ON THE NONLINEAR CONVECTION-DIFFUSION-REACTION PROBLEM IN A THIN DOMAIN WITH A WEAK BOUNDARY ABSORBTION

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Abstract. Motivated by the applications from chemical engineering, in this paper we present a derivation of the effective model describing the convection-diffusion-reaction process in a thin domain. The problem is described by a nonlinear elliptic problem with nonlinearity appearing both in the governing equation as well in the boundary condition. Using rigorous analysis in appropriate functional setting, we show that the starting singular problem posed in a two-dimensional region can be approximated with one which is regular, one-dimensional and captures the effects of all relevant physical processes which took place in the original problem.

1. Introduction

Convection-diffusion-reaction problems naturally appear if physical processes in chemical engineering are modeled and, thus, are of great interest both from the theoretical and practical point of view. Depending on the problem we want to model, different types of equations and boundary conditions can be considered. In the present paper, we study a stationary convection-diffusion equation in a thin (or long) channel with nonlinear reaction term concentrated in a narrow (oscillating) strip near one part of the channel wall. On the opposite part of the channel boundary, a nonlinear condition is prescribed modeling the reaction catalyzed by the wall. This type of elliptic boundary value problem describes, for instance, a transport of the solute by convection and diffusion where the solute particles undergo an irreversible chemical reaction on the one part of the boundary[1] and react among themselves in the vicinity of the other one. Our goal is to rigorously derive the effective model described by the one-dimensional boundary-value problem providing a good approximation of the governing problem when the ratio between channel's thickness and its length is small.

The study of the solute transport problem goes back to the celebrated work of Taylor [1] who first discussed the dispersion of a passive solute in a laminar flow. Extending Taylor's analysis, Aris [2] formally derived the effective equations describing the problem in the absence of the chemical reaction. Rigorous derivation of the asymptotic model for a solute transport in the presence of the first-order (linear) chemical reaction on the channel wall was given in [3]. With same type of boundary condition, a general model of convection-diffusion with reaction was treated in [4] via homogenization. The effects of the curved geometry and fluids microstructure on solute dispersion were investigated in [5, 6]. Let us also mention some contributions in the engineering literature as [7, 8, 9].

In the above mentioned papers, the problems under consideration were linear. In the present paper, we deal with a nonlinear elliptic problem with nonlinearity appearing both in the governing equation as well in the boundary condition. Diffusion problems with reaction terms concentrating in the neighborhood of the boundary were successfully addressed in recent papers by the second author of this paper. Combining techniques from geometric theory of parabolic problems, perturbation of linear operators and concentrated

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[1]The reaction mechanism is assumed to be weak ensuring that the loss of the solute at the boundary is not considerable.

integrals the authors have discussed in [10, 11] the continuity of the dynamics given by dissipative reaction-diffusion equations posed in fixed domains (independent of the small parameter). In thin channels and with homogeneous Neumann boundary condition, a nonlinear diffusion elliptic problem has been considered in [12]. It is important to emphasize that here we consider a more general situation. On the one hand we allow convection, diffusion and nonlinear reactions concentrated close to a portion of the boundary. On the other hand, we combine homogeneous and nonlinear Neumann boundary conditions on the domain boundary. Our main result provides a way how to replace a singular nonlinear elliptic problem posed in a two-dimensional region with one which is regular, one-dimensional and captures the effects of all physical processes which took place in the original problem. As far as we know, this is the first attempt to carry out such rigorous analysis and we believe that the result could be instrumental for creating more efficient numerical algorithms for approximating the solution of the convection-diffusion-reaction problems.

2. Formulation of the problem and the statement of the main result

We study the the asymptotic behavior of a family of solutions given by the nonlinear elliptic equation

$$-D\Delta w^\epsilon + Q_\epsilon(y)\frac{\partial w^\epsilon}{\partial x} + c\, w^\epsilon = \frac{1}{\epsilon^\alpha}\chi_{\theta_\epsilon}\, f(w^\epsilon) \quad \text{ in } R^\epsilon \tag{2.1}$$

with the following boundary conditions

$$D\frac{\partial w^\epsilon}{\partial \nu^\epsilon} = \epsilon\, g(w^\epsilon) \;\text{ on } \Gamma \quad \text{ and } \quad \frac{\partial w^\epsilon}{\partial \nu^\epsilon} = 0 \;\text{ on } \partial R^\epsilon \setminus \Gamma. \tag{2.2}$$

The domain R^ϵ is a simple thin (or long) channel given by

$$R^\epsilon = \{(x,y) \in \mathbb{R}^2 \;:\; x \in (0,1), \quad 0 < y < \epsilon\, H\}, \quad 0 < \epsilon \ll 1\,. \tag{2.3}$$

We denote by $\Gamma \subset \partial R^\epsilon$ the lower wall of the channel, namely

$$\Gamma = \{(x,0) \in \mathbb{R}^2 \;:\; x \in (0,1)\}. \tag{2.4}$$

In the governing equation, the given velocity field is assumed to be incompressible. Since we are studying the process in a thin domain, it is reasonable to take the velocity to be unidirectional implying $Q_\epsilon = Q_\epsilon(y)$, due to the incompressibility condition. We set

$$Q_\epsilon(y) = Q(y/\epsilon)\,,$$

where $Q \in L^\infty(0,H)$ is a non-negative function. $D > 0$ is the molecular diffusion, c is the reaction coefficient, the vector $\nu^\epsilon = (\nu_1^\epsilon, \nu_2^\epsilon)$ is the unit outward normal to ∂R^ϵ and $\frac{\partial}{\partial \nu^\epsilon}$ is the outside normal derivative. Observe that the reaction mechanism on the boundary Γ is taken to be weak and we model that by assuming that the wall absorbtion parameter is of order $\mathcal{O}(\epsilon)$ (see $(2.2)_1$). In case of the weak wall absorbtion, the loss of the solute at Γ is not considerable and, consequently, the effects of the reaction at the boundary remain in the limit problem.[2]

Nonlinearities f and $g : \mathbb{R} \mapsto \mathbb{R}$ are supposed to be $\mathcal{C}^2$-functions with bounded derivatives. Indeed, under the point of view of investigating the asymptotic behavior of problems as (2.1)-(2.2), to assume f and g bounded with bounded derivatives does not imply any restriction since we are interested here in solutions uniformly bounded in L^∞-norms.

The function

$$\chi_{\theta_\epsilon} : \mathbb{R}^2 \mapsto \mathbb{R}$$

is the characteristic function of the narrow strip θ_ϵ defined by (see Fig. 1)

$$\theta_\epsilon = \{(x,y) \in \mathbb{R}^2 \;:\; x \in (0,1), \quad \epsilon\,(H - \epsilon^\alpha\, G_\epsilon(x)) < y < \epsilon H)\},$$

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[2] If we write the problem in non-dimensional form (see e.g. [3]), such assumption would imply that the Damkohler number Da^ϵ is of order $\mathcal{O}(\epsilon)$. It can be easily verified that $Da^\epsilon = \mathcal{O}(\epsilon)$ is, in fact, the critical (and the most interesting) case. Indeed, if we took $Da^\epsilon \ll \mathcal{O}(\epsilon)$, the effects of the (chemical) reaction on Γ would disappear from the effective model. On the other hand, by assuming $Da^\epsilon \gg \mathcal{O}(\epsilon)$, the reaction on the wall would dominate the process keeping almost all solute in a small region near the left end of the channel.

where α is a positive parameter, $G_\epsilon : (0,1) \mapsto \mathbb{R}^+$ is smooth and non-negative satisfying $0 \le G_\epsilon(x) < H$ for all $x \in (0,1)$ and $\epsilon > 0$.

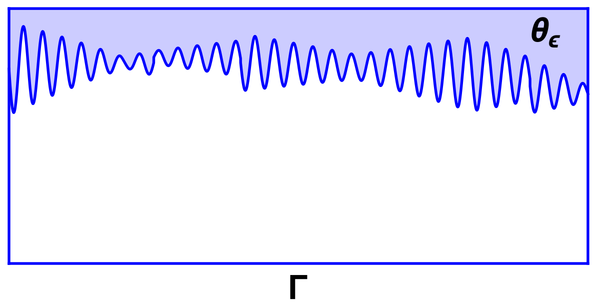

FIGURE 1. The domain under consideration.

We allow G_ϵ to oscillate when $\epsilon \to 0$ expressing it as

$$G_\epsilon(x) = G(x, x/\epsilon^\beta), \quad \text{for some } \beta > 0. \tag{2.5}$$

The function $G : (0,1) \times \mathbb{R} \mapsto \mathbb{R}$ is non-negative, continuous in x uniformly in the second variable y, that is, we suppose that for each $\eta > 0$, there exists $\delta > 0$ such that $|G(x,y) - G(x',y)| \le \eta$ for all x, $x' \in [0,1]$, $|x - x'| < \delta$, and $y \in \mathbb{R}$. We still assume that G is $l(x)$-periodic in y for each $x \in (0,1)$: $H(x, y + l(x)) = H(x,y)$, for all y, with the period function l positive and uniformly bounded, $0 < l_0 \le l(x) \le l_1$ in $(0,1)$.

Clearly the open set θ_ϵ is a neighborhood for the upper boundary of R^ϵ whose thickness and oscillatory behavior depend on the positive parameters α and β respectively. Note that α and β set the thickness and oscillating order when ϵ goes to zero. Also, if G only depends on the first variable x, then the function G_ϵ is independent of ϵ and the narrow strip θ_ϵ does not possess oscillatory behavior.

In order to model the concentration of reactions in the small region $\theta_\epsilon \subset R^\epsilon$, we will proceed as in [13, 14]. We will combine the characteristic function χ_ϵ, the parameter ϵ and the nonlinear reaction f by the term

$$\frac{1}{\epsilon^\alpha} \chi_{\theta_\epsilon} f \in L^\infty(R^\epsilon).$$

Moreover, since $R^\epsilon \subset (0,1) \times (-\epsilon\, b_1, \epsilon\, G_1)$ is thin and degenerates into the unit interval as ϵ goes to zero, it is reasonable to expect that the family of solutions w^ϵ converges to a solution of a one-dimensional equation capturing the variable profile of the oscillatory behavior of the narrow strip θ_ϵ as well as the effect of the nonlinear boundary condition.

We will show that the limit problem for (2.1)-(2.2) is the following one:

$$\begin{cases} -D u_{xx} + q\, u_x + c\, u = \dfrac{1}{H} \left(\mu(x) f(u) + g(u) \right) & \text{in } (0,1)\,, \\ u_x(0) = u_x(1) = 0\,, \end{cases} \tag{2.6}$$

where the constant q and the function $\mu : (0,1) \mapsto (0, \infty)$ are given by

$$\begin{gathered} q = \frac{1}{H} \int_0^H Q(y) dy, \\ \mu(x) = \frac{1}{l(x)} \int_0^{l(x)} G(x,y)\, dy. \end{gathered} \tag{2.7}$$

Notice that the positive constant q is the average of the velocity Q and the non-negative coefficient $\mu \in L^\infty(0,1)$ is related to the oscillating strip θ_ϵ set by the function G_ϵ. In view of that, we conclude that the asymptotic model (2.6) captures all the effects we seek for: the effects of convection, the reactions on the boundary and inside the oscillating strip and also the effect of the geometry of the region where those reactions take place.

In our analysis we combine the results from [12, 13, 15]. We apply methods from [15] to deal with the thin channel, and we use the concentrated integrals discussed in [14, 16, 12, 13] in order to obtain $\mu(x)$, which is the mean value of $G(x,\cdot)$ for each $x \in (0,1)$. The coefficient μ captures the oscillatory behavior and the geometry of the narrow strip where the reactions are concentrated. If G does not depend on the second variable y, then the narrow neighborhood does not have oscillatory behavior, and so, $\mu(x) = G(x)$ in $(0,1)$.

In order to study problem (2.1) in the thin domain R^ϵ, we perform a convenient change of variable obtaining the following problem

$$\begin{cases} -D\left(\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} + \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2}\right) + Q(x_2)\dfrac{\partial u^\epsilon}{\partial x_1} + c\, u^\epsilon = \dfrac{1}{\epsilon^\alpha}\chi_{o_\epsilon} f(u^\epsilon) \quad \text{ in } \Omega \\ \dfrac{\partial u^\epsilon}{\partial x_1}N_1 + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2}N_2 = 0 \quad \text{ on } \partial\Omega\setminus\Gamma \\ D\left(\dfrac{\partial u^\epsilon}{\partial x_1}N_1 + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2}N_2\right) = g(u^\epsilon) \quad \text{ on } \Gamma \end{cases} \tag{2.8}$$

where the function $\chi_{o_\epsilon} : \mathbb{R}^2 \mapsto \mathbb{R}$ is the characteristic function of the narrow strip o_ϵ given by

$$o_\epsilon = \{(x_1,x_2) \in \mathbb{R}^2 \,:\, x_1 \in (0,1), \quad (H - \epsilon^\alpha\, G_\epsilon(x_1)) < x_2 < H\}. \tag{2.9}$$

The vector $N = (N_1, N_2)$ is the outward unit normal to $\partial\Omega$ and $\Omega \subset \mathbb{R}^2$ is the set

$$\Omega = \{(x_1,x_2) \in \mathbb{R}^2 \,:\, x_1 \in (0,1), \quad 0 < x_2 < H\}.$$

The equivalence between problems (2.1)-(2.2) and (2.8) can be observed by changing the scale of the channel R^ϵ and the narrow strip θ_ϵ through the isomorphism $(x_1,x_2) \to (x_1, \epsilon^{-1}x_2)$ which consists in stretching the x_2-direction by a factor of ϵ^{-1}. The factor ϵ^{-2} establishes a very fast diffusion in the x_2-direction. Indeed, we have rescaled the neighborhood θ_ϵ into the strip $o_\epsilon \subset \Omega$ and substituted the thin domain R^ϵ for a domain Ω independent on ϵ, at a cost of introducing a very strong diffusion mechanism in the x_2-direction.

Due to this strong diffusion mechanism it is expected that solutions of (2.8) will become more and more homogeneous in the x_2-direction when ϵ decreases, such that the limit solution will not depend on x_2 and therefore the limit problem will be one dimensional. This is in fully agreement with the intuitive idea that an equation in a thin domain should approach one in a line segment.

Now we are in position to state our main result:

Theorem 2.1. *Assume the reaction coefficient c in problem (2.8) satisfies $c > \|Q\|^2_{L^\infty(0,H)}/4D$.*

a) *Then, if $\{u^\epsilon\}_{\epsilon>0}$ is a family of solutions of problem (2.8), there exists a subsequence, still defined by u^ϵ, and a function $u \in H^1(\Omega)$ with $u(x_1,x_2) = u(x_1)$, solution of the problem (2.6), such that*

$$\|u^\epsilon - u\|_{H^1(\Omega)} \to 0, \ \text{ as } \epsilon \to 0.$$

b) *On the other hand, if a solution u of (2.6) is hyperbolic, then there exists a sequence u^ϵ of solutions of problem (2.8) satisfying*

$$\|u^\epsilon - u\|_{H^1(\Omega)} \to 0, \ \text{ as } \epsilon \to 0.$$

Remark 2.2. *Recall that a steady state solution u of a nonlinear differential equation is called hyperbolic if $\lambda = 0$ is not an eigenvalue of the linearized problem around u. For instance, if u satisfies equation (2.6) and is hyperbolic, then $\lambda = 0$ is not an eigenvalue of the eigenvalue problem*

$$\begin{cases} -Dv_{xx} + q\, v_x + c\, v = \dfrac{1}{H}\left(\mu f'(u) + g'(u)\right) v + \lambda v \quad \text{ in } (0,1) \\ v_x(0) = v_x(1) = 0 \end{cases}.$$

Remark 2.3. *Let us call $\mathcal{E}_\epsilon = \{u^\epsilon \in H^1(\Omega) \,:\, u^\epsilon \text{ is a solution of } (2.8)\}$ for each $\epsilon > 0$. Thus assertions a) and b) at Theorem 2.1 respectively mean upper and lower semicontinuity of the equilibria set of the parabolic problem associated to (2.8) at $\epsilon = 0$. In this sense we are proving the continuity of the stead state solutions given by (2.8) at $\epsilon = 0$ which reach the limit equation (2.6) as $\epsilon \to 0$.*

3. Basic facts

In this section we state basic results, introducing notations and writing our problem in an abstract setting. We also discuss how concentrating integrals converge to boundary integrals taking results from [12, 13, 14, 16].

Throughout this work we call $H^1_\epsilon(U)$ the Hilbert space set by $H^1(U)$ with the equivalent norm

$$\|w\|^2_{H^1_\epsilon(U)} = \|w\|^2_{L^2(U)} + \left\|\frac{\partial w}{\partial x_1}\right\|^2_{L^2(U)} + \frac{1}{\epsilon^2}\left\|\frac{\partial w}{\partial x_2}\right\|^2_{L^2(U)} \tag{3.1}$$

defined by the inner product

$$(\phi,\varphi)_{H^1_\epsilon(U)} = \int_U \left\{ \frac{\partial \phi}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial \phi}{\partial x_2}\frac{\partial \varphi}{\partial x_2} + \phi\,\varphi \right\} dx_1 dx_2$$

where U is an arbitrary open set of $\mathbb{R}^2$. Note that $\|\cdot\|_{H^1_\epsilon(U)} \geq \|\cdot\|_{H^1(U)}$ wherever $\epsilon \in [0,1]$. As we will see, the strong diffusion mechanism in front of the second derivative makes this space a suitable one to deal with thin domain problems.

Remark 3.1. *Due to* (3.1) *it is clear that any sequence $u^\epsilon \in H^1_\epsilon(\Omega)$ with $\|u^\epsilon\|_{H^1_\epsilon(\Omega)} \leq C$ for some positive constant C independent of ϵ satisfies*

$$\left\|\frac{\partial u^\epsilon}{\partial x_2}\right\|_{L^2(\Omega)} \leq \epsilon C, \quad \forall \epsilon > 0.$$

Consequently we get

$$\left\|\frac{\partial u^\epsilon}{\partial x_2}\right\|_{L^2(\Omega)} \to 0, \text{ as } \epsilon \to 0.$$

Lemma 3.2. *If G_ϵ is defined as in (2.5), then*

$$G_\epsilon(\cdot) \to \mu(\cdot) = \frac{1}{l(\cdot)}\int_0^{l(\cdot)} G(\cdot, s)\, ds, \quad w^* - L^\infty(0,1).$$

Proof. See [13, Lemma 2.3] or [17, Lemma 4.2]. □

3.1. **Abstract settings and existence of solutions.** In order to write problems (2.6) and (2.8) in an abstract form, we introduce the bilinear forms $a_\epsilon : H^1_\epsilon(\Omega) \times H^1_\epsilon(\Omega) \mapsto \mathbb{R}$ and $a_0 : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$

$$\begin{aligned} a_\epsilon(u,v) &= \int_\Omega \left\{ D\left(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right) + Q\frac{\partial u}{\partial x_1}v + c\,u\,v \right\} dx_1 dx_2, \quad \text{for } \epsilon > 0, \\ a_0(u,v) &= H\int_0^1 \{D\, u_x\, v_x + q\, u_x v + c\, u\, v\}\, dx, \quad \text{at } \epsilon = 0, \end{aligned} \tag{3.2}$$

where the constant q is given by (2.7) and the constant H comes from the domain Ω.

It is not difficult to see that a_ϵ is continuous for all $\epsilon > 0$. Moreover, for each $\epsilon > 0$, we can define the linear operators $A_\epsilon : H^1_\epsilon \subset H^{-1}(\Omega) \mapsto H^{-1}(\Omega)$ by the expression

$$\langle A_\epsilon u, v\rangle_{-1,1} = a_\epsilon(u,v), \text{ for all } v \in H^1(\Omega).$$

Hence, we can write the problem (2.8) in the abstract form $A_\epsilon u = F_\epsilon(u)$, for $\epsilon > 0$, where

$$F_\epsilon : H^1_\epsilon(\Omega) \mapsto H^{-s}(\Omega), \quad 1/2 < s < 1,$$

is given by

$$\begin{gathered} F_\epsilon = F_{\epsilon,f} + F_g, \\ \langle F_{\epsilon,f}(u), v\rangle = \frac{1}{\epsilon^\alpha}\int_{o_\epsilon} f(u)\, v\, dx_1 dx_2 \quad \text{and} \quad \langle F_g(u), v\rangle = \int_\Gamma \gamma(g(u))\,\gamma(v)\, dS, \quad \text{for all } v \in H^s(\Omega). \end{gathered} \tag{3.3}$$

Recall that f and g are $\mathcal{C}^2$-nonlinearities, bounded with bounded derivatives, and o_ϵ is the narrow strip defined in (2.9). Here $\gamma : H^t(\Omega) \mapsto L^2(\Gamma)$ is the trace operator with $1/2 < t < 1$.

In a similar way we can write the limit problem (2.6) in an abstract form $A_0 u = F_0(u)$ where

$$F_0 : H^1(0,1) \mapsto H^{-s}(0,1), \quad 1/2 < s < 1,$$

is defined by

$$F_0 = F_{0,f} + F_{0,g}$$

$$\langle F_{0,f}(u), v\rangle = \int_0^1 \mu\, f(u)\, v\, dx \quad \text{and} \quad \langle F_{0,g}(u), v\rangle = \int_0^1 g(u)\, v\, dx, \quad \text{for all } v \in H^s(0,1), \tag{3.4}$$

where $\mu \in L^\infty(0,1)$ is the coefficient introduced in (2.7).

Remark 3.3. *Under our assumptions, it is known that functions F_ϵ and F_0 are Fréchet differentiable. The proof can be seen for example in* [10, Lemma 3.6 and 3.7].

Lemma 3.4. *The continuous bilinear form a_ϵ is uniformly coercive for all $c > \|Q\|^2_{L^\infty(0,H)}/4D$ and $\epsilon \in [0,1]$.*

Proof. We just prove the case a_ϵ with $\epsilon > 0$. An analogous argument shows the result to the bilinear form a_0 since $|q| \le \|Q\|_{L^\infty}$. Using Holder and Young's inequality, we can get

$$\begin{aligned}
\int_\Omega D\left\{\left(\frac{\partial u}{\partial x_1}^2 + \frac{1}{\epsilon^2}\frac{\partial u}{\partial x_2}^2\right) + c\, u^2\right\} dx_1 dx_2 &= a_\epsilon(u,u) - \int_\Omega Q\frac{\partial u}{\partial x_1} u\, dx_1 dx_2, \\
&\le a_\epsilon(u,u) + \|Q\|_{L^\infty}\left\|\frac{\partial u}{\partial x_1}\right\|_{L^2}\|u\|_{L^2} \\
&\le a_\epsilon(u,u) + \frac{1}{2}\|Q\|_{L^\infty}\left(\delta\left\|\frac{\partial u}{\partial x_1}\right\|^2_{L^2} + \frac{1}{\delta}\|u\|^2_{L^2}\right)
\end{aligned}$$

for all $u \in H^1(\Omega)$, ϵ and $\delta > 0$. Thus,

$$a_\epsilon(u,u) \ge \left(D - \frac{\delta}{2}\|Q\|_{L^\infty}\right)\left\|\frac{\partial u}{\partial x_1}\right\|^2_{L^2} + \frac{D}{\epsilon^2}\left\|\frac{\partial u}{\partial x_2}\right\|^2_{L^2} + \left(c - \frac{1}{2\delta}\|Q\|_{L^\infty}\right)\|u\|^2_{L^2}.$$

Now let us take $\delta > 0$ such that $c/\|Q\|_{L^\infty} > 1/2\delta > \|Q\|_{L^\infty}/4D$, and so, such that

$$c > \|Q\|_{L^\infty}/2\delta > \|Q\|^2_{L^\infty}/4D.$$

Then,

$$a_\epsilon(u,u) \ge \min\left\{D - \frac{\delta}{2}\|Q\|_{L^\infty}, D, c - \frac{1}{2\delta}\|Q\|_{L^\infty}\right\}\|u\|^2_{H^1_\epsilon(\Omega)}, \quad \forall u \in H^1(\Omega),$$

and we can conclude that there exists $k > 0$, independent of $\epsilon \in [0,1]$, such that

$$a_\epsilon(u,u) \ge k\|u\|^2_{H^1_\epsilon(\Omega)},\ \forall \epsilon \in [0,1] \text{ and } u \in H^1(\Omega).$$

□

Remark 3.5. *Since $\|\cdot\|_{H^1_\epsilon(\Omega)} \ge \|\cdot\|_{H^1(\Omega)}$ for all $\epsilon \in [0,1]$, we also get from Lemma 3.4 that*

$$a_\epsilon(u,u) \ge k\|u\|^2_{H^1(\Omega)},\ \forall \epsilon \in [0,1] \text{ and } u \in H^1(\Omega).$$

As a consequence of Lemma 3.4 we have that the unbounded operator A_ϵ is invertible, and then, we have that $u^\epsilon \in H^1(\Omega)$ is a solution of (2.8), if and only if satisfies $u^\epsilon = A_\epsilon^{-1}F_\epsilon(u^\epsilon)$. That is, u^ϵ must be a fixed point of the nonlinear map

$$A_\epsilon^{-1} \circ F_\epsilon : H^1_\epsilon(\Omega) \mapsto H^1_\epsilon(\Omega) \tag{3.5}$$

wherever $c > \|Q\|^2_{L^\infty(0,H)}/4D$. Under our assumptions the existence of solutions of (2.8) follows from Schauder's Fixed Point Theorem.

Analogously, we have the solutions of the limit problem (2.6) can be obtained as fixed points of the map

$$A_0^{-1} \circ F : H^1(0,1) \mapsto H^1(0,1). \tag{3.6}$$

Moreover, as a consequence of Theorem 2.1, the solutions of (2.6) also can be obtained as limits of the solutions of the perturbed problem (2.8).

Remark 3.6. *It follows from* [18, Theorem 3.2.1.3] *and Remark 3.5 that operator* A_ϵ^{-1} *defines an isomorphism from* $L^2(\Omega)$ *into* $H^2(\Omega)$. *Hence, due to the compact imbedding of* $H^{2-s}(\Omega)$ *into* $H^1(\Omega)$, *for* $0 < s < 1$, *and the continuity of* F_ϵ, *we can conclude that* $A_\epsilon^{-1} F_\epsilon$ *is a continuous and compact mapping for all* $\epsilon > 0$. *Note that we have the same to* $A_0^{-1} F_0$ *since a similar argument can be used to show that.*

3.2. **Concentrating integrals.** Next we colect some results that we need in order to prove the main result.

Lemma 3.7. *Suppose* $v \in H^s(\Omega)$ *with* $1/2 < s \le 1$ *and* $s - 1 \ge -1/q$. *Then, for small* ϵ_0, *there exists a constant* $C > 0$ *independent of* ϵ *and* v, *such that for any* $0 < \epsilon \le \epsilon_0$, *we have*

$$\frac{1}{\epsilon^\alpha} \int_{o_\epsilon} |v(\xi)|^q \, d\xi \le C \, \|v\|^q_{H^s(\Omega)} .$$

Proof. Performing a simple change of variable we get

$$\frac{1}{\epsilon^\alpha} \int_{o_\epsilon} |v(\xi)|^q \, d\xi = \frac{1}{\epsilon^\alpha} \int_0^1 \int_0^{\epsilon^\alpha H_\epsilon(x_1)} |v(x_1, G(x_1) - x_2)|^q \, dx_2 dx_1.$$

Now from [13, Lemma 2.1] we know that there exist ϵ_0 and $C > 0$ independent of ϵ such that

$$\frac{1}{\epsilon^\alpha} \int_{o_\epsilon} |v(\xi)|^q \, d\xi \le C \, \|w\|^q_{H^s(\tau^{-1}(\Omega))} , \quad \forall \epsilon \in (0, \epsilon_0),$$

where $w = v \circ \tau$ and $\tau : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is given by $\tau(x_1, x_2) = (x_1, G(x_1) - x_2)$. Hence, we can conclude the proof of the result using that norms $\|w\|_{H^s(\tau^{-1}(\Omega))}$ and $\|v\|_{H^s(\Omega)}$ are equivalents (for instance, see [14, Section 2]). □

Now we evaluate the convergence of the integrals with nonlinear terms.

Lemma 3.8. *Let* u^ϵ, $\varphi^\epsilon \in H^1_\epsilon(\Omega)$, *and* u, $\varphi \in H^1(0, 1)$ *satisfying* $u^\epsilon \rightharpoonup u$ *and* $\varphi^\epsilon \rightharpoonup \varphi$, $w - H^1(\Omega)$. *Then,*

$$\int_\Gamma \gamma(g(u^\epsilon)) \, \gamma(\varphi^\epsilon) \, dS \to \int_0^1 g(u) \, \varphi \, dx, \tag{3.7}$$

$$\frac{1}{\epsilon^\alpha} \int_{o_\epsilon} f(u^\epsilon) \, \varphi^\epsilon \, dx_1 dx_2 \to \int_0^1 \mu \, f(u) \, \varphi \, dx, \tag{3.8}$$

as $\epsilon \to 0$, *where* μ *is given by* (2.7) *and* $\gamma : H^t(\Omega) \mapsto L^2(\Gamma)$ *is the trace operator for any* $1/2 < t < 1$.

As a consequence we have that $\langle F_\epsilon(u^\epsilon), \varphi^\epsilon \rangle \to \langle F_0(u), \varphi \rangle$ *as* $\epsilon \to 0$.

Proof. Since g is bounded with bounded derivatives, it is clear that g is globally Lipschitz. Hence it is not difficult to see that $g(u^\epsilon) \to g(u)$ in $L^2(\Omega)$ as $\epsilon \to 0$. Moreover,

$$\int_\Gamma \gamma(g(u^\epsilon)) \, \gamma(\varphi^\epsilon) \, dS - \int_0^1 g(u) \, \varphi \, dx = \int_0^1 \gamma \left(g(u^\epsilon) - g(u) \right) \gamma(\varphi^\epsilon) dx - \int_0^1 \gamma \left(\varphi^\epsilon - \varphi \right) g(u) dx.$$

Thus, we obtain (3.7) combine continuity of the trace with $u^\epsilon \rightharpoonup u$ and $\varphi^\epsilon \rightharpoonup \varphi$, $w - H^1(\Omega)$.

Now let us evaluate

$$\frac{1}{\epsilon^\alpha} \int_{o_\epsilon} f(u^\epsilon) \, \varphi^\epsilon \, dx_1 dx_2 - \int_0^1 \mu \, f(u) \, \varphi \, dx = \frac{1}{\epsilon^\alpha} \int_{o_\epsilon} f(u^\epsilon) \, (\varphi^\epsilon - \varphi) \, dx_1 dx_2 + \frac{1}{\epsilon^\alpha} \int_{o_\epsilon} (f(u^\epsilon) - f(u)) \varphi \, dx_1 dx_2$$
$$+ \int_0^1 (G_\epsilon(x) - \mu(x)) f(u) \, \varphi \, dx = I_1 + I_2 + I_3$$

where I_i sets the integrals in an obvious way. We will get $I_i \to 0$ as $\epsilon \to 0$ for each $i = 1, 2, 3$ proving (3.8).

Due to Lemma 3.2, we have $I_3 \to 0$. On the other hand, Lemma 3.7 implies for any $s \in (1/2, 1]$ that

$$|I_1| \le C \, \|f\|_{L^\infty} \, \|\varphi^\epsilon - \varphi\|_{H^s(\Omega)}.$$

Thus, since from $\varphi^\epsilon \rightharpoonup \varphi$, $w - H^1(\Omega)$, we have $\varphi^\epsilon \to \varphi$, $s - H^s(\Omega)$, for all $0 \le s < 1$, then we get $I_1 \to 0$.

Finally, we use that f is globally Lipschitz, as well as Lemma 3.7 to obtain

$$|I_3| \quad \leq \quad \left(\frac{1}{\epsilon^{\alpha}}\|f(u^\epsilon)-f(u)\|^2_{L^2(o_\epsilon)}\right)^{\frac{1}{2}}\left(\frac{1}{\epsilon^{\alpha}}\|\varphi\|^2_{L^2(o_\epsilon)}\right)^{\frac{1}{2}} \leq L_f C\|u^\epsilon-u\|_{H^s(\Omega)}\|\varphi\|_{H^s(\Omega)}.$$

Hence, since $u^\epsilon \to u$, $s-H^s(\Omega)$ for all $0 \leq s < 1$, we also obtain $I_3 \to 0$ as $\epsilon \to 0$. □

Remark 3.9. *For now on, we will omit the trace operator symbol aiming to simplify the written.*

We still need to discuss some properties of the maps $A_\epsilon^{-1}F_\epsilon$. They will be necessary in order to prove the lower semicontinuity of the solutions at $\epsilon = 0$.

Lemma 3.10. *Let $A_\epsilon^{-1}F_\epsilon$ be the maps defined in* (3.5) *and* (3.6)*. Then*

i) *$A_\epsilon^{-1}F_\epsilon$ are compact for each fixed $\epsilon \geq 0$.*
ii) *$\{A_\epsilon^{-1}F_\epsilon(u^\epsilon)\}_{\epsilon\in[0,1)}$ is a pre-compact family whenever $\|u^\epsilon\|_{H^1_\epsilon(\Omega)}$ is uniformly bounded. Indeed, there exist a subsequence, still denoted by $A_\epsilon^{-1}F_\epsilon(u^\epsilon)$, and $u \in H^1(0,1)$, such that*
$$\|A_\epsilon^{-1}F_\epsilon(u^\epsilon) - A_0^{-1}F_0(u)\|_{H^1_\epsilon(\Omega)} \to 0, \qquad \text{as } \epsilon \to 0.$$
iii) *If $\|u^\epsilon - u\|_{H^1_\epsilon(\Omega)} \to 0$ as $\epsilon \to 0$, then $\|A_\epsilon^{-1}F_\epsilon(u^\epsilon) - A_0^{-1}F_0(u)\|_{H^1_\epsilon(\Omega)} \to 0$.*

Proof. Assertion $i)$ is given by Remark 3.6.

To prove ii) we take $u^\epsilon \in H^1_\epsilon(\Omega)$ such that $\|u^\epsilon\|_{H^1_\epsilon(\Omega)} \leq C$. Consequently,

$$\Big\|\frac{\partial u^\epsilon}{\partial x_1}\Big\|^2_{L^2(\Omega)} + \frac{1}{\epsilon^2}\Big\|\frac{\partial u^\epsilon}{\partial x_2}\Big\|^2_{L^2(\Omega)} + \|u^\epsilon\|^2_{L^2(\Omega)} \leq C^2,$$

and so, we have

$$\|u^\epsilon\|_{L^2(\Omega)}, \ \Big\|\frac{\partial u^\epsilon}{\partial x_1}\Big\|_{L^2(\Omega)} \text{ and } \frac{1}{\epsilon}\Big\|\frac{\partial u^\epsilon}{\partial x_2}\Big\|_{L^2(\Omega)} \leq C \quad \forall \epsilon > 0. \tag{3.9}$$

Thus, due to (3.9), we can extract a subsequence of solutions, still denoted by u^ϵ, such that as $\epsilon \to 0$

$$u^\epsilon \rightharpoonup u, \ w - H^1(\Omega), \quad \text{ and } \quad \frac{\partial u^\epsilon}{\partial x_2} \to 0, \ s - L^2(\Omega), \tag{3.10}$$

for some $u \in H^1(\Omega)$. Furthermore, from (3.10) we have that $u(x_1, x_2) = u(x_1)$ in Ω, and then, $u \in H^1(0,1)$. In fact, we get that $\frac{\partial u}{\partial x_2}(x_1,x_2) = 0$ a.e. Ω, since for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$

$$\int_\Omega u\,\frac{\partial \varphi}{\partial x_2}\,dx_1dx_2 = \lim_{\epsilon\to 0}\int_\Omega u^\epsilon\,\frac{\partial \varphi}{\partial x_2}\,dx_1dx_2 = -\lim_{\epsilon\to 0}\int_\Omega \frac{\partial u^\epsilon}{\partial x_2}\,\varphi\,dx_1dx_2 = 0. \tag{3.11}$$

Now let us call $w^\epsilon = A_\epsilon^{-1}F_\epsilon(u^\epsilon)$. Then, w^ϵ satisfies $A_\epsilon w^\epsilon = F_\epsilon(u^\epsilon)$, that is,

$$a_\epsilon(w^\epsilon, w^\epsilon) = \frac{1}{\epsilon^\alpha}\int_{o_\epsilon} f(u^\epsilon)w^\epsilon\,dx_1dx_2 + \int_\Gamma g(u^\epsilon)\,w^\epsilon\,dx_1dx_2.$$

Since we are taking f and g bounded with bounded derivatives, it follows from Lemma 3.4 and 3.7 that w^ϵ is a uniformly bounded sequence in $H^1_\epsilon(\Omega)$. Indeed, since f and g are bounded, it follows from Lemma 3.7

$$k\|w^\epsilon\|^2_{H^1_\epsilon(\Omega)} \leq a_\epsilon(w^\epsilon,w^\epsilon) \leq C^{1/2}\,H^{1/2}\,\|f(u^\epsilon)\|_{L^\infty}\,\|w^\epsilon\|_{L^2(\Omega)} + \|g(u^\epsilon)\|_{L^2(\Gamma)}\,\|w^\epsilon\|_{L^2(\Gamma)} \leq \hat{C}\|w^\epsilon\|_{H^1(\Omega)}.$$

where $\hat{C}(\Omega, f, g, C, k) = \hat{C} > 0$ is independent of ϵ. Hence, since $\|\cdot\|_{H^1_\epsilon(\Omega)} \geq \|\cdot\|_{H^1(\Omega)}$ for $\epsilon \in [0,1]$, we have

$$\|w^\epsilon\|_{L^2(\Omega)}, \ \Big\|\frac{\partial w^\epsilon}{\partial x_1}\Big\|_{L^2(\Omega)} \text{ and } \frac{1}{\epsilon}\Big\|\frac{\partial w^\epsilon}{\partial x_2}\Big\|_{L^2(\Omega)} \leq \hat{C}/k \quad \forall \epsilon \in (0,1].$$

Then, we can also argue as in (3.10) and (3.11) extracting a subsequence, still denoted by w^ϵ, such that

$$w^\epsilon \rightharpoonup w, \ w - H^1(\Omega), \quad \text{ and } \quad \frac{\partial w^\epsilon}{\partial x_2} \to 0, \ s - L^2(\Omega), \tag{3.12}$$

for some $w \in H^1(0,1)$. Hence, we can pass to the limit in $\langle A_\epsilon w^\epsilon, \varphi\rangle = \langle F_\epsilon(u^\epsilon), \varphi\rangle$ taking $\varphi \in H^1(0,1)$. In fact, due to (3.10), (3.12) and Lemma 3.8, we obtain $\langle A_0 w, \varphi\rangle = \langle F_0(u), \varphi\rangle$ for all $\varphi \in H^1(0,1)$, which implies $w = A_0^{-1} F_0(u)$. Moreover, using Lemma 3.8 we obtain

$$a_\epsilon(w^\epsilon, w^\epsilon) = \langle F_\epsilon(u^\epsilon), w^\epsilon\rangle \to \langle F_0(u), w\rangle = a_0(w, w) \text{ as } \epsilon \to 0,$$

proving ii).

Finally let us prove iii). Since we are supposing $\|u^\epsilon - u\|_{H^1_\epsilon(\Omega)} \to 0$, we have $\|u^\epsilon\|_{H^1_\epsilon(\Omega)} \le C$ for some $C > 0$ independent of ϵ. Hence, arguing as in the proof of item ii), for any subsequence, we still can extract another subsequence such that $\|A_\epsilon^{-1} F_\epsilon(u^\epsilon) - A_0^{-1} F_0(u)\|_{H^1_\epsilon(\Omega)} \to 0$ as $\epsilon \to 0$, with $\|u^\epsilon - u\|_{H^1_\epsilon(\Omega)} \to 0$. Thus, since this has been shown for any arbitrary sequence, the proof of item iii) is complete.

□

4. Continuity of the equilibria set

In order to prove Theorem 2.1, we break its two assertions concerning to upper and lower semicontinuity of the equilibria set $\mathcal{E}_\epsilon$ into Lemma 4.1 and 4.2. Initially we consider the upper semicontinuity of the solutions.

Lemma 4.1. *Let u^ϵ be a family of solutions of problem* (2.8).

Then there exist a subsequence, still denoted by u^ϵ, and a function $u \in H^1(\Omega)$, depending only on the first variable, that is, $u(x_1, x_2) = u(x_1)$, solution of the problem (2.6)*, such that*

$$\|u^\epsilon - u\|_{H^1(\Omega)} \to 0, \text{ as } \epsilon \to 0.$$

Proof. First we note that the solutions u^ϵ of (2.8) are uniformly bounded in $H^1(\Omega)$ with respect to ϵ. In fact, $u^\epsilon \in \mathcal{E}_\epsilon$ satisfies (2.8), if and only if

$$\int_\Omega D\left\{\left(\frac{\partial u}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u}{\partial x_2}\frac{\partial \varphi}{\partial x_2}\right) + Q\frac{\partial u}{\partial x_1}\varphi + c\,u\,\varphi\right\} dx_1 dx_2 = \frac{1}{\epsilon^\alpha}\int_{o_\epsilon} f(u^\epsilon)\varphi\, dx_1 dx_2 + \int_\Gamma g(u^\epsilon)\,\varphi\, dS \quad (4.1)$$

for all $\varphi \in H^1(\Omega)$. Hence, taking $\varphi = u^\epsilon$ in (4.1), we get from Lemma 3.4 and 3.7 that

$$k\|u^\epsilon\|^2_{H^1_\epsilon(\Omega)} \le C^{1/2}\, H^{1/2}\, \|f(u^\epsilon)\|_{L^\infty}\, \|u^\epsilon\|_{L^2(\Omega)} + \|g(u^\epsilon)\|_{L^2(\Gamma)}\, \|u^\epsilon\|_{L^2(\Gamma)} \le \hat{C}\|u^\epsilon\|_{H^1(\Omega)},$$

for some $\hat{C} > 0$ independent of ϵ. Recall that f and g are bounded functions. Thus, since $\|\cdot\|_{H^1_\epsilon(\Omega)} \ge \|\cdot\|_{H^1(\Omega)}$ for $\epsilon \in [0,1]$, we have

$$\|u^\epsilon\|_{L^2(\Omega)},\ \left\|\frac{\partial u^\epsilon}{\partial x_1}\right\|_{L^2(\Omega)} \text{ and } \frac{1}{\epsilon}\left\|\frac{\partial u^\epsilon}{\partial x_2}\right\|_{L^2(\Omega)} \le \hat{C}/k \quad \forall \epsilon \in (0,1].$$

Arguing as in (3.10) and (3.11), we can extract a subsequence, still denoted by u^ϵ, such that

$$u^\epsilon \rightharpoonup u,\ w - H^1(\Omega), \quad \text{ and } \quad \frac{\partial u^\epsilon}{\partial x_2} \to 0,\ s - L^2(\Omega), \tag{4.2}$$

for some $u \in H^1(0,1)$. Now we can easily pass to the limit in the variational formulation (4.1). Using Lemma 3.8 and (3.10) we obtain

$$\int_\Omega \{D\, u_x \varphi_x + Q(x_2) u_x\, \varphi + u\,\varphi\}\, dx_1 dx_2 = \int_0^1 \mu\, f(u)\,\varphi\, dx + \int_0^1 g(u)\,\varphi\, dx,$$

whenever $\varphi \in H^1(0,1)$. Hence, since u and φ do not depend on x_2, we have that

$$\int_0^1 H\,\{D\, u_x \varphi_x + q\, u_x\, \varphi + u\,\varphi\}\, dx_1 dx_2 = \int_0^1 \mu\, f(u)\,\varphi\, dx + \int_0^1 g(u)\,\varphi\, dx, \tag{4.3}$$

where q and μ are given by (2.7). (4.3) is the variational formulation of problem (2.6). Also, we note that $a_\epsilon(u^\epsilon, \varphi) \to a_0(u, \varphi)$, as $\epsilon \to 0$, for all $\varphi \in H^1(0,1)$.

Finally we prove convergence of the H^1-norm in order to show strong convergence in $H^1(\Omega)$. We use that the norm is lower semicontinuous with respect to the weak convergence:

$$\|u\|_{H^1(\Omega)} \le \liminf_\epsilon \|u^\epsilon\|_{H^1(\Omega)}. \tag{4.4}$$

In fact, from (4.1), (3.10), (4.3) and (4.4) we obtain

$$\begin{aligned}\int_0^1 H \frac{du}{dx}^2 dx = \int_\Omega |\nabla u|^2 \, dx_1 dx_2 &\leq \liminf_{\epsilon\in(0,1)} \int_\Omega |\nabla u^\epsilon|^2 \, dx_1 dx_2 \leq \limsup_{\epsilon\in(0,1)} \int_\Omega |\nabla u^\epsilon|^2 \, dx_1 dx_2 \\ &\leq \limsup_{\epsilon\in(0,1)} \int_\Omega \left\{ \frac{\partial u^\epsilon}{\partial x_1}^2 + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2}^2 \right\} dx_1 dx_2 \\ &\leq \frac{1}{D} \left\{ - \int_0^1 H \left\{ q \, u_x \, u + u^2 \right\} dx + \int_0^1 \left\{ p \, f(u) + \mu \, g(u) \right\} u \, dx \right\} \\ &= \int_0^1 H \frac{du}{dx}^2 dx.\end{aligned}$$

Therefore $\|u^\epsilon\|_{H^1(\Omega)} \to \|u\|_{H^1(\Omega)}$ and the proof is complete. $\square$

Finally we will show the lower semicontinuity of the state stead solutions u^ϵ at $\epsilon = 0$. As we will see, it is a direct consequence of Lemma 3.10 and [20, Theorem 3].

Lemma 4.2. *Let u be a hyperbolic solution of problem* (2.6).

Then there exists a sequence of solutions u^ϵ of problem (2.8) *such that*

$$\|u^\epsilon - u\|_{H^1(\Omega)} \to 0, \text{ as } \epsilon \to 0.$$

Proof. Note that u being a hyperbolic solution of (2.6) implies that u is an isolated equilibrium by Inverse Theorem applied to the map $\Phi : H^1(0,1) \mapsto H^{-1}(0,1) : u \to a_0(u,\cdot) - \langle F_0(u), \cdot \rangle$. Then, there exists $\delta > 0$ such that u is the unique solution of (2.6) in the open ball $B(u,\delta)$ of radius δ centered at $u \in H^1(0,1)$. Also, its *fixed point index*, relatively to map $A_0^{-1}F_0$, must satisfy $|\mathrm{ind}(u, A_0^{-1}F_0)| = 1$. For instance, we refer to [19] for an appropriated definition of fixed point index.

Now, since the family of compact operators $A_\epsilon^{-1}F_\epsilon$ satisfies items ii) and iii) of Lemma 3.10, and $H^1(0,1) \subset H^1(\Omega)$, it follows from [20, Theorem 3] that there exists $\epsilon_0 > 0$ such that the operator $A_\epsilon^{-1}F_\epsilon$ has at least one fixed point $u^\epsilon \in B(u,\delta)$, for each $\epsilon \in (0,\epsilon_0)$, such that $\|u^\epsilon - u\|_{H^1_\epsilon(\Omega)} \to 0$ as $\epsilon \to 0$. We finish the proof using that $\|\cdot\|_{H^1(\Omega)} \leq \|\cdot\|_{H^1_\epsilon(\Omega)}$ whenever $\epsilon \in (0,1)$. $\square$

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