On the Number of Perfect Matchings in a Bipartite Graph*

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Abstract

In this paper we show that, with 11 exceptions, any matching covered bipartite graph on n vertices, with minimum degree greater than two, has at least $2n - 4$ perfect matchings. Using this bound, which is best possible, and McCuaig's Theorem [8] on brace generation, we show that any brace on n vertices has at least $(n-2)^2/8$ perfect matchings. A bi-wheel on n vertices has $(n-2)^2/4$ perfect matchings. We conjecture that there exists an integer N such that every brace on $n \geq N$ vertices has at least $(n-2)^2/4$ perfect matchings.

Contents

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1 Matching Covered Graphs

For graph theoretical notation and terminology, we essentially follow Bondy and Murty [1]. The *order* of a graph is the number of its vertices, and its *size* is the number of its edges. We denote the number of perfect matchings in a graph G by $\Phi(G)$, and the minimum degree of vertices of G by $\delta(G)$.

A graph is *matching covered* if it is connected, has at least two vertices, and each of its edges lies in a perfect matching. In studying questions related to the number of perfect matchings in a graph one may clearly restrict oneself to matching covered graphs. Thus, all the graphs we consider here are matching covered graphs. The treatise by Lovász and Plummer [7], and the seminal work by Lovász [6] on the matching lattice, contain the basic theory of matching covered graphs.

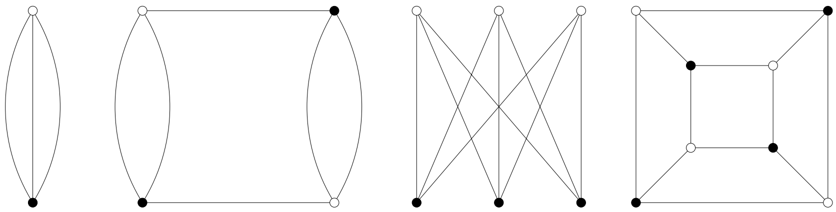
Let G be a bipartite matching covered graph on n vertices. Voorhoeve [10] showed that if G is 3-regular, then $\Phi(G) \geq (\frac{4}{3})^{n/2}$. (Establishing a more general conjecture of Lovász and Plummer, Esperet et al [5] have shown that any 2-connected cubic graph on n vertices has at least $2^{n/3656}$ perfect matchings.) As mentioned in the abstract, we shall show here that if $\delta(G) \geq 3$, and n is large enough, then $\Phi(G) \geq 2n - 4$. We shall also show that if G is a brace, then $\Phi(G) \geq \frac{(n-2)^2}{8}$. (For the definition of a brace, see Section 2.)

For the convenience of the reader, we shall briefly review here the terminology and results which are pertinent to this article.

1.1 Function $f(n)$

We denote the class of all bipartite matching covered graphs with minimum degree at least three by $\mathcal{F}$, and the subclass of those graphs on n vertices in $\mathcal{F}$ by $\mathcal{F}_n$. For each even integer $n \geq 2$, we define the function $f(n)$ as $\min\{\Phi(G)\}$, where the minimum is taken over all graphs in $\mathcal{F}_n$. It can be verified that $f(2) = 3$, $f(4) = 5$, $f(6) = 6$, and $f(8) = 9$.

A graph G_* in $\mathcal{F}_n$ is *extremal* if $\Phi(G_*) = f(n)$. Figure 1 shows extremal graphs on two, four, six and eight vertices.



(a) (b) (c) (d)

Figure 1: (a) The Theta graph (b) P_4 (c) $K_{3,3}$ (d) $B_8 = P_8$

For each even integer $n \geq 6$, let A_n be the graph obtained from a matching $\{u_1u_2, u_3u_4, \ldots, u_{n-3}u_{n-2}\}$, by adjoining two vertices, h_1 and h_2, and then joining h_1 to each of $u_1, u_3, \ldots, u_{n-3}$ by a pair of multiple edges, and h_2 to each of $u_2, u_4, \ldots, u_{n-2}$ by a pair of multiple edges. Figure 2 shows a drawing of A_{10}. (Compare the definition of A_n with the definition of bi-wheel B_n given in Section 2.1.)

We note that A_n is a matching covered graph with minimum degree three, that is, for $n \geq 6$, the graph A_n belongs to $\mathcal{F}_n$. In the graph A_n, vertex h_1 has degree $n-2$, and each edge incident with h_1 is in two perfect matchings. Therefore, $\Phi(A_n) = 2n - 4$. Thus:

$$f(n) \leq 2n - 4, \text{ for } n \geq 6 \tag{1}$$

This upper bound is not attained for $n < 14$. But we shall see later on that $f(n) = 2n - 4$, for all $n \geq 14$.

1.2 Tight cuts

Let G be a matching covered graph. We shall refer to a subset of edges of G of the form $\partial(X)$, where X is a nonempty proper subset of V, as a *cut* of G

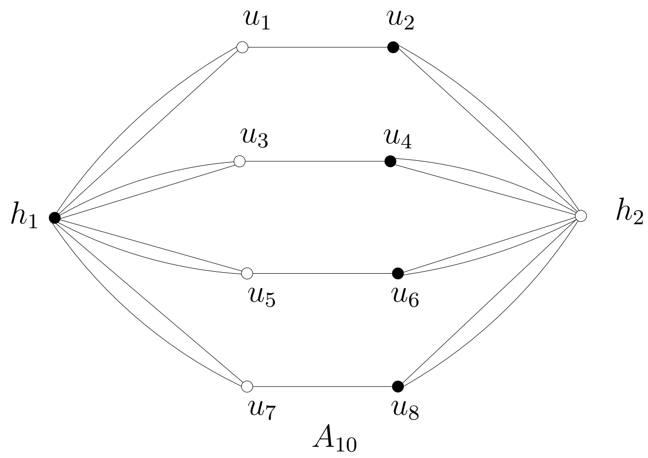


Figure 2: The graph A_{10}

with X and $\overline{X} = V \setminus X$ as its *shores*. A cut is *trivial* if one of its shores consists of exactly one vertex.

Given any cut $C := \partial(X)$ of G, one may obtain two graphs G/X and $G/\overline{X}$ by shrinking the shores of C to single vertices; they are called the C-*contractions* of G. A cut $C := \partial(X)$ of a matching covered graph G is *tight* if every perfect matching of G has exactly one edge in common with C. Trivial cuts are examples of tight cuts. As we shall see below, tight cuts in bipartite matching covered graphs may be described in a simple manner.

We denote a bipartite graph G with bipartition (A, B) by $G[A, B]$. Let $G := G[A, B]$ be a bipartite matching covered graph. If X is an odd subset of V then the larger of the two sets $|X \cap A|$ and $|X \cap B|$ is called the *majority part* of X and is denoted by X_+, and the other is called the *minority part* and is denoted by X_-. If X is a subset of V such that $|X_+| = |X_-| + 1$ and all edges of $\partial(X)$ have one end in X_+ and one end in $\overline{X}_+$, then $\partial(X)$ is a tight cut of G. It is not difficult to see that every tight cut of a bipartite matching covered graph is of this form.

Let F and H be two given matching covered graphs. A graph G is called a *splicing* of F and H, and is denoted by $F \odot H$, if there is a tight cut $\partial(X)$ of G such that $F \cong G/X$ and $H \cong G/\overline{X}$. In general, there may be many graphs G with this property. But in cases of interest to us here, $F \odot H$ is unique up to isomorphism. The two graphs in the top row of Figure 3 are $P_4 \odot K_{3,3}$ and $P_4 \odot B_8$, where P_4 is the graph shown in Figure 1(b), and B_8 is the cube, shown in Figure 1(d). The two graphs in the bottom row are

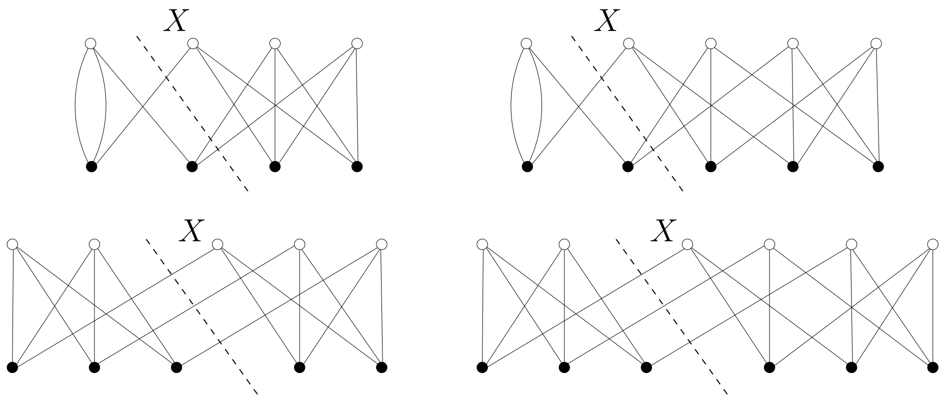


Figure 3: Top: $P_4 \odot K_{3,3}$ and $P_4 \odot B_8$; bottom: $K_{3,3} \odot K_{3,3}$ and $K_{3,3} \odot B_8$

$K_{3,3} \odot K_{3,3}$ and $K_{3,3} \odot B_8$.

Observation 1.1 *Since* $\Phi(P_4 \odot K_{3,3}) = 10 < 2 \times 8 - 4 = 12$, $\Phi(P_4 \odot B_8) = 15 < 2 \times 10 - 4 = 16$, $\Phi(K_{3,3} \odot K_{3,3}) = 12 < 2 \times 10 - 4 = 16$, *and* $\Phi(K_{3,3} \odot B_8) = 18 < 2 \times 12 - 4 = 20$, *we note that the four graphs in Figure 3 will have to be in the list of exceptions mentioned in the abstract.*

1.3 Bi-contractions and retracts

Vertices of degree two in a matching covered graph give rise to a very simple type of tight cuts. Let v be a vertex of degree two in a matching covered graph G, and let $X := \{v, v_1, v_2\}$, where v_1 and v_2 are the two neighbours of v. Then $\partial(X)$ is a tight cut of G, and the the graph G/X is said to be obtained from G by the *bi-contraction* of v.

Lemma 1.2
Let G be a graph on four or more vertices, let v be a vertex of degree two in G, and let H be the graph obtained from G by bi-contracting v. Then, $\Phi(G) = \Phi(H)$.

Proof: Every perfect matching of H has a unique extension to a perfect matching of G. Conversely, the restriction of any perfect matching of G is a perfect matching of H. □

If a graph G has at least three vertices and is not a cycle, then one can obtain a graph of minimum degree greater than two from G by means of bi-contractions. Up to isomorphism, the graph so obtained does not depend on the sequence of bi-contractions performed (see [2]). We refer to it as the *retract* of G, and denote it by $\widehat{G}$.

COROLLARY 1.3
For any graph G, $\Phi(G) = \Phi(\widehat{G})$. □

We denote the number of perfect matchings of a graph G containing a given edge e by $\Phi_e(G)$. If u and v are the two ends of e, then clearly $\Phi_e(G) = \Phi(G - \{u, v\})$. The following simple lemma plays a useful role in the proofs of many results.

LEMMA 1.4 (RECURSION LEMMA)
Let G be a graph and let $e = uv$ be any edge of G. Then

$$\Phi(G) = \Phi(\widehat{G - e}) + \Phi(G - \{u, v\}) \tag{2}$$

Proof: The number of perfect matchings not containing the edge e is equal to $\Phi(G - e)$, but $\Phi(G - e) = \Phi(\widehat{G - e})$. On the other hand, the number $\Phi_e(G)$ of perfect matchings containing e is $\Phi(G - \{u, v\})$. Hence the identity. □

LEMMA 1.5
Let G be a graph in $\mathcal{F}$, let v be a vertex of G, let d denote the degree of v. Then,

$$\Phi(G) = \frac{1}{d-1} \sum_{e \in \partial(v)} \Phi(\widehat{G - e}).$$

Proof: The following equalities hold:

$$\sum_{e \in \partial(v)} \Phi(\widehat{G - e}) = (d - 1) \sum_{e \in \partial(v)} \Phi_e(G) = (d - 1)\Phi(G).$$

□

2 Braces

A matching covered graph without nontrivial tight cuts is called a *brace* if it is bipartite, and a *brick* if it is nonbipartite. The theorem below provides a characterization of braces:

Theorem 2.1 (See [6], [7])
Let $G[A, B]$ be a bipartite matching covered graph. The following statements are equivalent:

(a) G is a brace;

(b) $G - a_1 - a_2 - b_1 - b_2$ has a perfect matching, for any two vertices a_1 and a_2 in A and any two vertices b_1 and b_2 in B;

(c) for every subset X of A such that $0 < |X| < |A| - 1$, we have $|N(X)| > |X| + 1$. □

A graph G is said to be 2-*extendable* if it has two non-adjacent edges and for any two non-adjacent edges e and f graph G has a perfect matching which includes $\{e, f\}$. Theorem 2.1 implies that every brace of order four or more is 2-extendable. It also implies that every vertex of a brace of order six or more has at least three distinct neighbors.

The following observation will be found useful later on. The only simple cubic bipartite graphs on six and eight vertices are, respectively, $K_{3,3}$ and the cube (see Figure 1(d)). Both of them happen to be braces. However, this not the case for larger orders. The Möbius ladder M_{10}, introduced in the next subsection, is a cubic brace of order 10, and there is a simple cubic bipartite graph of order 10 which is different from it.

Proposition 2.2
The only simple cubic bipartite graph of order 10 that is not a brace is $K_{3,3} \odot K_{3,3}$.

Proof: Let G be a simple cubic bipartite graph, and let $\partial(X)$ be a nontrivial tight cut of G. Since each vertex has three distinct neighbours, and the vertices of X_- are joined only to those in X_+, it follows that $|X| \geq 5$. Similarly, $|\overline{X}| \geq 5$. Thus, $|X| = 5 = |\overline{X}|$. It is now easy to deduce that G is $K_{3,3} \odot K_{3,3}$. □

2.1 McCuaig braces

There are three families of braces of special importance, namely prisms, Möbius ladders and bi-wheels. McCuaig [8] showed that all braces of order at least six may be generated from these special braces by using three elementary operations. (We shall briefly review McCuaig's result in Section 2.3.)

Prisms: The *prism* P_{2r} of order $2r$, $r \geq 2$, is the cartesian product of the r-cycle C_r and the complete graph K_2. For each even integer $r \geq 2$, the prism P_{2r} is a brace. Figures 1(b) and (d) depict the prisms P_4 and P_8. Figure 4(a) depicts the prism P_{12}.

Consider the sequence $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, \ldots$, of Fibonacci numbers defined by $F_1 = F_2 = 1$ and by the recursion $F_i = F_{i-1} + F_{i-2}$, for $i \geq 3$. Using simple inductive arguments, it is easy to see that the number of perfect matchings in the prism P_{2r} may be expressed in terms of Fibonacci numbers as follows:

$$\Phi(P_{2r}) = F_{r+1} + F_{r-1} + 2 \qquad (3)$$

In particular, $\Phi(P_{12}) = F_7 + F_5 + 2 = 13 + 5 + 2 = 20$.

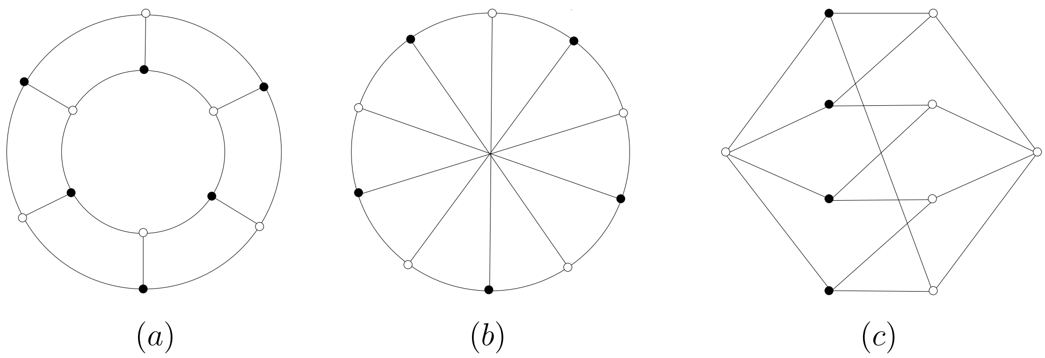


Figure 4: (a) P_{12} (b) M_{10} (c) B_{10}

Möbius ladders: The *Möbius ladder* M_{2r}, $r \geq 3$, is the cubic graph obtained from the $2r$-cycle by joining each vertex of the cycle to the vertex that is antipodal to it. For each odd integer $r \geq 3$, the Möbius ladder M_{2r} is a brace. Furthermore,

$$\Phi(M_{2r}) = F_{r+1} + F_{r-1} + 2 \qquad (4)$$

The Möbius ladder M_6 is isomorphic to $K_{3,3}$. Figure 4(b) depicts the Möbius ladder M_{10}. It follows from (4) that $\Phi(M_{10}) = F_6 + F_4 + 2 = 8 + 3 + 2 = 13$.

Bi-wheels: For $r \geq 4$, the *bi-wheel* B_{2r} of order $2r$ is the graph obtained from a $(2r-2)$-cycle $(u_1, u_2, \ldots, u_{2r-2}, u_1)$, called the *rim*, by adjoining two vertices, h_1 and h_2, called the *hubs*, and then joining h_1 to each of $u_1, u_3, \ldots, u_{2r-3}$, and h_2 to each of $u_2, u_4, \ldots, u_{2r-2}$. (Figure 4(c) depicts the bi-wheel B_{10}.) It is easy to verify that B_{2r} is a brace and that any edge incident with h_1 and any edge incident with h_2 are contained together in a unique perfect matching of B_{2r}. Consequently,

$$\Phi(B_{2r}) = (r-1)^2 \tag{5}$$

The cube, which we noted is a prism, is also a bi-wheel, the smallest bi-wheel B_8. Using the above formula, we have $\Phi(B_8) = (4-1)^2 = 9$.

Extended bi-wheels: The brace obtained by adding an edge to B_{2r} joining its two hubs is denoted by B_{2r}^+ and is called an *extended bi-wheel*. This auxiliary family of braces also plays an important role in McCuaig's work [8]. Figure 5 shows drawings of B_8^+ and B_{10}^+. It is easy to see that $\Phi(B_{2r}^+) = (r-1)^2 + 2$. In particular, $\Phi(B_8^+) = 9 + 2 = 11$.

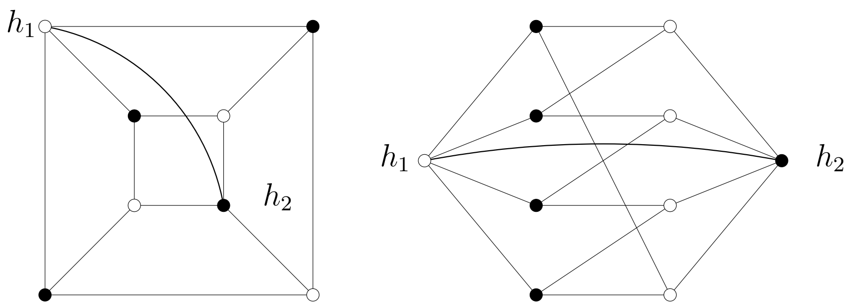


Figure 5: The extended bi-wheels B_8^+ and B_{10}^+

2.2 Removable edges and their indices

An edge e of a matching covered graph G is *removable* if $G-e$ is also matching covered. Suppose that the order of G is at least four. In this case, if e is removable, then both ends of e have degree at least three in G. The *index* of a removable edge is the number of its ends which have degree exactly three. Thus, the index of e is zero if both ends of e have degree greater then three, one if exactly one end of e has degree three, and two if both ends of e have degree three.

The following statement concerning braces may be deduced from Theorem 2.1.

Corollary 2.3
Every edge in a brace of order six or more is removable. □

If G is a bipartite matching covered graph, then it is known (see [7]) that the dimension of the linear space generated by the incidence vectors of perfect matchings of G is $|E(G)| - |V(G)| + 2$, implying that:

$$\Phi(G) \geq |E(G)| - |V(G)| + 2 \tag{6}$$

The proof of the next lemma uses this fact. However, the above lower bound is rarely attained. A characterization of bipartite graphs for which equality holds in (6) is given in [2].

Lemma 2.4
Let $G = G[A, B]$ be a brace of order $2r \geq 6$, and let $e = uv$ be a any edge of positive index of G, with $u \in A$ and $v \in B$. Then $H := G - \{u, v\}$ is matching covered, and $\Phi(H) \geq r - 1$.

Proof: The fact that H is matching covered follows from the fact that G is 2-extendable.

Since G has positive index, at least one of the ends of e has degree two in $G - e$. Without loss of generality, suppose that v has degree two in $G - e$, and let u_1 and u_2 denote the neighbours of v in $A - u$. Then $d_H(u_1) = d_G(u_1) - 1 \geq 2$, $d_H(u_2) = d_G(u_2) - 1 \geq 2$, and for $u' \in A - u - u_1 - u_2$, $d_H(u') = d_G(u') \geq 3$. We therefore have:

$$|E(H)| \geq 2 + 2 + 3(r - 3) = 3r - 5$$

Clearly, $|V(H)| = 2r - 2$. Thus, using (6), we have $\Phi(H) \geq (3r - 5) - (2r - 2) + 2 = r - 1$. □

We note that the assumption that e has positive index is necessary. For example, if G is the extended bi-wheel B_{2r}^{+}, and $e = h_1 h_2$ is the edge joining the two hubs, then $\Phi(B_{2r}^{+} - \{h_1, h_2\}) = 2$, for all r.

2.3 McCuaig's Theorem on brace generation

Let G be a brace on six or more vertices, let e be an edge of G. We note that the number of bi-contractions required to obtain $\widehat{G - e}$ from $G - e$ is equal to the index of e. Edge e is *thin* if $\widehat{G - e}$ is a brace, and is *strictly thin* if $\widehat{G-e}$ is a simple brace. The three edges e_0, e_1 and e_2 in Figure 6 are, respectively, thin edges of index zero, one and two in that brace. Edges e_0 and e_1 are strictly thin, but e_2 is not.

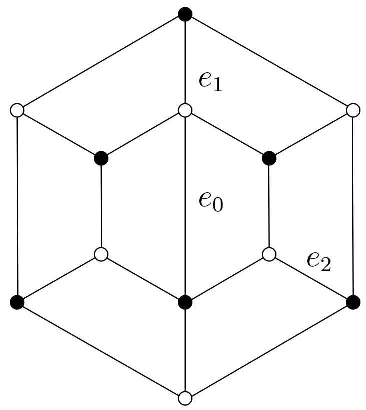


Figure 6: Edges e_0 and e_1 are strictly thin, but e_2 is not

Prisms, Möbius ladders, and bi-wheels do not have any strictly thin edges. We shall refer to braces in these families as *McCuaig braces*. In [8] McCuaig proved the following fundamental theorem. (In fact, he proved a stronger (splitter) version of the statement given here.)

Theorem 2.5 (The strictly thin edge theorem)
Every simple brace of order six or more which is not a McCuaig brace has a strictly thin edge.

A brace G is called an *extension* of *index* i of another brace H if G has a thin edge e of index i such that $H = \widehat{G-e}$. As an immediate consequence of the above theorem, we have:

Theorem 2.6 (Brace generation theorem)
Given any simple brace G of order at least six, there exists a sequence $(G_1, G_2, \ldots, G_r)$ of simple braces such that:

(i) G_1 *is a McCuaig brace,*

(ii) $G_r = G$,

(iii) for $1 \leq i \leq r-1$, G_{i+1} *is an extension of* G_i.

McCuaig braces do not have strictly thin edges, but they do have thin edges. Thus we may conclude from McCuaig's Theorem the following:

THEOREM 2.7 (THE THIN EDGE THEOREM)
Every brace of order six or more has a thin edge.

We gave a direct proof of the above theorem in [4] (unpublished) and showed how Theorem 2.5 may be deduced from it. (Our paper [3] contains the proof of an analogous theorem for bricks. An extension of that result has been published by Norine and Thomas [9]).

2.4 Braces of orders six to twelve

Braces of order six: The only simple brace of order six is $K_{3,3}$, and it has six perfect matchings.

OBSERVATION 2.8 *Since* $\Phi(K_{3,3}) = 6 < 2 \times 6 - 4 = 8$, *we note that* $K_{3,3}$ *will have to be in the list of exceptions mentioned in the abstract.*

Braces of order eight: Every simple brace of order eight is obtained by deleting the edges of a matching from $K_{4,4}$, and thus the sizes (numbers of edges) of such braces range from 12 to 16. The one of size twelve is the bi-wheel B_8 (which is the cube), the one of size thirteen is the extended bi-wheel B_8^+. We have seen that $\Phi(B_8) = 9$, and $\Phi(B_8^+) = 11$. The following proposition is easily verified.

PROPOSITION 2.9
Every simple brace of order eight, other than B_8, *and* B_8^+, *has at least 14 perfect matchings.*

OBSERVATION 2.10 *Since* $2 \times 8 - 4 = 12$, *we note that the bi-wheel* B_8, *with 9 perfect matchings, and the extended bi-wheel* B_8^+, *with 11 perfect matchings, will have to be in the list of exceptions.*

McCuaig's Stricly This Edge Theorem 2.5 plays a crucial role in the study of braces of order more than eight.

Braces of order 10:

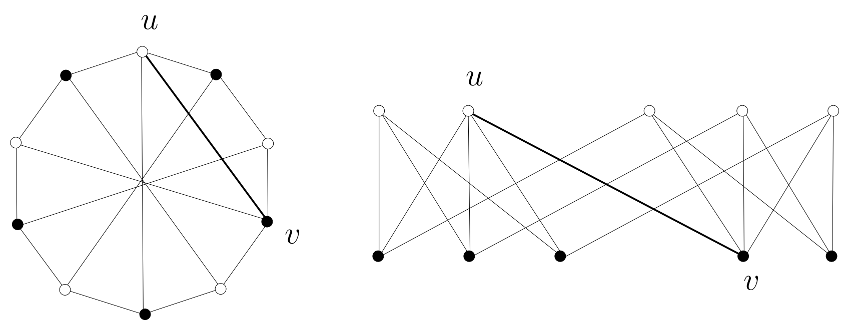
LEMMA 2.11
In a brace of order 10, no thin edge of index two is strictly thin.

Proof: Let G be a brace of order 10, and let e be a thin edge in G of index two. Then both ends of e have degree three in G, and $|V(\widehat{G-e})| = |V(G)| - 4 = 6$ and $|E(\widehat{G-e})| = |E(G)| - 5 \geq 10$. A simple bipartite graph on six vertices has at most nine edges. Hence, the brace $\widehat{G-e}$ is not simple, and e is not strictly thin. $\square$

COROLLARY 2.12
The Möbius ladder M_{10} is the only cubic brace on 10 vertices.

Proof: Let G be a cubic brace on 10 vertices. Since all degrees of G are three, G cannot have any strictly thin edges of index zero or of index one. By Lemma 2.11, G cannot have any strictly thin edges of index two either. Thus, by the Strictly Thin Edge Theorem, G is either M_{10} or B_{10}. But B_{10} is not cubic. $\square$

LEMMA 2.13
The only simple braces of order 10 and size 16 are M_{10}^{+} and $(K_{3,3} \odot K_{3,3})^{+}$ (shown in Figure 7), and B_{10}.



(a) (b)

Figure 7: (a) M_{10}^{+}, (b) $(K_{3,3} \odot K_{3,3})^{+}$

Proof: Let G be a simple brace of order 10 and size 16. Then, clearly, G has two vertices u and v of degree four, one in each part of the bipartition of G, and all the remaining vertices have degree three.

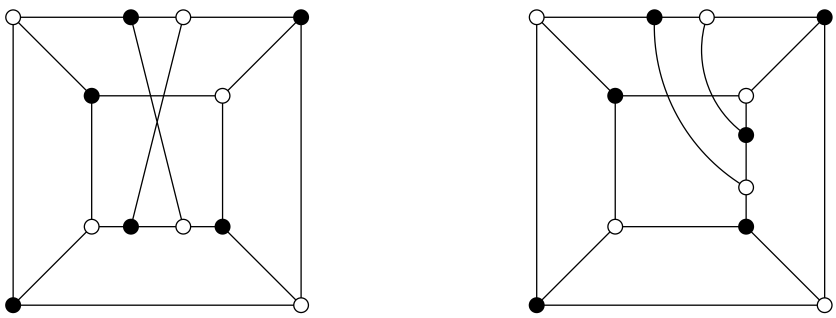
Suppose first that u and v are adjacent. If $G - uv$ is a brace then, by Corollary 2.12, G is isomorphic to the graph M_{10}^{+}. If $G - uv$ is not a brace then, by Proposition 2.2, $G - e$ is $= K_{3,3} \odot K_{3,3}$, and G is the graph $(K_{3,3} \odot K_{3,3})^+$ obtained by adding an edge to $K_{3,3} \odot K_{3,3}$.

Now suppose that u and v are not adjacent. It follows from Theorem 2.1(b) that graph $G - u - v$ is matching covered. As every vertex of $G - u - v$ has degree two, $G - u - v$ is a cycle on eight vertices. Thus G is the bi-wheel B_{10} with u and v as its hubs. $\square$

OBSERVATION 2.14 *Note that* $2 \times 10 - 4 = 16$. *Since* $\Phi(M_{10}) = 13$, *and* $\Phi((K_{3,3} \odot K_{3,3})^+) = 15$, *the two braces* M_{10} *and* $(K_{3,3} \odot K_{3,3})^+$ *will have to be included in the list of exceptions.*

Braces of order twelve:

LEMMA 2.15
There are three cubic braces of order twelve, namely the prism P_{12}, *and the two braces* G^1_{12} *and* G^2_{12} *shown in Figure 8.*



(a) (b)

Figure 8: (a) G^1_{12} (b) G^1_{12}

<u>Proof</u>: Let G be a cubic brace of order twelve. Clearly G cannot have any strictly thin edges of index zero or one. If it has no strictly thin edges of index two either, then by the Strictly Thin Edge Theorem, G is P_{12}. So, suppose that G has a strictly thin edge e of index two. Then $|V(\widehat{G-e})| = 12 - 4 = 8$, and $|E(\widehat{G-e})| = 18 - 5 = 13$. It follows that $\widehat{G-e}$ is the extended bi-wheel B_8^+. Thus G is an extension of index two of B_8^+. It can be verified that G^1_{12} and G^2_{12} are the only such extensions. $\square$

Observation 2.16 *The numbers of perfect matchings of* G^1_{12} *and* G^2_{12} *are, respectively, 17 and 18. Since* $2 \times 12 - 4 = 20$, *it follows that* G^1_{12} *and* G^2_{12} *will have to be included in the list of exceptions.*

3 A Linear Lower Bound for $f(n)$

3.1 The list $\mathcal{E}$ of exceptions

Recall that $\mathcal{F}$ is the class of all bipartite matching covered graphs with minimum degree at least three. We define $\mathcal{E}$ to be the family of 11 graphs in $\mathcal{F}$ consisting of:

- the brace $K_{3,3}$ of order six (Figure 1(c));
- the graph $P_4 \odot K_{3,3}$ (Figure 3, top left), and the braces B_8 (Figure 1(d)) and B_8^+ (Figure 5, left) of order eight;
- the graphs $P_4 \odot B_8$ (Figure 3, top right) and $K_{3,3} \odot K_{3,3}$ (Figure 3, bottom left), and the braces M_{10} (Figure 4(b)) and $(K_{3,3} \odot K_{3,3})^+$ (Figure 7(b)) of order ten; and
- the graph $K_{3,3} \odot B_8$ (Figure 3, bottom right), and the the braces G^1_{12} and G^2_{12} (Figure 8) of order twelve.

It would be useful to keep in mind that all members of $\mathcal{E}$, except B_8^+ and $(K_{3,3} \odot K_{3,3})^+$, are cubic.

We have observed that if $G \in \mathcal{E}$, then $\Phi(G) < 2|V(G)| - 4$. We wish to prove the following theorem:

Theorem 3.1
Any graph G in $\mathcal{F} \setminus \mathcal{E}$ has at least $2|V(G)| - 4$ perfect matchings.

In order to prove Theorem 3.1 by induction, it turns out to be convenient to prove a more general assertion. The notions of minimal graph and solitary edge play an important role in this more general setting.

3.2 Minimal graphs

A graph G in $\mathcal{F}$ is *minimal* if there is no edge e of G such that $G - e$ belongs to $\mathcal{F}$. As an immediate consequence of the definition we have:

LEMMA 3.2
If G is a minimal graph in $\mathcal{F}$ then every removable edge of G is incident with a vertex of degree three. $\square$

COROLLARY 3.3
If G is a minimal graph in $\mathcal{F}$ and e and f are multiple edges of G then at least one end of e has degree three in G. $\square$

LEMMA 3.4
Let G be a graph in $\mathcal{F}$. If G is minimal and simple then it has a vertex v of degree three such that each edge in $\partial(v)$ is removable in G.

<u>Proof</u>: The minimality of G implies that every removable edge of G is incident with a vertex of degree three. Consider first the case in which G is a brace. In that case, every edge of G is removable and the assertion holds.

Assume thus that G is not a brace. Let X be a minimal set of vertices of G such that $C := \partial(X)$ is a non-trivial tight cut of G. Then, $H := G/\overline{X}$ is a brace. Let X_+ and X_- denote, respectively, the majority and the minority parts of X. Every edge of G having both ends in X is removable in H, hence removable in G. To complete the proof, we now show that X_- has a vertex of degree three.

Consider first the case in which X_+ has a vertex w of degree four or more. Every vertex of X_+ is adjacent to some vertex of X_-. Every edge having both ends in X is removable. Every vertex of X_- adjacent to w has degree three. The assertion holds in this case.

Alternatively, assume that every vertex of X_+ has degree three. Then,

$$|\partial(X_-)| = |\partial(X_+)| - |C| \leq 3|X_+| - 2 = 3|X_-| + 1.$$

The simplicity of G implies that $|X_+| \geq 3$. Thus, $|X_-| \geq 2$. Then, the average degree of vertices in X_- is 3.5 or less. Consequently, at least one vertex of X_- has degree three. $\square$

LEMMA 3.5
Let G be a minimal brace in $\mathcal{F}$ of order $2r \geq 6$. If G has a vertex of degree at least four then $\Phi(G) \geq 2|V(G)| - 4$.

Proof: Let u be a vertex of degree at least four in G. By Corollary 2.3, every edge of G is removable. As G is minimal, every neighbor of u has degree three. Thus, every edge incident with u has index one. By Lemma 2.4, every edge incident with u lies in at least $r-1$ perfect matchings. Therefore,

$$\Phi(G) \geq 4(r-1) = 2(2r) - 4 = 2|V(G)| - 4.$$

□

3.3 Solitary edges

For an edge e of a graph G, recall that we denote the number of perfect matchings of G containing e by $\Phi_e(G)$. We say that an edge e of G is *solitary* if $\Phi_e(G) = 1$.

A brace of order six or more cannot have any solitary edges, but non-braces in $\mathcal{F}$ may have any number of them. Consider, for example, the graph $A_n^{(k)}$ obtained from A_n (described in Section 1) by joining the two vertices h_1 and h_2 by k multiple edges. Then each of those k edges is a solitary edge. The following property of solitary edges will be found to be very useful.

Lemma 3.6
Let G be a graph in $\mathcal{F}$ and let $e = uv$ be a solitary edge of G. If G has four or more vertices then there exists a vertex w in $V(G) - u - v$ that has degree one in $G - u - v$, and is joined to v by two or more parallel edges in G.

Proof: Since G has at least four vertices, $H := G - u - v$ has at least two vertices. By hypothesis e is solitary, hence H has precisely one perfect matching. Let M be the perfect matching of H, and let P be a maximal M-alternating path in H. Path P has odd length, its ends lie in distinct color classes of G. Moreover, both ends of P have degree one in H. Let w be the end of P that lies in the same colour class of u in G. Then, w is joined to v by two or more edges. □

Using the above lemma, we deduce that if G is a graph in $\mathcal{F}$ having solitary edges then G has two non-adjacent sets of parallel edges. One may then easily verify the following corollary:

Corollary 3.7

Let H be a graph in the family $\mathcal{E}$. Then both H and $H+e$, where $H+e$ is any graph obtained by adding an edge e joining two vertices of H, are free of solitary edges. $\square$

For a graph G, we denote by $\mu(G)$ the maximum multiplicity of solitary edges; in other words, $\mu(G)$ is the maximum number of parallel solitary edges joining any pair of vertices of G. If G has no solitary edges, then $\mu(G) = 0$.

In the graph $A_n^{(k)}$ described in Section 3.3, the k parallel edges joining h_1 to h_2 are solitary, but no other edge is solitary. Therefore, $\mu(A_n^{(k)}) = k$. Note that $\Phi(A_n^{(k)}) = 2n - 4 + \mu(A_n^{(k)})$.

3.4 A generalization and its proof

We are now ready to state and prove an assertion that is more general than Theorem 3.1.

Theorem 3.8

Any graph G in $\mathcal{F} \setminus \mathcal{E}$ has at least $2|V(G)| - 4 + \mu(G)$ perfect matchings.

Proof: By induction on $|E(G)|$. Let G be a graph of order n in $\mathcal{F}\setminus\mathcal{E}$. Assume inductively that if H is any graph in $\mathcal{F} \setminus \mathcal{E}$ with $|E(H)| < |E(G)|$, then $\Phi(H) \geq 2|V(H)| - 4 + \mu(H)$. We shall deduce that the asserted inequality holds for G by analyzing various cases.

Case 1 *Graph G is not minimal.*

By the definition of minimality, there exists an edge e of G such that $H := G - e \in \mathcal{F}$.

Case 1.1 *Suppose first that H is not in $\mathcal{E}$.*

Then, by induction hypothesis, $\Phi(H) \geq 2n - 4 + \mu(H)$. Every edge of G distinct from e that is solitary in G is also solitary in H. Thus, $\mu(H) \geq \mu(G) - 1$. Therefore,

$$\Phi(G) \geq \Phi(H) + 1 \geq 2n - 4 + \mu(H) + 1 \geq 2n - 4 + \mu(G),$$

showing that the assertion holds in this case.

Case 1.2 *Now suppose that H is in $\mathcal{E}$.*

If H is a graph in $\mathcal{E}$ other than B_8, $K_{3,3} \odot K_{3,3}$, M_{10} and G^1_{12} then, as we have seen, the difference between $\Phi(H)$ and $2|V(H)| - 4$ is at most two. By Corollary 3.7, $\mu(G) = 0$, that is, e is not solitary in G. Thus, $\Phi(G) \geq 2|V(G)| - 4$ in this case.

If $H = B_8$ then either $G = B_8^+$ and belongs to $\mathcal{E}$ or e is a parallel edge and $\Phi(G) = 12 = 2|V(G)| - 4$. If H is M_{10} or G^1_{12} then the difference between $\Phi(H)$ and $2|V(H)| - 4$ is three, and if H is $K_{3,3} \odot K_{3,3}$ then the difference between $\Phi(H)$ and $2|V(H)| - 4$ is four. One can now verify that if H is M_{10} or G^1_{12} then e lies in at least three perfect matchings, and if H is $K_{3,3} \odot K_{3,3}$ then G is $(K_{3,3} \odot K_{3,3})^+$, a member of $\mathcal{E}$, or e lies in at least four perfect matchings.

Case 2 *Graph G is minimal but has multiple edges.*

Let us first suppose that $\mu(G) > 1$. In this case, we assert that G is the theta graph (shown in Figure 1(a)). Let e' and e'' be two parallel edges that are solitary. By the minimality of G, at least one end of e' has degree three; otherwise $G - e' \in \mathcal{F}$, in contradiction to the minimality of G. By Lemma 3.6, G has only two vertices. We deduce that G is the theta graph in this case. Then, $\Phi(G) = \mu(G) = 3$ and the assertion holds.

We may thus assume that $\mu(G) \leq 1$. Then, G has more than two vertices. By the hypothesis of the case, G has multiple edges. Let e' and e'' be two parallel edges of G. If possible, choose e' and e'' so that they are adjacent to some solitary edge of G. Let u_1 and v_1 denote the ends of e' and e''. The minimality of G implies that at least one of u_1 and v_1 has degree three. Adjust notation so that u_1 has degree three. Then, e' and e'' are the only edges that join u_1 and v_1. Let u_1v_2 be the edge of G incident with u_1 but not with v_1 (Figure 9). Let $X := \{u_1, v_1, v_2\}$. Clearly, $C := \partial(X)$ is a tight cut of G. For $i = 1, 2$, let C_i denote the set of edges of C that are incident with vertex v_i.

Let $H := G/X$. Clearly, C_1 is non-empty and C_2 has two or more edges. Thus $H \in \mathcal{F}$.

Case 2.1 *Graph H does not lie in $\mathcal{E}$.*

For $i = 1, 2$, let s_i denote the number of edges in C_i that are solitary in H. Then,

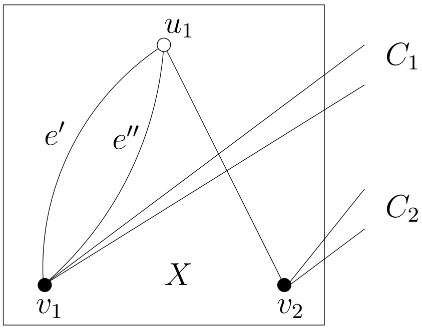


Figure 9: The case in which G has multiple edges

$$\Phi(G) = \Phi(H) + \sum_{f \in C_2} \Phi_f(H),$$

$$\sum_{f \in C_2} \Phi_f(H) \geq 2|C_2| - s_2 \geq 4 - s_2,$$

$$\Phi(H) \geq 2(n-2) - 4 + s_1 + s_2 = 2n - 8 + s_1 + s_2,$$

where the last inequality follows by induction. Addition and simplification yields

$$\Phi(G) \geq 2n - 4 + s_1.$$

To complete the analysis of the case, we prove that $s_1 \geq \mu(G)$. This inequality certainly holds if $\mu(G) = 0$. We have assumed that $\mu(G) \leq 1$. We may thus assume that $\mu(G) = 1$. By Lemma 3.6 and the criterion used for choosing e' and e'', it follows that a solitary edge of G is incident with one of u_1 and v_1. If u_1v_2 is solitary in G then v_1 also has degree three and the edge of C_1 is solitary. Alternatively, if u_1v_2 is not solitary then, by the choice of e' and e'', some edge of C_1 is solitary in G. In both alternatives, we may assume that some edge of C_1 is solitary in G. Every perfect matching of H is extendable to a perfect matching of G. Thus, every solitary edge of G that lies in $E(H)$ is solitary in H. We conclude that some edge of C_1 is solitary in H. In other words, $s_1 > 0$. The assertion holds in this case.

Case 2.2 *Graph H is in $\mathcal{E}$.*

By Corollary 3.7, H is free of solitary edges. Every edge of H that is solitary in G is also solitary in H. Edges e' and e'' are not solitary in G. Edge u_1v_2 is not solitary, otherwise C_1 would consist of a single edge, an edge of H solitary in G. We conclude that G is free of solitary edges. We must thus prove that $\Phi(G) \geq 2|V(G)| - 4$.

Define the parameter $\phi_2(H)$ to be the minimum of $\Phi_e(H)+\Phi_f(H)$, where the minimum is taken over all pairs $\{e, f\}$ of adjacent edges of H. Clearly,

$$\Phi(G) \geq \Phi(H) + \phi_2(H) \tag{7}$$

If H is either $K_{3,3}$ or B_8, then G also belongs to $\mathcal{E}$. So, we need only examine the other nine graphs in $\mathcal{E}$. For each of those graphs, the parameter ϕ_2 can be computed, and it can be verified that $\Phi(G) \geq 2|V(G)| - 4$. The details are included in Table 1.

Graph H	$\Phi(H)$	$\phi_2(H)$	$2\|V(G)\| - 4$
$P_4 \odot K_{3,3}$	10	6	16
B_8^+	11	5	16
$P_4 \odot B_8$	15	9	20
$K_{3,3} \odot K_{3,3}$	12	8	20
$(K_{3,3} \odot K_{3,3})^+$	15	7	20
M_{10}	13	8	20
$K_{3,3} \odot B_8$	18	12	24
G_{12}^1	17	11	24
G_{12}^2	18	12	24

Table 1: Values of $\Phi(H)$, $\phi_2(H)$ and $2|V(G)| - 4$

Case 3 *Graph G is minimal and free of multiple edges.*

We remark that as G is free of multiple edges it is also free of solitary edges, by Lemma 3.6. We must thus prove that $\Phi(G) \geq 2n - 4$. Every graph in $\mathcal{F}$ having fewer than six vertices has multiple edges. Thus, G has six or more vertices. The brace $K_{3,3}$ is the only simple graph on six vertices in $\mathcal{F}$. But $K_{3,3}$ is a member of $\mathcal{E}$. Thus, G has eight or more vertices. Every graph in $\mathcal{F}$ on eight vertices and free of multiple edges is a brace. Every brace on eight vertices includes the cube B_8 as a subgraph. The minimality of G then implies that G is B_8. But B_8 is a member of $\mathcal{E}$. We deduce that G has more than eight vertices.

Case 3.1 $n = 10$.

Consider first the case in which G has a non-trivial tight cut $C := \partial(X)$. The absence of multiple edges in G implies that $H := G/X$ is $K_{3,3}$, up to multiple edges in C. Every edge in C lies in two perfect matchings of H. Likewise, $H' := G/\overline{X}$ is $K_{3,3}$ up to multiple edges in C and each edge in C lies in two perfect matchings of H'. We deduce that every edge of C lies in four perfect matchings of G. As C is tight in G it follows that $\Phi(G) = 4|C|$. But $2n-4 = 16$. Thus, the asserted inequality holds, unless $|C| = 3$. In that case, G is $K_{3,3} \odot K_{3,3}$, which is a member of $\mathcal{E}$.

We consider now the case in which G is a brace. If G is cubic, then by Corollary 2.12, G is M_{10}. But M_{10} belongs to the list $\mathcal{E}$. If G is not cubic then by Lemma 3.5, $\Phi(G) \geq 2n - 4$.

Case 3.2 $n = 12$.

Consider first the case in which G is not a brace. The simplicity of G implies that it has a non-trivial tight cut $C := \partial(X)$, where $H := G/X$ is $K_{3,3}$, up to multiple edges of C. Let $H' := G/\overline{X}$.

- If $H' \notin \mathcal{E}$ then, by induction hypothesis, $\Phi(H') \geq 12$, and $\Phi(G) = 2\Phi(H') \geq 24 > 2n - 4$;
- If $H' = P_4 \odot K_{3,3}$ then $\Phi(H') = 10$ and $\Phi(G) = 2\Phi(H') = 20 = 2n - 4$;
- If $H' = B_8$ then G is the graph $K_{3,3} \odot B_8$ which is a member of $\mathcal{E}$;
- If $H' = B_8^+$, then $\Phi(H') = 11$, and $\Phi(G) \geq 2\Phi(H') = 22 > 20 = 2n-4$.

Hence the assertion holds for graphs of order 12 that are not braces.

We now consider the case in which G is a brace. If G is not cubic then, by Lemma 3.5, $\Phi(G) \geq 2n-4$. Assume thus G to be cubic. By Lemma 2.15, G is either P_{12}, or one of the two braces G^1_{12} and G^2_{12} shown in Figure 8. The number of perfect matchings of P_{12} is equal to the required lower bound $20 = 2n - 4$. On the other hand, the two braces G^1_{12} and G^2_{12} are members of $\mathcal{E}$. Hence the assertion holds for braces of order 12.

Case 3.3 $n \geq 14$.

By Lemma 3.4, G has a vertex v of degree three such that every edge in $\partial(v)$ is removable in G. Let e be any edge in $\partial(v)$. The retract $\widehat{G-e}$ of $G-e$ is not cubic and has $n-4$ or more vertices. Thus, either $\widehat{G-e}$ does not lie in $\mathcal{E}$ or it is $(K_{3,3} \odot K_{3,3})^+$. By induction hypothesis,

$$\Phi(\widehat{G-e}) \geq 2(n-4) - 5 = 2n - 13,$$

with equality only if (i) $n = 14$, (ii) edge e has index two and (iii) $\widehat{G-e} = (K_{3,3} \odot K_{3,3})^+$. Assume that equality does not hold, for any edge e in $\partial(v)$. In that case, by Lemma 1.5,

$$\Phi(G) \geq \frac{3 \cdot (2n-12)}{2} = 3n - 18 \geq 2n - 4.$$

To complete the analysis of the case, we must consider the situation in which $n = 14$ and $\partial(v)$ has an edge e of index two such that the retract of $G - e$ is $(K_{3,3} \odot K_{3,3})^+$. Up to isomorphism, there are two graphs of order fourteen, denoted by G^1_{14} and G^2_{14}, which are expansions of index two of $(K_{3,3} \odot K_{3,3})^+$. They are shown in Figure 10.

The first graph G^1_{14} has 25 perfect matchings, the second graph G^2_{14} has 24. We may now conclude that the assertion holds, by induction, for every minimal simple graph. □

We may deduce from the proof of the above theorem that $f(6) = 6$, $f(8) = 9$, $f(10) = 12$, and $f(12) = 17$, and that the unique extremal graphs of orders six, eight, 10, and twelve are, respectively, $K_{3,3}$, B_8, $K_{3,3} \odot K_{3,3}$, and G^1_{12}. The value of $f(14)$ is 24, and A_{14}, G^1_{14} are extremal graphs of order fourteen. The well-known Heawood graph, which is an expansion of B_{10}, is also an extremal graph of order fourteen.

4 A Quadratic Lower Bound for $b(n)$

We denote the class of all braces of order n by $\mathcal{B}_n$. Analogous to $f(n)$, we define $b(n)$ to be $\min\{\Phi(G) : G \in \mathcal{B}_n\}$. Clearly, $b(4) = 2$. Moreover, $\mathcal{B}_n \subseteq \mathcal{F}_n$ for $n \geq 6$. Thus:

PROPOSITION 4.1
For all $n \geq 6$, $f(n) \leq b(n)$.

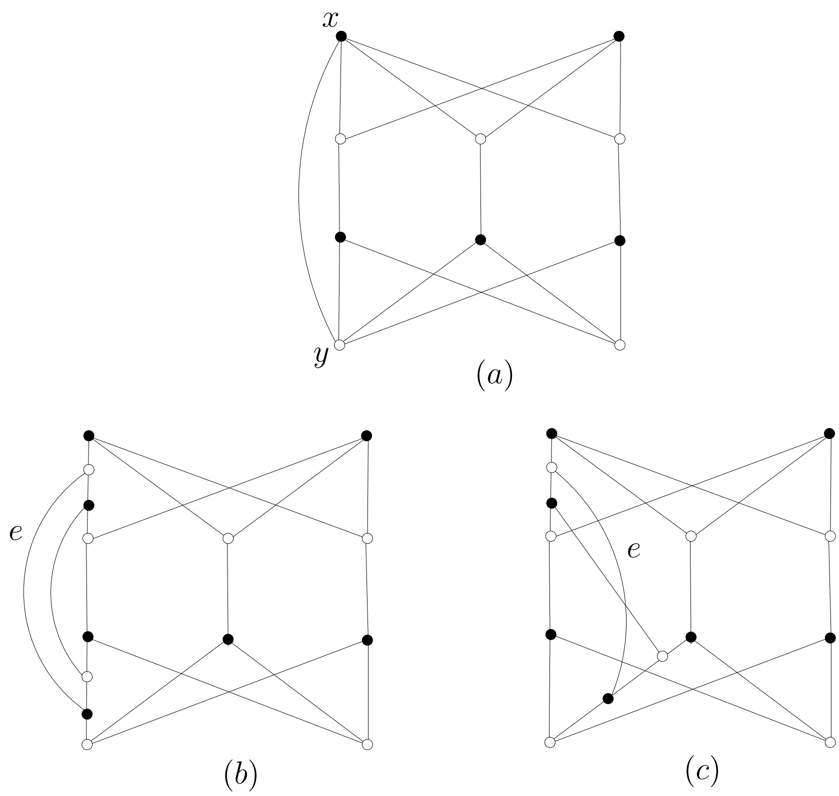


Figure 10: (a) $(K_{3,3} \odot K_{3,3})^+$, (b) G^1_{14} and (c) G^2_{14}

It so happens that, for $n = 6, 8$, the values of $f(n)$ and $b(n)$ coincide. But the brace of order 10 with the fewest number of perfect matchings is M_{10}, and it has thirteen perfect matchings, that is, $b(10) = 13$, whereas $f(10) = 12$.

A brace B_* of order n is *extremal* if $\Phi(B_*) = b(n)$. By Proposition 4.1, if an extremal graph of order n happens to be a brace, then it is also an extremal brace of order n. Thus, $b(6) = 6$, $b(8) = 9$, $b(10) = 13$, and $b(12) = 17$.

Since the bi-wheel of order n is a brace and has $(n-2)^2/4$ perfect matchings, it follows that, for $n \geq 8$,

$$b(n) \leq (n-2)^2/4 \tag{8}$$

As mentioned in the abstract, we shall show that, for all $n \geq 2$, $b(n) \geq (n-2)^2/8$. We find it compelling to believe that, for large enough n, $b(n) = (n-2)^2/4$.

Theorem 4.2
A brace of order n, where $n \geq 4$, has at least $\ell(n) := \frac{(n-2)^2}{8}$ perfect matchings.

Proof: Let us compare $\ell(n)$ and $b(n)$ for small values of n. The following table shows the values of $\lceil \ell(n) \rceil$ and $b(n)$ for $4 \leq n \leq 12$.

n	$\lceil \ell(n) \rceil$	$b(n)$
4	1	2
6	2	6
8	5	9
10	8	13
12	13	17

Thus $\ell(n) \leq b(n)$, for $4 \leq n \leq 12$. This inequality also holds for $14 \leq n \leq 18$. To see this, first observe that, in this range, $\ell(n) \leq 2n - 4$. By Theorem 3.1, $2n - 4 \leq f(n)$, for $n \geq 14$. On the other hand, by Proposition 4.1, $f(n) \leq b(n)$. Thus, for $14 \leq n \leq 18$, we have $\ell(n) \leq b(n)$.

We shall prove the validity of the inequality for $n \geq 20$ by induction on the number of edges. Consider any brace G of order n, where $n \geq 20$. By Theorem 2.7, G has a thin edge, say, $e = uv$.

Case 0 *Index of e is zero.*

In this case, $\widehat{G-e} = G - e$ is a brace on n vertices. By the induction hypothesis, $\Phi(G-e) \geq \ell(n)$, implying that $\Phi(G) > \ell(n)$.

Case 1 *Index of e is one.*

In this case, $\widehat{G-e}$ has $n-2$ vertices, and by induction hypothesis,

$$\Phi(\widehat{G-e}) \geq \ell(n-2) = \ell(n) - \frac{4n-12}{8} \tag{9}$$

On the other hand, by Lemma 2.4,

$$\Phi(G - \{u, v\}) \geq \frac{n-2}{2} = \frac{4n-8}{8} \tag{10}$$

Adding inequalities (9) and (10), and using Lemma 1.4 we deduce that $\Phi(G) > \ell(n)$.

Case 2 *Index of e is two.*

In this case, $\widehat{G-e}$ has $n-4$ vertices. By the induction hypothesis,

$$\Phi(\widehat{G-e}) \geq \ell(n-4) = \ell(n) - n + 4 \tag{11}$$

The graph $G - \{u, v\}$ is a matching covered graph on $n-2 \geq 18$ vertices and at least 25 edges. It has at most four vertices of degree two. Let H denote the retract of $G - \{u, v\}$. Each bi-contraction decreases the number of vertices and edges by two. Since H is obtained from $G - \{u, v\}$ by at most four bi-contractions, the number of vertices of H is at least $n - 10 \geq \frac{n}{2}$.

If at most two bi-contractions were required to obtain H from $G-\{u, v\}$, then $|V(H)| \geq 14$. If three bi-contractions were required to obtain H then H would have 12 vertices and 19 edges. The members of $\mathcal{E}$ on 12 vertices (graphs G^1_{12} and G^2_{12}) have 18 edges. If four bi-contractions were required to obtain H then H would have 10 vertices and 17 edges. The members of $\mathcal{E}$ on 10 vertices have at most 16 edges. Hence H belongs to $\mathcal{F}$, but is not a member of $\mathcal{E}$. Consequently, by Theorem 3.1,

$$\Phi(G - \{u, v\}) = \Phi(H) \geq 2\,\frac{n}{2} - 4 = n - 4 \tag{12}$$

Now, adding the two inequalities (11), and (12), and using Lemma 1.4 we deduce that $\Phi(G) \geq \ell(n)$. □

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