RIGHT COIDEAL SUBALGEBRAS IN $U_q^+(\mathfrak{so}_{2n+1})$

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QUANTUM GROUP $U_q^+(\mathfrak{so}_{2n+1})$

It is generated by $x_1, x_2, \ldots, x_n; g_1, g_2, \ldots, g_n$ with

$$\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i,\ \Delta(g_i) = g_i \otimes g_i,$$
$$g_j^{-1} x_i g_j = p_{ij} x_i,$$

where the parameters p_{ij} satisfy the relations

$$p_{nn} = q, p_{ii} = q^2,\ \ p_{i\,i+1} p_{i+1\,i} = q^{-2},\ 1 \leq i < n;$$
$$p_{ij} p_{ji} = 1,\ |i-j| > 1,$$

while the generators are related by

$$[x_i, [x_i, x_{i+1}]] = 0,\ 1 \leq i < n;\ [x_i, x_j] = 0,\ \ |i-j| > 1;$$
$$[[x_i, x_{i+1}], x_{i+1}] = [[[x_{n-1}, x_n], x_n], x_n] = 0,\ 1 \leq i < n-1.$$

Here the (skew)commutator is defined as follows

$$[u, v] = uv - p(u, v)vu,\ p(x_i, x_j) = p_{ij},$$
$$p(uv, w) = p(u, w)p(v, w),\ p(u, vw) = p(u, v)p(u, w).$$

The Weyl basis of $\mathfrak{so}_{2n+1}^+$ with the skew brackets in place of the Lie operation is a set of PBW-generators of $U_q^+(\mathfrak{so}_{2n+1})$ over $\mathbf{k}[G]$:

$$u[k,m] = [\dots[[x_k, x_{k+1}], x_{k+2}], \dots, x_m], \;\; m \leq 2n-k,$$

where by definition $x_i = x_{2n-i+1}$ for $i > n$.

Theorem 1. The coproduct on the PBW-generators has the following explicit form:

$$\Delta(u[k,m]) = u[k,m] \otimes 1 + g_k g_{k+1} \cdots g_m \otimes u[k,m]$$
$$+ \sum_{i=k}^{m-1} \tau_i (1 - q^{-2}) g_k g_{k+1} \cdots g_i \, u[i+1,m] \otimes u[k,i],$$

where $\tau_i = 1$ with only one exception being $\tau_n = q$.

The idea of the proof. First, we show that in the Shuffle representation we have

$$u[k,m] \sim x_m \otimes x_{m-1} \otimes \cdots \otimes x_{k+1} \otimes x_k.$$

Then, we use the definition of the braided coproduct in the Shuffle algebra. Next, using relation between coproduct and braided coproduct by routine calculations we find the coefficients.

Interestingly, this formula differs from that in $U_{q^2}^+(\mathfrak{sl}_{2n+1})$ by just one term.

RIGHT COIDEAL SUBALGEBRAS

Right coideal is a subspace U such that

$$\Delta(U) \subseteq U \otimes H.$$

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Theorem 2. There exists a bijection between all sequences $(\theta_1, \theta_2, \ldots, \theta_n)$ such that

$$0 \le \theta_i \le 2n - 2i + 1, \quad 1 \le i \le n$$

and the set of all right coideal subalgebras of $U_q^+(\mathfrak{so}_{2n+1})$ that contain the coradical.

Steps of the proof.

A. PBW-generators have the following form

$$\Phi^S(k,m) = u[k,m] - (1-q^2) \sum_{i=1}^{r} \alpha_i \Phi^S(1+s_i, m) u[k, s_i],$$

where $k \le s_1 < s_2 < \ldots < s_r < m$,

$$S \cap [k, m-1] = \{s_1, s_2, \ldots, s_r\}.$$

The root sequence: $r(\mathbf{U}) = (\theta_1, \theta_2, \ldots, \theta_n)$. The number θ_i is the maximal m such that for some S the value of $\Phi^S(i,m)$ belongs to $\mathbf{U}$, while the degree $x_i + x_{i+1} + \ldots + x_m$ of $\Phi^S(i,m)$ is not a sum of other nonzero degrees of elements from $\mathbf{U}$.

We show that $r(\mathbf{U})$ uniquely defines $\mathbf{U}$.

B. To each sequence of integer numbers

$$\theta = (\theta_1, \theta_2, \dots, \theta_n),\ 0 \le \theta_i \le 2n - 2i + 1,$$

we associate a r. c. s. $\mathbf{U}_\theta$ such that $r(\mathbf{U}_\theta) = \theta$.

B1. Subsets R_k, T_k and a binary predicate $\mathbf{P}$.
By definition $R_k, = T_k = \emptyset$ if $k > n$. Suppose that R_i, T_i, $k < i \le 2n$ are already defined. We put

$$\mathbf{P}(i, j) \rightleftharpoons j \in T_i \vee 2n - i + 1 \in T_{2n-j+1}.$$

If $\theta_k = 0$, then we set $R_k = T_k = \emptyset$. If $\theta_k \neq 0$, then by definition R_k contains $\tilde{\theta}_k = k + \theta_k - 1$ and all m satisfying the following three properties

$a)$ $k \le m < \tilde{\theta}_k$;

$b)$ $\neg\mathbf{P}\,(m + 1, \tilde{\theta}_k)$;

$c)$ $\forall r (k \le r < m)\ \ \mathbf{P}\,(r + 1, m) \Longleftrightarrow \mathbf{P}\,(r + 1, \tilde{\theta}_k)$.

Further, we define an auxiliary set

$$T'_k = R_k \cup \bigcup_{s \in R_k} \{a \mid s < a \le 2n - k,\ \mathbf{P}(s + 1, a)\},$$

and put

$$T_k = \begin{cases} T'_k, & \text{if } (2n - R_k) \cap T'_k = \emptyset; \\ T'_k \cup \{2n - k + 1\}, & \text{otherwise.} \end{cases}$$

B2. The subalgebra $\mathbf{U}_\theta$ by definition is generated over $\mathbf{k}[G]$ by values in $U_q^+(\mathfrak{so}_{2n+1})$ of the polynomials $\Phi^{T_k}(k, m)$, $1 \le k \le m$ with $m \in R_k$.

SOME PROPERTIES OF RIGHT C. SUBALGEBRAS

Denote by $\Sigma(U)$ the set of all (multi)degrees of elements from U.

Theorem 3. *Each right coideal subalgebra* $U \supset \mathbf{k}[G]$ *is uniquely defined by the set of degrees* $\Sigma(U)$. *More precisely*

$$U_1 \subseteq U_2 \text{ if and only if } \Sigma(U_1) \subseteq \Sigma(U_2).$$

Corollary. *Right coideal subalgebras are differentially closed in the following sense. Let* f *be a polynomial in* $x_1, x_2, \ldots, x_n$, *such that* $Deg(f) \in \Sigma(U)$. *If* $\partial_i(f) \in U$, $1 \leq i \leq n$, *then* $f \in U$.

DUALITY FOR MINIMAL PBW-GENERATORS

A PBW-generator is said to be *minimal* if it is not a polynomial in other PBW-generators.

$$\Phi^S(k, m) \sim \Phi^T(\psi(m), \psi(k)).$$

Here $\psi(i) = n - i + 1$, and T is complement of $\{\psi(s) - 1 \mid s \in S\}$ with respect to the interval $[\psi(m), \psi(k))$.

A natural conjecture:

$$\mathbf{U} = U_{\theta'} \otimes_{\mathbf{k}[F]} \mathbf{k}[H] \otimes_{\mathbf{k}[G]} U_{\theta}. \tag{1}$$

To describe necessary conditions when (1) is a subalgebra we display the element $\Psi^{\mathbf{S}}(k,m)$ schematically:

$$\begin{array}{ccccccccc} k{-}1 & k & k{+}1 & k{+}2 & k{+}3 & & m{-}2 & m{-}1 & m \\ \circ & \circ & \circ & \bullet & \circ & \cdots & \bullet & \circ & \bullet \end{array}$$

Consider the minimal PBW-generators $\Psi^{T_k}(k,m)$ and $\Psi^{T_i'}(i,j)$:

$$\begin{array}{cccccccccc} k{-}1 & & i{-}1 & i & i{+}1 & & m & & j \\ \circ & \cdots & \bullet & \bullet & \circ & \cdots & \bullet & & \cdot \\ & & \circ & \circ & \bullet & \cdots & \bullet & \cdots & \bullet \end{array} \tag{2}$$

We prove that if (1) is a subalgebra then for each pair of minimal PBW-generators one of the following two options is fulfilled:

a) No one of the possible four representations (2) has fragments of the form

$$\begin{array}{ccc} t & & l \\ \circ & \cdots & \bullet \\ \circ & \cdots & \bullet \end{array}$$

b) One of the possible four representations (2) has the form

$$\begin{array}{ccccccccc} k{-}1 & & & & & & & & m \\ \circ & \cdots & \circ & \cdots & \bullet & \cdots & \bullet \\ \circ & \cdots & \bullet & \cdots & \circ & \cdots & \bullet \end{array}$$

where no one of the intermediate columns has points of the same color.

$$\Phi^S(k,m) \sim \Phi^T(\psi(m), \psi(k)).$$

FURTHER HYPOTHESIS AND PROBLEMS

The total number of coideal subalgebras in $U_q^+(\mathfrak{so}_{2n+1})$ over $\mathbf{k}[G]$ equals $(2n)!!$, the order of the Weyl group.

Although there is no theoretical explanation why there appears the Weyl group, we state the following general hypothesis.

Conjecture 1. *Let $\mathfrak{g}$ be a simple Lie algebra defined by a finite root system R. The number of different right coideal subalgebras that contain the coradical in a quantum Borel algebra $U_q^+(\mathfrak{g})$ equals the order of the Weyl group defined by the root system R provided that q is not a root of 1.*

Yes: G_2 (B.S. Pogorelsky), A_n, B_n.

?: $C_n, D_n, F_4, E_6, E_7, E_8$.

A general approach for Kac-Moody algebras.

α_i – a simple root $\leftrightarrow$ a Weyl generator x_i.

$\alpha = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_m}$ – a root

$\leftrightarrow$ a basis element $[\ldots[[x_{i_1}, x_{i_2}], x_{i_3}], \ldots, x_{i_m}]$.

We may define in $U_q^+(\mathfrak{g})$ analogue of $\Phi^S(k, m)$:

$$\Phi^S(\alpha) = [\ldots[x_{i_5}, [[x_{i_3}, [x_{i_1}, x_{i_2}]], x_{i_4}]], \ldots, x_{i_m}],$$

where $i_3, i_5, \ldots \in S$, $i_2, i_4, \ldots, i_m \notin S$.

Conjecture 2. *Every r.c.s of $U_q^+(\mathfrak{g})$ has a set of PBW-generators of the form $\Phi^S(\alpha)$.*