Funções Exponenciais e Logarítmicas

$a \in \mathbb{R}$, $a \neq 0$ e $a \neq 1$

•) $n \in \mathbb{N}$, $a^n = \underbrace{a.a \dots a}_{n \text{ vezes}}$ $\quad \begin{cases} a^0 = 1 \\ a^n = a.a^{n-1} \end{cases}$

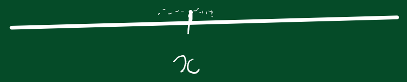
•) $n \in \mathbb{Z}$, $a^n = \begin{cases} a \dots a, & n \geq 0 \\ \frac{1}{a^{-n}}, & n < 0 \end{cases}$

•) $r \in \mathbb{Q} \Rightarrow r = p/q$: $a^r = \sqrt[q]{a^p}$
$q > 0$

•) $x \in \mathbb{R} \Rightarrow a^x = ?$ $\quad 2^{\sqrt{2}} = ?$

$\Big\}$ $a > 0$

$(-2^{1/2} = ?)$
$\notin \mathbb{R}$



$A = \{ r \in \mathbb{Q} : r \leq x \}$

existe sup A

$a^x = \sup \{ a^r : r \in A \}$

$x \longmapsto a^x$ $\qquad f(x) = a^x$ (função exponencial de base a)

•) injetora

•) Imagem: $\mathbb{R}^+$

•) Propriedades: $f(x+y) = f(x).f(y)$

$f(0) = 1$

$f(x.y) = f(x)^y$

•) $a > 1$ crescente

•) $a < 1$ decrescente

$f : \mathbb{R} \to \mathbb{R}^+$ é bijetora

$f^{-1} : \mathbb{R}^+ \to \mathbb{R}$

$x \mapsto f^{-1}(x) = \log_a(x) \quad \Rightarrow \quad y = \log_a x$

$x = a^y$

Logaritmos na base a:

•) injetora

•) Imagem: $\mathbb{R}$

•) Propriedades: $f^{-1}(x.y) = f^{-1}(x) + f^{-1}(y)$

$f^{-1}(1) = 0 \qquad f^{-1}(x^y) = y . f^{-1}(x)$.

Cálculo:

- $f(x) = a^x$ é contínua
- $f^{-1}(x) = \log_a x$ é contínua

$$\lim_{x \to +\infty} a^x = \begin{cases} +\infty, & a > 1 \\ 0, & a < 1 \end{cases}$$

$$\lim_{x \to +\infty} \log_a x = \begin{cases} +\infty, & a > 1 \\ 0, & a < 1 \end{cases}$$

$$\lim_{x \to 0} \log_a x = \begin{cases} -\infty, & a > 1 \\ +\infty, & a < 1 \end{cases}$$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{a^x - a^{x_0}}{x - x_0} \quad \text{ou} \quad f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{a^{x_0+h} - a^{x_0}}{h}$$

$$= \lim_{h \to 0} a^{x_0} \frac{(a^h - 1)}{h}$$

$$= a^{x_0} \cdot \boxed{\lim_{h \to 0} \frac{a^h - 1}{h}}$$

- Existe $a > 0$, $a \neq 1$ tq $f'(0) = 1$?

Se existir, $a = \lim_{x \to \infty} (1 + 1/x)^x$ ou $a = \lim_{x \to 0} (1 + x)^{1/x}$

Ele existe! Indicamos por e. (Veremos depois)

Se $f(x) = e^x$ então $f'(0) = 1 = \lim_{h \to 0} \frac{e^h - 1}{h}$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= e^{x_0} = f(x_0) \therefore f(x) = e^x$$

$$\Downarrow$$

$$\boxed{f'(x_0) = f(x_0)}$$

P/ o logaritmo: $f(x) = e^x$; $f^{-1}(x) = \log_e x = \ln x$

$$f(f^{-1}(x)) = x \Rightarrow f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$$

$$\Rightarrow \underbrace{e^{\ln x}}_{x} \cdot (f^{-1})'(x) = 1$$

$$\Rightarrow (f^{-1})'(x) = \frac{1}{x}$$

Outras bases? $a^b = e^{b \ln a}$

$$e^{b \ln a} = e^{\ln(a^b)} = a^b$$

$$x \longmapsto x \cdot \ln a \to e^{x \cdot \ln a}$$

$$f(x) = a^x = e^{x \cdot \ln a}$$

$$\Downarrow$$

$$f'(x_0) = e^{x_0 \cdot \ln a} \cdot \ln a$$

$$\boxed{f'(x_0) = \ln a \cdot a^{x_0}}$$

$$g(x) = \log_a x$$

$$f(g(x)) = x \Rightarrow f'(g(x_0)) \cdot g'(x_0) = 1$$

$$\Rightarrow \ln a \cdot \underbrace{a^{\log_a x_0}}_{x_0} \cdot g'(x_0) = 1$$

$$\Rightarrow \boxed{g'(x_0) = \frac{1}{x_0 \ln a}}$$

P.A: $a_0, a_0+r, a_0+2r, \ldots, a_0+nr \Rightarrow f(n) = a_0 + rn$

$\Rightarrow f(x) = a_0 + rx$ (função linear)

$f(n+1) - f(n) = a_0 + r(n+1) - (a_0 + rn) = r.$

P.G: $a_0, a_0 \cdot r, a_0 \cdot r^2, \ldots, a_0 r^n \Rightarrow f(n) = a_0 r^n$

$\Rightarrow f(x) = a_0 r^x$

$$f(n+1) - f(n) = a_0 r^{n+1} - a_0 r^n = \underbrace{a_0 r^n}_{f(n)} (r-1) = f(n) \cdot (r-1)$$

$$\Rightarrow f(n+1) = f(n)(r-1) + f(n) = f(n) \cdot r$$

N = nº de cont.

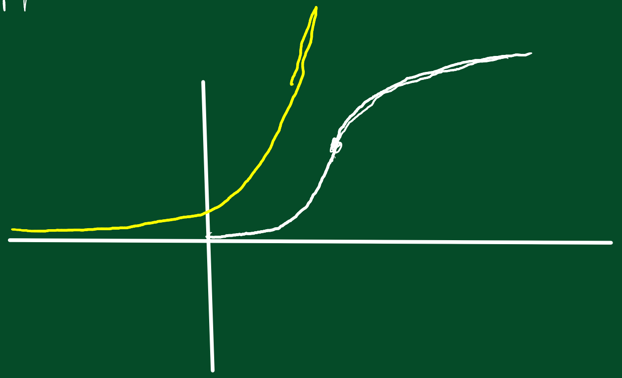
P = prob. de pegar qdo entra em contato

E = espalhar a doença.

$\left(1 - \frac{N_n}{pop.}\right)$

$\Delta N = N_{n+1} - N_n = P \cdot E \cdot N_n$

$$\frac{df}{dt} = c \cdot \left(1 - \frac{f}{pop}\right) \cdot f.$$



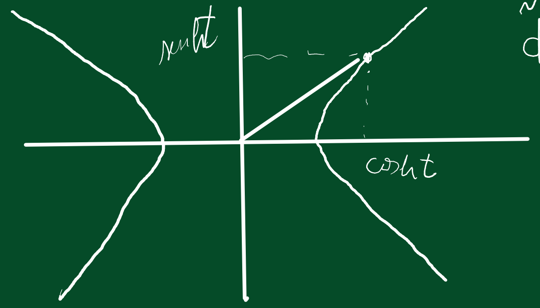
Exercícios;

1) $f(x) = \dfrac{e^{x} + e^{-x}}{2} = \cosh x$

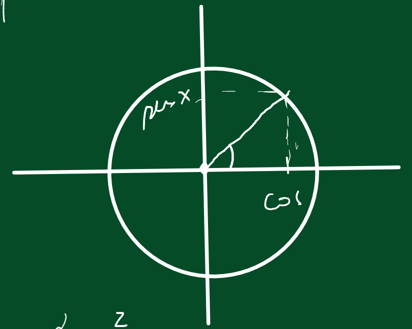
$f'(x) = \dfrac{e^{x} - e^{-x}}{2} = \operatorname{senh} x$

$\cosh^2 x - \operatorname{senh}^2 x = 1$ $\qquad x^2 - y^2 = 1$

$\tilde{d}^{2}(x,y) = x^2 - y^2$



$\operatorname{sen}^2 x + \cos^2 x = 1$



$d^2(x,y) = x^2 + y^2$

2) $f(x) = 2^{x^2} + 3^{2x}$

$f'(x) = (\ln 2).\, 2^{x^2} .\, 2x + (\ln 3).\, 3^{2x} .\, 2$

$= 2 \ln 2 \,.\, x \,.\, 2^{x^2} + 2 \ln 3 \,.\, 3^{2x}$

$\cancel{f(x) = 2^{x}}$

$\cancel{f'(x) = x .\, 2^{x-1}}$

$x \xmapsto{(\ )^2} x^2 \xmapsto{2^{(\ )}} 2^{x^2}$

$2(\ ) \qquad \ln 2 \,.\, 2^{(\ )}$

3) $f(x) = x^{\pi} + \pi^{x} \Rightarrow f'(x) = \pi .\, x^{\pi - 1} + \ln \pi \,.\, \pi^{x}$

4) $f(x) = \dfrac{\ln(x^3 + 2^{x^3})}{x^2 + e^{\cos x}}$