Lost and Found: An Unpublished $\zeta(2)$-Proof

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Letter from Leonhard Euler to Daniel Bernoulli

St-Petersburg, April 16, 1768

Hochedelgebohrener Hochgeehrtester Herr Professor.

I finally managed to simplify my method to calculate the sum

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}$$

As your worship may remember, I found a derivation that this sum equals $\frac{\pi^2}{6}$ which I myself liked a lot. It was published in the *Journal littéraire d'Allemagne* some years ago. But I wasn't very happy with one thing: The derivation led to the series

The paper Euler refers to appeared in 1743 in the Journal littéraire d'Allemagne, de Suisse et du Nord, 2:1, pp. 115–127, *under the title* Démonstration de la somme de cette suite [6].

$$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\text{etc.}$$

and then the missing terms had to be added to get the result. In that same paper, a direct proof was given. But there I used differential equations; now I have found a more elegant approach, using only the integration by parts result given below.

We start with the well-known formula

$$\int \sin^n \phi\, d\phi=-\frac{1}{n}\sin^{n-1}\phi\cos\phi+\frac{n-1}{n}\int \sin^{n-2}\phi\, d\phi. \tag{1}$$

This formula can be found in Euler's book Institutionem calculi integralis (1766). *But it goes back to James Gregory (1638-1675) who in 1673 proved it on the back of a letter he received from his friend John Collins*.

It will be used from right to left, as your worship can see here:

$$\int \sin^n \phi\, d\phi=\frac{1}{n+1}\sin^{n+1}\phi\cos\phi+\frac{n+2}{n+1}\int \sin^{n+2}\phi\, d\phi.$$

From this formula all the formulas in the following table may be extracted:

$$\begin{aligned}
\int \sin^0 \phi\, d\phi &= \sin\phi\cos\phi+\frac{2}{1}\int \sin^2\phi\, d\phi\\
\int \sin^0 \phi\, d\phi &= \sin\phi\cos\phi+\frac{2}{3}\sin^3\phi\cos\phi+2\frac{4}{3}\int \sin^6\phi\, d\phi\\
\int \sin^0 \phi\, d\phi &= \sin\phi\cos\phi+\frac{2}{3}\sin^3\phi\cos\phi+\frac{2\cdot 4}{3\cdot 5}\sin^5\phi\cos\phi\\
&\quad +2\frac{4\cdot 6}{3\cdot 5}\int \sin^8\phi\, d\phi
\end{aligned}$$

and so on.

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Continuing in the same manner, we see that

$$\phi = \sin\phi\cos\phi + \frac{2}{3}\sin^3\phi\cos\phi + \frac{2\cdot4}{3\cdot5}\sin^5\phi\cos\phi + \frac{2\cdot4\cdot6}{3\cdot5\cdot7}\sin^7\phi\cos\phi + \frac{2\cdot4\cdot6\cdot8}{3\cdot5\cdot7\cdot9}\sin^9\phi\cos\phi + \text{etc.} \tag{2}$$

If we replace ϕ in this formula by arctan t, *we get a series that was already known to Newton (1643-1727). It can be found in his* De Computo Serierum (1684). *Newton applied a finite difference transformation to the well-known Madhava-Gregory series for* arctan.

Here the integral on the left has been replaced by its value.

The terms in this sum are particularly easy to integrate:

$$\frac{\phi^2}{2} = \frac{1}{2}\sin^2\phi + \frac{2}{3}\frac{1}{4}\sin^4\phi + \frac{2\cdot4}{3\cdot5}\frac{1}{6}\sin^6\phi + \frac{2\cdot4\cdot6}{3\cdot5\cdot7}\frac{1}{8}\sin^8\phi + \frac{2\cdot4\cdot6\cdot8}{3\cdot5\cdot7\cdot9}\frac{1}{10}\sin^{10}\phi + \text{etc.} \tag{3}$$

Johann Bernoulli (1667-1748) also found Newton's series for arctan. Dividing both sides by $1 + t^2$ *and integrating, he found this formula* [1, p. 25]. *This series was found independently in 1722 by Takebe Katahiro* [5].

Now we write down the integral of this sum without computing the right-hand side:

$$\frac{\phi^3}{6} = \frac{1}{2}\int\sin^2\phi\,d\phi + \frac{2}{3}\frac{1}{4}\int\sin^4\phi\,d\phi + \frac{2\cdot4}{3\cdot5}\frac{1}{6}\int\sin^6\phi\,d\phi + \frac{2\cdot4\cdot6}{3\cdot5\cdot7}\frac{1}{8}\int\sin^8\phi\,d\phi + \frac{2\cdot4\cdot6\cdot8}{3\cdot5\cdot7\cdot9}\frac{1}{10}\int\sin^{10}\phi\,d\phi + \text{etc.} \tag{4}$$

and we replace ϕ by $\frac{\pi}{2}$. Then the integrals in the sum may be calculated from the first formula I used. Your worship can see the results in the following table:

From here on Euler repeats his arguments from the Journal littéraire.

$$\int\sin^2\phi\,d\phi = \frac{1}{2}\frac{\pi}{2}$$
$$\int\sin^4\phi\,d\phi = \frac{3\cdot1}{4\cdot2}\frac{\pi}{2}$$
$$\int\sin^6\phi\,d\phi = \frac{5\cdot3\cdot1}{6\cdot4\cdot2}\frac{\pi}{2}$$
$$\int\sin^8\phi\,d\phi = \frac{7\cdot5\cdot3\cdot1}{8\cdot6\cdot4\cdot2}\frac{\pi}{2}$$

and so on.

These formulas can already be found in the book Arithmetica Infinitorum (1656) *by John Wallis (1616-1703).*

If we multiply every integral by its coefficient in the series for $\frac{\phi^3}{6}$ or in this case $\frac{\pi^3}{48}$, then we will have

$$\frac{\pi^3}{48} = \frac{1}{2}\frac{1}{2}\frac{\pi}{2} + \frac{2}{3}\frac{1}{4}\frac{3\cdot1}{4\cdot2}\frac{\pi}{2} + \frac{2\cdot4}{3\cdot5}\frac{1}{6}\frac{5\cdot3\cdot1}{6\cdot4\cdot2}\frac{\pi}{2} + \frac{2\cdot4\cdot6}{3\cdot5\cdot7}\frac{1}{8}\frac{7\cdot5\cdot3\cdot1}{8\cdot6\cdot4\cdot2}\frac{\pi}{2} + \text{etc.}$$

Dividing both sides by $\frac{\pi}{2}$ and multiplying by 4 gives us:

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.}$$

So happy was I with this result, that when our new house in stone was finally finished I asked the stonemason to use it in our housenumber. Your worship can admire it in this picture:

I hope everything is well in Basel? My wife Katharina and I wish to bestow upon your worship our sincerest regards. In the meanwhile, I remain your faithful friend and humble and obedient servant.

Leonhard Euler.

This very straightforward derivation in this (apocryphal) letter of the sum of the series defining $\zeta(2)$ cannot be found in this form in Euler's papers. It may be used in a basic calculus class to find the sum of the series in an informal way. But can it be made into a valid proof? Indeed it can. In what follows I will fill in the missing details.

(A) Does the series in (2) converge to the left-hand side? Obviously it does for $\phi = 0$. I will prove that the series converges to ϕ for $-\pi/2 < \phi < \pi/2$.
To do this, we need the Wallis formulas:

$$\int_0^{\pi/2} \sin^{2n-1} t dt = \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdot \ldots \cdot \frac{4}{5} \cdot \frac{2}{3}$$

which can easily be deduced from (1). With them we can rewrite the series in (2) as

$$\cos\phi \cdot \sum_{n=1}^{\infty} \left(\int_0^{\pi/2} \sin^{2n-1} t dt \right) \sin^{2n-1} \phi. \tag{5}$$

Interchanging the sum and the integral is allowed since the series

$$\sum_{n=1}^{\infty} \sin^{2n-1}\phi \cdot \sin^{2n-1} t \tag{6}$$

converges uniformly for all t if $\phi \in (-\pi/2, \pi/2)$. We can prove this with the Weierstrass M-test [2, p. 535][1]: For all t

$$|\sin^{2n-1}\phi \cdot \sin^{2n-1} t| \le M_n = |\sin^{2n-1}\phi|.$$

Furthermore, the series $\sum M_n$ converges: It is a geometric series with ratio $|\sin^2\phi| < 1$ if $\phi \in (-\pi/2, \pi/2)$.

Note that the series (6) itself is a geometric series, with sum

$$\sum_{n=1}^{\infty} \sin^{2n-1}\phi \cdot \sin^{2n-1} t = \frac{\sin\phi \cdot \sin t}{1 - \sin^2\phi \cdot \sin^2 t}.$$

Hence we get for the series (5)

$$\begin{aligned} &\cos\phi \cdot \int_0^{\pi/2} \left(\sum_{n=1}^{\infty} \sin^{2n-1}\phi \cdot \sin^{2n-1} t \right) dt \\ &= \cos\phi \cdot \int_0^{\pi/2} \frac{\sin\phi \cdot \sin t}{1 - \sin^2\phi \cdot \sin^2 t} dt. \end{aligned}$$

Using $\sin^2 t = 1 - \cos^2 t$, we can calculate the integral on the right:

$$\begin{aligned} &\cos\phi \cdot \int_0^{\pi/2} \frac{\sin\phi \cdot \sin t}{\cos^2\phi + (\sin\phi \cdot \cos t)^2} dt \\ &= \cos\phi \cdot \frac{1}{\cos\phi} \left[-\arctan\left(\frac{\sin\phi \cdot \cos t}{\cos\phi} \right) \right]_0^{\pi/2} \\ &= \arctan(\tan\phi) = \phi. \end{aligned}$$

The last step follows from $\phi \in (-\pi/2, \pi/2)$, and it concludes this part of the proof.

(B) If we want to integrate the series (2) term by term, we need to prove uniform convergence. This can be done for the interval $[-t, t]$ with $0 < t < \pi/2$ by using the inequality

$$\left| \frac{2}{3} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{2n-2}{2n-1} \sin^{2n-1}\phi \cdot \cos\phi \right| \le M_n = \sin^{2n-1} t$$

which holds for $|\phi| \le t$. Hence term-by-term integration between 0 and t is allowed.

(C) We now need uniform convergence of the series (3) in $[0, \pi/2]$. Again the Weierstrass M-test does the trick. In this interval we have the following inequality:

$$\begin{aligned} &\left| \frac{2}{3} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{2n-2}{2n-1} \cdot \frac{1}{2n} \sin^{2n}\phi \right| \le M_n \\ &= \frac{2n-2}{2n-1} \cdot \ldots \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{1}{2n}. \end{aligned}$$

It follows from Raabe's (or Gauss's) convergence test (see [2, p. 566]) that the series $\sum M_n$ converges. Hence convergence in (3) is uniform and term-by-term integration between 0 and $\pi/2$ is allowed, resulting in (4).

This concludes the proof.

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REFERENCES

[1] J. Bernoulli, *Opera Omnia Vol. 4*, Lausanne, Genf, 1742 (reprinted Hildesheim, 1968).

[2] R. Courant and F. John, *Introduction to Calculus and Analysis Vol. I*, Springer-Verlag, New York, 1989.

[1] The Weierstrass M-test requires us to find an upper bound M_n on the terms of this series, with M_n independent of t. If $\sum M_n$ converges, then the original series is uniformly convergent.

[3] C. E. Sandifer, ''Euler's solution of the Basel Problem – the longer story'', in *Euler at 300: An Appreciation*, R. E. Bradley, L. A. D'Antonio, and C. E. Sandifer, eds., The MAA Tercentenary Euler Celebration, The Mathematical Association of America, 2007, 105–118.

[4] D. T. Whiteside, ed., *The Mathematical Papers of Isaac Newton 1674-1684*, Cambridge University Press, Cambridge, 1971.

[5] R. Roy, ''The Discovery of the Series Formula for Pi by Leibniz, Gregory and Nilakantha'', *Math. Mag*. **63**, 5 (1990), 291–306.

[6] P. Stäckel, ''Eine vergessene Abhandlung Eulers'', *Bibl. Math*. (3)**8** (1907-1908), 37–54.

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PAUL LEVRIE obtained a doctorate in mathematics from the Catholic University of Leuven in 1987 in the field of numerical analysis. Since then he has been busy teaching and trying to find easy proofs of well-known mathematical results. He coauthors a Dutch blog entitled Wiskunde is Sexy (Mathematics is Sexy), an attempt to make Mathematics more popular in Flanders. For some time now he has been working on the problem of how to get an apparently infinite number of mathematics books into a finite house.

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