Resolvability properties of certain topological spaces

István Juhász

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Sao Paulo, Brasil, August 2013
DEFINITION. (Hewitt, 1943; Pearson, 1963)
– A topological space $X$ is \( \kappa \)-resolvable iff it has \( \kappa \) disjoint dense subsets. (resolvable $\equiv$ 2-resolvable)
– $X$ is maximally resolvable iff it is $\Delta(X)$-resolvable, where $\Delta(X) = \min\{|G| : G \neq \emptyset \text{ open in } X\}$.

EXAMPLES:
– $\mathbb{R}$ is maximally resolvable.
– Compact Hausdorff, metric, and linearly ordered spaces are maximally resolvable.

QUESTION. What happens if these properties are relaxed?

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Sao Paulo 2013 2 / 18
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Every crowded countably compact $T_3$ space $X$ is $\omega_1$-resolvable.

NOTE. This fails for $T_2$!

PROOF. (Not Pytkeev's)
Tkachenko (1979): If $Y$ is countably compact $T_3$ with $\text{ls}(Y) \leq \omega$ then $Y$ is scattered.
But every open $G \subset X$ includes a regular closed $Y$, hence $\text{ls}(G) \geq \text{ls}(Y) \geq \omega_1$.
So, any maximal disjoint family of dense left separated subsets of $X$ must be uncountable.

PROBLEM. Is every crowded countably compact $T_3$ space $X$ $c$-resolvable?

NOTE: $\Delta(X) \geq c$.  

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3 / 18
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PROBLEM. (Malychin, 1995) Is a Lindelöf $T_3$ space $X$ with $\Delta(X) > \omega$ resolvable?

NOTE. Malychin constructed Lindelöf irresolvable Hausdorff ($T_2$) spaces, and Pavlov Lindelöf irresolvable Uryson ($T_{2.5}$).}

THEOREM. (Filatova, 2004) YES, every Lindelöf $T_3$ space $X$ with $\Delta(X) > \omega$ is 2-resolvable. This is the main result of her PhD thesis.

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NOTE. For $\Delta(X) > \omega$ regular these suffice. If $\Delta(X) = \lambda$ is singular, we need something extra. For $\Delta(X) = \lambda > \alpha(X)$ we automatically get that $X$ is $<\lambda$-resolvable. But now $\Delta(X) = \lambda > \text{s}(X) +$, so we may use Pavlov's Thm (i). For $\Delta(X) = \lambda > \text{e}(X) +$ we may use Pavlov's Thm (ii).
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If \( X \) is \( T_3 \), \( \Delta(X) \geq \kappa = \text{cf}(\kappa) > \omega \) and \( X \) has no closed discrete subset of size \( \kappa \) then \( X \) is \( \omega \)-resolvable.

NOTE. For \( \Delta(X) > \omega \) regular these suffice. If \( \Delta(X) = \lambda \) is singular, we need something extra.

For \( \Delta(X) = \lambda > s(X) \) we automatically get that \( X \) is \( < \lambda \)-resolvable.

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For any $\kappa \geq \lambda = \text{cf}(\lambda) > \omega$ there is a dense $X \subset D(\mathcal{2})^{\kappa}$ with $\Delta(X) = \kappa$ that is $<\lambda$-resolvable but not $\lambda$-resolvable.

NOTE. This solved a problem of Ceder and Pearson from 1967.

We used the general method of constructing $D$-forced spaces.

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Is this true for each singular $\lambda$?
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THEOREM. (J-S-Sz, 2006)

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DEFINITION. The space $X$ is monotonically normal (MN) iff it is $T_1$ (i.e. all singletons are closed) and it has a monotone normality operator $H$ that "halves" neighbourhoods: $H$ assigns to every $\langle x, U \rangle$, with $x \in U$ open, an open set $H(x, U)$ such that (i) $x \in H(x, U) \subset U$, and (ii) if $H(x, U) \cap H(y, V) \not= \emptyset$ then $x \in V$ or $y \in U$.

FACT. Metric spaces and linearly ordered spaces are MN.

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(i) \( D \subset X \) is strongly discrete if there are pairwise disjoint open sets \( \{ U_x : x \in D \} \) with \( x \in U_x \) for \( x \in D \).

EXAMPLE: Countable discrete sets in \( T_3 \) spaces are SD.

(ii) \( X \) is an SD space if every non-isolated point \( x \in X \) is an SD limit.

THEOREM. (Sharma and Sharma, 1988) Every \( T_1 \) crowded SD space is \( \omega \)-resolvable.

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István Juhász (Rényi Institute)
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Clearly, MN spaces are DSD.

Main results of [J-S-Sz] –

1. If $\kappa$ is measurable then there is a MN space $X$ with $\Delta(X) = \kappa$ that is $\omega_1$-irresolvable.

2. If $X$ is DSD with $|X| < \aleph_\omega$ then $X$ is maximally resolvable.

3. From a supercompact cardinal, it is consistent to have a MN space $X$ with $|X| = \Delta(X) = \aleph_\omega$ that is $\omega_2$-irresolvable.

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DEFINITION.
– An ultrafilter $F$ is $\mu$-descendingly complete iff for any descending $\mu$-sequence $\{A_\alpha : \alpha < \mu\} \subset F$ we have $\bigcap \{A_\alpha : \alpha < \mu\} \in F$.

$\mu$-descendingly incomplete is (now) called $\mu$-decomposable.

– $\Delta(F) = \min\{|A| : A \in F\}$.

– $F$ is maximally decomposable iff it is $\mu$-decomposable for all (infinite) $\mu \leq \Delta(F)$.

FACTS.
– Any "measure" is countably complete, hence $\omega$-indecomposable.
– [Donder, 1988] If there is a not maximally decomposable ultrafilter then there is a measurable cardinal in some inner model.
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Filtration spaces

DEFINITION. – A filtration $F$ is a filtration if $\text{dom}(F) = T$ is an infinitely branching tree (of height $\omega$) and, for each $t \in T$, $F(t)$ is a filter on $S(t)$ that contains all co-finite subsets of $S(t)$.

– The topology $\tau_F$ on $T$: For $G \subset T$, $G \in \tau_F$ iff $t \in G \Rightarrow G \cap S(t) \in F(t)$.

– $X(F) = \langle T, \tau_F \rangle$ is called a filtration space.

FACT. [J-S-Sz] Every filtration space $X(F)$ is MN. Moreover, filtration spaces determine the resolvability behavior of all MN (or DSD) spaces.
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THEOREM. [J-S-Sz]
If $F$ is an ultrafiltration and $\mu \geq \omega$ is a regular cardinal s.t.
$F(t)$ is $\mu$-descendingly complete for all $t \in T = \text{dom}(F)$,
then $X(F)$ is hereditarily $\mu^+$-irresolvable.

COROLLARY. [J-S-Sz]
If $F \in \text{un}(\kappa)$ is a measure and $F(t) = F$ for all $t \in \text{dom}(F) = \kappa < \omega$,
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If $\mathcal{F} \in \text{un}(\kappa)$ is a measure and $F(t) = \mathcal{F}$ for all $t \in \text{dom}(F) = \kappa^{<\omega}$, then $X(F)$ is hereditarily $\omega_1$-irresolvable.
DEFINITION. \([J-M]\)

\(F\) is a \(\lambda\)-filtration if

1. \(\text{dom}(F) \subset \lambda < \omega\),
2. for each \(t \in T\) there is \(\omega \leq \mu_t \leq \lambda\) s.t. \(S(t) = \{\alpha : \alpha_\lambda \alpha_\mu < \mu_t\}\) and \(F(t) \in \text{un}(\mu_t)\),
3. moreover, for any \(\mu < \lambda\) and \(t \in T\): \(\{\alpha : \mu < \alpha_\mu < \mu_t\} \in F(t)\).

NOTE. If \(F\) is a \(\lambda\)-filtration then
\(|X(F)| = \Delta(X(F)) = \lambda\).

- The \(\lambda\)-filtration \(F\) is full if \(\text{dom}(F) = \lambda < \omega\), i.e. \(\mu_t = \lambda\) for all \(t \in \lambda < \omega\).

Full \(\lambda\)-filtrations were considered in \([J-S-Sz]\).
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the two steps of reduction

Lemma 1. [J-S-Sz]
If \( \lambda \) is regular, \( X \) is DSD with \( |X| = \Delta(X) = \lambda \), and there are "dense many" points in \( X \) that are not CAPs of any SD set of size \( \lambda \), then \( X \) is \( \lambda \)-resolvable.

Lemma 2. [J-S-Sz]
For any \( \lambda \geq \omega \), if \( X \) is any space s.t. every point in \( X \) is the CAP of some SD set of size \( \lambda \), then there is a full \( \lambda \)-filtration \( F \) and a one-one continuous map \( g : X(F) \rightarrow X \).

This takes care of the case when \( \lambda \) is regular. The singular case (proved in [J-M]) is similar but more complicated.
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The singular case (proved in [J-M]) is similar but more complicated.
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If $\kappa \leq \lambda$ and $F$ is a $\lambda$-filtration s.t.

(i) for every $t \in T = \text{dom}(F)$, if $\mu_t \geq \kappa$ then $F(t)$ is $\kappa$-decomposable,

(ii) for every $t \in T = \text{dom}(F)$ and $\mu \leq \kappa$, \{ $\alpha < \mu$ : $F(t \uparrow \alpha)$ is $\mu$-decomposable \} $\in F(t)$,

then $X(F)$ is $\kappa$-resolvable.

COROLLARY [J-M]

If every $F \in \text{un}(\mu)$ is maximally decomposable whenever $\omega \leq \mu \leq \lambda$,

then $X(F)$ is $\lambda$-resolvable for any $\lambda$-filtration $F$.
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István Juhász (Rényi Institute) 
Resolvability 
Sao Paulo 2013
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2. For every $t \in T = \text{dom}(F)$ and $\mu \leq \kappa$, \(\{\alpha < \mu \mid F(t \upharpoonright \alpha) \text{ is } \mu\text{-decomposable}\} \in F(t)\),

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\{ \alpha < \mu_t : F(t \cap \alpha) \text{ is } \mu \text{-decomposable} \} \in F(t),
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\( \lambda \)-resolvability of \( \lambda \)-filtration spaces

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\(\lambda\)-resolvability of \(\lambda\)-filtration spaces

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**COROLLARY [J-M]**

If every \(F \in \text{un}(\mu)\) is maximally decomposable whenever \(\omega \leq \mu \leq \lambda\), then \(X(F)\) is \(\lambda\)-resolvable for any \(\lambda\)-filtration \(F\).
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COROLLARY [J-M]

If every $F \in \text{un}(\mu)$ is maximally decomposable whenever $\omega \leq \mu \leq \lambda$, then $X(F)$ is $\lambda$-resolvable for any $\lambda$-filtration $F$. 