Jordan gradings on exceptional simple Lie algebras

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Jordan subgroups

Jordan subgroups

Definition (Alekseevskiĭ 1974)

Given a simple Lie algebra $\mathfrak{g}$ and a complex Lie group G with $\mathrm{Int}(\mathfrak{g}) \leq G \leq \mathrm{Aut}(\mathfrak{g})$, an abelian subgroup A of G is a *Jordan subgroup* if:

(i) its normalizer $N_G(A)$ is finite,

(ii) A is a minimal normal subgroup of its normalizer, and

(iii) its normalizer is maximal among the normalizers of those abelian subgroups satisfying (i) and (ii).

Jordan gradings

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The classification of Jordan subgroups by Alekseevskiĭ splits in two types: classical and exceptional.

Jordan subgroups: classical cases

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$\mathfrak{g}$	A
A_{p^n-1}	$\mathbb{Z}_p^{2n}$
B_n $(n \geq 3)$	$\mathbb{Z}_2^{2n}$
C_{2^n-1} $(n \geq 2)$	$\mathbb{Z}_2^{2n}$
D_{n+1} $(n \geq 3)$	$\mathbb{Z}_2^{2n}$
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The dimension of all nonzero homogeneous spaces is always 1 in these classical cases, which are well-known.

Jordan subgroups: exceptional cases

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$\mathfrak{g}$	A	$\dim \mathfrak{g}_\alpha$ $(\alpha \neq 0)$
G_2	$\mathbb{Z}_2^3$	2
F_4	$\mathbb{Z}_3^3$	2
E_8	$\mathbb{Z}_5^3$	2
D_4	$\mathbb{Z}_2^3$	4
E_8	$\mathbb{Z}_2^5$	8
E_6	$\mathbb{Z}_3^3$	3

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Models of these gradings?

Composition algebras

Composition algebras

Definition

A *composition algebra* over a field $\mathbb{F}$ is a triple $(C, \cdot, n)$ where

- C is a vector space over $\mathbb{F}$,
- $x \cdot y$ is a bilinear multiplication $C \times C \to C$,
- $n : C \to \mathbb{F}$ is a multiplicative nondegenerate quadratic form:
    - its polar $n(x, y) = n(x + y) - n(x) - n(y)$ is nondegenerate,
    - $n(x \cdot y) = n(x)n(y)\ \forall x, y \in C$.

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The unital composition algebras will be called *Hurwitz algebras*.

Hurwitz algebras

Hurwitz algebras

Hurwitz algebras form a class of degree two algebras:

$$x^{\cdot 2} - n(x,1)x + n(x)1 = 0$$

for any x.
They are endowed with an antiautomorphism, the *standard conjugation*:

$$\bar{x} = n(x,1)1 - x,$$

satisfying

$$\bar{\bar{x}} = x, \quad x + \bar{x} = n(x,1)1, \quad x \cdot \bar{x} = \bar{x} \cdot x = n(x)1.$$

Cayley-Dickson doubling process

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Let $(B, \cdot, n)$ be an associative Hurwitz algebra, and let λ be a nonzero scalar in the ground field $\mathbb{F}$. Consider the direct sum of two copies of B:

$$C = B \oplus Bu,$$

with the following multiplication and nondegenerate quadratic form that extend those on B:

$$(a + bu) \cdot (c + du) = (a \cdot c + \lambda \bar{d} \cdot b) + (d \cdot a + b \cdot \bar{c})u,$$
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Notation: $CD(A, \mu, \lambda) := CD\big(CD(A, \mu), \lambda\big).$

Generalized Hurwitz Theorem

Generalized Hurwitz Theorem

Theorem

Every Hurwitz algebra over a field $\mathbb{F}$ is isomorphic to one of the following:

(i) *The ground field $\mathbb{F}$ if its characteristic is $\neq 2$.*

(ii) *A quadratic commutative and associative separable algebra $K(\mu) = \mathbb{F}1 + \mathbb{F}v$, with $v^2 = v + \mu$ and $4\mu + 1 \neq 0$. The norm is given by its generic norm.*

(iii) *A quaternion algebra $Q(\mu, \beta) = CD(K(\mu), \beta)$. (These four dimensional algebras are associative but not commutative.)*

(iv) *A Cayley algebra $C(\mu, \beta, \gamma) = CD(K(\mu), \beta, \gamma)$. (These eight dimensional algebras are alternative, but not associative.)*

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Hence if the characteristic of the ground field $\mathbb{F}$ is $\neq 2$, any Cayley algebra appears as

$$\begin{aligned} C = CD(\mathbb{F}, \alpha, \beta, \gamma) &= Q \oplus Qz \\ &= (K \oplus Ky) \oplus (K \oplus Ky)z \\ &= (\mathbb{F} \oplus \mathbb{F}x) \oplus (\mathbb{F} \oplus \mathbb{F}x)y \\ &\qquad \oplus \Big((\mathbb{F} \oplus \mathbb{F}x) \oplus (\mathbb{F} \oplus \mathbb{F}x)y\Big)z, \end{aligned}$$

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and it is naturally graded over $\mathbb{Z}_2^3$, with

$$C_{(\bar{1},\bar{0},\bar{0})} = \mathbb{F}x, \quad C_{(\bar{0},\bar{1},\bar{0})} = \mathbb{F}y, \quad C_{(\bar{0},\bar{0},\bar{1})} = \mathbb{F}z.$$

Symmetric composition algebras

Symmetric composition algebras

Definition

A composition algebra $(S, *, n)$ is said to be *symmetric* if the polar form of its norm is associative:

$$n(x * y, z) = n(x, y * z),$$

for any $x, y, z \in S$.

This is equivalent to the condition:

$$(x * y) * x = n(x)y = x * (y * x),$$

for any $x, y \in S$.

Examples

Examples

- *Para-Hurwitz algebras*: Given a Hurwitz algebra $(C, \cdot, n)$, its para-Hurwitz counterpart is the composition algebra $(C, \bullet, n)$, where

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- *Okubo algebras*: Assume $\operatorname{char} \mathbb{F} \neq 3$ and $\exists \omega \neq 1 = \omega^3$ in $\mathbb{F}$. Consider the algebra A_0 of zero trace elements in a central simple degree 3 associative algebra with multiplication

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \operatorname{tr}(xy)1,$$

and norm $n(x) = -\frac{1}{2} \operatorname{tr}(x^2)$.
(There is a more general definition valid over arbitrary fields.)

Classification

Classification

Theorem (E.-Myung 93, E. 97)

Any symmetric composition algebra is either:

- *a para-Hurwitz algebra,*
- *a form of a two-dimensional para-Hurwitz algebra without idempotent elements (with a precise description),*
- *an Okubo algebra.*

Gradings on para-Hurwitz algebras

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Theorem

Gradings on para-Hurwitz algebras of dimension 4 or 8

$$\updownarrow$$

Gradings on their Hurwitz counterparts.

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Gradings on their Hurwitz counterparts.

Therefore, any para-Cayley algebra over a field of characteristic $\neq 2$ is endowed with a $\mathbb{Z}_2^3$-grading.

Gradings on Okubo algebras

Gradings on Okubo algebras

Assuming $\mathbb{F}$ is a field of characteristic $\neq 3$ containing a primitive third root ω of 1, then the matrix algebra $\mathrm{Mat}_3(\mathbb{F})$ is generated by the order 3 matrices:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and the assignment

$$\deg(x) = (\bar{1}, \bar{0}), \qquad \deg(y) = (\bar{0}, \bar{1}),$$

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Over algebraically closed fields, any grading on an Okubo algebra is a coarsening of either the natural $\mathbb{Z}^2$-grading (Cartan grading) or this $\mathbb{Z}_3^2$-grading.

Triality Lie algebra

Triality Lie algebra

Assume from now on that $\operatorname{char}\mathbb{F} \neq 2, 3$ and $\omega \in \mathbb{F}$.

Let $(S, *, n)$ be any symmetric composition algebra and consider the corresponding orthogonal Lie algebra:

$$\mathfrak{o}(S, n) = \{d \in \operatorname{End}_{\mathbb{F}}(S) : n\big(d(x), y\big) + n\big(x, d(y)\big) = 0 \ \forall x, y \in S\},$$

and the subalgebra of $\mathfrak{o}(S, n)^3$ (with componentwise multiplication):

$$\mathfrak{tri}(S, *, n) = \{(d_0, d_1, d_2) \in \mathfrak{o}(S, n)^3 : d_0(x*y) = d_1(x)*y + x*d_2(y) \ \forall x, y\}$$

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The map: $\theta : \mathfrak{tri}(S, *, n) \to \mathfrak{tri}(S, *, n)$, $(d_0, d_1, d_2) \mapsto (d_2, d_0, d_1)$ is an automorphism of order 3.

Principle of Local Triality

Principle of Local Triality

Theorem (Principle of Local Triality)

Let $(S, *, n)$ *be an eight dimensional symmetric composition algebra. Then the projection*

$$\begin{aligned} \pi_0 : \mathfrak{tri}(S, *, n) &\longrightarrow \mathfrak{o}(S, n) \\ (d_0, d_1, d_2) &\mapsto \quad d_0, \end{aligned}$$

is an isomorphism of Lie algebras.

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Theorem

- *A $\mathbb{Z}_2^3$-grading of a para-Cayley algebra $(\bar{C}, \bullet, n)$ induces a $\mathbb{Z}_2^3 \times \mathbb{Z}_3$-grading of the orthogonal Lie algebra $\mathfrak{o}(C, n)$ of type $(14, 7)$.*
- *The standard $\mathbb{Z}_3^2$-grading on an Okubo algebra $(\mathcal{O}, *, n)$ induces a $\mathbb{Z}_3^3$-grading on the orthogonal Lie algebra $\mathfrak{o}(\mathcal{O}, n)$ of type $(24, 2)$.*

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Remark

A $\mathbb{Z}_2^3$-grading of a para-Cayley algebra $(\bar{C}, \bullet, n)$ also induces a $\mathbb{Z}_2^3$-grading of its Lie algebra of derivations (which is an exceptional simple Lie algebra of type G_2). The type of this grading is $(0, 7)$.

Freudenthal Magic Square

Freudenthal Magic Square

Let $(S, *, n)$ and $(S', \star, n')$ be two symmetric composition algebras. One can construct a Lie algebra as follows:

$$\mathfrak{g} = \mathfrak{g}(S, S') = \big(\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')\big) \oplus \Big(\oplus_{i=0}^{2} \iota_i(S \otimes S')\Big),$$

with bracket given by:

- the Lie bracket in $\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')$, which thus becomes a Lie subalgebra of $\mathfrak{g}$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i\big(d_i(x) \otimes x'\big)$,
- $[(d_0', d_1', d_2'), \iota_i(x \otimes x')] = \iota_i\big(x \otimes d_i'(x')\big)$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}\big((x * y) \otimes (x' \star y')\big)$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = n'(x', y')\theta^i(t_{x,y}) + n(x, y)\theta'^i(t'_{x',y'})$,

Freudenthal Magic Square

Freudenthal Magic Square

		dim S'			
$\mathfrak{g}(S,S')$		1	2	4	8
dim S	1	A_1	A_2	C_3	F_4
	2	A_2	$A_2 \oplus A_2$	A_5	E_6
	4	C_3	A_5	D_6	E_7
	8	F_4	E_6	E_7	E_8

Gradings on the Freudenthal Magic Square

The Lie algebra $\mathfrak{g}(S, S')$ is naturally $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded with

$$\mathfrak{g}_{(\bar{0},\bar{0})} = \mathfrak{tri}(S) \oplus \mathfrak{tri}(S'),$$

$$\mathfrak{g}_{(\bar{1},\bar{0})} = \iota_0(S \otimes S'), \qquad \mathfrak{g}_{(\bar{0},\bar{1})} = \iota_1(S \otimes S'), \qquad \mathfrak{g}_{(\bar{1},\bar{1})} = \iota_2(S \otimes S').$$

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Also, the order 3 automorphisms θ and θ' extend to an order 3 automorphism Θ of $\mathfrak{g}(S, S')$. The eigenspaces of Θ constitute a $\mathbb{Z}_3$-grading of $\mathfrak{g}(S, S')$.

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- The $\mathbb{Z}_3^2$-grading on the Okubo algebra $\mathcal{O}$ induces a $\mathbb{Z}_3^3$-grading on both the simple Lie algebra $\mathfrak{g}(\mathbb{F}, \mathcal{O})$ of type F_4 and the simple Lie algebra $\mathfrak{g}(S, \mathcal{O})$ (for the two-dimensional para-Hurwitz algebra S) of type E_6.
  In both cases $\mathfrak{g}_0 = 0$ and $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is a Cartan subalgebra of $\mathfrak{g}$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$.

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  In both cases $\mathfrak{g}_0 = 0$ and $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is a Cartan subalgebra of $\mathfrak{g}$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$.
- The $\mathbb{Z}_2^3$-grading on a para-Cayley algebra $\bar{C}$ induces a $\mathbb{Z}_2^5$-grading on the simple Lie algebra $\mathfrak{g}(\bar{C}, \bar{C})$ of type E_8.
  Moreover, $\mathfrak{g}_0 = 0$ and $\mathfrak{g}_\alpha$ is a Cartan subalgebra of $\mathfrak{g}$ for any $0 \neq \alpha \in \mathbb{Z}_2^5$.

Exceptional Jordan gradings

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Theorem

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are exceptional Jordan gradings.

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Let V_1 and V_2 be two vector spaces over $\mathbb{F}$ of dimension 5, and consider the $\mathbb{Z}_5$-graded vector space

$$\mathfrak{g} = \oplus_{i=0}^{4} \mathfrak{g}_{\bar{\imath}},$$

where

$$\begin{aligned}
\mathfrak{g}_{\bar{0}} &= \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2),\\
\mathfrak{g}_{\bar{1}} &= V_1 \otimes \textstyle\bigwedge^2 V_2,\\
\mathfrak{g}_{\bar{2}} &= \textstyle\bigwedge^2 V_1 \otimes \bigwedge^4 V_2,\\
\mathfrak{g}_{\bar{3}} &= \textstyle\bigwedge^3 V_1 \otimes V_2,\\
\mathfrak{g}_{\bar{4}} &= \textstyle\bigwedge^4 V_1 \otimes \bigwedge^3 V_2.
\end{aligned}$$

This is a $\mathbb{Z}_5$-graded Lie algebra in a unique way: the exceptional simple Lie algebra of type E_8.

The missing exceptional Jordan grading

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Up to conjugation in $\operatorname{Aut}\mathfrak{g}$, there is a unique order 5 automorphism of the simple Lie algebra $\mathfrak{g}$ of type E_8 such that the dimension of the subalgebra of fixed elements is 48.

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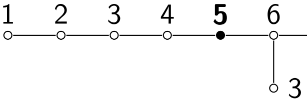

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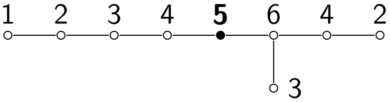

The uniqueness shows us that, up to conjugation, this is the automorphism of $\mathfrak{g}$ such that its restriction to $\mathfrak{g}_{\bar{i}}$ is ξ^i times the identity, where ξ is a fixed primitive fifth root of unity.

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Consider the following automorphisms $\sigma_1, \sigma_2, \sigma_3$ of $\mathfrak{g}$:

$$\begin{aligned}
&\sigma_1(x) = \xi^i x \quad \text{for any } x \in \mathfrak{g}_{\bar{\imath}} \text{ and } 0 \leq i \leq 4,\\
&\sigma_2|_{\mathfrak{g}_{\bar{1}}} = b_1 \otimes \wedge^2 b_2,\\
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where on fixed bases of V_1 and V_2, the coordinate matrices of b_1, c_1, b_2, c_2 are:

$$b_1 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 & 0 \\ 0 & 0 & \xi^2 & 0 & 0 \\ 0 & 0 & 0 & \xi^3 & 0 \\ 0 & 0 & 0 & 0 & \xi^4 \end{pmatrix}, \quad c_1 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$b_2 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 & 0 \\ 0 & 0 & \xi^4 & 0 & 0 \\ 0 & 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & 0 & \xi^3 \end{pmatrix}, \quad c_2 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

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The grading of E_8 induced by the order 5 commuting automorphisms $\sigma_1, \sigma_2, \sigma_3$ is the Jordan grading over $\mathbb{Z}_5^3$.

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There are models of the Jordan gradings of F_4 and E_6 over $\mathbb{Z}_3^3$ constructed along the same lines.

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That's all. Thanks