Functorial constructions in paratopological groups reflecting separation axioms

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In honor of Ofelia T. Alas
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Paratopological and semitopological groups

A semitopological group is an abstract group $G$ with topology $\tau$ such that the left and right translations in $G$ are continuous or, equivalently, multiplication in $G$ is separately continuous.

$G^\prime = (G, \tau^{-1})$ is also a paratopological group and the inversion in $G$ is a homeomorphism of $(G, \tau)$ onto $(G, \tau^{-1})$.

Let $\tau^\ast = \tau \lor \tau^{-1}$ be the least upper bound of $\tau$ and $\tau^{-1}$. Then $G^\ast = (G, \tau^\ast)$ is a topological group associated to $G$.

For the Sorgenfrey line $S$, the topological group $S^\ast$ is discrete.
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Let $(G, \tau)$ be a paratopological group and $\tau^{-1} = \{U^{-1} : U \in \tau\}$ be the conjugate topology of $G$. Then $G' = (G, \tau^{-1})$ is also a paratopological group and the inversion in $G$ is a homeomorphism of $(G, \tau)$ onto $(G, \tau^{-1})$. 

For the Sorgenfrey line $S$, the topological group $S^*$ is discrete.
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be the **conjugate** topology of $G$. Then $G' = (G, \tau^{-1})$ is also a paratopological group and the inversion in $G$ is a homeomorphism of $(G, \tau)$ onto $(G, \tau^{-1})$.

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For the Sorgenfrey line $\mathbb{S}$, the topological group $\mathbb{S}^*$ is discrete.
Theorem 1.1 (Alas–Sanchis, 2007).

Let $G$ be a $T_1$ paratopological group. Then the diagonal $\Delta = \{(x, x) : x \in G\}$ is a closed subgroup of $G \times G'$ and, when considered with the topology induced from $G \times G'$, the diagonal $\Delta$ is a Hausdorff topological group topologically isomorphic to the group $G^*$ associated to $G$. 

Corollary 1.2.

Let $H$ be a $T_1$ paratopological group. Then:

a) $H$ is $\sigma$-compact $\iff$ $H^*$ is $\sigma$-compact.

b) $H$ has a countable network $\iff$ $H^*$ has a countable network.

c) If $H$ is second countable, so is $H^*$.

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Corollary 1.3 (Reznichenko, 2005).

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Idea of the proof: If \(G\) is a precompact paratopological group, then the non-empty open sets in \(G_*\) form a \(\pi\)-base for \(G\).
Regularization of paratopological groups

Given a space $X$, let $X_{sr}$ be the underlying set $X$ endowed with the topology whose base is formed by the regular open sets in $X$:

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For any paratopological group $G$, the semiregularization $G_{sr}$ of $G$ is a $T_3$ paratopological group. Hence the semiregularization of a Hausdorff paratopological group is a regular paratopological group.
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The group $G_{sr}$ will be called the regularization of $G$ and denoted by $G_r$. 
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Discussion

Taking the associated topological group $G^*$, the group reflection $G_*$, and the regularization $G_r$ of a paratopological group $G$ are, in fact, covariant functors in the category of paratopological groups and their continuous homomorphisms.
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Another useful functor in the category of topological groups:

$$G \to G/\overline{\{e\}},$$

where $\overline{\{e\}}$ is the closure of the identity $e$ in $G$.
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Question. Is a similar construction possible in paratopological or semitopological groups?
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Further, **if $f : G \rightarrow X$ is a continuous mapping of a topological group $G$ to a $T_1$-space $X$, then there exists a continuous mapping $\bar{f} : T_1(G) \rightarrow X$ such that $f = \bar{f} \circ \pi_G$.**
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Consider the real line \( \mathbb{R} \) with the ‘topology’ \( \tau = \{(r, \infty) : r \in \mathbb{R} \} \). Then \((G, \tau)\) is a \( T_0 \) paratopological group, but \( \{0\} = (-\infty, 0] \).
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Let \( \mathcal{P} \) be a (topological) property and \( G \) a semitopological group. A semitopological group \( H \) is called a \( \mathcal{P} \)-reflection of \( G \) if there exists a continuous homomorphism \( \varphi_G : G \to H \) onto \( H \) satisfying the following conditions:
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The definition of a \( \mathcal{P} \)-reflection in the class of paratopological groups is the same (\( H \) must be a paratopological group).
Reflection of separation axioms

**Theorem 2.2 (Tk., 2013).**

For every $k = 0, 1, 2, 3, 3.5$, there exists a covariant functor $T_k$ in the category of semitopological groups such that $T_k(G)$ is the $T_k$-reflection of $G$, for an arbitrary semitopological group $G$. 

If $k = 0, 1, 2$, then the corresponding continuous homomorphism $\phi_{G,k} : G \to T_k(G)$ is open, so $T_k(G)$ is a quotient group of $G$.

'Top-reflection' means the reflection in the class of spaces satisfying the $T_k$ separation axiom.

Two more functors: $T_1 \to \text{Reg}$ and $T_1 \to \text{Tych}$.

**Corollary 2.3.**

For every semitopological (paratopological) group $G$ and every $k \in \{0, 1, 2, 3, 3.5\}$, there exists a continuous homomorphism $\phi_{G,k} : G \to H$ onto a semitopological (paratopological) group $H$ satisfying the $T_k$ separation axiom such that for every continuous mapping $f : G \to X$ to a $T_k$-space $X$, one can find a continuous mapping $h : H \to X$ with $f = h \circ \phi_{G,k}$. [R stands for regularity.]
Reflection of separation axioms

**Theorem 2.2 (Tk., 2013).**

For every $k = 0, 1, 2, 3, 3.5$, there exists a covariant functor $T_k$ in the category of semitopological groups such that $T_k(G)$ is the $T_k$-reflection of $G$, for an arbitrary semitopological group $G$. If $k = 0, 1, 2$, then the corresponding continuous homomorphism $\varphi_{G,k} : G \to T_k(G)$ is open, so $T_k(G)$ is a quotient group of $G$. 

**Corollary 2.3.**

For every semitopological (paratopological) group $G$ and every $k \in \{0, 1, 2, 3, R\}$, there exists a continuous homomorphism $\varphi_{G,k} : G \to H$ onto a semitopological (paratopological) group $H$ satisfying the $T_k$-separation axiom such that for every continuous mapping $f : G \to X$ to a $T_k$-space $X$, one can find a continuous mapping $h : H \to X$ with $f = h \circ \varphi_{G,k}$. [R stands for regularity.]
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‘Internal’ description of the groups $T_0(G)$

The canonical homomorphism $\varphi_{G,k} : G \to T_k(G)$ is continuous, open, and surjective for $k = 0, 1, 2$ (Theorem 2.2). Hence $T_k(G)$ is a quotient group of $G$ in this case.
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**Conclusion:** To describe the group $T_k(G)$ for $k = 0, 1, 2$ in ‘internal’ terms, it suffices to determine the kernel $N_k$ of the homomorphism $\varphi_{G,k}$. Then $T_k(G) \cong G/N_k$ and $\varphi_{G,k}$ is simply the quotient homomorphism $\pi_k : G \to G/N_k$. 

Let us start with $k = 0$.

**Theorem 3.1.** Let $G$ be an arbitrary semitopological group and $N(e)$ the family of open neighborhoods of the neutral element $e$ in $G$. Then $N_0 = \bigcap_{x \in G} N(e)$, where $P = \bigcap_{x \in G} N(e)$. Hence $T_0(G) \cong G/N_0$.

**Warning:** The subgroup $N_0$ of $G$ is not necessarily closed in $G$.
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Let us start with $k = 0$.

**Theorem 3.1.**

Let $G$ be an arbitrary semitopological group and $\mathcal{N}(e)$ the family of open neighborhoods of the neutral element $e$ in $G$. Then $N_0 = P \cap P^{-1}$, where $P = \bigcap \mathcal{N}(e)$. Hence $T_0(G) \cong G/N_0$. 

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Given a semitopological group \( G \), it is tempting to conjecture that \( N_1 = \bigcap \mathcal{N}(e) \). Unfortunately, this candidate for \( N_1 \) can easily fail to be a subgroup!
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*Let $G$ be an arbitrary semitopological group. Then $N_1$ is the smallest closed subgroup of $G$. Hence $T_1(G) \cong G/N_1$.***
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"Internal’ description of the groups $T_1(G)$"
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**TRY IT!** (A hint follows.)
‘Internal’ description of the groups $T_2(G)$
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**Open problem.** Give an internal description of the kernel $N_2$ of the canonical homomorphism $\varphi_{G,2} : G \to T_2(G)$, for an arbitrary semitopological group $G$. 
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We solve the problem for paratopological groups:

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*Let $G$ be a paratopological group and $\mathcal{N}(e)$ the family of open neighborhoods of the neutral element $e$ in $G$. Then*

\[
N_2 = \bigcap_{U \in \mathcal{N}(e)} \overline{U}
\]

*or, equivalently,*

\[
N_2 = \bigcap_{U \in \mathcal{N}(e)} UU^{-1}.
\]

*Hence $T_2(G) \cong G/N_2$.**
‘Internal’ description of the groups $T_3(G)$ and $\text{Reg}(G)$

Again, we do not know any description of $T_3(G)$ or $\text{Reg}(G)$, for a semitopological group $G$. 
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**Lemma 3.4.**

For every semitopological group $G$, the canonical homomorphism $\varphi_{G,3} : G \to T_3(G)$ is a continuous bijection. Hence the kernel $N_3$ of $\varphi_{G,3}$ is trivial.
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Sometimes the functor $T_3$ ‘collapses’ the topology of a paratopological group $G$: 
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**Example 3.5.**

Let $(\mathbb{R}, +)$ be the additive group of reals and

$$V_n = \{0\} \cup [n, \infty).$$
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Again, we do not know any description of $T_3(G)$ or $\text{Reg}(G)$, for a semitopological group $G$.

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Let $(\mathbb{R}, +)$ be the additive group of reals and

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Then $\{V_n : n \in \mathbb{N}\}$ is a local base at zero for a paratopological group topology $\mathcal{T}$ on $\mathbb{R}$ and the group $G = (\mathbb{R}, \mathcal{T})$ satisfies the $T_1$ separation axiom.
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Theorem 3.6.

$T_3(G)$ is the regularization of $G$, i.e., $T_3(G) \cong G_r$, for every paratopological group $G$. 

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Thus the groups $T_3(G)$ and $G$ coincide algebraically, while the regular open sets in $G$ constitute a base for the topology of $T_3(G)$. 

Theorem 3.7 admits a more general functorial form:

$\text{Reg} \cong T_3 \circ T_2$. 

‘Internal’ description of the groups $T_3(G)$ and $\text{Reg}(G)$

**Theorem 3.6.**

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Here is a two-step description of the groups $\text{Reg}(G)$:

**Theorem 3.7.**

Let $G$ be an arbitrary paratopological group. Then $\text{Reg}(G)$ is the regularization of the paratopological group $T_2(G)$. Therefore, $\text{Reg}(G) \cong (G/N_2)_r$. 
Theorem 3.6.
*T*₃(*G*) is the regularization of *G*, i.e., *T*₃(*G*) ∼ *G*ᵣ, for every paratopological group *G*.

Thus the groups *T*₃(*G*) and *G* coincide algebraically, while the regular open sets in *G* constitute a base for the topology of *T*₃(*G*).

Here is a two-step description of the groups Reg(*G*):

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Theorem 3.7 admits a more general functorial form:

Reg ∼ *T*₃ ∘ *T*₂.
Properties of the functors $T_k$’s

Regularity = $T_1 + T_3$. Does this imply that $\text{Reg} \simeq T_3 \circ T_1$ or $\text{Reg} \simeq T_1 \circ T_3$ in the category of paratopological groups?
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**Theorem 4.1.**

The functors $\text{Reg}$, $T_0 \circ T_3$, $T_1 \circ T_3$ and $T_2 \circ T_3$ are naturally equivalent in the category of semitopological groups.
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$T_2 \circ T_3 \cong T_3 \circ T_2$, i.e., the functors $T_2$ and $T_3$ 'commute' in the category of paratopological groups.
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**Open problem.** Do the functors $T_2$ and $T_3$ commute in the category of semitopological groups?
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Which of the ‘equalities’

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$T_1 \circ T_3 \not\cong T_3 \circ T_1$. Indeed, let $G$ be the group in Example 3.5. We know that $G$ is a $T_1$-space with $|G| = |\mathbb{R}| = 2^\omega$ and $T_3(G)$ is the same group $G$ endowed with the anti-discrete topology.
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$$T_3(T_1(G)) \cong T_3(G)$$

is an infinite group, while $T_1(T_3(G))$ is a trivial one-element group.
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Which of the ‘equalities’

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is an infinite group, while $T_1(T_3(G))$ is a trivial one-element group. Concluding, $|T_3(T_1(G))| = 2^\omega > 1 = |T_1(T_3(G))|$. Similarly, $T_0 \circ T_3 \not\cong T_3 \circ T_0$. 
Products and functors

Let $\Pi = \prod_{i \in I} G_i$ be a product of semitopological (paratopological) groups. We wonder whether the ‘equality’

$$T_k(\Pi) \cong \prod_{i \in I} T_k(G_i)$$

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**Theorem 5.1.**
The functors $T_0$, $T_1$, and $T_2$ commute with arbitrary products of semitopological groups.
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**Theorem 5.1.**

The functors $T_0$, $T_1$, and $T_2$ commute with arbitrary products of semitopological groups.

For each of the functors $T_0$, $T_1$, $T_2$, the proof of Theorem 5.1 is ‘individual’, depending on the form of $N_k = \ker \varphi_{G,k}$ for $k = 0, 1, 2$. 

Products and functors

The case of products of paratopological groups:

**Theorem 5.2.**

*The functors $T_3$ and $\text{Reg}$ commute with arbitrary products of paratopological groups.*
Products and functors

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Sketch of the proof. It is well-known that every product of topological spaces satisfies

$$\left(\prod_{i \in I} X_i\right)_{sr} \cong \prod_{i \in I} (X_i)_{sr}$$

where the subscript ‘$sr$’ stands for the semiregularization.
Products and functors

The case of products of paratopological groups:

**Theorem 5.2.**
The functors $T_3$ and $\text{Reg}$ commute with arbitrary products of paratopological groups.

**Sketch of the proof.** It is well-known that every product of topological spaces satisfies

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Applications of $T_k$-reflections

Extension of Reznichenko’s theorem (Every $\sigma$-compact Hausdorff paratopological group has countable cellularity):
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**Proof of Theorem 6.1.** $G$ is $\sigma$-compact $\implies T_2(G)$ is $\sigma$-compact. Hence, by Lemma 6.2, $c(G) = c(T_2(G)) \leq \omega$. \qed
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In fact, the conclusion of Theorem 6.1 can be strengthened: *Every $\sigma$-compact paratopological group has the Knaster property.*
Applications of $T_k$-reflections

A space $X$ is **Moscow** if every regular closed set in $X$ is the union of a family of $G_\delta$-sets in $X$. 
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Any product of locally feebly compact paratopological groups is a Moscow space.

**Proof.**
Let $G = \prod_{i \in I} G_i$ be a product of locally feebly compact paratopological groups.
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Let $G = \prod_{i \in I} G_i$ be a product of locally feebly compact paratopological groups. We know that $T_2(G) \cong \prod_{i \in I} T_2(G_i)$ (Theorem 5.1).
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On the existence of $T_k$-reflections

**Theorem 6.6 (Pontryagin, $\simeq$ 1935).**

For every continuous real-valued function $f$ on a compact topological group $G$, one can find a continuous homomorphism $\pi : G \to H$ onto a compact metrizable topological group $H$ and a continuous function $g$ on $H$ such that $f = g \circ \pi$. 

Pontryagin's idea: Given a continuous function $f$ on $G$ as above, consider the set $N_f = \{ x \in G : f(AXB) = f(x) \text{ for all } a, b \in G \}$.

Then $N_f$ is a closed invariant subgroup of $G$ and $f$ is constant on each coset of $N_f$ in $G$.

Crucial step: Let us forget about both the compactness of $G$ and topological group structure of $G$ and then apply Pontryagin’s formula directly to a continuous mapping $f : G \to X$ defined on a semitopological group $G$. 

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NEW AMAZING RESULTS TO YOU!!