Extended abstract

On minimal non-Pfaffian graphs

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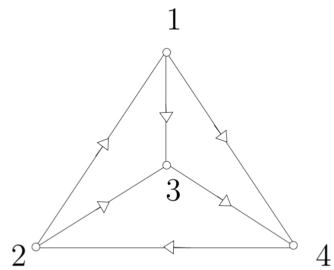
Based on work with Marcelo Carvalho and Cláudio Lucchesi

Lovász and Plummer's book "Matching Theory" is the basic reference for this subject.

Pfaffian Orientations

1. Sign of a perfect matching: If $M := \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ is a perfect matching of a digraph G, then $\mathrm{sgn}(M)$ is the sign of the permutation $\pi(M) := (u_1, v_1, u_2, v_2, \ldots, u_k, v_k)$.



$\pi(M)$	$\mathrm{sgn}(M)$
(2, 1, 3, 4)	−
(1, 3, 4, 2)	+
(1, 4, 2, 3)	+

2. Pfaffian of the adjacency matrix: $\mathrm{Pf}(\mathbf{A}) = \sum_{M \in \mathcal{M}} \mathrm{sgn}(M)\ 1,$

where $\mathcal{M}$ is the set of all perfect matchings.

3. Classical Identity: $\det(\mathbf{A}) = (\mathrm{Pf}(\mathbf{A}))^2$.

4 Alternating cycle: A even cycle in which the edges belong alternately to two different perfect matchings.

5. Oddly-oriented cycle: A even cycle in which there are an odd number of edges whose directions agree with any chosen sense of traversal.

6. Pfaffian orientation: An orientation D of G is *Pfaffian* if all alternating cycles are oddly-oriented. A graph G is *Pfaffian* if it admits a Pfaffian orientation.

7. Theorem 8.3.2, (Lovász and Plummer): An orientation D of G is Pfaffian iff all perfect matchings of G have the same sign in D.

8. Corollary: If D is a Pfaffian orientation of G, then the determinant of the adjacency matrix of D is the square of number of perfect matchings of G.

9. Two Problems: POP: Given a graph G, decide if G is Pfaffian. PREP: Given an orientation D of G, decide if D is Pfaffian. (Vazirani and Yannakakis (1989): The two problems are polynomially-equivalent.)

Tutte (1947) was the first to use Pfaffians in matching theory. (A delightful account of how Tutte was led to Pfaffians is given in his mathematical autobiography: *Graph Theory As I Have Known It.*)

11. Kasteleyn (1963) showed that every planar graph has a Pfaffian orientation.

He showed an orientation of a plane graph in which each cycle bounding a finite face (odd or even) has an odd number of edges directed in the clockwise direction is a Pfaffian orientation of the graph. (An account of Kasteleyn work can be found in Lovász-Plummer.)

12. Similar orientations: Two orientations D and D' of a graph G are *similar* if one can be obtained from the other by reversing the orientations of edges in a cut.

13. Any two Pfaffian orientations of a Pfaffian bipartite matching covered graph are similar.

14. $K_{3,3}$ is not Pfaffian.

Matching Covered Graphs
(1-extendable graphs in Lovász-Plummer)

1. A *matching covered graph* is a nontrivial connected graph in which each edge is in some perfect matching.

Using Tutte's theorem it can be shown that every 2-connected cubic graph is matching covered.

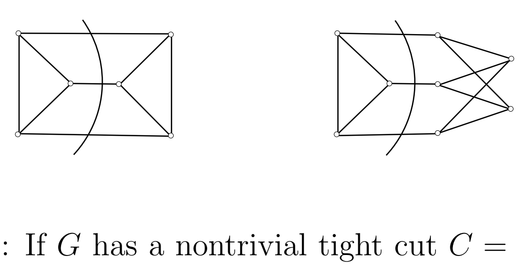
2. Cuts: For a subset X of the vertex set V of a graph G, the set $\partial(X)$ of all edges with exactly one end in X is called a *cut* of G with X and $\overline{X} = V \setminus X$ as *shores*. (For graph theoretical notation, we follow Bondy and Murty's book *Graph Theory.*)

3. Cut-contractions: If $C := \partial(X)$ is a cut of G, the two graphs G/X and $G/\overline{X}$ (obtained from G by shrinking, respectively, X and $\overline{X}$ to single vertices) are called the two C-contractions of G.

4. Separating cuts: A cut C of a matching covered graph G is *separating* if both C-contractions are also matching covered.

5. Tight cuts: A cut C is *tight* if $|C \cap M| = 1$, for each perfect matching M of G.

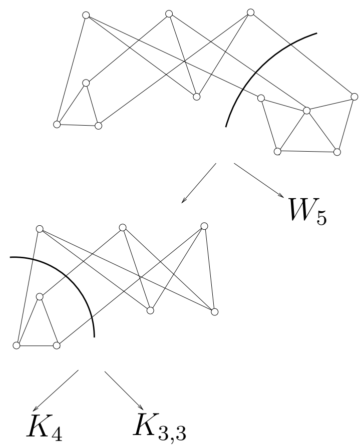
Every tight cut is also a separating cut, but the converse is not true. (Both the cuts shown in the figure below are separating cuts, but only the second cut is tight.)



6. Tight cut decomposition: If G has a nontrivial tight cut $C = \partial(X)$, then G/X and $G/\overline{X}$ are smaller matching covered graphs than G.

If either G/X or $G/\overline{X}$ has a nontrivial tight cut, that graph may be decomposed into even smaller matching covered graphs.

In this manner any matching covered graph may be decomposed into graphs free of nontrivial tight cuts.



7. Bricks and Braces: Of the graphs resulting from a tight cut decomposition of a graph G, those which are non-bipartite are called *bricks* and those which are bipartite are called *braces*.

8. Bricks are 3-connected and *bi-critical* ($G - \{u, v\}$ has a perfect matching for any two distinct vertices u and v).

9. Theorem (Lovász, 1987): Any two tight cut decompositions of a matching covered graph yield the same list of bricks and braces (up to multiple edges).

In particular, the number of bricks of a graph is an invariant of the graph. (The dimension of the linear space generated by the incidence vectors of perfect matchings of m.c. a graph is $|E| - |V| + 2 - b(G)$, where $b(G)$ is the number of bricks of G.)

10. Solid Graphs. A brick does not have any nontrivial tight cuts, but it may have nontrivial separating cuts. (For example, the Petersen graph, which is a brick, has six nontrivial separating cuts.)

A graph G is *solid* if every separating cut is tight.

All bipartite graphs are solid.

11. Solid bricks: A brick G is *solid* iff it is free of nontrivial separating cuts.

Read-Wakabayashi, CLM (2004): A brick G is solid iff, for any two disjoint odd cycles C_1 and C_2, the graph $G - (V(C_1) \cup V(C_2))$ has no perfect matching.

CLM (2004): A brick is solid if and only if its perfect matching polytope is defined by the following set of inequalities:

$$\begin{aligned} x_e &\geq 0, \quad \forall\, e \in E \\ \textstyle\sum_{e \in \partial(v)} x_e &= 1 \quad\ \ \forall\, v \in V. \end{aligned}$$

(Edmonds' description of the perfect matching polytope of a graph includes the so called *odd set constraints*: $\sum_{e\in\partial(S)} x_e \geq 1$, for each odd subset $S \subset V$. What we have shown is that we do not need to consider all odd subsets; we only need to consider subsets S for which $\partial(S)$ is a separating cut. For example, to describe the perfect matching polytope of the Petersen graph, we only need six odd set constraints.)

12. Examples of Solid Bricks: A brick G is *odd-intercyclic* if any two odd cycles of G have at least one vertex in common.

Odd wheels and Möbius ladders are examples of odd-intercyclic bricks—(The Möbius ladder M_{4n} is obtained from a cycle of length $4n$ by joining each vertex to its antipodal vertex.)

Odd-intercyclic bricks are solid by the Reed-Wakabayashi theorem. Not every solid brick is odd inter-cyclic. (First graph on page six is an example.)

We showed that odd wheels are the only simple planar solid bricks.

SOBREP is the problem of deciding if a given brick is solid. We do not know if this problem is in $\mathcal{NP}$. (It is in co-$\mathcal{NP}$ by the Reed-Wakabayashi theorem.)

Cuts and Orientations

Theorem: Let $C := \partial(X)$ be a separating cut of G. If G is Pfaffian, then both G/X and $G/\overline{X}$ are also Pfaffian.

The converse is false in general. (Let G be the Petersen graph, and let X be the vertex set of a pentagon. Then, both G/X and $G/\overline{X}$, being planar, are Pfaffian. But G itself is not Pfaffian.)

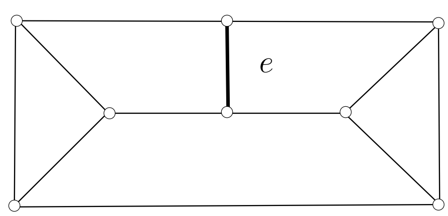
Little and Rendl (1991): The converse is true if C is tight.

A matching covered graph is Pfaffian iff each of its bricks and braces is Pfaffian.

Removable edges and doubletons

An edge e of a matching covered graph G is *removable* if $G - e$ is also matching covered.

The brick shown in the following figure has just one removable edge, namely e.



A pair $\{e, f\}$ of edges of a matching covered graph G is called a *removable doubleton* if neither e, nor f, is individually removable, but $G - \{e, f\}$ is matching covered.

(Möbius ladders M_{4n} have many pairs of removable doubletons.)

Ear Decompositions (Will not be mentioned in the talk.)

A matching covered graph G may not have removable edges or doubletons. For example, this is the case if $|V(G)| \geq 4$, and every edge of G is incident with a vertex of degree two. In such cases, one needs to speak about 'ears' rather than edges. An *ear* in a graph G is a path of odd length all of whose internal vertices have degree two. If P is an ear in G, then $G - P$ denotes the graph obtained from G by deleting all edges and internal vertices of P. An ear P in a matching covered graph G is *removable* if $G - P$ is also matching covered; in this case G is said to be obtained from $G - P$ by *adding* the ear P. Any bipartite matching covered graph G may be obtained from K_2 by a sequence of ear additions. More precisely, there exists a sequence $(G_1, G_2, \ldots, G_r)$ of matching covered subgraphs of G such that (i) $G_1 = K_2$, (ii) $G_r = G$, and (iii) for $1 \leq i \leq r - 1$, the graph G_{i+1} is obtained from G_i by adding an ear. Such a sequence is called an *ear decomposition* of G.

A removable *double ear* in a matching covered graph G is a pair $\{P_1, P_2\}$ of two vertex-disjoint ears such that neither P_1 nor P_2 is individually removable, but $G - P_1 - P_2$ is matching covered; in this case G is said to be obtained from $G - P$ by *adding* the double ear $\{P_1, P_2\}$.

A well-known theorem of Lovász and Plummer states that, given any matching covered graph G, there exists a sequence $(G_1, G_2, \ldots, G_r)$ of matching covered subgraphs of G such that (i) $G_1 = K_2$, (ii) $G_r = G$, and (iii) for $1 \leq i \leq r - 1$, the graph G_{i+1} is obtained from G_i by adding a single or a double ear.

An ear decomposition of a non-bipartite matching covered graph requires at least one double edge addition. But some graphs require the addition of more than one double ear. For example, every ear decomposition of the Petersen graph requires two double ear additions.

A non-bipartite matching covered graph G is near-bipartite if it has an ear decomposition $(G_1, G_2, \ldots, G_r)$ in which the first $r - 1$ graphs are bipartite. (Equivalently, all ear additions, except the last one, are single ear additions.)

S-Minors

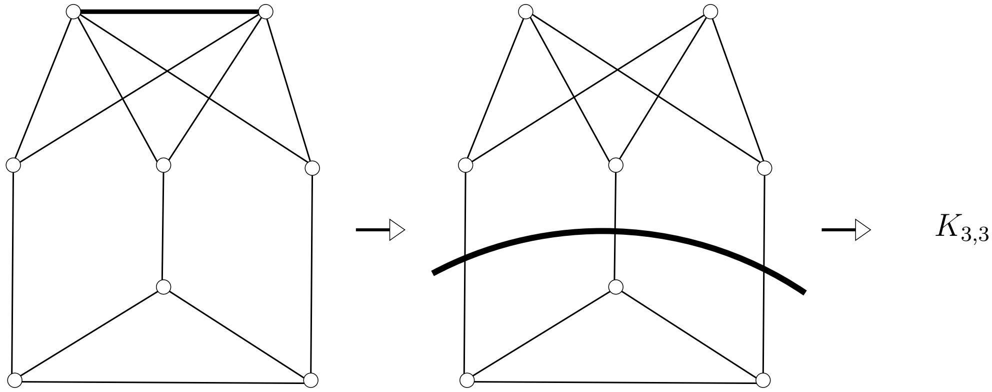
A matching covered graph H is a *separation deletion minor* or an *S-minor* of a matching covered graph G if H can be obtained from G by a sequence of deletions of removable edges or doubletons, and contractions of shores of separating cuts.

Minimal non-Pfaffian Graphs

Theorem: If G is Pfaffian, then every S-minor of G is also Pfaffian.

Definition: A non-Pfaffian matching covered graph G is minimal if every proper S-minor of G is Pfaffian.

We seek to characterize minimal non-Pfaffian graphs à la Kuratowski-Wagner.



Previous work—bipartite graphs

Little (1973): Every bipartite non-Pfaffian graph has $K_{3,3}$ as an S-minor.

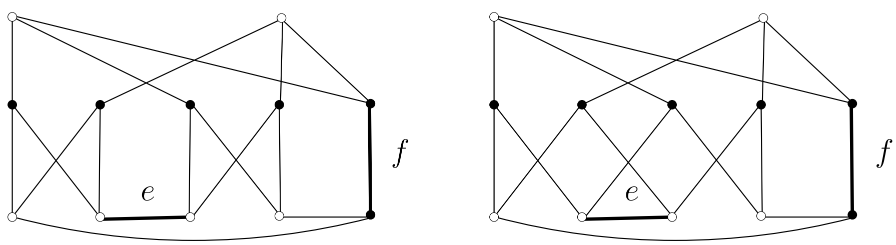
Robertson, Seymour, and Thomas (1999), McCuaig (2004): Polynomial-time algorithm.

The Pfaffian Orientation Problem (POP) turns out to be equivalent to many seemingly unrelated problems.

For example, a digraph D has an even directed cycle if and only if a related bipartite digraph is not Pfaffian.

Previous work—near-bipartite graphs

Fischer and Little (2001): Every near-bipartite non-Pfaffian graph has either $K_{3,3}$, or Γ_1, or Γ_2 as an S-minor. (In each of Γ_1 and Γ_2, the pair $\{e, f\}$ is a removable doubleton, and the deletion of $\{e, f\}$ results in a bipartite graph.)



Lucchesi-Miranda (2008): Polynomial-time algorithm.

Norine and Thomas (2008): 'Generalizing' the above graphs constructed an infinite family of minimal non-Pfaffian bricks.

Our work

Theorem 1.: No minimal non-Pfaffian brick G is solid, therefore:

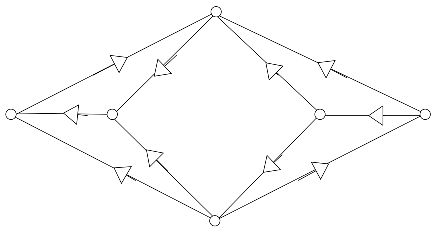
Every minimal non-Pfaffian brick must contain two disjoint odd cycles C_1 and C_2 such that $G - (V(C_1) \cup V(C_2))$ has a perfect matching.

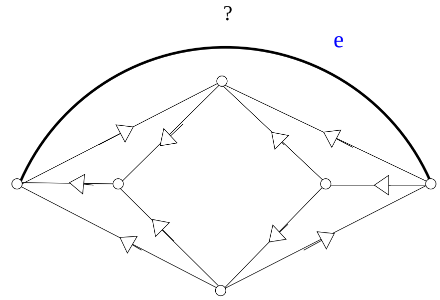
Theorem 2: Every non-Pfaffian solid graph must contain $K_{3,3}$ as an S-minor.

A key ingredient in our proof

If G is a Pfaffian bipartite matching covered graph, and e is *any* removable edge of G, then *any* Pfaffian orientation of $G - e$ may be extended to a Pfaffian orientation of G.

The analogous property does not, in general, hold in case of non-bipartite graphs.





b-Invariant edges: A removable edge of a brick G is *b-invariant* if $G - e$ has exactly one brick.

K_4 and the triangular prism have no removable edges at all.

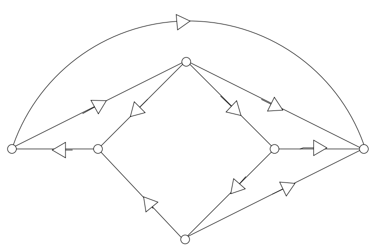
The deletion of any edge from the Petersen graph results in a graph with two bricks!

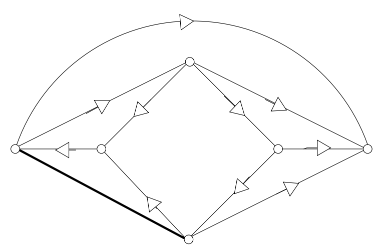
Some years ago we showed:

Theorem (CLM, 2000): Any brick different from K_4, the triangular prism, and the Petersen graph has a b-invariant edge.

The relevance of b-invariant edges to Pfaffian orientations is the following:

Theorem: If G is a Pfaffian brick, and e is *any b-invariant edge* of G, then *any* Pfaffian orientation of $G - e$ may be extended to a Pfaffian orientation of G.





In a solid brick, every removable edge is also b-invariant.

A second key ingredient

Let G be a minimal non-Pfafian bipartite graph, and let e be a removable edge of G. Then, it is easy to see that there is an orientation D of G such that:

- $D - e$ is a Pfaffian orientation of $G - e$;

- there are two perfect matchings M_1 and M_2 of G, both containing e, which have different signs.

We observed the following:

Theorem: Every removable edge of $G - e$ lies in $M_1 \cup M_2$.

Ananlogous statement, with *removable* replaced by *b-invariant* holds for non-bipartite graphs.

Solid bricks have “too many” b-invariant edges which makes it very “inconvenient” for them to be minimal non-Pfaffian graphs.

Some References

I have already mentioned the book *Matching Theory* by Lovász and Plummer.

Tight cut decomposition, and its uniqueness were established by Lovász in a paper entitled *Matching Structure and the Matching Lattice* which appeared in JCT-B in 1987. This paper has been the single most important source of inspiration for our work.

Our results on the existence of b-invariant edges appeared in JCT-B in 2002.

Our results relating solid bricks to the perfect matching polytope appeared in JCT-B in 2004.

This talk is based our paper *A Generalization of Little's Theorem on Pfaffian Orientations* which has been accepted for publication in JCT-B.

The paper by Norine and Thomas on minimal non-Pfaffian graphs appeared in JCT-B in 2008.

They do not restrict themselves to matching covered graphs. So, the operations they use look more complicated. We came up with our notion of minimality independently; the first draft of our paper was written in 2004. When restricted to matching covered graphs, their notion of minimality and ours are the same.

They present an infinite family of minimal non-Pfaffian bricks, and conjecture that every minimal non-Pfaffian graph is either $K_{3,3}$, or the Petersen graph, or is a member of the infinite family they define.