An (l, u)-Transversal Theorem for Bipartite Graphs

Cláudio L. Lucchesi
Department of Computer Science
University of Campinas (UNICAMP)
C. Postal 6065
13081 Campinas, SP - Brasil

Daniel H. Younger*
Department Of Combinatorics and Optimization
University of Waterloo
Waterloo, Ont - Canada N2L 3G1

September 5, 1991

*Research partly supported by CNPq, grant 403103/89, and by FAPESP, grant 91/1494, while visiting UNICAMP.

Abstract

Continuing the work begun by Philip Hall in 1935, we here give necessary and sufficient conditions for the existence, in a bipartite graph, of a set of edges satisfying specified lower and upper bounds. Here the graph is directed bipartite; lower and upper bounds are specified by integer-valued functions, l and u, on the collection of all directed sets of vertices, or perhaps on some subcollection, such as the collection of singletons. We require these functions to be super- and sub-modular, respectively. An (l, u)-*transversal* is a set t of edges that satisfies these bounds. A second restriction, $q \subseteq t \subseteq r$, for edge sets q and r, is also permitted.

One might hope to give necessary and sufficient conditions for the existence of a general (l, u)-transversal. In this paper, this is done for the special case in which the domain of one of the functions, say u, is restricted to singletons. Graph G contains an (l, u)-transversal t such that $q \subseteq t \subseteq r$ if and only if for each X in $Dom\, l$ and each subset N of VG, $lX \leq uN + [q, r](X \oplus N)$. This function $[q, r]$, when applied to a set Y of vertices of G, is the number of edges of r directed away from Y minus the number of edges of q directed toward Y.

This work is motivated by the Woodall Conjecture, which states: in any directed graph, a largest packing of transversals of directed coboundaries is equal in size to a smallest directed cut. We observe that the domain of this Conjecture can be reduced to directed bipartite graphs. For such graphs, the partial (l, u)-theory developed here is used to show that the edge set of any directed bipartite graph can be partitioned into two subsets, one a transversal of directed coboundaries, the other a $(k-1)$-transversal of the vertex coboundaries. In this application we require the supermodularity of the size of a maximum partition of a directed coboundary into directed cuts.

1 Woodall's Conjecture and Directed Bipartite Graphs

Let G be a graph with vertex set VG and edge set eG. For any set X of vertices in G, $\overline{X}$ denotes set $VG \setminus X$. The *coboundary* of a set X of vertices in G is the set of edges that each have one end in X and one end in $\overline{X}$. A set d of edges is a coboundary if there is a set X of vertices such that $d = \delta X$. A *cut* is a minimal nonnull coboundary.

For directed graph G, each edge α of G *leaves* its *positive end* $p\alpha$ and *enters* its *negative end* $n\alpha$. The *constituents* of a coboundary δX are $\delta^+ X := \{\alpha \in \delta X : p\alpha \in X\}$ and $\delta^- X := \{\alpha \in \delta X : n\alpha \in X\}$. Vertex set X is *outdirected* if $\delta^- X = \emptyset$, *indirected* if $\delta^+ X = \emptyset$; in either case X is *directed*. For X a directed set of vertices, δX is *a directed coboundary.* A *directed cut* is a cut that is a directed coboundary.

A *transversal* of directed cuts in G is a set t of edges that intersects (has a nonnull intersection with) each directed cut.

Woodall's Conjecture *In directed graph G, let T^* be a maximum packing of directed cut tranversals; let d_* be a minimum directed cut: then $|T^*| = |d_*|$.*

Woodall [11] described this Conjecture as the Menger dual of the

Lucchesi-Younger Theorem *A minimum directed cut transversal has size equal to that of a maximum packing of directed cuts: $|t_*| = |D^*|$.* [7, 8, 6]

A graph G is *directed bipartite* if its edge set eG is a directed coboundary in G. Equivalently, G has a *directed bipartition*, a bipartition (V^+, V^-) such that each edge α has $p\alpha$ in V^+ and $n\alpha$ in V^-.

The Woodall Conjecture can be reduced, without loss of generality, to directed bipartite graphs. Let G be any directed graph, with maximum packing T^* and minimum directed cut d_*. Let $k := |d_*|$. For each vertex v in G that is not a source or sink, replace v in G by two vertices, a source v^+ and sink v^-, whose incident edges are those in G with positive end v and negative end v, respectively. Add k edges directed from v^+ to v^-. The sources and sinks are left unchanged. Let G' denote the directed bipartite graph thus obtained. Now, each directed cut of G is a directed cut of G'. And each directed cut of G' that is not in G has at least k edges. Moreover, for every packing of transversals of G', the images of these transversals in G make up a packing of directed cut transversals of G. So the minimax equality holds in G if and only if it holds in directed bipartite graph G'.

2 f-Coverings and Supermodular Functions

In this section, we characterize when an f-covering in a directed graph exists in terms of the existence of a supermodular function satisfying a certain inequality.

For directed graph G, let f be a real-valued function defined on some subsets of VG. An *f-covering* is a set t of edges of G such that for each X in the domain $Dom\, f$ of f, $fX \le |t \cap \delta^+ X| - |t \cap \delta^- X|$. Given two real-valued functions f and g, we say that $f \le g$ if $Dom\, f \subseteq Dom\, g$ and, for all X in $Dom\, f$, $fX \le gX$. Note that $\le$ is transitive.

Whenever the domain of f is closed under intersection and union, f is *supermodular* if $f(X \cap Y) + f(X \cup Y) \ge fX + fY$ for each X, Y in $Dom\, f$; f is *submodular* if the reverse inequality holds, and *modular* if equality holds.

For q and r subsets of eG, let $[q,r] := [r]^+ - [q]^-$, where $[r]^+X := |\delta^+ X \cap r|$ and $[q]^- X := |\delta^- X \cap q|$, for all subsets X of VG. Let $[t]$ abbreviate $[t,t]$. Then t is an f-covering if and only if $f \le [t]$.

Proposition 1 *For edge sets q, r and vertex sets X, Y of G,*

$$\begin{aligned}[q,r]X + [q,r]Y &= [q,r](X \cap Y) + [q,r](X \cup Y)\\ &+ |\delta^+ X \cap \delta^- Y \cap r| + |\delta^- X \cap \delta^+ Y \cap r|\\ &- |\delta^+ X \cap \delta^- Y \cap q| - |\delta^- X \cap \delta^+ Y \cap q|.\end{aligned}$$

Proof. The asserted equation follows from

$$\begin{aligned}[r]^+X + [r]^+Y &= [r]^+(X \cap Y) + [r]^+(X \cup Y)\\ &+ |\delta^+ X \cap \delta^- Y \cap r| + |\delta^- X \cap \delta^+ Y \cap r|,\end{aligned}$$

and the similar relation for q.□

Corollary $[q,r]$ *is submodular if $q \subseteq r$ and modular if $q = r$.*□

The main result of this section is

Theorem 2 *For directed graph G, let f be a real-valued function on subsets of the vertex set of G; let q and r be subsets of eG such that $q \subseteq r$. There is an f-covering t satisfying $q \subseteq t \subseteq r$ if and only if there exists a supermodular integer-valued function h such that $f \leq h \leq [q, r]$.*

Proof. For necessity, let t be an f-covering satisfying $q \subseteq t \subseteq r$. Then $f \leq [t] = [t, t] \leq [q, r]$. The inequality is satisfied with $[t]$ in the role of h. By the above Corollary, $[t]$ is modular.

For sufficiency, let h be an integer-valued supermodular function such that $f \leq h \leq [q, r]$. We use induction on $r \setminus q$. If this difference is null, i.e., $q = r$, then $[q, r] = [q, q] = [q]$, whereupon $f \leq [q]$, i.e., q is an f-covering.

Assume then that $r \setminus q$ is nonnull.

Lemma *For each edge α in $r \setminus q$, either $h \leq [q, r \setminus \{\alpha\}]$ or $h \leq [q \cup \{\alpha\}, r]$.*

Proof. Since $h \leq [q, r]$, either

(a) $h \leq [q, r \setminus \{\alpha\}]$, or
(b) $\exists X \subseteq VG$ such that $hX = [q, r]X$ and $\alpha \in \delta^+X$.

Likewise, either

(a) $h \leq [q \cup \{\alpha\}, r]$, or
(b) $\exists Y \subseteq VG$ such that $hY = [q, r]Y$ and $\alpha \in \delta^-Y$.

Suppose that in each case, alternative (b) holds. Then $\alpha \in \delta^+X \cap \delta^-Y$, whence by Proposition 1,

$$[q, r]X + [q, r]Y > [q, r](X \cap Y) + [q, r](X \cup Y).$$

The left side is equal to $hX + hY$; the right side is at least as large as $h(X \cap Y) + h(X \cup Y)$. This contradicts the supermodularity of h. So at least one of alternatives (a) holds. □

Under each of the alternatives of the Lemma, there is by induction hypothesis an f-covering t such that $q \subseteq t \subseteq r$. The Theorem follows by induction. □

3 (l, u)-Transversals for Directed Bipartite Graphs

Consider a graph G with directed bipartition (V^+, V^-). We seek necessary and sufficient conditions for the existence of a set of edges in G satisfying lower and upper bounds, l and u, on directed coboundaries.

The first such Theorem we take to be Hall's [5]: the lower bound is 1 on each vertex in V^+; the upper bound is 1 on each vertex in VG. There exists such an (l, u)-transversal in G if and only if, for each subset X of V^+, $|X| \leq |N|$, where N is the neighbor set of X.

Hall's Theorem has been generalized to arbitrary integer-valued functions l and u on the vertices of G. As a notational device, the domains of l and u are extended to subsets of VG by $lX := \sum\{lv : v \in X\}$ and likewise for u. There exists an (l, u)-transversal in G if and only if, for each X and N such that one of X and N is a subset of V^+ and the other a subset of V^-, the following inequality holds:

$$lX \leq uN + |\delta X \setminus \delta N| \,. \qquad (1)$$

There have been results which extend this Theorem further, so that the domain of one of l and u includes directed subsets of VG other than singletons. Theorems of this type have been found by McWhirter-Younger [9], Rolle [10], Edmonds-Giles [1] and Feofiloff [2, 3]. The generalization described here is easy to relate to special cases, even to Hall's Theorem.

We begin with a general definition of (l, u)-transversal. Let $\mathcal{V}^+$ and $\mathcal{V}^-$ be the collections of all outdirected and indirected subsets of VG, respectively. Let l be an integer-valued function on some subcollection $Dom\, l$ of $\mathcal{V}^+$ that is closed under intersection and union; for X in $\mathcal{V}^+ \cap \mathcal{V}^-$, we take $lX = 0$; finally, l is supermodular. Let u have the same defining properties over $\mathcal{V}^-$, except that u is submodular rather than supermodular. Subset t of eG is an *(l, u)-transversal* if $lX \leq |\,\delta^+ X \cap t\,|$ for each X in $Dom\, l$, and $|\,\delta^- X \cap t\,| \leq uX$ for each X in $Dom\, u$. More compactly, set t of edges is an (l, u)-transversal if $l \leq [t]$ and $-u \leq [t]$. Note that an (l, u)-transversal is a directed cut transversal if $lX \geq 1$ for each X in $\mathcal{V}^+ \setminus \mathcal{V}^-$.

Let the *join* of l and $-u$ be a function $\langle l, -u \rangle$ whose domain is $Dom\, l \cup Dom\, u$ and whose value is lX if $X \in Dom\, l$, and $-uX$ if $X \in$

$Dom\,u$. Set t of edges is an (l,u)-transversal if and only if $\langle l, -u\rangle \leq [t]$. Consequently, an (l,u)-transversal is an $\langle l,-u\rangle$-covering, and conversely.

In the following Theorem, function u is restricted to singletons and co-singletons: $Dom\,u := \{\{v\} : v \in V^-\} \cup \{\overline{\{v\}} : v \in V^+\}$. Under these restrictions, we adopt the following notational conventions: $uv := u\{v\}$ for $v \in V^-$ and $uv := u\overline{\{v\}}$ for $v \in V^+$. Moreover, for N a subset of VG, $uN := \sum\{uv : v \in N\}$. We also assume that $u \geq 0$.

Theorem 3 *Let G be a graph with directed bipartition (V^+, V^-). Let q and r be subsets of eG such that $q \subseteq r$. There exists an (l,u)-transversal t such that $q \subseteq t \subseteq r$ if and only if for each X in $Dom\,l$ and subset N of VG,*

$$lX \leq uN + [q,r](X \oplus N).$$

Remark To see that this inequality generalizes (1), observe that

$$\begin{array}{llllll} \text{if} & X \subseteq V^+, N \subseteq V^-, & \text{then} & \delta X \setminus \delta N & = \delta^+(X \cup N) & = \delta^+(X \oplus N); \\ \text{if} & X \subseteq V^-, N \subseteq V^+, & \text{then} & \delta X \setminus \delta N & = \delta^+(\overline{X} \setminus N) & = \delta^+(\overline{X} \oplus N). \end{array}$$

Proof of Theorem 3. To prove necessity, consider any X in $Dom\,l$ and subset N of VG. Let $N_1 := N \cap X$ and $N_2 := N \setminus X$. Let t be an (l,u)-transversal. By hypothesis, $l \leq [t]$; by the Corollary of Proposition 1, $[t]$ is modular. Thus,

$$lX \;\leq\; [t]X = [t](X \oplus N) + [t]N_1 - [t]N_2.$$

By hypothesis, $[t] \leq u$ and $0 \leq u$, whence

$$\begin{array}{rcccccc} [t]N_1 & \leq & [t]^+N_1 & \leq & u(V^+ \cap N_1) & \leq & uN_1 \\ -[t]N_2 & \leq & [t]^-N_2 & \leq & u(V^- \cap N_2) & \leq & uN_2. \end{array}$$

Using these inequalities, we conclude that

$$\begin{array}{rcl} lX & \leq & uN_1 + uN_2 + [t](X \oplus N) \\ & \leq & uN + [t](X \oplus N). \end{array}$$

By hypothesis, $q \subseteq t \subseteq r$, whence $[t] \leq [q,r]$: the cited inequality holds.

To prove sufficiency, assume that the inequality stated in the Theorem holds. Define function $h : 2^{VG} \to Z$ by

$$hY := max\{lX - uN : Y = X \oplus N, X \in Dom\,l, N \subseteq VG\}.$$

Function h is well-defined since for every subset Y of VG, it is the case that $Y = X \oplus Y$ for $X = \emptyset \in Dom\, l$. We claim that this function h satisfies $\langle l, -u\rangle \leq h \leq [q, r]$ and is supermodular. Assuming this, since h is integer-valued, there is, by Theorem 2, an $\langle l, -u\rangle$-covering t such that $q \subseteq t \subseteq r$. An $\langle l, -u\rangle$-covering is an (l, u)-transversal. So the proof is completed by verifying this claim.

(i) $\langle l, -u\rangle \leq h \leq [q, r]$.

For X in $Dom\, l$, $X = X \oplus \emptyset$, whence $hX \geq lX - u\emptyset = lX$. For $Y \subseteq VG$, $Y = \emptyset \oplus Y$, whence $hY \geq l\emptyset - uY = -uY$. Thus $\langle l, -u\rangle \leq h$.

For Y a subset of VG, there exist X in $Dom\, l$ and subset N of VG, where $Y = X \oplus N$, such that $hY = lX - uN$. Since $lX - uN \leq [q, r](X \oplus N) = [q, r]Y$, there follows $h \leq [q, r]$.

(ii) *h is supermodular.*

Let Y, Y' be subsets of VG. There exist X, X' in $Dom\, l$ and subsets N, N' of VG, where $Y = X \oplus N$, $Y' = X' \oplus N'$, for which

$$\begin{array}{rcrcl} hY & = & lX & - & uN \\ hY' & = & lX' & - & uN'. \end{array}$$

Let $X_I := X \cap X'$, $X_U := X \cup X'$. By the supermodularity of l,

$$lX + lX' \leq lX_I + lX_U.$$

Let $Y_I := Y \cap Y'$, $Y_U := Y \cup Y'$. There is just one pair N_I, N_U of subsets of VG such that $Y_I = X_I \oplus N_I, Y_U = X_U \oplus N_U$, namely, $N_I := X_I \oplus Y_I$, $N_U := X_U \oplus Y_U$. A look at the Venn Diagrams (Figure 1) verifies that $N \cap N' \supseteq N_I \cap N_U$ and $N \cup N' \supseteq N_I \cup N_U$; since u is nonnegative,

$$uN + uN' \geq uN_I + uN_U.$$

Therefore

$$hY + hY' \leq lX_I - uN_I + lX_U - uN_U.$$

The right side of this inequality is at most $hY_I + hY_U$. We conclude that h is supermodular.$\Box$

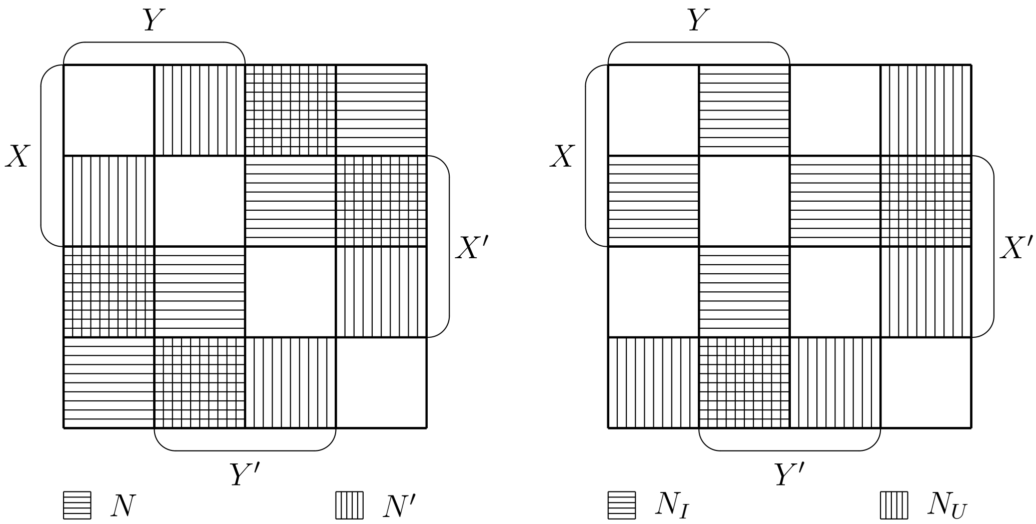

Figure 1: A comparison of N and N' with N_I and N_U.

4 An Application of the (l, u)-Transversal Theorem

Proposition 4 *For directed bipartite graph G, let k be the size of a minimum directed cut. There is a transversal t of directed cuts such that $eG \setminus t$ is a $(k-1)$-transversal of vertex coboundaries.*

A subset t of eG is a $(k-1)$-*transversal of vertex coboundaries* if each nonisolated vertex of G has at least $k-1$ edges of t incident.
Proof of Proposition 4. For each set of vertices X in $\mathcal{V}^+$, denote by μX the cardinality of a maximum partition of δX into nonnull directed coboundaries. Frank, Sebö and Tardos [4] showed that function μ is supermodular. Let l be this function μ. Let u be the function on vertices which assigns each vertex v the value $|\delta\{v\}| - (k-1)$ if v is nonisolated, and 0 otherwise.

Consider the conditions of the (l, u)-transversal Theorem, with $q := \emptyset$ and $r := eG$. If $X \in \mathcal{V}^+$ and $N \subseteq VG$, let $N^+ := N \cap V^+$ and $N^- := N \cap V^-$. By the hypotheses on k and r, and the definition of l and u,

$$\begin{array}{rcl} k \cdot lX & \leq & |\delta X|\,, \\ |\delta N^+| & \leq & k \cdot uN^+, \\ |\delta N^-| & \leq & k \cdot uN^-. \end{array}$$

Every edge of δX either has one end in N or lies in $\delta^+(X \oplus N)$, whence

$$|\delta X| \leq \left|\delta N^+\right| + \left|\delta N^-\right| + \left|\delta^+(X \oplus N)\right|.$$

From these inequalities we conclude that

$$k \cdot lX \leq k\left[uN^+ + uN^- + \left|\delta^+(X \oplus N)\right|/k\right],$$

whence $lX \leq uN + [\emptyset, eG](X \oplus N)$, i.e., the conditions of Theorem 3 are satisfied. By that Theorem, G has an (l, u)-transversal t. For these values of l and u, set t is a transversal of directed cuts and $eG \setminus t$ is a $(k-1)$-transversal of vertex coboundaries.□

While this result is modest, it suggests that a strengthened (l, u)-transversal Theorem could have useful implications for the Woodall Conjecture.