Group-graded identities of PI-algebras.

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We will consider associative algebras over a field F of zero characteristic graded by a finite abelian group G, and satisfying a nontrivial non-graded polynomial identity.

F is a field , char $F = 0$.
A is an associative F-algebra.
G is a finite abelian group,
$|G| = m, \quad G = \{g_1, \ldots, g_m\}$.

Definition 1 *A is a G-graded algebra if $A = \bigoplus_{g \in G} A_g$,
A_g are vector subspaces of A ($g \in G$) such that
$A_g A_h \subseteq A_g h \quad$ for all $g,\ h \in G$.*

Example 1.
$E = \langle\ e_i \mid e_i e_j = -e_j e_i, \quad i, j \in \mathbf{N}\ \rangle$ is the Grassmann algebra of infinite rank.
$E = E_0 \bigoplus E_1$ is $\mathbf{Z}_2$-graded,
$E_0 = \mathrm{Span}_F\{e_{i_1} \cdots e_{i_{2k}}\}$ is the subspace of E generated by all words of even lenghts on the generators,
$E_1 = \mathrm{Span}_F\{e_{i_1} \cdots e_{i_{2k=1}}\}$ is the subspace of E generated by all words of odd lenghts on the generators.

Let us denote by
$X^G = \{\ x_{ig} \mid i \in \mathbf{N}, g \in G\ \}$ a countable set of indeterminants, we will call them graded variables.
The variables of the set $X_g = \{x_{ig} | i \in \mathbf{N}\}$ are called homogeneous variables of the graded degree g, or g-variables, $\quad X^G = \bigcup_{g \in G} X_g$.

$F\langle X^G\rangle$ is the free associative generated by X^G.
Then $\mathcal{F} = F\langle X^G\rangle$ is G-graded with the grading
$\mathcal{F} = \oplus_{g\in G}\mathcal{F}_g$, where $\mathcal{F}_g = \mathrm{Span}_F\{x_{i_1g_1}x_{i_2g_2}\cdots x_{i_sg_s} | g = g_1g_2\cdots g_s\}$.

Let A be a G-graded F-algebra, $f(x_{1g_1},\ldots,x_{ng_n}) \in F\langle X^G\rangle$ be a G-graded polynomial then

$f(x_{1g_1},\ldots,x_{ng_n}) = 0$ is the graded identity of A if

$f(a_{1g_1},\ldots,a_{ng_n}) = 0$ in A for all $a_{ig_i} \in A_{g_i}$
(i.e. if f vanishes for all substitutions instead of the graded variables of any homogeneous elements of the algebra A of the same graded degrees).

A is a PI-algebra
(an algebra satisfying a non-graded polynomial identity $h(x_1,\ldots,x_k) = 0$, $h \in F\langle X\rangle$) if

$h(a_1,\ldots,a_k) = 0$ in A for all $a_i \in A$
(if some nontrivial polynomial h vanishes for the substitutions instead of the variables of all elements of the algebra A).

Due to the results of **Bergen, Cohen (1986)** for finite abelian groups, and of **Bahturin, Giambruno, Riley (1998)** for all (not necessary abelian) finite groups:
A G-graded algebra A is PI if and only if A satisfies a graded identity of the type $f(x_{1e},\ldots,x_{ne}) = 0$, where e is the unit element of the group G (f depends only of the variables of the graded degree e)

Example 2.
The Grassmann algebra E satisfies graded identities
$[y_1,y_2] = y_1y_2 - y_2y_1 = 0,$ (*)
$[y_1,z_1] = y_1z_1 - z_1y_1 = 0,$
$z_1^2 = 0,$
where $y_1,y_2 \in X_0$ (are even variables), $z_1 \in X_1$ (is an odd variable).
Thus E is PI (since it satisfies (*)).

$Id(A) = \{h \in F\langle X\rangle | A$ satisfies $h = 0\}$ is the ideal of all ordinary (non-graded) identities of the algebra A.
$Id^G(A) = \{f \in F\langle X^G\rangle | A$ satisfies $f = 0\}$ is the ideal of G-graded identities of the algebra A.
$Id^G(A)F\langle X^G\rangle$.

Specht problem: Is it true that any algebra has the finite basis of identities (the finite set of identities which imply all identities of the algebra)?

This problem (the same as its various interpretations) is one of the central question of the area.

Kemer A.R. (1987-1988) - positive solution for associative PI-algebras over a field of zero characteristic.

The solution is the corollary of the next two remarkable theorems:

Theorem 1 (Kemer A.R.) *If A is a finitely generated (f.g.) associative PI-algebra over a field of zero characteristic then $Id(A) = Id(B)$, where B is a finite dimensional (f.d.) algebra.*

Theorem 2 (Kemer A.R.) *A is any associative PI-alebra over a field of zero characteristic. Then $Id(A) = Id(E(B))$, where $B = B_0 \oplus B_1$ is f.d. $\mathbf{Z}_2$-graded algebra, $E(B) = B_0 \otimes E_0 \bigoplus B_1 \otimes E_1$.*

$E(B) = B_0 \otimes E_0 \bigoplus B_1 \otimes E_1$ is the Grassmann envelope of the $\mathbf{Z}_2$-graded algebra B.

Thus any associative algebra has exactly the same polynomial identities as the Grassmann envelope of some finite dimentional $\mathbf{Z}_2$-graded algebra. And any finitely generated associative algebra has exactly the same polynomial identities as a finite dimentional algebra.

It is very nice that the similar to these two remarkable facts results also take place for graded identities of a G-graded associative algebra.

1 G-graded identities

Let us assume that

G is a finite abelian group,

F is a splitting field of G, char $F = 0$.

It follows from the Kemer's works that the finite dimensional algebra B in Theorems 1, 2 really has more specific form. For G-graded algebras the analogue looks like the following

More precise:

Definition 2 *A G-graded finite dimensional F-algebra B is GPI-reduced if*
$B = (C_1 \times \cdots \times C_p) \oplus J$ *with* $C_1 J C_2 J \cdots J C_p \neq 0,$ *where*

C_i $(i = 1, \dots, p)$ are G-graded simple finite dimensional algebras,
J is the Jacobson radical of B.

Notice, that the frase "G-graded simple" means here that the algebra has no any nontrivial G-graded ideal (simple as G-graded algebra). A G-graded simple algebra may be not simple in general, but it is always semisimple.

Theorem 3 (IS) *Let G be a finite abelian group, F be a splitting field of G,* char $F = 0$.
For any G-graded PI-algebra A holds
$\underline{Id^G(A) = Id^G(E(B_1) \times \dots \times E(B_s))},$
where B_i $(i = 1, \dots, s)$ are $(G \times \mathbf{Z}_2)$-graded, and $(G \times \mathbf{Z}_2)$PI-reduced f.d. algebras,
$E(B_i) = \bigoplus_{g \in G}(E_0 \otimes B_{(g,0)} \bigoplus E_1 \otimes B_{(g,1)}).$

$E(B_i) = \bigoplus_{g \in G}(E_0 \otimes B_{(g,0)} \bigoplus E_1 \otimes B_{(g,1)})$ is the Grassmann envelope of the $(G \times \mathbf{Z}_2)$-graded algebra B. It has the natural G-grading.

In the case of finitely generated algebra we have

Theorem 4 (IS) *Let G be a finite abelian group, F be a splitting field of G,* char $F = 0$.
If A is a finitely generated G-graded PI-algebra over F then
$\underline{Id^G(A) = Id^G(B_1 \times \dots \times B_s)},$
where B_i $(i = 1, \dots, s)$ are GPI-reduced f.d. algebras.

It is clear that an ideal of G-graded identities Γ is the ideal of graded identities of a finitely generated (or finite dimensional) G-graded PI-algebra over F if and only if Γ contains a Capelli polynomial.

Corollary 1 *If G is a finite abelian group, and* char $F = 0$ *then any G-graded PI-algebra over F has a finite bases of graded identities.*

Thus the next natural questions arise.

Question 1 *To find the effective algorithm producing the list of the corresponding GPI-reduced algebras B_i for any given G-graded algebra A (for any GT-ideal containing a T-ideal of ordinary identities).*

The proof of the theorems is semi-constructive in some sense. It gives some algorithm how to construct these algebras for arbitrary given algebra. But we can not garantee any bound for the number of the steps of this procedure. But it does not mean that does not exist another way to define the list of the algebras B_i. Because their parameters of the structure should be strictly defined by the graded identitties.

Question 2 *To find the algorithm generating a finite basis of graded identities of a G-graded algebra (of a GT-ideal containing a T-ideal of ordinary identities).*

The problem 2 is extremly complicated for T-ideals of ordinary identites. But usually the graded identities have more understandable and reasonable structure. And the last problem may be much simpler to resolve for graded identities of some concrete algebras. Also the solution of this question may really help to study the ordinary (non-graded) identities of the algebras.

Corollary 2 *The similar results (Theorems 3, 1, Corollary 1) take place for identities with automorphisms of a PI-algebra if the automorphismos of the identities generate a finite abelian group.*

A. Kanel-Belov, E. Aljadeff (*arXiv:0903.0362v3 [math.RA] 6 May 2009*) have announced the proofs of the similar results for PI-algebras graded by any finite group (not necessary abelian).

Question 3 *Do the analogous theorems hold for identities with automorphisms and antiautomorphisms in general case?*

The Corollary 2 is true because there exist the strict relations between the gradings of associative algebras and the group actions on them in the case of an abelian group [2]. It implies the similar correspondence of the graded identities and of the identities with automorphisms of an algebra in this case (see [3, 4]). But in general case the situation is different and is much more complicated. That is why the question 3 is non-trivial and very interesting.

References

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