Computing with rational functions and applications to symmetric functions, invariant theory and PI-algebras

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Let K be a field of characteristic 0 and let R be a K-algebra (associative, Lie, Jordan, etc.) Further we consider the case of associative algebras only, the definitions for other classes of algebras are similar. Let $K\langle X\rangle = K\langle x_1, x_2, \ldots\rangle$ be the free associative algebra (the algebra of polynomials in noncommuting variables).

Definition

The polynomial $f(X) = f(x_1, \dots, x_d) \in K\langle X\rangle$ is a *polynomial identity* for the algebra R if

$$f(r_1, \dots, r_d) = 0 \quad \text{for all } r_1, \dots, r_d \in R.$$

If R satisfies a nontrivial polynomial identity ($f(X) \neq 0$ in $K\langle X\rangle$), then R is called a *PI-algebra* (PI = Polynomial Identity).
The two-sided ideal of $K\langle X\rangle$

$$T(R) = \{f(X) \in K\langle X\rangle \mid f(X) \text{ is a polynomial identity for } R\}$$

is called the *T-ideal of the polynomial identities* of R.

Important:

Since the characteristic of the base field is equal to 0, every T-ideal is generated (as a T-ideal) by its multilinear elements, i.e., the polynomials in

$$P_n = \left\{ \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \alpha_\sigma \in K \right\}.$$

The symmetric group S_n acts on the vector P_n by

$$\tau : x_{\sigma(1)} \cdots x_{\sigma(n)} \to x_{\tau\sigma(1)} \cdots x_{\tau\sigma(n)}, \quad \sigma, \tau \in S_n,$$

and $P_n \cap T(R)$ is an S_n-submodule of P_n.

Definition

The sequence of S_n-characters

$$\chi_n(R) = \chi_{S_n}(P_n/(P_n \cap T(R))) = \sum_{\lambda \vdash n} m_\lambda(R)\chi_\lambda$$

is called the *cocharacter sequence* of the T-ideal of the polynomial identities of R. Here χ_λ is the irreducible S_n-character indexed by the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, and the nonnegative integers $m_\lambda(R)$ are called the *multiplicities* of the cocharacters of R.

Important problem in PI-theory:

Describe the multiplicities $m_\lambda(R)$ for important PI-algebras!

Approach:

$$F_d(\text{var}(R)) = K\langle x_1, \ldots, x_d\rangle / (K\langle x_1, \ldots, x_d\rangle \cap T(R))$$

is the *relatively free algebra of the variety of algebras* $\mathrm{var}(R)$ *generated by* R. It is a $GL_d(K)$-module with the natural action of the general linear group $GL_d(K)$.

$$F_d(\mathrm{var}(R)) = \bigoplus_{\lambda} m'_\lambda(R) W_d(\lambda),$$

where $W_d(\lambda)$ is the irreducible polynomial $GL_d(K)$-module indexed by $\lambda = (\lambda_1, \ldots, \lambda_d)$.

Theorem

(Berele, Drensky, early 1980's) *The multiplicities in the cocharacter sequence and in the* $GL_d(K)$*-module decomposition of* $F_d(\mathrm{var}(R))$ *are the same for all* $\lambda = (\lambda_1, \ldots, \lambda_d)$*:*

$$m'_\lambda(R) = m_\lambda(R).$$

The algebra $F_d(\mathrm{var}(R))$ is a $\mathbb{Z}^d$-graded vector space. The homogeneous component $F_d(\mathrm{var}(R))^{(n_1,\ldots,n_d)}$ of degree $(n_1,\ldots,n_d)$ consists of all polynomials which are homogeneous of degree n_i in x_i. Its *Hilbert series* is

$$H(F_d(\mathrm{var}(R)),t_1,\ldots,t_d)=\sum_{n_i\geq 0}\dim\left(F_d(\mathrm{var}(R))^{(n_1,\ldots,n_d)}\right)t_1^{n_1}\cdots t_d^{n_d}.$$

It is a symmetric formal power series and has the decomposition

$$H(F_d(\mathrm{var}(R)),t_1,\ldots,t_d)=\sum_{\lambda}m_\lambda(R)S_\lambda(t_1,\ldots,t_d),$$

where $S_\lambda(t_1,\ldots,t_d)$ is the Schur function indexed by $\lambda=(\lambda_1,\ldots,\lambda_d)$.

Theorem

(Belov, 1997) *For any PI-algebra R the Hilbert series $H(F_d(var(R)), t_1, \dots, t_d)$ is a rational function of the form*

$$H(F_d(var(R)), t_1, \dots, t_d) = p(t_1, \dots, t_d) \prod \frac{1}{(1 - t_1^{a_1} \cdots t_d^{a_d})^{b_a}},$$

where $a_i \geq 0$, $b_a > 0$ and $p(t_1, \dots, t_d)$ is a polynomial.

Problem.

Given a rational symmetric function $f(t_1, \ldots, t_d)$ of this form, find the multiplicities of the Schur functions in the decomposition

$$f(t_1, \ldots, t_d) = \sum_{\lambda} m_\lambda S_\lambda(t_1, \ldots, t_d).$$

Definition

(Drensky, Genov, 2003)
The generating function

$$M(f; t_1, \ldots, t_d) = \sum_{\lambda} m_\lambda t_1^{\lambda_1} \cdots t_d^{\lambda_d}$$

is called the *multiplicity series of f*.

Definition

(Berele) Rational functions of this kind are called *nice rational functions*.

Nice rational functions appear in many places in mathematics: They play a key role in the theory of linear systems of homogeneous diophantine equations, (Elliott, 1903).
If W is any finitely generated graded module of a finitely generated graded commutative algebra, the Hilbert – Serre theorem gives that the Hilbert series of W is a nice rational function.

Theorem

(Berele, 2006 – 2008) *If $f(t_1,\dots,t_d)$ is a nice rational symmetric function, then the multiplicity series $M(f;t_1,\dots,t_d)$ is also a nice rational function.*

The proof of Berele does not give an algorithm to find $M(f)$.

Problem

Find an efficient algorithm for this purpose.

Lemma

If $f(t_1,\dots,t_d)\in K[[t_1,\dots,t_d]]$ *is a symmetric function and*

$$f(t_1,\dots,t_d)\prod_{1\le i<j\le d}(t_i-t_j)=\sum_{n_i\ge 0}\alpha_n t_1^{n_1}\cdots t_d^{n_d},\quad \alpha_n\in K,$$

then

$$M(f;t_1,\dots,t_d)=\frac{1}{t_1^{d-1}\cdots t_{d-2}^2 t_{d-1}}\sum_{n_i>n_{i+1}}\alpha_n t_1^{n_1}\cdots t_d^{n_d}.$$

Problem

Given a nice rational function

$$f(t_1,\dots,t_d)=\sum_{n_i\ge 0}\alpha_n t_1^{n_1}\cdots t_d^{n_d},$$

find an algorithm to express the sum

$$\sum_{n_1\ge n_2}\alpha_n t_1^{n_1}\cdots t_d^{n_d}.$$

Idea.

Illustration for two variables:

$$f(x,y)=\sum_{i,j\geq 0}\alpha_{ij}x^iy^j,\quad g(x,y)=\sum_{i\geq j}\alpha_{ij}x^iy^j.$$

Consider the Laurent series with respect to t

$$f(xt,\frac{y}{t})=\sum_{m=-\infty}^{\infty}h_m(x,y)t^m,\quad h_m(x,y)\in K[[x,y]].$$

Then

$$r(x,y;t)=\sum_{m=0}^{\infty}h_m(x,y)t^m,\quad g(x,y)=r(x,y;1).$$

Equation of Elliott:

$$\frac{1}{(1-A)(1-B)}=\frac{1}{1-AB}\left(\frac{1}{1-A}+\frac{1}{1-B}-1\right).$$

Applied to one pair

$$\frac{1}{(1-x^a y^b t^c)(1-x^i y^j/t^d)}$$

in the fraction

$$f\left(xt,\frac{y}{t}\right)=p\left(xt,\frac{y}{t}\right)\prod\frac{1}{1-x^a y^b t^c}\prod\frac{1}{1-x^i y^j/t^d}$$

the equation of Elliott replaces the product of one factor of the form $1-x^a y^b t^c$ and one of the form $1-x^i y^j/t^d$ in the denominator of $f(xt,y/t)$ with the sum of three fractions

$$\frac{1}{(1-x^{a+i}y^{b+j}t^{c-d})(1-x^ay^bt^c)},$$

$$\frac{1}{(1-x^{a+i}y^{b+j}t^{c-d})(1-x^iy^j/t^d)}, \quad \frac{1}{1-x^{a+i}y^{b+j}t^{c-d}}.$$

If $c \geq d$, then the first summand contains positive degrees of t only, in the second summand the positive c-th degree of t is replaced with $(c-d)$-th degree and in the third summand we have a nonnegative degree of t only. The case $c < d$ is similar. Continuing in this way, we replace $f(xt, y/t)$ with a linear combination of fractions of the form

$$x^ay^bt^k\prod\frac{1}{1-x^cy^dt^l}, \quad \frac{x^ay^b}{t^k}\prod\frac{1}{1-x^cy^dt^l},$$

$$x^ay^bt^k\prod\frac{1}{1-x^cy^d/t^l}, \quad \frac{x^ay^b}{t^k}\prod\frac{1}{1-x^cy^d/t^l}.$$

Expanded as a formal power series with respect to t, the first type of fractions contain nonnegative degrees of t only; the second summand contains only a finite number of negative degrees of t; the third summand contains a finite number of positive degrees of t; the last summand contains negative degrees of t only. Hence we are able to separate the part with nonnegative degrees of t in the expansion of $f(xt, y/t)$ for the part with negative degrees.
This provides also an algorithm to find the nonnegative part in t of the expansion of $f(xt, y/t)$.

Second algorithm. Example.

$$f(x,y)=\frac{x+y}{(1-xy)(1-x^2y)^2(1-xy^3)},$$

$$f(xt,yt^{-1})=u_1(x,y,t)+u_2(x,y,t)+u_3(x,y,t),$$

$$u_1(x,y,t)=\frac{x^8y^8+3x^5y^5+x^3y^3-1}{xy(1-xy)(1-x^5y^5)^2(1-x^2yt)},$$

$$u_2(x,y,t)=\frac{1+x^3y^3}{xy(1-xy)(1-x^5y^5)(1-x^2yt)^2},$$

$$u_3(x,y,t)=\frac{y(1+x^2y^2)(2x^3y^4+(1+x^5y^5)t)}{(1-xy)(1-x^5y^5)^2(t^2-xy^3)},$$

$$g(x,y,t)=u_1(x,y,t)+u_2(x,y,t)$$

$$=\frac{x(-(x^8y^8+3x^5y^5+x^3y^3-1)t+2xy^2(x^2y^2+1))}{(1-xy)(1-x^5y^5)^2(1-x^2yt)^2}.$$

Applications to PI-algebras

Hilbert series of relatively free algebras which we know or know how to compute:
For $R = E$ (the Grassmann algebra), $R = M_2(K)$ and $R = E \otimes E$.
In these cases we know the multiplicities.
For $T(R) = T(R_1)T(R_2)$, when we know $T(R_1)$ and $T(R_2)$.
Application to upper triangular matrices and block triangular matrices with entries from the Grassmann algebra and 2×2 matrices.

More important case:

Pure and mixed generic trace algebras.

$$x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \quad y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$$

are two generic 3×3 matrices. The pure trace algebra C_{32} is generated by all traces of products $\mathrm{tr}(z_1 \cdots z_k)$, $z_i = x$ or $z_i = y$. Similar result holas for the pure trace algebra C_{nd} generated by d matrices of size $n \times n$.

The Hilbert series of C_{32} is (Teranishi, 1986)

$$H(C_{32}, t, u) = \frac{1 + t^3u^3}{q_1(t,u)q_2(t,u)q_3(t,u)q_4(t,u)}$$

$$q_1(t,u) = (1-t)(1-u), \quad q_2(t,u) = (1-t^2)(1-tu)(1-u^2),$$

$$q_3(t,u) = (1-t^3)(1-t^2u)(1-tu^2)(1-u^3), \quad q_4(t,u) = 1-t^2u^2.$$

Theorem

(Drensky – Genov, 2003) *The multiplicity series of $H(C_{32},t,u)$ is*

$$M(H(C_{32},t,v);t,v) = \frac{1}{(1-v^2)(1-v^3)^2}\times$$

$$\times\left(\frac{(1+v^2+v^4)((1+v^2)(1-t^2v)+2tv(1-v))}{3(1-v)(1-v^2)^3(1-t)^2(1-t^2)}+\right.$$

$$+\frac{(1-v)(1+tv)}{3(1-v^2)(1-t)(1-t^2)}+\frac{(1-v^2)(1-tv)}{3(1-v^3)(1-t^3)}-$$

$$\left.-\frac{v^3((1-v+v^2)(1-t^2v^2)+tv(1-v^2))}{(1-v)(1-v^2)^2(1-v^4)(1-t)(1-t^2)(1-tv)}\right),$$

where $v = tu$.

Other cases when the Hilbert series of C_{nd} is known $(n, d \geq 2)$:
$n = 2$, any d;
$n \leq 6$, $d = 2$; $n = d = 3$.
Multiplicities are known for:
$n = 2$, all λ;
$n = 3$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. In this case the nominator of the multiplicity series has several thousand summands.

Applications

In many cases results from commutative algebra inspire results in noncommutative algebra. Very rarely results in noncommutative algebra have an impact on commutative algebra.
Counterexamples to this statement:
The Nagata – Higman theorem to invariant theory: The polynomial identity $x^n = 0$ implies nilpotency $x_1 \cdots x_N = 0$. The minimal N with this property gives an exact upper bound for the degree of the generators of the algebra C_{nd}.
The theorem of Shestakov – Umirbaev that the Nagata automorphism of $K[x, y, z]$ is wild. The proof involves free Poisson algebras.

The present combinatorial results on the multiplicities are inspired by problems in noncommutative algebra. But they have applications to combinatorics of symmetric functions, representation theory of the general linear group, classical invariant theory, etc.

Combinatorics of symmetric functions

Problem

Let W be a finite dimensional polynomial $GL_d(K)$-module. Find the multiplicities in the symmetric algebra $K[W]$.

Solution. Let

$$W = W_d(\lambda^{(1)}) \oplus \cdots \oplus W_d(\lambda^{(k)}),$$

and let

$$\sum_{i=1}^{k} S_{\lambda_i}(t_1, \dots, t_d) = \sum a_n t_1^{n_1} \cdots t_d^{n_d}, \quad a_n \in \mathbb{N} \cup \{0\}.$$

Then

$$H(K[X], t_1, \dots, t_d) = \prod \frac{1}{(1 - t_1^{n_1} \cdots t_d^{n_d})^{a_n}} = \sum_{\lambda} m_\lambda S_\lambda(t_1, \dots, t_r)$$

and the multiplicity series can be found with our prescriptions.

Examples

$d = 2$, the Hilbert series is $H(K[W], x, y)$, $v = xy$:
$M(H(K[W(3)], x, y); x, y)$

$$= M\left(\frac{1}{(1-x^3)(1-x^2y)(1-xy^2)(1-y^3)}; x, y\right)$$

$$= \frac{1 - vx + v^2x^2}{(1-v^6)(1-x^3)(1-vx)};$$

$$M(H(K[W(4)], x, y); x, y) = \frac{1 - vx^2 + v^2x^4}{(1-v^4)(1-v^6)(1-x^4)(1-vx^2)};$$

$$M(H(K[W(5)],x,y);x,y)$$

$$= \frac{p(x,v)}{(1-v^{10})(1-v^{15})(1-v^{20})(1-x^5)(1-vx^3)(1-v^2x)},$$

$$p(x,v) = (1-v^{15}+v^{30})(1-v^4x^7)$$

$$-v^2x((1-v^5-v^{15})(1+v^5-v^{15})-(1-v^{10}-v^{15})(1+v^{10}-v^{15})x^5)$$

$$+v^4x^2((1-v^{15}+v^{20})-v^6x^3(1-v^5+v^{20}))$$

$$-vx^3(1-v^{10})((1-v^{15}-v^{20})-(1+v^5-v^{20})v^2x);$$

$$M(H(K[W(2)\oplus W(2)],x,y);x,y)$$

$$=M'\left(\frac{1}{(1-x^2)^2(1-xy)^2(1-y^2)^2}\right)=\frac{1+x^2v}{(1-v^2)^3(1-x^2)^2};$$

$$M(H(K[W(3)\oplus W(2)],x,y);x,y)$$

$$=\frac{p(x,v)}{(1-v^2)(1-v^3)(1-v^4)(1-v^6)(1-x^2)(1-x^3)(1-xv)},$$

$$p(x,v)=(1-v^3+v^6)(1-v^2x^4)$$

$$-vx((1-v-v^2-v^3+v^6)-x^2(1-v^3-v^4-v^5+v^6))+v^2(1-v^4)x^2.$$

For $d = 3$, $u = x$, $v = xy$, $w = xyz$:

$$M(H(K[W(3)]); x, y, z) = \frac{1}{q(u, v, w)} \sum_{i=0}^{8} h_i(v, w) u^i,$$

$$q = (1 - w^4)(1 - w^6)(1 - u^3)(1 - u^3 w^2)$$
$$(1 - u^3 w^3)(1 - uv)(1 - v^6)(1 - v^3 w)(1 - w^3 v^3),$$

$$
\begin{aligned}
h_0 &= w^6v^9+1,\\
h_1 &= -v(w^5(w+1)v^9 - w^2(w^2+1)v^6 - w^2(w^3+w^2+1)v^3+1),\\
h_2 &= v^2(w^5v^9 - w^2(w^2+1)v^6 + w^5(w-1)v^3 + (w^4+w^2+1)),\\
h_3 &= -w^2(w^4(w^3+w^2+1)v^9 + v^6 - w(w^3v^3-1)),\\
h_4 &= w^2v(w^4(w^3+2w^2+1)v^9 - w(w^4+w^3+2w^2+w+1)v^6\\
&\quad -w^2(w^4+w^3+2w^2+w+1)v^3 + (w^3+2w+1)),\\
h_5 &= -w^2v^2(w^6v^9 - w^3v^6 + w^7v^3 + w^3+w+1),\\
h_6 &= w^5(w^2(w^4+w^2+1)v^9 - (w-1)v^6 - w^2(w^2+1)v^3 + w),\\
h_7 &= -w^5v(w^6v^9 - w(w^3+w+1)v^6 - w^2(w^2+1)v^3 + w+1),\\
h_8 &= w^5v^2(w^6v^9+1).
\end{aligned}
$$

Invariant theory of the unitriangular group

Unitriangular matrix

$$g = \begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & 1 & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ 0 & 0 & 1 & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Problem

Let W be a finite dimensional polynomial $GL_d(K)$-module. Find the Hilbert series of the algebra of invariants $K[W]^{UT_d(K)}$.

Solution.

Every irreducible polynomial $GL_d(K)$-module $W_d(\lambda)$ has a one-dimensional $UT_d(K)$-invariant. Let

$$H(K[W], t_1, \ldots, t_d; z) = \prod \frac{1}{(1 - t_1^{n_1} \cdots t_d^{n_d} z)^{a_n}}.$$

The extra variable z counts the degree of the invariants with respect to their natural degree (the elements of W are of first degree). Let the corresponding multiplicity function be $M(H[K[W]; t_1, \ldots, t_d)$. Then

$$H(K[W]^{UT_d(K)}, z) = M(H(K[W], 1, \ldots, 1; z).$$

Examples.

$$H(K[W(1)]^{UT_2(K)}, z) = \frac{1}{1-z};$$

$$H(K[W(2)]^{UT_2(K)}, z) = \frac{1}{(1-z)(1-z^2)};$$

$$H(K[W(3)]^{UT_2(K)}, z) = \frac{1+z^3}{(1-z)(1-z^2)(1-z^4)};$$

$$H(K[W(4)]^{UT_2(K)}, z) = \frac{1+z^3}{(1-z)(1-z^2)^2(1-z^3)};$$

$$H(K[W(2) \oplus W(2)]^{UT_2(K)}, z) = \frac{1+z^2}{(1-z)^2(1-z^2)^3};$$

$$H(K[W(3) \oplus W(3)]^{UT_2(K)}, z)$$

$$= \frac{1+3z^2+6z^3+6z^4+6z^5+6z^6+6z^7+3z^8+z^{10}}{(1-z)^2(1-z^2)^2(1-z^4)^3};$$

$d = 3$: $H(K[W(3)]^{UT_3(K)}, z)$

$$= \frac{1 + z^3 + 2z^4 + 3z^5 + 3z^6 + 3z^7 + 2z^8 + z^9 + z^{12}}{(1-z)(1-z^2)(1-z^3)^2(1-z^4)^2(1-z^5)};$$

$$H(K[W(2,1)]^{UT_3(K)}, z) = \frac{1 + z^2 + z^4}{(1-z)(1-z^2)(1-z^3)^3};$$

$$H(K[W(1,1,1)]^{UT_3(K)}, z) = \frac{1}{1-z}.$$

Invariant theory of the special linear group

Problem

Let W be a finite dimensional polynomial $GL_d(K)$-module. Find the Hilbert series of the algebra of invariants $K[W]^{SL_d(K)}$.

This is a classical problem. The first results are due to Cayley and Sylvester. For example, Sylvester and Franklin, 1879, computed these series for $d = 2$, $\lambda = (n, 0)$, $n \leq 10$, $n = 12$.
Solution. The irreducible polynomial $GL_d(K)$-module $W_d(\lambda)$ has a one-dimensional $SL_d(K)$-invariant if and only if $\lambda_1 = \cdots = \lambda_d$. Hence in the multiplicity series of $H(K[W], t_1, \ldots, t_d; z)$ we have to count only such partitions.

Examples

$d = 2$

$$H(K[W(1)]^{SL_2(K)}, z) = 1;$$

$$H(K[W(2)]^{SL_2(K)}, z) = \frac{1}{1 - z^2};$$

$$H(K[W(3)]^{SL_2(K)}, z) = \frac{1}{1 - z^4};$$

$$H(K[W(4)]^{SL_2(K)}, z) = \frac{1}{(1 - z^2)(1 - z^3)};$$

$$H(K[W(5)]^{SL_2(K)}, z) = \frac{1 - z^6 + z^{12}}{(1 - z^4)(1 - z^6)(1 - z^8)};$$

$$H(K[W(2) \oplus W(2)]^{SL_2(K)}, z) = \frac{1}{(1 - z^2)^3};$$

$$H(K[W(3) \oplus W(3)]^{SL_2(K)}, z) = \frac{(1 - z^2 + z^4)(1 + z^4)}{(1 - z^2)^2(1 - z^4)^3};$$

$$H(K[W(3)]^{SL_3(K)}, z) = \frac{1}{(1-z^4)(1-z^6)};$$

$$H(K[W(2,1)]^{SL_3(K)}, z) = \frac{1}{(1-z^2)(1-z^3)};$$

$$H(K[W(1,1,1)]^{SL_3(K)}, z) = \frac{1}{1-z}.$$