $E_\epsilon$ maps the family of spaces $W^{1,p}(\Omega_1)$ into $W^{1,p}(\Omega^\epsilon)$ and satisfies $\frac{\partial}{\partial y}(E_\epsilon u) = 0$.

It is easy to see from Fubini-Tonelli Theorem and Hölder inequality that the operators $M_\epsilon : L^p(\Omega^\epsilon) \rightarrow L^p(\Omega_1)$ satisfy $\|M_\epsilon\|_{L(L^p(\Omega^\epsilon), L^p(\Omega_1))} = 1$. In fact, let $u \in L^p(\Omega^\epsilon)$,
\[\|M_\epsilon u\|_{L^p(\Omega_1; \omega)} = \left[ \int_{\Omega_1} \omega(x) |M_\epsilon u(x)|^p \, dx \right]^{1/p} = \left[ \int_{\Omega_1} \frac{1}{\omega(x)^{p-1}} \int_{\Gamma_x} u(x, y)^p \, dy \, dx \right]^{1/p} \leq \left[ \int_{\Omega_1} \frac{1}{\omega(x)^{p-1}} \omega(x)^{p-1} \int_{\Gamma_x} |u(x, y)|^p \, dy \, dx \right]^{1/p} = \left[ \int_{\Omega} |u(x, y)|^p \, dx \, dy \right]^{1/p} = \|u\|_{L^p(\Omega)}.
\]
The equality holds taking $u$ independent of $y$ in $\Omega$.

3.1. Convergence. Similarly to the previous Section, we notice that $u^\epsilon \in W^{1,p}(\Omega)$ is solution of (3.3) if and only if $u^\epsilon$ satisfies
\[
\frac{1}{p} \|u^\epsilon\|_\epsilon^p - \int_\Omega f^\epsilon u^\epsilon \, dxdy = \min_{\varphi \in W^{1,p}(\Omega)} \left\{ \frac{1}{p} \|\varphi\|_\epsilon^p - \int_\Omega f^\epsilon \varphi \, dxdy \right\}.
\]
**Proposition 3.2.** Assume that for all \( x \in \pi_1(\Omega) \), \( \Gamma_x \) is connected. Let \( f^c \in L^{p'}(\Omega) \) be a family of functions uniformly bounded with respect to \( \epsilon > 0 \) such that
\[
M_c f^c \xrightarrow{c \to 0} f^0, \quad w - L^{p'}(\Omega_1).
\]
If we define
\[
\lambda^c = \min_{\varphi \in W^{1,p}(\Omega)} \left\{ \frac{1}{p} \| \varphi \|_p^p - \int_{\Omega} f^c \varphi \, dx \, dy \right\} \quad \text{and} \quad \lambda^0 = \min_{\varphi \in W^{1,p}(\Omega_1)} \left\{ \frac{1}{p} \| \varphi \|^p_0 - \int_{\Omega_1} \omega f^0 \varphi \, dx \right\},
\]
then \( \lambda^c \to \lambda^0 \).

**Proof.** Taking \( \varphi = \varphi(x) \) we get
\[
\lambda^c \leq \frac{1}{p} \int_{\Omega} \left( \| \nabla_x \varphi \|_p^p + \| \epsilon^{-1} \nabla_y \varphi \|_p^p + | \varphi |^p \right) \, dx \, dy - \int_{\Omega} f^c \varphi \, dx \, dy
\]
\[
= \frac{1}{p} \int_{\Omega_1} \omega \left( \| \nabla_x \varphi \|_p^p + | \varphi |^p \right) \, dx - \int_{\Omega_1} \omega M_c f^c \varphi \, dx
\]
\[
= \frac{1}{p} \int_{\Omega_1} \omega \left( \| \nabla_x \varphi \|_p^p + | \varphi |^p \right) \, dx - \int_{\Omega_1} \omega f^0 \varphi \, dx + \int_{\Omega_1} \omega (f^0 - M_c f^c) \varphi \, dx.
\]
Taking the infimum over all \( \varphi \in W^{1,p}(\Omega_1) \) we obtain that \( \limsup_{c \to 0} \lambda^c \leq \lambda^0 \).

Recalling the previous Section, since \( f^c \in L^{p'}(\Omega) \) is uniformly bounded, if \( u^c \) is the solution of (3.3), we can obtain a function \( u^0 \in W^{1,p}(\Omega) \) such that \( \nabla_y u = 0 \) a.e. in \( \Omega \) and
\[
u \]
\[
\lambda^c \overset{c \to 0}{\longrightarrow} \lambda^0, \quad \text{weakly in } W^{1,p}(\Omega) \text{ and strongly in } L^p(\Omega). \tag{3.7}
\]
From the weak convergence in \( W^{1,p}(\Omega) \) and strong convergence in \( L^p(\Omega) \) we obtain that
\[
\liminf_{c \to 0} \lambda^c = \liminf_{c \to 0} \left\{ \frac{1}{p} \| u^c \|_p^p - \int_{\Omega} f^c u^c \, dx \, dy \right\}
\]
\[
\geq \liminf_{c \to 0} \left\{ \frac{1}{p} \| u^c \|_{W^{1,p}(\Omega)}^p - \int_{\Omega} f^c u^c \, dx \, dy \right\}
\]
\[
\geq \frac{1}{p} \| u^0 \|_0^p - \int_{\Omega_1} \omega f^0 u^0 \, dx \geq \lambda^0, \tag{3.8}
\]
which proves the statement. \( \square \)

**Remark 3.3.** Notice that we use \( u^0 \) as a test function in the conclusion of the proof of Proposition 3.2. This was possible thanks to the hypothesis \( \Gamma_x \) connected.

**Remark 3.4.** By (3.8)
\[
\lambda^c \geq \frac{1}{p} \| u^0 \|_0^p - \int_{\Omega_1} \omega f^0 u^0 \, dx \geq \lambda^0,
\]
consequently, we derive
\[
\lambda^0 = \frac{1}{p} \| u^0 \|_0^p - \int_{\Omega_1} \omega f^0 u^0 \, dx.
\]
Therefore \( u^0 \) is characterized as solution of the equation
\[
- \frac{1}{\omega} \partial_\omega (\omega (| \nabla u |^{p-2} \nabla u) + | u |^{p-2} u ) = f^0, \quad \text{in } \Omega_1, \tag{3.9}
\]
\[
\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial \Omega_1,
\]
where \( \Omega_1 = \pi_1(\Omega) \subset \mathbb{R}^m \) and \( \nu \) denotes the outward unitary normal vector field to \( \partial \Omega_1 \).
In the next result we strong the convergence obtained in Proposition 3.2

**Theorem 3.5.** Let \( f^\epsilon, u^\epsilon, u^0 \) and \( f^0 \) be as before. Then

\[
\lim_{\epsilon \to 0} u^\epsilon = u^0, \text{ strongly in } W^{1,p}(\Omega).
\]

**Proof.** From the weak convergence \( u^\epsilon \rightharpoonup u^0 \) in \( W^{1,p}(\Omega) \) and Proposition 3.2, we have that

\[
\int_{\Omega} \left( \|\nabla_x u^0\|^p + \|u^0\|^p \right) dx dy \leq \liminf_{\epsilon \to 0} \int_{\Omega} \left( \|\nabla_x u^\epsilon\|^p + \|\nabla_y u^\epsilon\|^p + |u^\epsilon|^p \right) dx dy
\]

\[
\leq \limsup_{\epsilon \to 0} \int_{\Omega} \left( \|\nabla_x u^\epsilon\|^p + \|\nabla_y u^\epsilon\|^p + |u^\epsilon|^p \right) dx dy
\]

\[
\leq \lim_{\epsilon \to 0} \int_{\Omega} \left( \|\nabla_x u^\epsilon\|^p + \|\epsilon^{-1}\nabla_y u^\epsilon\|^p + |u^\epsilon|^p \right) dx dy
\]

\[
= \int_{\Omega} \omega f^0 u^0 \ dy dx = \int_{\Omega} \omega \left( \|\nabla_x u^0\|^p + \|u^0\|^p \right) dy dx
\]

\[
= \int_{\Omega} \left( \|\nabla_x u^0\|^p + \|u^0\|^p \right) dx dy.
\]

Now the result follows from the uniform convexity of \( W^{1,p}(\Omega) \). \( \square \)

**Corollary 3.6.** Let \( u^\epsilon \) and \( u^0 \) be as in Theorem 3.5. Then

\[
\|u^\epsilon - E_\epsilon u^0\| \to 0.
\]

**Proof.** Since we have strong convergence of \( u^\epsilon \) to \( u^0 \) in \( W^{1,p}(\Omega) \), the identity

\[
\int_{\Omega} \left( \|\nabla_x u^0\|^p + \|u^0\|^p \right) dx dy = \lim_{\epsilon \to 0} \int_{\Omega} \left( \|\nabla_x u^\epsilon\|^p + \frac{1}{\epsilon^p} \|\nabla_y u^\epsilon\|^p + |u^\epsilon|^p \right) dx dy,
\]

obtained in the proof of Theorem 3.5, implies that \( \epsilon^{-1} \|\nabla_y u^\epsilon\|_{L^p(\Omega)} \to 0 \). \( \square \)

### 3.2. Nonlinearities

In the case of Neumann boundary conditions if we consider a continuous nonlinearity \( f : \mathbb{R} \to \mathbb{R} \), the correspondent problem assume the form

\[
\int_{\Omega} \left( \|\nabla_x u, \frac{1}{\epsilon} \nabla_y u\|^{p-2} (\nabla_x u \cdot \nabla_x \varphi + \frac{1}{\epsilon^2} \nabla_y u \cdot \nabla_y \varphi) + |u|^{p-2} u \varphi \right) dx dy = \int_{\Omega} f(u) \varphi dx dy, \quad (3.10)
\]

for all \( \varphi \in W^{1,p}(\Omega) \).

Assuming that the nonlinearity \( f \) also satisfies (2.11), we have similarly to (2.12) that the family of solutions \( u^\epsilon \) of (3.10) is uniformly bounded in \( W^{1,p}(\Omega) \). Therefore we still have the existence of a suitable limit \( u^0 \in W^{1,p}(\Omega) \) (see (3.7)), which satisfies the nonlinear equation

\[
\int_{\Omega} \omega \left( \|\nabla_x u\|^{p-2} \nabla_x u \cdot \nabla_x \varphi + |u|^{p-2} u \varphi \right) dx = \int_{\Omega} \omega f(u) \varphi dx,
\]

for all \( \varphi \in W^{1,p}(\Omega) \). As \( f(u^\epsilon) \to f(u^0) \) in \( L^p(\Omega) \), we obtain, mutatis mutandis the proof of Proposition 3.5, strong convergence in \( W^{1,p}(\Omega) \), and therefore a similar convergence as stated in Corollary 3.6.
4. A SPECIFIC EXAMPLE

In this section we present the case of a thin domain considered by Hale and Raugel in the seminal paper \cite{14} in the case \( p = 2 \). Let \( \Omega_1 \) be a smooth bounded domain in \( \mathbb{R}^m, n \geq 1 \), and \( g \in C^2(\Omega_1; \mathbb{R}) \) a positive function. We define the family of thin domains \( \Omega^\epsilon \subset \mathbb{R}^{m+1} \) as

\[ \Omega^\epsilon := \{ (x, y) \in \mathbb{R}^{m+1} : x \in \Omega_1, 0 < y < \epsilon g(x) \} \]

In \( \Omega^\epsilon \) we consider the family of elliptic equations

\[
-\Delta_p u + |u|^{p-2}u = h^\epsilon, \quad \text{in } \Omega^\epsilon, \\
\frac{\partial u}{\partial \eta^\epsilon} = 0, \quad \text{on } \partial \Omega^\epsilon,
\]

where \( h^\epsilon \in L^{p'}(\Omega^\epsilon) \) and \( \eta^\epsilon \) denotes the outward unitary normal vector field to \( \partial \Omega^\epsilon \).

Considering the change of coordinates \( \Phi^\epsilon : \mathbb{R}^m+1 \to \Omega^\epsilon, (x, y) \mapsto (x, \epsilon y) \), where \( \mathbb{R}^m+1 = \{ (x, y) \in \mathbb{R}^{m+1} : x \in \Omega_1, 0 < y < g(x) \} \), equation (4.1) becomes

\[
-\Delta_p^\epsilon u + |u|^{p-2}u = f^\epsilon, \quad \text{in } \Omega, \\
\nabla_x u \cdot \eta_x + \frac{1}{\epsilon^2} \nabla_y u \cdot \eta_y = 0, \quad \text{on } \partial \Omega,
\]

where \( \Delta_p^\epsilon \) is the operator introduced in (2.4), \( f^\epsilon(x, y) := h^\epsilon(x, \epsilon y) \) and \( (\eta_x, \eta_y) \) is the outward unitary normal vector field to \( \partial \Omega \).

As proved in Section 3, the limiting problem is

\[
-\frac{1}{g} \text{div}(g|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = f^0, \quad \text{in } \Omega_1, \\
\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial \Omega_1,
\]

where \( f^0 \in L^{p'}(\Omega_1) \) is the weak limit (in \( L^{p'}(\Omega_1) \)) of the family \( M^\epsilon f^\epsilon \). This agree with the limiting problem founded in \cite{14} with \( p = 2 \),

\[
-\frac{1}{g} \text{div}(g\nabla u) + u = f^0, \quad \text{in } \Omega_1, \\
\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial \Omega_1.
\]

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