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ON TWO UNSOLVED PROBLEMS CONCERNING MATCHING COVERED GRAPHS*

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Dedicated to the memory of Professor W. T. Tutte on the occasion of the centennial of his birth

Abstract. A cut $C := \partial(X)$ of a matching covered graph G is a *separating cut* if both its C-contractions G/X and $G/\overline{X}$ are also matching covered. A brick is *solid* if it is free of nontrivial separating cuts. Three of us (Carvalho, Lucchesi, and Murty [*J. Combin. Theory Ser. B*, 92 (2004), pp. 319–324]) showed that the perfect matching polytope of a brick may be described without recourse to odd set constraints if and only if it is solid, and we proved (Carvalho, Lucchesi, and Murty [*Discrete Math.*, 306 (2006), pp. 2383–2410]) that the only simple planar solid bricks are the odd wheels. The problem of characterizing nonplanar solid bricks remains unsolved. A *bisubdivision* of a graph J is a graph obtained from J by replacing each of its edges by paths of odd length. A matching covered graph J is a *conformal minor* of a matching covered graph G if there exists a bisubdivision H of J which is a subgraph of G such that $G - V(H)$ has a perfect matching. For a fixed matching covered graph J, a matching covered graph G is *J-based* if J is a conformal minor of G and, otherwise, G is *J-free*. A basic result due to Lovász [*Combinatorica*, 3 (1983), pp. 105–117] states that every nonbipartite matching covered graph is either K_4-based or is $\overline{C_6}$-based or both, where $\overline{C_6}$ is the triangular prism. Two of us (Kothari and Murty [*J. Graph Theory*, 82 (2016), pp. 5–32]) showed that, for any cubic brick J, a matching covered graph G is J-free if and only if each of its bricks is J-free. We also found characterizations of planar bricks which are K_4-free and those which are $\overline{C_6}$-free. Each of these problems remains unsolved in the nonplanar case. In this paper we show that the seemingly unrelated problems of characterizing nonplanar solid bricks on the one hand, and on the other of characterizing nonplanar $\overline{C_6}$-free bricks, are essentially the same. We do this by establishing that a simple nonplanar brick, other than the Petersen graph, is solid if and only if it is $\overline{C_6}$-free. In order to prove this, we first show that any nonsolid brick has one of the four graphs $\overline{C_6}$, the bicorn, the tricorn, and the Petersen graph as a conformal minor. Then, using a powerful theorem due to Norine and Thomas [*J. Combin. Theory Ser. B*, 97 (2007), pp. 769–817], we show that the bicorn, the tricorn, and the Petersen graph are dead-ends in the sense that any simple nonplanar nonsolid brick which contains any one of these three graphs as a proper conformal minor also contains $\overline{C_6}$ as a conformal minor.

Key words. perfect matchings, matching covered graphs, solid graphs, prism-based graphs, conformal minors

AMS subject classifications. 05C70, 05C83

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1. Background and preliminaries.

1.1. Matching covered graphs. For graph theoretical terminology and notation, we essentially follow the book by Bondy and Murty [1]. All graphs considered in this paper are loopless.

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A graph is *matchable* if it has a perfect matching. Tutte [21] established the following fundamental theorem.

THEOREM 1.1 (Tutte's theorem). *A graph G is matchable if and only if*

$$o(G - S) \leq |S|$$

for any subset S of $V(G)$, where $o(G - S)$ denotes the number of odd components of $G - S$.

An edge e of a graph G is *admissible* if there is some perfect matching of G that contains it. A *matching covered* graph is a connected graph of order at least two in which every edge is admissible. A simple argument shows that a matching covered graph cannot have a cut vertex. Tutte [21] used Theorem 1.1 to strengthen a classical result of Petersen [19], by showing that every 2-connected cubic graph is matching covered.

A subset B of the vertex set $V(G)$ of a matchable graph G is a *barrier* if $o(G-B) = |B|$. Using Tutte's theorem, one may easily deduce the following characterization of inadmissible edges.

PROPOSITION 1.2. *An edge e of a matchable graph G is inadmissible if and only if there exists a barrier of G that contains both ends of e.* □

The above proposition yields the following characterization of matching covered graphs.

COROLLARY 1.3. *A connected matchable graph G is matching covered if and only if every barrier of G is a stable set.*

There is an extensive theory of matching covered graphs and its applications. In the book by Lovász and Plummer [16], matching covered graphs are referred to as 1-*extendable graphs*. The terminology we use here was introduced by Lovász in his seminal work [15] and in the follow-up work by three of us in [4]. This work relies on a number of notions introduced and results proved by us and, among others, Lovász and Norine and Thomas. For the benefit of the readers, we shall describe them and provide references. For uniformity, we have found it necessary, in some cases, to change the notation and terminology used in the original sources.

Certain cubic graphs play special roles in this theory. They include the complete graph K_4, and the four graphs shown in Figure 1, namely, the triangular prism which is denoted by $\overline{C_6}$ because it is the complement of the 6-cycle, the bicorn denoted by R_8, the tricorn denoted by R_{10} (as they resemble, in our imagination, the two-cornered and three-cornered hats worn by pirates), and the ubiquitous Petersen graph, which we denote by $\mathbb{P}$.

1.2. Bisubdivisions. A *bisubdivision* of an edge e of a graph J consists of subdividing it by inserting an even number of vertices. A graph H obtained from J by bisubdividing each edge, in any subset of the edges, is called a *bisubdivision* of J. (The term "bisubdivision" is due to McCuaig [17]. The same notion has been called an "even subdivision" by some authors and an "odd subdivision" by some others.) If J is a matching covered graph, then any bisubdivision H of J is also matching covered; in fact, there is a one-to-one correspondence between the sets of perfect matchings of J and of H.

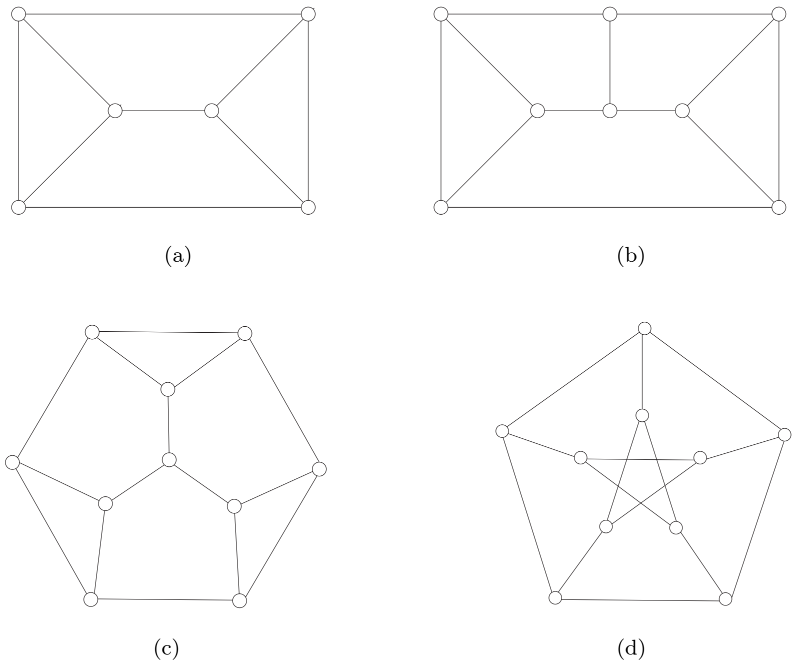


FIG. 1. (a) $\overline{C_6}$, (b) *the bicorn* R_8, (c) *the tricorn* R_{10}, *and* (d) $\mathbb{P}$.

1.3. Splicing and separation.

1.3.1. The operation of splicing. Let G_1 with a specified vertex u, and G_2 with a specified vertex v, be two disjoint graphs. Suppose that the degree of u in G_1 and the degree of v in G_2 are the same and that π is a bijection between the set $\partial_1(u)$ of edges of G_1 incident with u, and the set $\partial_2(v)$ of edges of G_2 incident with v. We denote by $(G_1 \odot G_2)_{u,v,\pi}$ the graph obtained from the union of $G_1 - u$ and $G_2 - v$ by joining, for edge e in $\partial_1(u)$, the end of e in $G_1 - u$ to the end of $\pi(e)$ in $G_2 - v$, and refer to it as the graph obtained by *splicing* G_1 *at* u with G_2 *at* v with respect to the bijection π. The proof of the following proposition is straightforward.

PROPOSITION 1.4. *The graph* $(G_1 \odot G_2)_{u,v,\pi}$ *obtained by splicing two matching covered graphs* G_1 *and* G_2 *is also matching covered.* ∎

In general, the result of splicing two graphs G_1 and G_2 depends on the choices of u, v, π. (Both the pentagonal prism and the Petersen graph can be realized as splicings of two copies of the 5-wheel at their hubs.) However, if H is a vertex-transitive cubic graph, then the result of splicing $G_1 = K_4$ with $G_2 = H$ does not depend, up to isomorphism, on the choices of u, v, and π, and we denote it simply by $K_4 \odot H$. More generally, for any cubic graph H, the result of splicing K_4 and H depends, up to isomorphism, only on the orbit of the automorphism group of H to which v belongs (and the choices of u and π are immaterial); and we denote it simply by $(K_4 \odot H)_v$.

For example, since both K_4 and $\overline{C_6}$ are vertex-transitive, there is only one way of splicing K_4 with itself or with $\overline{C_6}$. Thus $K_4 \odot K_4 = \overline{C_6}$, and $K_4 \odot \overline{C_6}$ is the bicorn. But the automorphism group of the bicorn has three orbits and, consequently, three different graphs (one of which is the tricorn) can be produced by splicing K_4 with the bicorn (Figure 2). The automorphism group of the tricorn also has three orbits and splicing K_4 with the tricorn yields three different graphs (Figure 3).

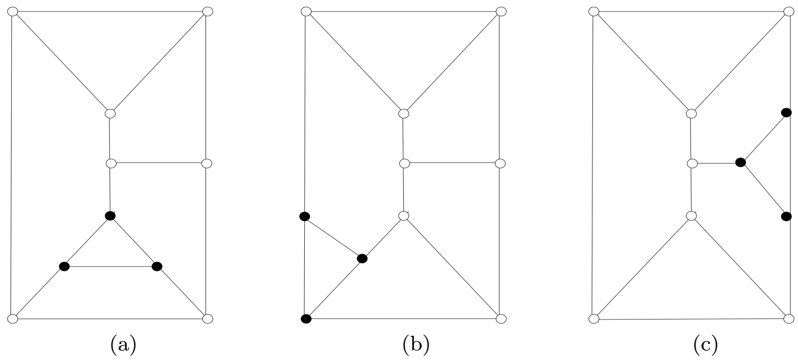


FIG. 2. *Cases of splicing K_4 and the bicorn R_8.*

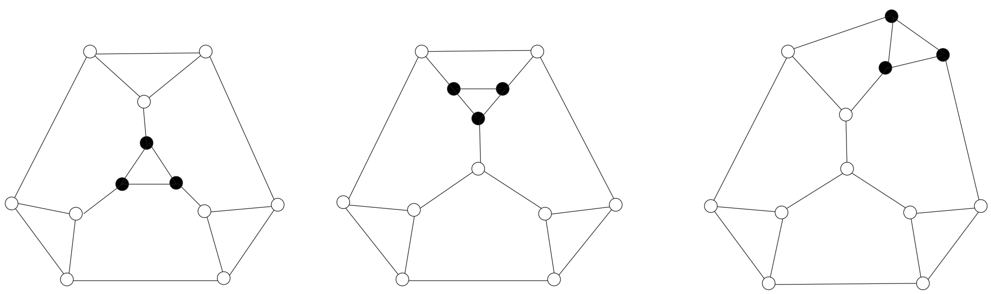


FIG. 3. *Cases of splicing K_4 and the tricorn R_{10}.*

1.3.2. Cuts and cut-contractions. For a subset X of the vertex set $V(G)$ of a graph G, we denote the set of edges of G which have exactly one end in X by $\partial(X)$ and refer to it as the *cut* of X. (For a vertex v of G, we simplify the notation $\partial(\{v\})$ to $\partial(v)$.) If G is connected and $C := \partial(X) = \partial(Y)$, then $Y = X$ or $Y = \overline{X} = V - X$, and we refer to X and $\overline{X}$ as the *shores* of C.

For a cut C of a matching covered graph, the parities of the cardinalities of the two shores are the same. Here, we shall only be concerned with those cuts that have shores of odd cardinality. A cut is *trivial* if either shore has just one vertex and is *nontrivial* otherwise.

Given any cut $C := \partial(X)$ of a graph G, one can obtain a graph by shrinking X to a single vertex x (and deleting any resulting loops); we denote it by $G/(X \to x)$ and refer to the vertex x as its *contraction vertex*. The two graphs $G/(X \to x)$ and $G/(\overline{X} \to \overline{x})$ are the two C-*contractions* of G. When the names of the contraction vertices are irrelevant we shall denote the two C-contractions of G simply by G/X and $G/\overline{X}$.

1.3.3. Separating cuts. A cut $C := \partial(X)$ of a matching covered graph G is *separating* if both the C-contractions of G are also matching covered. All trivial cuts are clearly separating cuts. Figures 4(a) and 4(b) show examples of separating cuts, but the cut indicated in Figure 4(c) is not a separating cut.

The following proposition provides a necessary and sufficient condition under which a cut in a matching covered graph is a separating cut and is easily proved.

PROPOSITION 1.5 (see [4, Lemma 2.19]). *A cut C of a matching covered graph G is a separating cut if and only if, given any edge e, there is a perfect matching M_e of G such that $e \in M_e$ and $|C \cap M_e| = 1$.* ∎

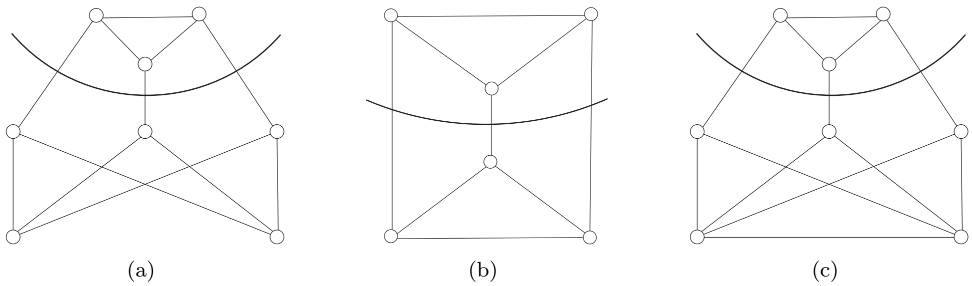


Fig. 4. *Cuts shown in* (a) *and* (b) *are separating cuts, but the one in* (c) *is not.*

Let G_1 and G_2 be two disjoint matching covered graphs. Then, as noted before, any graph $G = (G_1 \odot G_2)_{u,v,\pi}$ obtained by splicing G_1 and G_2 is also matching covered. Clearly the cut $C := \partial(V(G_1) - u) = \partial(V(G_2) - v)$, which we refer to as the *splicing cut*, is a separating cut of G, and G_1 and G_2 are the two C-contractions of G. Conversely, if $C := \partial(X)$ is a separating cut of a matching covered graph G, then G can be recovered from its two C-contractions $G_1 := G/(\overline{X} \to \overline{x})$ and $G_2 := G/(X \to x)$, by splicing them at the contraction vertices with respect to the identity mapping between $\partial_1(\overline{x})$, which is equal to C, and $\partial_2(x)$, which is also equal to C. Thus, a matching covered graph G has a nontrivial separating cut if and only if it can be obtained by splicing two smaller matching covered graphs G_1 and G_2.

1.4. Bricks and braces.

1.4.1. Tight cuts, bricks and braces. A cut C in a matching covered graph G is a *tight cut* of G if $|C \cap M| = 1$ for every perfect matching M of G. It follows from Proposition 1.5 that every tight cut of G is also a separating cut of G. However, the converse does not always hold. For example, the cut shown in Figure 4(b) is a separating cut, but it is not a tight cut.

A matching covered graph, which is free of nontrivial tight cuts, is a *brace* if it is bipartite and is a *brick* if it is nonbipartite.

1.4.2. Tight cut decompositions. Given any matching covered graph G one may apply to it a procedure, called a *tight cut decomposition*, to produce a list of bricks and braces. If G is free of nontrivial tight cuts, then this list simply consists of G. If not, we choose a nontrivial tight cut, say, C, of G and obtain the two C-contractions, say, G_1 and G_2, of G. Since C is nontrivial, the two graphs G_1 and G_2 are matching covered graphs of smaller order than G. We may now apply the tight cut decomposition procedure recursively to each of G_1 and G_2, and then combine the resulting lists to produce a tight cut decomposition of G.

Clearly, any application of the tight cut decomposition procedure on a given matching covered graph produces a list of bricks and braces. Lovász [15] proved the following striking property of this procedure.

Theorem 1.6 (uniqueness of the tight cut decomposition). *Any two applications of the tight cut decomposition procedure on a matching covered graph yield the same list of bricks and braces (up to multiple edges).*

In other words, the list of underlying simple graphs of bricks and braces produced by an application of the tight cut decomposition procedure does not depend on the choices of tight cuts made during the course of the application. In particular, any two applications of the tight cut decomposition procedure on a matching covered graph G yield the same number of bricks; we denote this invariant by $b(G)$ and refer to it as the *number of bricks* of G.

1.4.3. Barrier cuts and 2-separation cuts. Let G be a matching covered graph. If B is a barrier of G, then, for any perfect matching M of G and any odd component K of $G - B$, a simple counting argument shows that $|M \cap \partial(V(K))| = 1$ (and also that $G - B$ has no even components). Consequently, $\partial(V(K))$ is a tight cut of G for any component K of $G - B$. Tight cuts of G which arise in this manner are called *barrier cuts associated with the barrier* B (see Figure 4(a)).

We shall refer to a vertex cut $\{u, v\}$ of G which is not a barrier as a 2-*separation* of G. When $\{u, v\}$ is a 2-separation of G, the fact that $\{u, v\}$ is not a barrier implies that each component of the disconnected graph $G - u - v$ is even. Let S denote the vertex set of the union of a nonempty proper subset of the components of $G - u - v$. It can then be verified that both $C := \partial(S \cup \{u\})$ and $D := \partial(S \cup \{v\})$ are tight cuts of G. Tight cuts which arise in this manner are called 2-*separation cuts*.

An *ELP-cut* in a matching covered graph is a tight cut which is either a barrier cut or is a 2-separation cut. A theorem due to Edmonds, Lovász, and Pulleyblank [11] states that if a matching covered graph has nontrivial tight cuts, then it has an ELP-cut. The following characterization of bricks is a consequence of that basic result.

THEOREM 1.7 (the ELP theorem). *A matching covered graph is a brick if and only if it is* 3*-connected and is free of nontrivial barriers.*

Characterizations of braces can be found in [15] and [16].

1.4.4. Bicontractions and retracts. Suppose that v_0 is a vertex of degree two in a matching covered graph G of order four or more, and let v_1 and v_2 denote the two neighbors of v_0. Then $\{v_1, v_2\}$ is a barrier of G and the barrier cut $\partial(X)$ associated with this barrier, where $X := \{v_0, v_1, v_2\}$, is a tight cut of G. The graph $G/\overline{X}$ is a brace on four vertices, and G/X is a matching covered graph on $|V(G)| - 2$ vertices and is said to be obtained by *bicontracting* the vertex v_0 in G.

Let G be any matching covered graph which has order four or more and is not an even cycle. Then one can obtain a sequence $(G_1, G_2, \ldots, G_r)$ of graphs such that (i) $G_1 = G$, (ii) G_r has no vertices of degree two, and (iii) for $2 \leq i \leq r$, the graph G_i is obtained from G_{i-1} by bicontracting some vertex of degree two in it. Then, up to isomorphism, the graph G_r does not depend on the sequence of bicontractions performed (see [7, Proposition 3.11]). We denote it by $\widehat{G}$ and refer to it as the *retract* of G. The retract of a bisubdivision H of a brick J is J itself. The notion of the retract of a matching covered graph will play an important role in the last section of this article.

1.4.5. Five special families of bricks. We now describe five families of bricks that are of particular interest in this work.

(i) *Wheels.* For each odd integer $k \geq 3$, the wheel W_k with k spokes is a brick.

(ii) *Prisms.* For each integer $k \geq 3$, the k-prism, which we denote by P_{2k}, is the Cartesian product of the k-cycle C_k and K_2. For each odd $k \geq 3$, the prism P_{2k} is a brick (and for each even $k \geq 4$, the prism P_{2k} is a brace). McCuaig [17] and Norine and Thomas [18] refer to prisms as planar ladders.

(iii) *Möbius ladders.* For each integer integer $k \geq 2$, the Möbius ladder of order $2k$, which we denote by $\mathbb{M}_{2k}$, is the cubic graph obtained from the cycle C_{2k} by joining each vertex to the one that is antipodal to it. The Möbius ladder $\mathbb{M}_{2k}$ is a brick if k is even and is a brace if k is odd.

(iv) *Truncated biwheels.* Let $(v_1, v_2, \ldots, v_{2k})$ be a path of odd length, where $k \geq 2$, and let h and h' be two vertices (*hubs*) not on that path. We shall refer to the graph obtained by joining the hub h to the k vertices in $\{v_1, v_3, \ldots, v_{2k-1}\} \cup \{v_{2k}\}$, and joining the hub h' to the k vertices in $\{v_1\} \cup \{v_2, v_4, \ldots, v_{2k}\}$ as a *truncated biwheel*. We denote it by T_{2k+2}. The smallest truncated biwheel is isomorphic to $\overline{C_6}$. Every truncated biwheel is a brick. Norine and Thomas [18] refer to truncated biwheels as lower prismoids. The reason we have chosen to call them truncated biwheels is because they are closely related to braces known as biwheels. See [9].

(v) *Staircases.* Let $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_k)$ be two disjoint paths of length at least one. The graph obtained from the union of these two paths by adjoining two new vertices x and y, and joining u_i to v_i, for $1 \leq i \leq k$, and, in addition joining x to u_1 and v_1, y to u_k and v_k, and x and y to each other, is referred to as a *staircase* by Norine and Thomas [18]. We denote it by S_{2k+2}. The staircase on six vertices is isomorphic to the triangular prism. Every staircase is a brick.

For illustrations and for further details concerning these families of bricks, we refer the reader to [9] and [13].

1.4.6. Near-bricks. Let G be a matching covered graph and let $C := \partial(X)$ be a separating cut of G such that the subgraph $G[X]$ induced by X is bipartite. As C is a separating cut, by definition, the C-contraction $G_1 := G/(\overline{X} \to \overline{x})$ is matching covered and thus $|X|$ is odd. So, one of the color classes of $G[X]$ is larger than the other. We denote the larger color class by X_+ and the smaller color class by X_- and refer to them, respectively, as the *majority* and *minority* parts of X. If the contraction vertex $\overline{x}$ were joined by an edge to a vertex v in the minority part, then the graph $G_1 - \{\overline{x}, v\}$ would be a bipartite graph with color classes of different cardinalities, implying that there is no perfect matching of G_1 which contains the edge $\overline{x}v$. This is impossible because G_1 is matching covered. The following results may be easily deduced from this observation.

PROPOSITION 1.8. *Let $C := \partial(X)$ be a separating cut of a matching covered graph G such that the subgraph $G[X]$ is bipartite. Then the majority part X_+ of X is a barrier of G, and C is a tight cut associated with this barrier.* □

COROLLARY 1.9. *A cut of a bipartite matching covered graph is separating if and only if it is tight.* □

COROLLARY 1.10. *Let G be a matching covered graph. Then, G is bipartite if and only if $b(G) = 0$.* □

COROLLARY 1.11. *Let G be a matching covered graph with $b(G) = 1$, and let C be a tight cut of G. Then one of the C-contractions of G is bipartite, the majority part of that shore is a barrier of G, and C is a barrier cut associated with that barrier.*

We refer to a matching covered graph G with $b(G) = 1$ as a *near-brick*. Properties of near-bricks are in many ways akin to those of bricks and, in trying to prove statements concerning bricks by induction, it is often convenient to try to prove the corresponding statements for near-bricks.

1.5. Solid bricks. A matching covered graph is *solid* if every separating cut of G is a tight cut. In particular, any bipartite matching covered graph is solid. That is, in a bipartite matching covered graph, every separating cut is also a tight cut (Corollary 1.9). However, nonbipartite graphs, even bricks, may have separating cuts which are not tight. (For example, see Figure 4(b).) *Solid bricks* are precisely those bricks which are free of nontrivial separating cuts. It can be verified that the graph shown in Figure 4(c) is a solid brick. The following theorem is a consequence of [4, Theorem 2.25].

THEOREM 1.12. *Let G be a matching covered graph and let G_1 and G_2 be the two C-contractions with respect to a tight cut C of G. Then G is solid if and only if both G_1 and G_2 are solid.* ☐

COROLLARY 1.13. *A matching covered graph G is solid if and only if each of its bricks is solid.* ☐

The notion of solid matching covered graphs was introduced in [4] by three of us (CLM—Carvalho, Lucchesi, and Murty). We noted there that certain special properties that are enjoyed by bipartite graphs are shared by the more general class of solid matching covered graphs and exploited these properties in establishing the validity of a conjecture due to Lovász.

In a later paper [6], we (CLM) showed that bipartite matching covered graphs and solid near-bricks share the property that their perfect matching polytopes may be defined without using the odd set inequalities.

The problem of recognizing solid bricks is in co-$\mathcal{NP}$, since any nontrivial separating cut serves as a certificate for demonstrating that a brick is nonsolid. In the same paper mentioned above, we presented a proof of the following (unpublished) theorem due to Reed and Wakabayashi [20] that provides another succinct certificate for demonstrating that a brick is nonsolid.

THEOREM 1.14 (Reed and Wakabayashi). *A brick G has a nontrivial separating cut if and only if it has two vertex-disjoint odd cycles C_1 and C_2 such that $G-(V(C_1)\cup V(C_2))$ has a perfect matching.* ☐

We showed in [8] that the only simple planar solid bricks are the odd wheels. The solid-brick-recognition problem remains unsolved for nonplanar graphs.

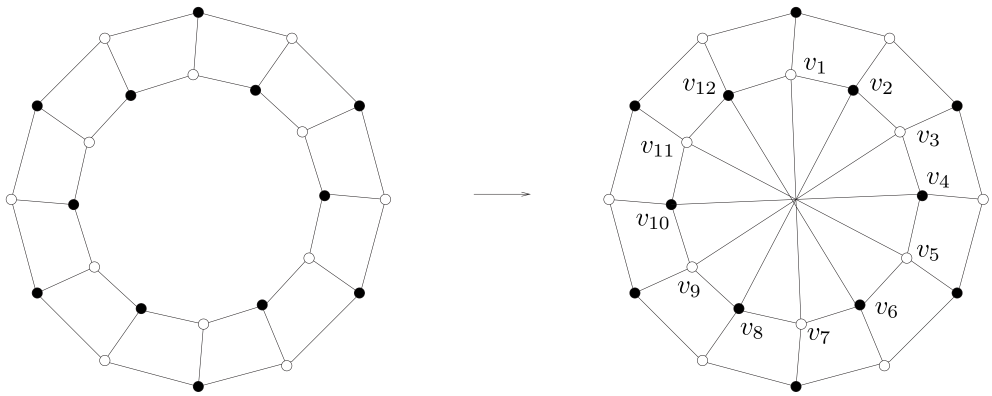
UNSOLVED PROBLEM 1.15. *Characterize solid bricks. (Is the problem of deciding whether or not a given brick is solid in the complexity class $\mathcal{NP}$? Is it in $\mathcal{P}$?)*

As stated in the abstract, the objective of this paper is to establish a connection between this unsolved problem and another basic problem (Problem 1.34) concerning matching covered graphs.

A graph is *odd-intercyclic* if any two odd cycles in it have a vertex in common. (Odd wheels and Möbius ladders of order $4k$, $k \geq 1$, are examples of odd-intercyclic bricks.) It follows from Theorem 1.14 that every odd-intercyclic brick is solid.

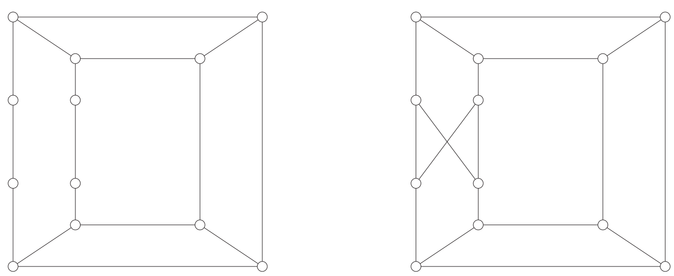
Kawarabayashi and Ozeki [12] showed that an internally 4-connected graph G is odd-intercyclic if and only if it satisfies one of the following conditions: (i) $G - v$ is bipartite for some $v \in V$, (ii) $G - \{e_1, e_2, e_3\}$ is bipartite for some three edges e_1, e_2, and e_3 which constitute the edges of a triangle of G, (iii) $|V| \leq 5$, or (iv) G can be embedded in the projective plane so that each face boundary has even length.

The above result leads to a polynomial-time algorithm for deciding whether or not a given internally 4-connected graph is odd-intercyclic. However, not all solid bricks are odd-intercyclic, nor are they necessarily internally 4-connected. For example, the graph shown in Figure 4(c) is a solid brick which is not odd-intercyclic. We describe below a general procedure for constructing odd-intercyclic graphs of type (iv) mentioned in the previous paragraph and then present two infinite families of solid bricks which are not odd-intercyclic.



H G

FIG. 5. *A construction of odd-intercyclic graphs.*



(a) H (b) G

FIG. 6. *An example of a cubic solid brick that is not a Möbius ladder.*

An infinite family of odd-intercyclic bricks. An infinite class of odd-intercyclic graphs may be obtained as follows. Let H be a 2-connected planar bipartite graph and let $(v_1, v_2, \ldots, v_{2k})$ be a facial cycle of H, where k is even. Obtain G from H by joining, for $1 \leq i \leq k$, the vertices v_i and v_{i+k} by a new edge. Such a graph G has an embedding on the projective plane so that all faces are even and it is not too difficult to see that it is odd-intercyclic. See Figure 5 for an example, where $k = 6$.

In the above construction, if H is just a cycle of length $2k$, where $k \geq 2$ is an even integer, the resulting brick is the Möbius ladder $\mathbb{M}_{2k}$.

To obtain an odd-intercyclic brick using the above construction, it is not always necessary to add all the chords $v_i v_{i+k}$; in some cases, it is adequate to add just "two crossing chords" as illustrated in Figure 6. The graph G shown in that figure is a cubic odd-intercyclic brick. It can be checked that it is not isomorphic to the Möbius ladder $\mathbb{M}_{12}$. (The Möbius ladder $\mathbb{M}_{12}$ does not have 5-cycles whereas the brick G shown in Figure 6 has 5-cycles.)

A family of solid bricks obtained from bipartite Möbius ladders. Let $n \geq 8$ be an integer which is divisible by four. Then $k = (n-2)/2$ is an odd integer. Consider the Möbius ladder $\mathbb{M}_{2k}$ (a brace) which is obtained from the $2k$-cycle $(0, 1, 2, \ldots, 2k-2, 2k-1)$ by joining each vertex i to its antipode $i + k$ for

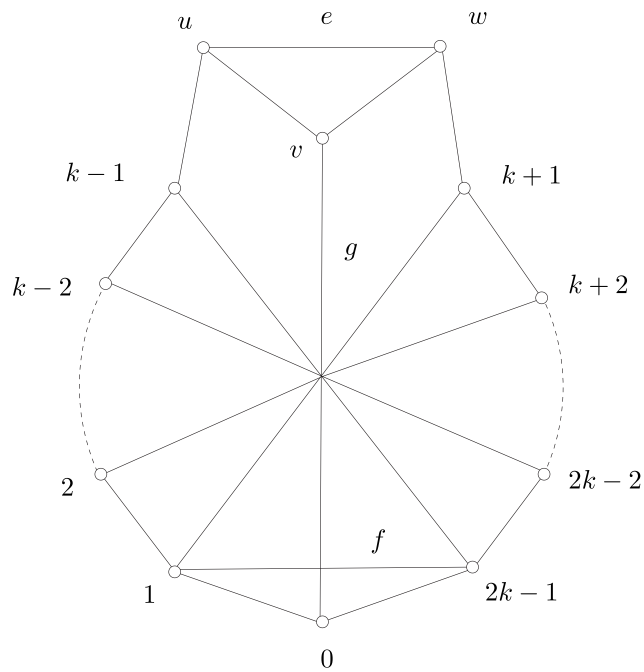


FIG. 7. *The solid brick* $G := \Sigma_n$.

$0 \leq i \leq k-1$. Let H denote the cubic graph obtained from $\mathbb{M}_{2k}$ by deleting the vertex k, adding three new vertices u, v, and w, and joining u to $k-1$, v to 0 by an edge labeled g, vertex w to $k+1$, and u and w to each other by an edge labeled e. (Thus H is obtained by splicing $\mathbb{M}_{2k}$ and K_4.) Now add to H an edge joining 1 and $2k-1$ by an edge labeled f to obtain the graph of order $n = 2k+2$, which we shall denote Σ_n. This construction is illustrated in Figure 7. (Note that Σ_8 is the same as the brick shown in Figure 4(c).) It is straightforward to show that the deletion of any two vertices from $G := \Sigma_n$ results in a connected graph with a perfect matching and deduce, using Theorem 1.7, that G is a brick for all integers $n \geq 8$ which are divisible by four.

A somewhat more involved argument is necessary to show that G is solid. Toward this end, let C_1 and C_2 be any two vertex-disjoint odd cycles of G. As may easily be verified, the graph $G - \{e, f, g\}$ is bipartite. Also any cycle that passes through g also passes through at least one end of e and at least one end of f. Thus, we may assume without loss of generality that $e \in E(C_1)$, $f \in E(C_2)$ and $g \notin E(C_1) \cup E(C_2)$. Now we note that the graph $G - \{v, 0\}$ is a bisubdivision of the odd-intercyclic brick $\mathbb{M}_{n-4}$. So, $(V(C_1) \cup V(C_2)) \cap \{v, 0\}$ is not empty. If this intersection consists of just one of the two vertices v and 0, then the other is an isolated vertex of $G - (V(C_1) \cup V(C_2))$ implying that this graph has no perfect matching. On the other hand, if both the vertices v and 0 belong to $V(C_1) \cup V(C_2)$, then C_1 would have to be the triangle (u, v, w, u) and C_2 would have to be the triangle $(1, 0, 2k-1, 1)$ and, as can be easily checked, the graph $G - (V(C_1) \cup V(C_2))$ has no perfect matching. Now it follows from Theorem 1.14 that G is solid.

A family of solid bricks obtained from odd wheels. Let $n \geq 12$ be an even integer. If we set $k := (n-6)/2$, then $2k+1 \geq 7$ is an odd integer. Let W_{2k+1}

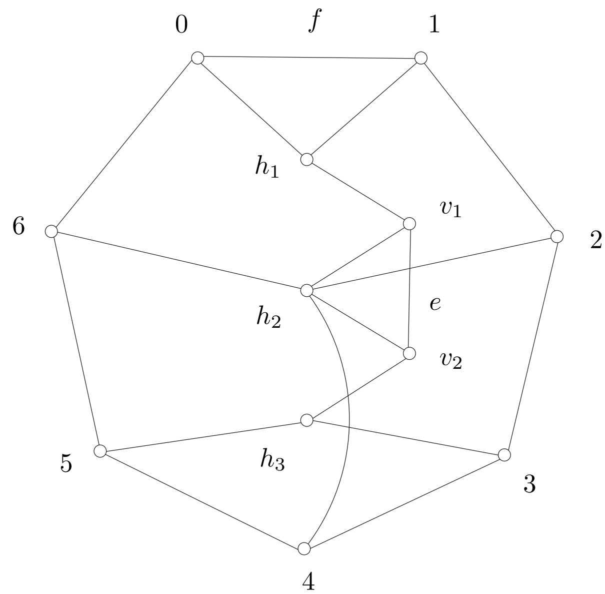


FIG. 8. *The solid brick* $G := \mathcal{T}_{12}$.

denote the wheel whose rim is the cycle $(0, 1, 2, \ldots, 2k-1, 2k, 0)$. First split the hub h of W_{2k+1} into three pairwise nonadjacent vertices h_1, h_2, and h_3 and distribute the spokes of the wheel so that

- h_1 is adjacent to the vertices 0 and 1;
- h_2 is adjacent to $2, 4, \ldots, 2k$; and
- h_3 is adjacent to $3, 5, \ldots, 2k-1$.

Now add two new vertices v_1 and v_2, and join v_1 to h_1 and h_2, v_2 to h_2 and h_3, and v_1 and v_2 to each other, to obtain a graph, which we shall denote by $\mathcal{T}_n$, of order $2k+6 = n$. (The graph $G := \mathcal{T}_{12}$ is depicted in Figure 8.) This construction can be better understood in terms of the expansion operation (expansion of a vertex by a barrier of size three) defined on p. 25 of [8]. Theorem 36 of the same article implies that $\mathcal{T}_n$ is a brick for all even integers $n \geq 12$. Using fairly simple case analysis, one can show that if C_1 and C_2 are any two vertex-disjoint odd cycles of $G := \mathcal{T}_n$, then the graph $G - (V(C_1) \cup V(C_2))$ has no perfect matching, and then, by appealing to Theorem 1.14, deduce that G is solid.

No graph in the two families of bricks described above is cubic. We have not been able to find a cubic solid brick that is not odd-intercyclic and would be surprised if the following were not true.

CONJECTURE 1.16. *Every cubic solid brick is odd-intercyclic.*[1]

We conclude this section by defining an important parameter related to each nontrivial separating cut C of a nonsolid brick G. Since G is free of nontrivial tight cuts, it follows that some perfect matching M meets C in at least three edges. We define the *characteristic of* C to be the minimum value of $|M \cap C|$, where the minimum is taken over all perfect matchings M of G that meet C in at least three edges. (In particular, the characteristic of a nontrivial separating cut in a brick is at least three.)

[1] An infinite family of counterexamples to Conjecture 1.16 appears in a recent manuscript by Guantao Chen, Xing Feng, Fuliang Lu and Lianzhu Zhang.

1.6. Removable classes.

1.6.1. Removable edges and doubletons. Let G be a matching covered graph and let e and f be two edges of G. We say that e *depends* on f, and write $e \Rightarrow f$, if every perfect matching of G that contains e also contains f. Edges e and f are *mutually dependent* if $e \Rightarrow f$ and $f \Rightarrow e$, and we write $e \Leftrightarrow f$ to signify this. It is easy to see that $\Leftrightarrow$ is an equivalence relation on the edge set $E(G)$ of G. In general, the cardinality of an equivalence class may be arbitrarily large. (For example, C_{2k}, the cycle of length $2k$, has two equivalence classes of size k each.) However, in a brick, an equivalence classes has at most two edges (see Theorem 1.18).

The relation $\Rightarrow$ may be visualized by means of the directed graph on the edge set $E(G)$ of G, where there is an arc with e as tail and f as head whenever $e \Rightarrow f$. From this digraph we obtain a new digraph, denoted by $D(G)$, by identifying equivalence classes under the relation $\Leftrightarrow$. Clearly $D(G)$ is acyclic. Moreover, there is an arc from a class R to a class S if and only if every edge of R depends on every edge of S. Thus, the acyclic digraph $D(G)$ suggests an obvious partial order on the set of equivalence classes of the precedence relation. For this reason, we refer to the equivalence classes that correspond to the sources of $D(G)$ as *minimal classes*. For any edge e of G, a source Q of D that contains an edge f that depends on e is said to be a *minimal class induced by* e. (Here we admit the possibility that e and f may be the same.)

If R is a minimal class of G, then every edge of $G - R$ is admissible. Moreover, if $G - R$ happens to be connected, then $G - R$ is matching covered; in this case, we shall say that R is a *removable class*.

An edge e of a matching covered graph G is a *removable edge* if $G - e$ is matching covered, and a pair $\{e, f\}$ of edges of G is a *removable doubleton* if neither e nor f is individually removable, but the graph $G - \{e, f\}$ is matching covered. In the former case, $\{e\}$ is a minimal class, and in the latter, $\{e, f\}$ is a minimal class. A *removable class* is either a removable singleton (consisting of a removable edge) or a removable doubleton.

The result below concerning braces will prove to be useful.

THEOREM 1.17 (see [3, Lemma 3.2]). *Every edge in a brace of order six or more is removable.* ∎

1.6.2. Removable classes in bricks. A matching covered graph G is *near-bipartite* if it has a removable doubleton R such $G - R$ is a bipartite matching covered graph.

THEOREM 1.18 (see [3, Lemma 2.3], [15, Lemma 3.4]). *Any equivalence class R in a brick G has cardinality at most two. Moreover, if $|R| = 2$, say, $R = \{e, f\}$, then $G - e - f$ is a bipartite matching covered graph, both ends of e are in one part of the bipartition of $G - e - f$, and both ends of f are in the other part.*

In particular, every removable class of a brick is either a removable edge or is a removable doubleton. It follows from the above theorem that every brick with a removable doubleton is indeed near-bipartite. Truncated biwheels, prisms of order 2 (modulo 4), Möbius ladders of order 0 (modulo 4), and staircases are examples of near-bipartite bricks. The bicorn (Figure 1(b)) has two removable doubletons and also a unique removable edge. The two bricks K_4 and $\overline{C_6}$ have three removable doubletons each, but have no removable edges; the following was established by Lovász [15].

THEOREM 1.19. *Every brick distinct from K_4 and $\overline{C_6}$ has a removable edge.*

There is an extensive discussion of removable edges in bricks in our paper [10]. We now present a technical result which will turn out to be useful in the proof of the main theorem (Theorem 2.1) in section 2.

For a fixed vertex v_0 of a matching covered graph G, a subset M of the edges of G is a *v_0-matching* if $|M \cap \partial(v)| = 1$ for each vertex v distinct from v_0, and if $|M \cap \partial(v_0)| > 1$. A simple counting argument shows that $|M \cap \partial(v_0)|$ is odd. The following may also be easily verified.

PROPOSITION 1.20. *Let $G[A, B]$ be a bipartite graph such that $|A| = |B|$. Then G does not have a v_0-matching for any vertex v_0.* □

LEMMA 1.21. *Let G be a brick, let v_0 be a vertex of G, and let M be a v_0-matching. Let e be an edge in $\partial(v_0) - M$ and let Q be a minimal class of G induced by e. Then, Q is a singleton which is disjoint from M.*

Proof. If e is the only member of Q, then there is nothing to prove. Let f be an edge of Q such that $f \neq e$. Then, $f \Rightarrow e$ in G. As $G - e$ has a perfect matching, and f is inadmissible, by Proposition 1.2, $G - e$ has a barrier B containing both ends of f, and $G - e - B$ has exactly $|B|$ odd components, two of which contain the ends of e. In particular, v_0 lies in an odd component K of $G - e - B$.

As $|V(G)|$ is even, $|M \cap \partial(v_0)|$ is odd and so, $|M \cap \partial(V(K))|$ is also odd. Moreover, $|M \cap \partial(V(K'))| \geq 1$ for any other odd component K' of $G - e - B$. By simple counting, and taking into account that $e \notin M$, we conclude that $|M \cap \partial(V(K'))| = 1$ for each odd component K' of $G - e - B$ and that each vertex of B is matched by M with a vertex in an odd component of $G - e - B$. Thus, $f \notin M$. As f is an arbitrary edge of $Q - \{e\}$, and since $e \notin M$, we conclude that the minimal class Q does not meet M.

It remains to argue that $|Q| = 1$. By Theorem 1.18, Q has at most two edges. Suppose that $|Q| = 2$. By Theorem 1.18, $G - Q$ is a bipartite matching covered graph. Since $Q \cap M$ is empty, M is a v_0-matching of the bipartite graph $G - Q$, and this contradicts Proposition 1.20. Thus, $|Q| = 1$. □

1.6.3. Removable classes in solid graphs. Here we state some useful results regarding the properties and existence of removable edges in solid graphs.

THEOREM 1.22 (see [4, Theorem 2.2.8]). *For any removable edge e of a solid matching covered graph G, the graph $G - e$ is also solid.*

We now proceed to prove a result which we shall refer to as the lemma on odd wheels (Lemma 1.24), which will play a crucial role in the proof of the main theorem of section 2. (A weaker version of this result appeared in [5].) The proof of this lemma relies on the following.

THEOREM 1.23 (see [10, Theorem 6.11]). *Let G be a solid brick, let v be a vertex of G, let n be the number of neighbors of v, and let d be the degree of v. Enumerate the d edges of $\partial(v)$ as $e_i := vv_i$ for $i = 1, 2, \ldots, d$. Assume that neither e_1 nor e_2 is removable in G. Then, $n = 3$ and, for $i = 1, 2$, there exists an equipartition (B_i, I_i) of $V(G)$ such that*

(i) *e_i is the only edge of G that has both ends in I_i,*
(ii) *every edge that has both ends in B_i is incident with v_3, and*
(iii) *the subgraph H_i of G, obtained by the removal of e_i and each edge having both ends in B_i, is matching covered and bipartite with bipartition $\{B_i, I_i\}$.*

Moreover, $B_1 = (I_2 - v) \cup \{v_3\}$ and $B_2 = (I_1 - v) \cup \{v_3\}$. (See Figure 9 for an illustration.)

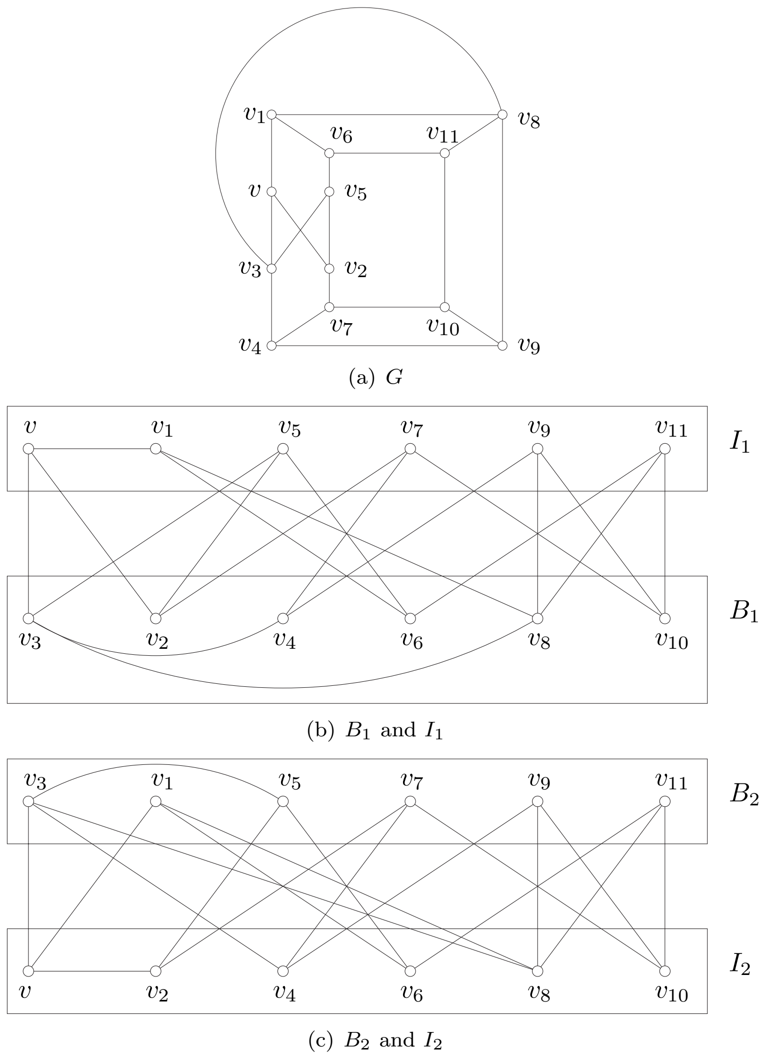


FIG. 9. *The solid brick G and the pairs B_1, I_1 and B_2, I_2.*

We say that G is a v_0-*wheel* if G is a wheel having v_0 as a hub.

LEMMA 1.24 (lemma on odd wheels). *Let G be a simple solid brick, let v_0 be a vertex of G, and let M_0 be a v_0-matching. Then either G is a v_0-wheel or G has a removable edge $e \notin M_0 \cup \partial(v_0)$.*

Proof.
Case 1. *The brick G has a vertex $v \neq v_0$ that has degree four or more in G.* As G is simple, at least two edges, e_1 and e_2, are not in $M_0 \cup \partial(v_0)$ but are incident with v. By Theorem 1.23, one of e_1 and e_2 is removable in G.

We may thus assume that every vertex $v \neq v_0$ has degree three in G.

Case 2. *Every vertex of $G - v_0$ is adjacent to v_0.* Since every vertex $v \neq v_0$ has degree three in G and is adjacent to v_0, every vertex distinct from v_0 has degree two in $G - v_0$. Then $G - v_0$ is a collection of cycles. By the 3-connectivity of G, it follows that $G - v_0$ is a cycle and, consequently, G is a v_0-wheel.

Case 3. *The previous cases are not applicable.* Every vertex $v \neq v_0$ of G has degree three in G. Moreover, G has a vertex, $v \neq v_0$, that is not adjacent to v_0. Let $e_i := vv_i$, $i = 1, 2, 3$, be the three edges incident with v. Adjust notation so that $e_3 \in M_0$.

1.24.1. *One of the edges e_1 and e_2 is removable in G.*

Proof. Assume the contrary. By Theorem 1.23, G has an equipartition (B_1, I_1) such that e_1 is the only edge having both ends in I_1 and every edge having both ends in B_1 is incident with v_3. Moreover, the bipartite graph H obtained from G by the removal of e_1 and each edge having both ends in B_1 is matching covered. Vertex v_3, a vertex adjacent to v, is distinct from v_0. Thus, v_3 has degree three. As H is matching covered, precisely one edge of $\partial(v_3)$, say, f, has both ends in B_1. But e_3 is the only edge of M_0 incident with v and its end v is in I_1. Thus, $f \notin M_0$ and $e_1 \notin M_0$. In particular, M_0 is a v_0-matching of H, and this contradicts Proposition 1.20. ∎

The proof of the lemma on odd wheels is complete. ∎

1.7. Ear decompositions.

1.7.1. Deletions and additions of ears. A path $P := v_0v_1 \ldots v_\ell$ of odd length in a graph G is a *single ear* in G if each of its internal vertices $v_1, v_2, \ldots, v_{\ell-1}$ has degree two in G. If P_1 and P_2 are two vertex-disjoint single ears in G, then $\{P_1, P_2\}$ is a *double ear* with P_1 and P_2 as its *constituent single ears.* The *deletion* of a single ear P from G consists of deleting all the edges and internal vertices of P, and the graph obtained by deleting P from G is denoted by $G - P$. Likewise, the deletion of a double ear $\{P_1, P_2\}$ consists of deleting each of its constituent single ears P_1 and P_2.

A single ear P in a matching covered graph G is *removable* if the graph $G - P$ obtained by deleting P from G is also matching covered. If P_1 and P_2 are two vertex-disjoint single ears neither of which is removable, but the graph $G - P_1 - P_2$ is matching covered, then the double ear $\{P_1, P_2\}$ is *removable.* When the length of a single ear is one, then we identify it with its only edge. An *ear decomposition* of a matching covered graph G is a sequence $(G_1, G_2, \ldots, G_r)$ of matching covered subgraphs of G such that

(i) $G_1 = K_2$ and $G_r = G$; and
(ii) for $2 \leq i \leq r$, the graph G_{i-1} is obtained from G_i by the deletion of either a removable single ear or of a removable double ear.

The following basic result was proved by Lovász and Plummer [16].

Theorem 1.25 (the two-ear theorem). *Every matching covered graph G has an ear decomposition.*

1.7.2. Conformal subgraphs. A matching covered subgraph H of a matching covered graph G is *conformal* if the graph $G - V(H)$ has a perfect matching. It is easily seen that this notion obeys transitivity.

Proposition 1.26. *Any conformal subgraph of a conformal subgraph of a matching covered graph G is also a conformal subgraph of G.* ∎

Conformal subgraphs have been referred to by various other names ("nice" subgraphs, "central" subgraphs, and "well-fitted" subgraphs) in the literature. The following result is due to Lovász and Plummer [16].

Theorem 1.27. *A matching covered subgraph H of a matching covered graph G is conformal if and only if there is some ear decomposition $\mathcal{G} := (G_1, G_2, \ldots, G_r)$ of G such that H is one of the graphs in $\mathcal{G}$.* ∎

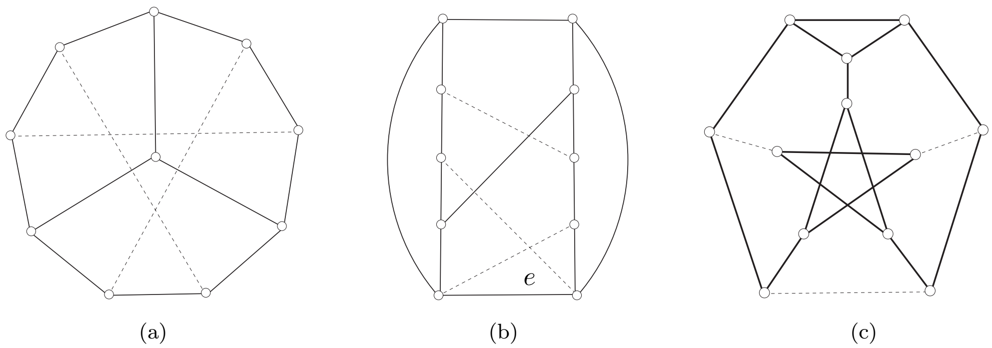


FIG. 10. *Conformal minors:* (a) K_4 *of* $\mathbb{P}$*,* (b) $\overline{C_6}$ *of* $\mathbb{P}+e$*,* (c) $\overline{C_6}$ *of* $K_4 \odot \mathbb{P}$.

Every ear decomposition of a bipartite matching covered graph involves only single ear additions. However, in an ear decomposition of a nonbipartite matching covered graph there must be at least one double ear addition. Lovász established the following fundamental result concerning nonbipartite graphs.

THEOREM 1.28 (see [14]). *Every nonbipartite matching covered graph has an ear decomposition such that either the third graph is a bisubdivision of* K_4 *or the fourth graph is a bisubdivision of* $\overline{C_6}$.

This theorem gives rise to natural questions which are described in terms of special types of minors of matching covered graphs which we now proceed to discuss.

1.7.3. Conformal minors and matching minors. Let G be a matching covered graph. A matching covered graph J is a *conformal minor* of G if some bisubdivision H of J is a conformal subgraph of G. Figure 10(a) shows that K_4 is a conformal minor of $\mathbb{P}$, the Petersen graph. It is not too difficult to show that $\overline{C_6}$ is not a conformal minor of $\mathbb{P}$. However, if $\mathbb{P}+e$ is any graph obtained by adding an edge e to $\mathbb{P}$ joining two nonadjacent vertices, then $\overline{C_6}$ is a conformal minor of $\mathbb{P}+e$ as illustrated in Figure 10(b), and also of $K_4 \odot \mathbb{P}$ as illustrated in Figure 10(c).

A bicontraction of a vertex v of degree two is *restricted* if at least one of the two neighbors of v also has degree two. It follows from Theorem 1.27 that if H is a conformal matching covered subgraph of a matching covered graph G, then H can be obtained from G by a sequence of deletions of removable ears (single or double). But the deletion of an ear amounts to first reducing that ear to one of length one by means of restricted bicontractions and then deleting the only edge of that ear. Likewise, if H is a bisubdivision of a graph J, then J can in fact be obtained from H by restricted bicontractions. These observations imply the following consequence.

COROLLARY 1.29. *A matching covered graph* J *is a conformal minor of a matching covered graph* G *if and only if* J *can be obtained from* G *by restricted bicontractions of vertices of degree two and deletions of removable classes.* ∎

Norine and Thomas [18] call a matching covered graph J a *matching minor* of a matching covered graph G if J can be obtained from a conformal subgraph H of G by means of bicontractions. It follows that a matching covered graph J is a matching minor of a matching covered graph G if and only if J can be obtained from graph G by bicontractions and deletions of removable classes.

COROLLARY 1.30. *Every conformal minor of a matching covered graph* G *is a matching minor of* G. ∎

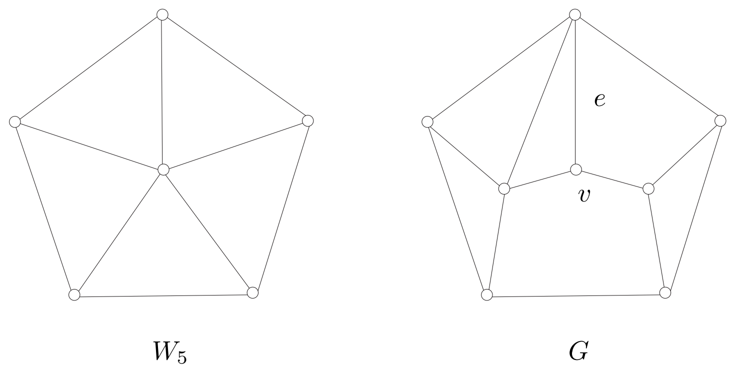


FIG. 11. *The wheel W_5 is a matching minor of G.*

The converse is not true in general, due to the fact that unrestricted bicontractions are permissible in obtaining a matching minor of G. For example, the wheel W_5 is a matching minor of the graph G shown in Figure 11 because W_5 can be obtained from G by first deleting the edge e and then bicontracting the vertex v in the resulting graph. But it is not a conformal minor of G for the simple reason that G has no vertices of degree greater than four.

However, it is easily seen that if a cubic matching covered graph J is obtained from a matching covered graph H by means of bicontractions, then H must be a bisubdivision of J. Consequently, we have the following.

COROLLARY 1.31. *A cubic matching covered graph J is a matching minor of a matching covered graph G if and only if J is a conformal minor of G.*

Given a fixed matching covered graph J, we say that a matching covered graph G is *J-based* if J is a conformal minor of G, and otherwise G is *J-free*. For example, the Petersen graph $\mathbb{P}$ is K_4-based but is $\overline{C_6}$-free, and $\mathbb{P}+e$ depicted in Figure 10(b) is both K_4-based and $\overline{C_6}$-based.

Theorem 1.28 implies that every nonbipartite matching covered graph is either K_4-based or is $\overline{C_6}$-based (or both), and it raises two natural problems: characterize those matching covered graphs that are K_4-free and those that are $\overline{C_6}$-free. Two of us (KM–Kothari and Murty) showed that it suffices to solve these problems for bricks by establishing the following result concerning cubic bricks.

THEOREM 1.32 (see [13]). *Suppose that J is a cubic brick and that C is a tight cut of a matching covered graph G. Then G is J-free if and only if each C-contraction of G is J-free. (In particular, G is J-free if and only if each brick of G is J-free.)*

The restriction that J be a cubic brick is crucial for the validity of the above statement. (Curiously, it is not valid even for cubic braces. For example, consider the graph $G := K_4 \odot K_{3,3}$. If C denotes the unique nontrivial tight cut in G, one of the C-contractions of G is the brace $K_{3,3}$. However, $K_{3,3}$ is not a conformal minor of G!)

In light of Theorem 1.32, it suffices to solve the following problems.

UNSOLVED PROBLEM 1.33. *Characterize K_4-free bricks.*

UNSOLVED PROBLEM 1.34. *Characterize $\overline{C_6}$-free bricks.*

Using the brick generation theorem of Norine and Thomas, which will be described later on, we (KM) were able to resolve Problems 1.33 and 1.34 in the special case of planar bricks by proving the following results. (By a well-known theorem of Whitney

[22], every simple 3-connected planar graph has a unique embedding in the plane [1, Theorem 10.28]).

THEOREM 1.35 (see [13]). *A simple planar brick is K_4-free if and only if its (unique) planar embedding has precisely two odd faces.*

THEOREM 1.36 (see [13]). *The only simple planar $\overline{C_6}$-free bricks are the odd wheels, staircases of order $4k$, and the tricorn.*

In the case of nonplanar bricks, Problems 1.33 and 1.34 remain unsolved.

We conclude this section by observing that matching minors of solid matching covered graphs inherit their solidity.

LEMMA 1.37. *Every matching minor of a solid matching covered graph is a solid matching covered graph.*

Proof. Let G be a solid matching covered graph and let J be a matching minor of G. We prove that J is solid by induction on $|E(G)|$. If $J = G$, we are done. Assume then that J and G are distinct. Thus, by definition, J may be obtained from G by means of bicontractions (of vertices of degree two) and deletions of removable classes. In particular, J is a matching minor of a matching covered graph G' such that either (i) G' is obtained from G by a bicontraction (of a vertex of degree two) or (ii) $G' = G - R$, where R is a removable class of G. In each case, $|E(G')| < |E(G)|$, and thus it suffices to prove that G' is solid. From Corollary 1.13, we deduce immediately the following property.

1.37.1. *For each tight cut C of G, both C-contractions of G are solid.* □

Suppose that G' is obtained from G by a bicontraction (of a vertex of degree two). Then, G has a nontrivial tight cut C such that G' is one of its C-contractions. In that case, by 1.37.1, we deduce that G' is solid. The assertion holds in this case.

We may thus assume that $G' = G - R$, where R is a removable class of G. Every bipartite matching covered graph is solid. If $G - R$ is bipartite, then it is solid. We may thus assume that $G - R$ is not bipartite. This implies that G is not bipartite. If G is a brick, then, by Theorem 1.18, R is a singleton. In that case, $G - R$ is solid, by Theorem 1.22.

We may thus assume that G is not a brick hence G has a nontrivial tight cut C. Let G_1 and G_2 be the two C-contractions of G. By 1.37.1, G_1 is solid. The set $C - R$ is a tight cut of $G - R$. The graph $G_1 - R$, a $(C - R)$-contraction of $G - R$, is matching covered. Moreover, $G_1 - R$ is a matching minor of G_1. By induction, $G_1 - R$ is solid. Likewise, $G_2 - R$ is also solid. By Theorem 1.12, $G' = G - R$ is solid. This completes the proof. □

COROLLARY 1.38. *Every conformal minor of a solid matching covered graph is a solid matching covered graph.* □

However, not every conformal minor of a nonsolid graph is nonsolid. For example, the bicorn is nonsolid, but K_4, which is solid, is a conformal minor of the bicorn. Moreover, the bicorn is $\overline{C_6}$-free.

Since $\overline{C_6}$ is nonsolid, we have the following consequence.

COROLLARY 1.39. *Every solid matching covered graph is $\overline{C_6}$-free.* □

1.8. Robust cuts in nonsolid bricks. A nontrivial separating cut $C := \partial(X)$ of a nonsolid brick G is *robust* if both the C-contractions G/X and $G/\overline{X}$ of G are near-bricks. We say that a robust cut C is *k-robust* if C has characteristic k. We were able to prove the following fundamental result.

THEOREM 1.40 (see [5, Theorem 4.1]). *Every simple nonsolid brick distinct from the Petersen graph has a* 3*-robust cut. The Petersen graph has only* 5*-robust cuts.*

This was one of the main tools used by us (CLM) in our proof of a conjecture due to Lovász (see [4] and [5]).[2]

2. Conformal minors of nonsolid bricks. We shall refer to $\overline{C_6}$, the bicorn, the tricorn, and the Petersen graph as the *basic* nonsolid bricks.

THEOREM 2.1 (main theorem). *Every nonsolid matching covered graph contains a basic nonsolid brick as a conformal minor.*

Proof. Let G be any nonsolid matching covered graph. We shall prove the validity of the assertion by induction on the number of edges of G.

It follows from Theorem 1.28 that the smallest nonsolid matching covered graph is $\overline{C_6}$, which is a basic nonsolid brick. For the general case, we adopt as the inductive hypothesis that every nonsolid matching covered graph with fewer edges than G has one of the four basic nonsolid bricks as a conformal minor.

Case 1. *G contains a proper conformal subgraph H that is a nonsolid matching covered graph.* By the induction hypothesis, H contains a basic nonsolid brick as a conformal minor. Hence, by Proposition 1.26, G also contains a basic nonsolid brick as a conformal minor.

Note that this case applies when G has multiple edges.

Case 2. *Graph G has a nontrivial tight cut C.* By Corollary 1.13, G has a C-contraction, G_1, that is nonsolid. By the induction hypothesis, G_1 contains a basic nonsolid brick as a conformal minor. Since each of the basic nonsolid bricks is cubic, it follows from Theorem 1.32 that G also contains a basic nonsolid brick as a conformal minor.

Case 3. *The previous cases do not apply.* The graph G is free of nontrivial tight cuts, and hence G is either a brick or a brace. Every bipartite graph is solid. Thus, G is a brick. In fact, G is a simple nonsolid brick, free of nonsolid proper conformal minors. In sum, the following holds.

LEMMA 2.2. *Let R be a nonempty set of edges of G. If $G-R$ is matching covered, then it is solid.* ☐

We shall prove that G is one of the four basic nonsolid bricks. If G is the Petersen graph, then we are done. We may thus assume that G is not the Petersen graph. We shall prove that G is either $\overline{C_6}$, the bicorn, or the tricorn.

As G is not the Petersen graph, then, by Theorem 1.40, G has 3-robust cuts. Let $C := \partial(X)$ be a 3-robust cut of G and let M_0 be a perfect matching of G such that $|M_0 \cap C| = 3$. Let $G_1 := G/(\overline{X} \to \overline{x})$ and $G_2 := G/(X \to x)$ be the two C-contractions of G obtained by contracting $\overline{X}$ and X to single vertices $\overline{x}$ and x, respectively. As C is robust, the graphs G_1 and G_2 are near-bricks.

LEMMA 2.3. *Let R be a nonempty set of edges of G. If $G_1 - R$ and $G_2 - R$ are both matching covered, then the graphs $G_1 - R$ and $G_2 - R$ are both solid and $C - R$ is tight in $G - R$.*

[2]In a later unpublished paper [2], Campos and Lucchesi showed that, in fact, in every simple brick distinct from the Petersen graph, every nontrivial separating cut has characteristic three. In particular, in every simple brick distinct from the Petersen graph, every robust cut is 3-robust.

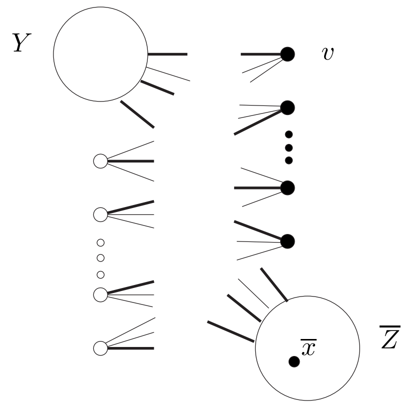


FIG. 12. *The brace* $J_1 := G/(Y \to y)/(\overline{Z} \to \overline{z})$.

Proof. Suppose that $G_1 - R$ and $G_2 - R$ are both matching covered. Then, $G - R$ is matching covered, and the cut $C - R$ is separating in $G - R$. By Lemma 2.2, the graph $G - R$ is solid. Thus, $C - R$ must be tight in $G - R$. Moreover, by Corollary 1.13, both $(C - R)$-contractions of $G - R$ must be solid. That is, $G_1 - R$ and $G_2 - R$ are both solid. ∎

COROLLARY 2.4. *Let e be an edge of G. If $G_1 - e$ and $G_2 - e$ are both matching covered, then $e \in M_0$ and $G_1 - e$ and $G_2 - e$ are both solid.*

Proof. Suppose that the graphs $G_1 - e$ and $G_2 - e$ are both matching covered. By Lemma 2.3, the cut $C - e$ is tight in $G - e$. Thus, M_0 is not a perfect matching of $G - e$, and hence $e \in M_0$. Moreover, also by Lemma 2.3, the graphs $G_1 - e$ and $G_2 - e$ are both solid. ∎

LEMMA 2.5. *The graphs G_1 and G_2 are bricks.*

Proof. Suppose that G_1 is not a brick. As G_1 is a near-brick that is not a brick, it has nontrivial tight cuts. Moreover, by Corollary 1.11, for any tight cut D of G_1, one of the D-contractions of G_1 must be bipartite, and the other must be a near-brick.

2.5.1. *Let D be a nontrivial tight cut of G_1, and let Y be its nonbipartite shore. The vertices $\overline{x}$ and y lie in distinct parts of the bipartition of $J := G_1/(Y \to y)$.*

Proof. The graph J is bipartite and matching covered. If one of the parts in the bipartition of J contains neither of the contraction vertices, then that part would be a barrier of G. Since G is a brick, this is not possible. It follows that $\overline{x}$ and y belong to different parts of the bipartition of J. ∎

Now let Y be a minimal nontrivial subset of $X = V(G_1) - \overline{x}$ such that $\partial(Y)$ is a tight cut of G_1. Then the minimality of Y, together with 2.5.1, implies that $G_0 := G/(\overline{Y} \to \overline{y})$ is a brick and $G_1/(Y \to y)$ is a bipartite matching covered graph with the two contraction vertices y and $\overline{x}$ in different parts of its bipartition. Moreover, $|M_0 \cap \partial(Y)| = |M_0 \cap C| = 3$, and hence $M_0 \cap E(G_0)$ is a $\overline{y}$-matching of G_0. In general $G_1/(Y \to y)$ need not be a brace. However, if we take Z to be a minimal subset of X which properly contains Y and is such that $\partial(Z)$ is a nontrivial tight cut of G_1, then the graph $J_1 := (G/(Y \to y))/(\overline{Z} \to \overline{z})$ is a brace of order four or more. See Figure 12.

By 2.5.1, the vertices y and $\overline{z}$ lie in distinct parts of the bipartition of J_1. Let v denote a vertex in the same part that contains $\overline{z}$ but is distinct from $\overline{z}$. Then, no edge incident with v is in C because the contraction vertex $\overline{x}$ of G_1 is in $\overline{Z}$.

Consider first the case in which v is adjacent to at most one vertex of Y. As G is a brick, vertex v is adjacent to three or more vertices of the brace J_1. Thus, J_1 has six or more vertices. By Theorem 1.17, all the edges of J_1 are removable. In particular, one of the edges of J_1 incident with v is not in $M_0 \cup \partial(Y)$. Thus, G_1 has a removable edge that does not lie in $M_0 \cup C$. This is a contradiction of Corollary 2.4.

Alternatively, suppose that v is adjacent to two vertices of Y, say, w_1 and w_2. The edges vw_1 and vw_2 are multiple edges in G/Y, and hence removable in G_1/Y. At least one of the edges vw_1 and vw_2 is not in M_0. Adjust notation so that $vw_1 \notin M_0$. Since $M_0 \cap E(G_0)$ is a $\overline{y}$-matching of G_0, it follows, by Lemma 1.21, that either vw_1 is removable in the brick G_0 or G_0 has an edge that is removable and does not lie in $M_0 \cup \partial(Y)$. In both cases, G_1 has a removable edge that does not lie in $M_0 \cup C$, a contradiction of Corollary 2.4.

In all cases considered, we have derived a contradiction. We deduce that G_1 is a brick. Likewise, a similar argument may be used to prove that G_2 is also a brick. ∎

LEMMA 2.6. $C \subseteq M_0$.

Proof. Suppose, to the contrary, that $C - M_0$ contains an edge, e. By Corollary 2.4, at least one of the graphs $G_1 - e$ and $G_2 - e$ is not matching covered. Adjust notation so that $G_1 - e$ is not matching covered. That is, the edge e is not removable in the brick G_1. By Lemma 1.21, G_1 has a removable edge, f, that does not lie in $M_0 \cup C$. Thus, $G_1 - f$ and $G_2 - f = G_2$ are both matching covered, a contradiction of Corollary 2.4. Indeed, $C \subseteq M_0$. ∎

LEMMA 2.7. *If a C-contraction G_i of G is solid, then $G_i = K_4$.*

Proof. Suppose that G_1 is solid. Assume, to the contrary, that $G_1 \neq K_4$. The cut C consists only of three edges in M_0 (by Lemma 2.6) and G_1 is a brick (by Lemma 2.5). Thus, G_1 is simple but is not a wheel having $\overline{x}$ as a hub. By Lemma 1.24 (lemma on odd wheels), G_1 has a removable edge e that does not lie in $M_0 \cup C$. In this case, $G_1 - e$ and $G_2 - e = G_2$ are both matching covered and $e \notin M_0$. This is a contradiction of Corollary 2.4. We conclude that $G_1 = K_4$. Likewise, if G_2 is solid, then $G_2 = K_4$. ∎

LEMMA 2.8. *If a C-contraction G_i of G is not solid, then G_i is one of the four basic nonsolid bricks.*

Proof. Suppose that the C-contraction G_1 is not solid. By induction, G_1 has a conformal minor J that is one of the four basic nonsolid bricks. Thus, some bisubdivision H of J is a conformal subgraph of G_1. As G_1 is a brick, if $G_1 = H$, then $G_1 = J$. In this case, the assertion holds.

We may thus assume that H is a proper subgraph of G_1. By Theorem 1.27, G_1 has a removable ear R such that H is a conformal subgraph of $G_1 - R$. Furthermore, as G_1 is a brick, it follows that the edges of R constitute either a removable edge or a removable doubleton.

If R and C are disjoint, then $G_1 - R$ is matching covered and nonsolid (by Corollary 1.38), and $G_2 - R = G_2$ is matching covered. This is a contradiction of Lemma 2.3. Thus, R contains an edge, e, in C. Clearly, since all edges of C are incident with the contraction vertex $\overline{x}$ of G_1, edge e is the only edge of R in C. Let S be a minimal class of the dependence relation in G_2 induced by edge e. As G_2 is a brick, $G_2 - S$ is matching covered. If $e \in S$, then $G_1 - (R \cup S)$ is $G_1 - R$, a

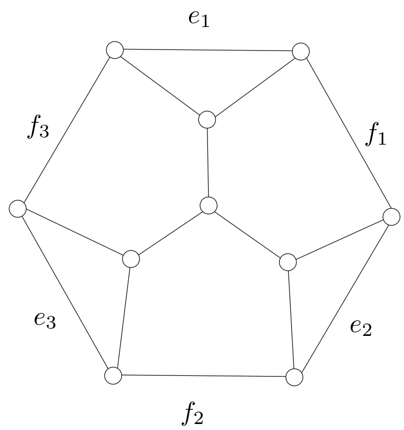


FIG. 13. *The three removable edges of the tricorn:* e_1*,* e_2*, and* e_3*.*

nonsolid matching covered graph, and $G_2 - (R \cup S)$ is $G_2 - S$, a matching covered graph. Alternatively, if $e \notin S$, then $G_1 - S = G_1$ is nonsolid and matching covered, and $G_2 - S$ is matching covered. In both alternatives, we derive a contradiction of Lemma 2.3. We deduce that G_1 is one of the four basic nonsolid bricks. Likewise, if G_2 is not solid, then it is one of the four basic nonsolid bricks. ∎

Let us denote the bicorn by R_8 and the tricorn by R_{10}. From the preceding two lemmas, we now know that G_1 and G_2 are both in the set $\{K_4, \overline{C_6}, R_8, R_{10}, \mathbb{P}\}$. We now proceed to show that, in fact, no C-contraction of G is the Petersen graph and no C-contraction of G is the tricorn.

LEMMA 2.9. *Neither* G_1 *nor* G_2 *is in* $\{\mathbb{P}, R_{10}\}$.

Proof. Assume, to the contrary, that G_2 is the Petersen graph. Every one of the 15 edges of $\mathbb{P}$ is removable and 9 of them do not lie in $M_0 \cup C$, a contradiction of Corollary 2.4. As asserted, $G_2 \neq \mathbb{P}$.

Suppose now, to the contrary, that G_2 is the tricorn. The tricorn has three removable edges, e_i, $i = 1, 2, 3$. The three removable edges of the tricorn, together with the edges f_i, $i = 1, 2, 3$, constitute the edge set of a hexagon which we shall denote by H. See Figure 13.

If M_0 does not contain at least one of these three edges, then again we get a contradiction of Corollary 2.4. We may thus assume that $\{e_1, e_2, e_3\} \subset M_0$. In this case, the contraction vertex x of G_2 cannot be in the vertex set of the hexagon H, and hence H is an M_0-alternating cycle. We may then replace M_0 by its symmetric difference with $E(H)$ and again obtain a contradiction of Corollary 2.4.

We conclude that $G_2 \notin \{R_{10}, \mathbb{P}\}$. The same conclusion holds for G_1. ∎

LEMMA 2.10. *At least one* C*-contraction of* G *is solid.*

Proof. Assume the contrary. By Lemma 2.8, both C-contractions of G are basic nonsolid bricks. By Lemma 2.9, G_1 and G_2 are both in $\{\overline{C_6}, R_8\}$.

Assume that $G_2 = \overline{C_6}$. The brick $\overline{C_6}$ has three removable doubletons $R_i := \{e_i, f_i\}$, $i = 1, 2, 3$. See Figure 14.

No vertex of $\overline{C_6}$ is incident to edges of all three doubletons. Thus, G_2 has a removable doubleton R disjoint with C, and hence $G_1 - R = G_1$ is nonsolid and $G_2 - R$ is matching covered. This is a contradiction of Lemma 2.3.

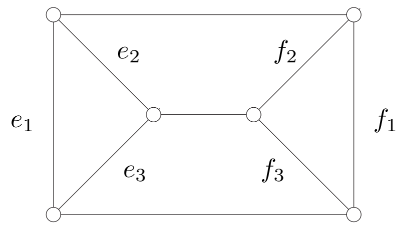


FIG. 14. *The three removable doubletons of* $\overline{C_6}$*:* $\{e_i, f_i\}$, $i = 1, 2, 3$.

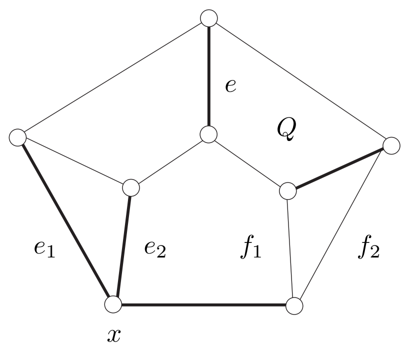


FIG. 15. *The three removable classes of* R_8*: the doubletons* $\{e_i, f_i\}$, $i = 1, 2$*, and the edge* e.

Alternatively, assume that $G_2 = R_8$. The graph R_8 has three removable classes, the two doubletons $\{e_i, f_i\}$, $i = 1, 2$, and the edge e. See Figure 15.

The edge e and $\{e_1, f_1\}$ are disjoint. Thus, G_2 has a removable class R disjoint with C, and hence $G_1 - R = G_1$ is nonsolid and $G_2 - R$ is matching covered. This is a contradiction of Lemma 2.3.

In all cases considered, we derived a contradiction. Indeed, at least one C-contraction of G is solid. ∎

If G_1 and G_2 are both solid, then, by Lemma 2.7, G_1 and G_2 are both K_4 and therefore G is $\overline{C_6}$. The assertion holds in this case. We may thus assume that at least one of G_1 and G_2 is nonsolid. Adjust notation so that G_2 is nonsolid. By Lemma 2.10, the brick G_1 is solid. By Lemmas 2.7 to 2.9, the brick G_1 is K_4 and the brick G_2 is either $\overline{C_6}$ or the bicorn.

If G_2 is $\overline{C_6}$, then G is the bicorn. Consider next the case in which G_2 is the bicorn. If x is not incident with the only removable edge e of G_2 but $e \in M_0$ (see Figure 15), then there is only one possibility, up to automorphisms. The edge e lies in an M_0-alternating quadrilateral Q and we may replace M_0 by its symmetric difference with $E(Q)$, in contradiction of Corollary 2.4. Thus, x is an end of e. In that case, G is the tricorn.

Indeed, if G is not the Petersen graph, then G is either $\overline{C_6}$, the bicorn, or the tricorn. The proof of Theorem 2.1 is complete. ∎

3. Equivalence of Problems 1.15 and 1.34.

3.1. Thin and strictly thin edges. Motivated by the problem of recursively generating bricks, we were led to the notion of thin edges. An edge e of a brick G is *thin* if the retract of $G - e$ is also a brick. (Our definition of a thin edge in [8] was

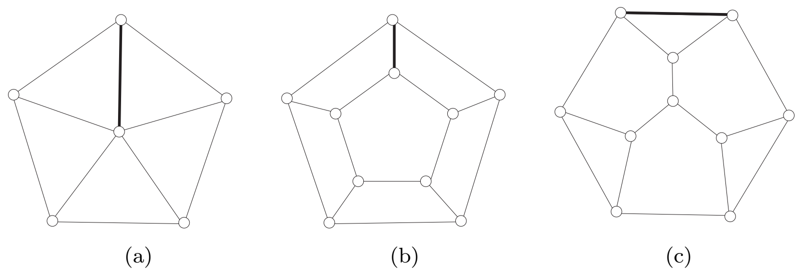


FIG. 16. (a) *The wheel* W_5, (b) *the pentagonal prism,* (c) *the tricorn.*

phrased in terms of sizes of barriers, but is equivalent to the one given here.) We (CLM) proved in [8] the following assertion.

THEOREM 3.1. *Every brick distinct from* K_4, $\overline{C_6}$, *and* $\mathbb{P}$ *(the Petersen graph) has a thin edge.* ∎

The above theorem implies the following corollary which has the flavor of Theorem 1.25.

COROLLARY 3.2. *Given any brick* G, *there exists a sequence* $(G_1, G_2, \ldots, G_r)$ *of bricks such that*

(i) $G_r = G$ *and* $G_1 \in \{K_4, \overline{C_6}, \mathbb{P}\}$*; and*

(ii) *for* $1 < i \leq r$, *the brick* G_i *has a thin edge* e_i *such that* G_{i-1} *is the retract of* $G_i - e_i$.

This corollary is the basis of a recursive procedure for generating bricks described in [8]. We showed that there exist four elementary "expansion operations" which can be used to build any brick starting from one of K_4, $\overline{C_6}$, and $\mathbb{P}$. (The simplest of these operations consists of just adding an edge joining two distinct vertices of a given brick G. The other three involve bisplitting vertices and adding edges.)

We associate with each thin edge a number called its index, as defined below. Let G be a brick and let e be a thin edge of G. Then the retract of $G - e$, by definition, is a brick. The *index* of e is

- *zero* if both ends of e have degree four or more in G;
- *one* if exactly one end of e has degree three in G;
- *two* if both ends of e have degree three in G and edge e does not lie in a triangle;
- *three* if both ends of e have degree three in G and edge e lies in a triangle.

Examples of thin edges of indices one, two, and three are indicated by solid lines in the three bricks, respectively, shown in Figure 16.

The following consequence of Theorem 1.32 will be useful later.

PROPOSITION 3.3 (see [13]). *Let* G *be a brick and* e *be a thin edge of* G. *For any cubic brick* J, *if the retract of* $G - e$ *is* J*-based, then* G *is also* J*-based.* ∎

In order to establish recursive procedures for generating simple bricks, one needs the notion of a strictly thin edge. An edge e of a simple brick G is *strictly thin* if e is thin and the retract of $G-e$ is simple. There are five infinite families of bricks that are free of strictly thin edges; these are (i) odd wheels, (ii) prisms of order 2 (modulo 4), (iii) Möbius ladders of order 0 (modulo 4), (iv) staircases, and (v) truncated biwheels. We refer to bricks in these five families together with the Petersen graph as *Norine–Thomas bricks.* (Note that K_4 is the Möbius ladder of order four, and $\overline{C_6}$ is the prism of order six.) For brevity, we shall denote the family of all Norine–Thomas bricks by $\mathcal{NT}$.

Norine and Thomas established the following strengthening of Theorem 3.1.

THEOREM 3.4 (see [18]). *Every simple brick G which is not a Norine–Thomas brick has a strictly thin edge.* $\square$

The work of Norine and Thomas is independent of our work and uses entirely different methods. After learning about the statement of their result, we were able to show that it is possible to derive Theorem 3.4 from our Theorem 3.1. Our proof appears in an unpublished report [9]. As an immediate consequence of the above theorem, we have the following.

COROLLARY 3.5. *Given any simple brick G, there exists a sequence*

$$(G_1, G_2, \ldots, G_r)$$

of simple bricks such that

(i) *$G_r = G$ and G_1 is in $\mathcal{NT}$; and*

(ii) *for $1 < i \leq r$, the brick G_i has a strictly thin edge e_i such that G_{i-1} is the retract of $G_i - e_i$.* $\square$

In the same paper [18], Norine and Thomas have also proved the following powerful generalization of Theorem 3.4; it belongs to a class of theorems in structural graph theory known as "splitter theorems." To state this generalization, we need to define a new class of graphs. The graph T_{2k}^{+} is obtained from the truncated biwheel T_{2k} by joining its hubs. The extended Norine–Thomas family $\mathcal{NT}^{+}$ is the union of $\mathcal{NT}$ and $\{T_{2k}^{+} :\ k \in \mathbb{Z}, k \geq 3\}$.

THEOREM 3.6. *Let G be a simple brick which is not in $\mathcal{NT}^{+}$ and let J be a simple brick that is distinct from K_4 and $\overline{C_6}$. If J is a matching minor of G, then there there exists a sequence $G_1, G_2, \ldots, G_r$ of simple bricks such that*

(i) *$G_r = G$ and $G_1 = J$, and*

(ii) *for $1 < i \leq r$, the brick G_i has a strictly thin edge e_i such that G_{i-1} is the retract of $G_i - e_i$.*

Since any cubic brick which is a conformal minor of G is also a matching minor of G, the above theorem is applicable to the case in which J is a cubic brick, distinct from K_4 and $\overline{C_6}$, that happens to be a conformal minor of G.

3.2. Proof of the equivalence of Problems 1.15 and 1.34. By Corollary 1.39, every solid brick is $\overline{C_6}$-free. A natural question then is to determine which nonsolid bricks are $\overline{C_6}$-free. The staircases of order 0 (modulo 4) are nonsolid and $\overline{C_6}$-free. The tricorn and the ubiquitous Petersen graph are also nonsolid and $\overline{C_6}$-free. It turns out that these are the only simple nonsolid bricks that are $\overline{C_6}$-free. Indeed, in this section we prove that the family of simple nonsolid bricks consists of precisely the following bricks:

(i) the $\overline{C_6}$-based simple bricks,

(ii) the staircases of order 0 (modulo 4),

(iii) the tricorn, and

(iv) the Petersen graph.

By Theorem 1.36, the only simple planar bricks that are $\overline{C_6}$-free are the odd wheels, the staircases of order 0 (modulo 4), and the tricorn. The odd wheels are solid. This establishes the result for planar bricks. We now establish the equivalence of Problems 1.15 and 1.34 for nonplanar simple bricks.

THEOREM 3.7. *The Petersen graph is the only brick that is simple, nonplanar, nonsolid, and $\overline{C_6}$-free.*

Proof. Let G be a simple, nonplanar, nonsolid brick distinct from the Petersen graph. The only nonplanar members of the family $\mathcal{NT}^+$ are the Möbius ladders of order 0 (modulo 4) and the Petersen graph. The Möbius ladders are solid. It follows that $G \notin \mathcal{NT}^+$. Also, by Theorem 2.1, G contains one of the four basic nonsolid bricks as a conformal minor. To complete the proof, it suffices to show that if G has either the bicorn, the tricorn, or the Petersen graph as a conformal minor, then it also has $\overline{C_6}$ as a conformal minor. Toward this end, let J be one of the above-mentioned bricks (that is, bicorn, tricorn, or the Petersen graph) such that J is a conformal minor of G.

By Theorem 3.6, there exists a sequence $G_1, G_2, \ldots, G_r$ of simple bricks such that (i) $G_r = G$ and $G_1 = J$, and (ii) for $1 < i \le r$, the brick G_i has a strictly thin edge e_i such that G_{i-1} is the retract of $G_i - e_i$. In particular, $G_1 = J$ is the retract of $G_2 - e_2$. Since J is a cubic brick, it follows that e_2 is a strictly thin edge of index zero and, hence, that $J = G_2 - e_2$. In other words, G_2 is obtained from J by joining two nonadjacent vertices by an edge. By Proposition 3.3, every cubic brick that is a conformal minor of G_2 is also a conformal minor of G. Thus, in order to complete the proof, all we need to do is to show that any brick obtained from either the bicorn, the tricorn, or the Petersen graph $\mathbb{P}$ by adding an edge e joining two nonadjacent vertices contains a bisubdivision of $\overline{C_6}$ as a conformal subgraph. This is routine. (Figure 10(b) shows the relevant conformal subgraph of $\mathbb{P}+e$. Propositions 6.5 and 6.6 in [13] deal, respectively, with bricks obtained by adding an edge to the bicorn and the tricorn.) ∎

COROLLARY 3.8. *The only simple nonsolid $\overline{C_6}$-free bricks are the staircases of order* 0 (modulo 4)*, the tricorn, and the Petersen graph.* ∎

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