We give a construction of polynomial integrable systems in \( \mathbb{C}^{2N} \) (or on \( \mathbb{R}^{2N} \), if the base field is \( \mathbb{R} \)) using the algebra-geometric structure of the space \( \text{Sym}^{N}(\mathbb{C}^2) \). It is based on a canonical transformation \( \varphi : \mathbb{C}^{2N} \to \mathbb{C}^{2N} \) from variables \( (x, y) \in \mathbb{C}^{2N}, \ x = (x_1, \ldots, x_N), \ y = (y_1, \ldots, y_N) \) to \( q = (q_1, \ldots, q_N), \ p = (p_1, \ldots, p_N) \) given by the generating function

\[
G = \sum_{i, n=1}^{N} \frac{1}{n} x_i^n p_n \Rightarrow \quad q_n = \frac{\partial G}{\partial p_n} = \frac{1}{n} \sum_{i=1}^{N} x_i^n, \quad y_i = \frac{\partial G}{\partial x_i} = \sum_{n=1}^{N} x_i^{n-1} p_n.
\]

The canonical transformation \( \varphi \) can be decomposed in the projection \( \pi : \mathbb{C}^{2N} \to \text{Sym}^{N}(\mathbb{C}^2) \) and a bi-rational isomorphism \( \text{Sym}^{N}(\mathbb{C}^2) \to \mathbb{C}^{2N} \). The projection \( \pi \) gives a branching covering of \( \text{Sym}^{N}(\mathbb{C}^2) \).

With any polynomial \( F(x, y) \in \mathbb{C}[x, y] \) such that \( \partial_y F(x, y) \neq 0 \) we associate \( N \) compatible Stäkel type integrable Hamiltonian systems in \( \mathbb{C}^{2N} \)

\[
\frac{dx_i}{dt_k} = \frac{\partial H_k(x, y)}{\partial y_i}, \quad \frac{dy_i}{dt_k} = -\frac{\partial H_k(x, y)}{\partial x_i}, \quad i, k \in \{1, \ldots, N\},
\]

where \( H_k(x, y) = \sum_{i=1}^{N} W_{k, i} F(x_i, y_i) \) and \( W_{k, i} \) is the inverse Vandermonde matrix. The intersection of the level sets \( H_s(x, y) = h_s, \ h_s \in \mathbb{C}, \ s = 1, \ldots, N, \) is a quasi-projective algebraic variety in \( \mathbb{C}^{2N} \)

\[
\mathcal{G} = \{(x, y) \in \mathbb{C}^{2N} \mid x_i \neq x_j \text{ if } i \neq j, \text{ and } F(x_i, y_i) = \sum_{s=1}^{N} h_s x_i^{s-1}, \ i = 1, \ldots, N\}.
\]

which is \( S_N \) invariant with the free action of \( S_N \).

We show that the functions \( H_k(q, p), \ k = 1, \ldots, N, \) defined by \( \phi^* H_k(q, p) = H_k(x, y) \) are polynomials. They are functionally independent. It leads us to one of our main result:

In the space \( \mathbb{C}^{2N} \) there are \( N \) commuting polynomial Hamiltonian systems corresponding to the Hamiltonians \( \mathcal{H}_1(q, p), \ldots, \mathcal{H}_N(q, p) \).

It follows from the Liouville theorem that all Hamiltonian systems obtained are completely integrable. In the results obtained we do not impose any condition on the genus of the curve

\[
\Gamma = \{(x, y) \in \mathbb{C}^2 \mid F(x, y) = \sum_{s=1}^{N} h_s x^{s-1}\}
\]

neither request that the curve \( \Gamma \) is regular.
Application of this construction to \( N \)-th symmetric power of a plane algebraic curve \( \Gamma \) of genus \( g \) leads to \( N \) integrable Hamiltonian systems on \( \mathbb{C}^{2N} \). In the case of a non-singular hyperelliptic curves \( \Gamma \) of genus \( g \) and \( N = g \) our systems represent integrable hierarchies of equations which had been discovered in the theory of finite gap solutions (algebra-geometric integration) of the Korteweg-de-Vries equation.

For \( N = 2, 3 \) and \( g = 1, 2, 3 \) we present explicit examples of our polynomial systems and discuss the problem of their integration. These results were announced in: