Answer Set Programming

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Outline

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- (Propositional) rules with negation in the body.

$$\underbrace{p}_{head} \leftarrow \underbrace{L_1, \ldots, L_n}_{body}$$

$n \geq 0$, p is an atom and L_i are literals, that is, an atom q or its default negation *not q*.

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- The ordering is irrelevant. We can generally write the rule as:

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- The rule is positive when $m = n$ (no negations).
- When $n = 0$, the rule is called a fact, and we usually omit the $\leftarrow$.

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- Example: given P below, $T_P(\{b, p, s\}) = \{p, q, r, a\}$

$$\begin{array}{lll} p & & \\ q & & \\ r & \leftarrow & p, s \end{array} \qquad \begin{array}{lll} s & \leftarrow & q \\ a & \leftarrow & b, p \end{array} \qquad \begin{array}{lll} b & \leftarrow & s, a \\ a & \leftarrow & c \end{array}$$

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$$q \qquad a \leftarrow b, p \qquad a \leftarrow c$$
$$r \leftarrow p, s$$

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Positive programs

- A set of atoms I is a model of a program P, $I \models P$, when $I \models q_1 \wedge \ldots q_m \wedge \neg q_{m+1} \wedge \cdots \wedge \neg q_n \rightarrow p$ for any rule (1) in P.

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the models of P are $\{p, q, r, s\}$, $\{p, q, r, s, a, b\}$, $\{p, q, r, s, a, b, c\}$.

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  the models of P are $\{p, q, r, s\}$, $\{p, q, r, s, a, b\}$, $\{p, q, r, s, a, b, c\}$.
- Exercise: prove it.

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- However, this rule is classically equivalent to $q \vee p$ and has three models: $\{p, q\}, \{p\}, \{q\}$, being the last two minimal.
- Furthermore, $q \vee p$ is also equivalent (in classical logic) to:

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  whose "expected" behavior should be obviously different.

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Assume, say, *not* q … q, our assumption was inconsistent.

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Assume now *not* p . . . q, and the first two rules become redundant.

Adding negation: stable models

- Gelfond, M., and Lifschitz, V. 1988. The stable model semantics for logic programming. In ICLP'88, 1070-1080.

Definition (program reduct)

We define the reduct of a program P with respect to an interpretation (set of atoms) I, written P^I, as the set of rules:

$$P^I \stackrel{def}{=} \{ \quad (p \leftarrow q_1, \ldots, q_m) \mid (p \leftarrow q_1, \ldots, q_m, not\ q_{m+1}, \ldots, not\ q_n) \in P \text{ and } q_j \notin I, \text{ for all } j = m+1, \ldots, n \}$$

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Definition (stable model)

An interpretation I is a stable model of a program P iff $\boxed{I = \Gamma_P(I) = LM(P^I)}$. □

Stable models: some properties

Proposition (Stable models are models)

If I is a stable model of P then $I \models P$.

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Exercise: prove the above theorems.

Stable models

- An example of default. Try this program:

$$\begin{aligned} \mathit{flies} &\leftarrow \mathit{bird}, \mathit{not}\ \mathit{ab} \\ \mathit{bird} & \end{aligned}$$

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- This program has these three models:

I	P^I	$LM(P^I)$
$\{bird, ab\}$		
$\{bird, ab, flies\}$		
$\{bird, flies\}$		

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$\{bird, flies\}$	$flies \leftarrow bird$ $bird$	

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$\{bird, flies\}$	$flies \leftarrow bird$ $bird$	$\{bird, flies\}$ stable!

Stable models

- Adding new information:

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Stable models: some properties

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- Typically use: (1) generate multiple solutions (even cycles like P_1) and (2) prune undesired models (odd cycles like P_2).
- Constraints. Example: to avoid a model where p holds but q doesn't:

$$aux \leftarrow p, not\ q, not\ aux$$

where aux is a new fresh atom. Usually written: $\leftarrow p, not\ q$

Stable models vs Default Logic

- Very close to Default Logic. A rule like:

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just corresponds to the default:

$$\frac{q_1 \wedge \cdots \wedge q_m : \neg q_{m+1}, \ldots, \neg q_n}{p}$$

- So, it's like playing with defaults where we mostly deal with atoms.

Stable models vs answer sets

- We can sometimes be interested in a second negation, strong or explicit negation (originally called "classical"). Example:

  $$cross \leftarrow not\ train$$

  risky! we cross the railway tracks when no information on train approaching is available. Compare to:

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- Stable models with strong negation are called answer sets. We just compute stable models and reject those where p, $-p$ occur.

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- The well-founded model (WFM) of P is a three-valued interpretation such that:
  - atoms in $lfp(\Gamma_P^2)$ are called well-founded;
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- A. Van Gelder, K.A. Ross and J.S. Schlipf. The Well-Founded Semantics for General Logic Programs. Journal of the ACM 38(3) pp. 620—650, 1991.

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Corollary

If there are no undefined atoms, then $lfp(\Gamma_P^2)$ is the only stable model of P.

- Checking whether P has a stable model is an NP-complete problem [Eiter & Gottlob 93]. Computing $WFM(P)$ takes polynomial time (quadratic).

Well-founded model

- Computing the WFM: when P is finite, we can just iterate Γ^2_P on $\emptyset$.
  Example: try with program P_3

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- When we exhaust these rules, we get the program remainder.

Proposition

The facts of the program remainder are the well-founded atoms; the non-head atoms are the unfounded atoms.

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- Fitting's model=$\langle\{c\},\{g,h\}\rangle$. The final program remainder is:

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and so, $WFM = \langle \{c, f\}, \{g, h, d, e\} \rangle$. Stable models $\{c, f, a\}$ and $\{c, f, b\}$.

ASP

- Most ASP solvers (DLV, smodels, clasp) alternate computation of WFM and nondeterministic choice with backtracking.
  1. Compute the WFM
  2. If no undefined atoms: stable model found.
  3. Else: select an undefined atom *p* (using some heuristics) and branch: *p*; *not p*. Simplify the program accordingly to the choice and go to 1.

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- [K. L. Clark 1978] $COMP[P]$ is the classical theory consisting of:

$$p \leftrightarrow B_1 \vee \cdots \vee B_n$$

  for each atom p, and all rules $p \leftarrow B_i$ in P.

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An empty disjunction is $\bot$.

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- Example: let *P* be the program

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$$

whose only model is $\{p, q, t\}$.

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- Semantic counterpart: supported models.

Definition (Supported model)

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$T_P(\{p, q, t\}) = \{p, q, t\}$ (supported), whereas, for instance $T_P(\{p, s\}) = \{p, t\}$ (non-supported).

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- Yes, supported and stable models coincide! Is this general?

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- No: they differ in positive cycles. Consider this extremely simple example:

$$p \leftarrow p$$

Completion would be $p \leftrightarrow p$ which has two supported models, $\emptyset$ and $\{p\}$. Only $\emptyset$ is stable.

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- Tight programs [Fages 1994][Babovich, Erdem, Lifschitz 2000].

Definition (Tight programs)

P is tight on a set I of atoms if there is some partial ordinal mapping $\lambda : X \rightarrow N$ such that all $\lambda(B_i) < \lambda(H)$ for any rule in P like:

$$H \leftarrow B_1, \ldots, B_n, not\ C_1, \ldots, not\ C_m$$

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Theorem

If I is supported model of P and P tight on I, then I is stable model of P.

Clark's completion

- Example:

$$p \leftarrow not\ q \qquad q \leftarrow not\ p \qquad r \leftarrow r \qquad p \leftarrow r$$

the completion

$$p \leftrightarrow \neg q \vee r \qquad q \leftrightarrow \neg p \qquad r \leftrightarrow r$$

has models $\{p\}$, $\{q\}$ and $\{p, r\}$, but $\{p, r\}$ is not tight - only the first 2 ones are stable.

Loop formulas

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- A loop L is a set of atoms forming a Strongly Connected Component in G.
- Given a loop $L = \{p_1, \ldots, p_n\}$ its loop formula $LF(L)$ is defined as:

$$\neg(BB_1 \vee \cdots \vee BB_n) \rightarrow \neg p_1 \wedge \cdots \wedge \neg p_n$$

where BB_i is the disjunction $B_1 \vee \cdots \vee B_{m_i}$ of all bodies for rules in P like

$$p_i \leftarrow B_j$$

such that not atom in L occurs in the positive body of B_j.

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Completion $COMP[P]$ is:

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- Loops $L_1 = \{a, b\}$ and $L_2 = \{c, d\}$. Loop Formulas:

$$LF(L_1) : c \rightarrow \neg a \wedge \neg b$$
$$LF(L_2) : a \rightarrow \neg c \wedge \neg d$$

Adding them to $COMP[P]$ leaves $\{a, b\}$ and $\{c, d\}$ as only stable models.

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- This restriction is being overcome:
  - `lparse` allows functors, but their nesting is limited (no lists, for instance).
  - More recently, `DLV complex` allows functions (lists, sets, etc) for finitely ground programs, a class of programs with finitely many answer sets that are finite.

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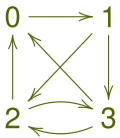
- We have the extensional database describing the graph

```
node(0).  node(1).  node(2).  node(3).
edge(0,1).  edge(1,2).  edge(1,3).
edge(2,0).  edge(2,3).  edge(3,2).  edge(3,0).
```



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```

- Only one outgoing node, only one incoming node:

```
:- in(X,Y), in(X,Z), Y!=Z.
:- in(X,Z), in(Y,Z), X!=Y.
```

Introducing variables

- Predicate `in(X,Y)` points out that and edge `X` $\longrightarrow$ `Y` is in the cycle. We generate arbitrary choices with an auxiliary predicate `out`.

```
in(X,Y)  :- edge(X,Y), not out(X,Y).
out(X,Y) :- edge(X,Y), not in(X,Y).
```

- Only one outgoing node, only one incoming node:

```
:- in(X,Y), in(X,Z), Y!=Z.
:- in(X,Z), in(Y,Z), X!=Y.
```

- Disregard disconnected cycles. We use a predicate `reached(X)` meaning that `X` can be reached from an arbitrary fixed node, say 0.

```
reached(X) :- in(0,X).
reached(Y) :- reached(X), in(X,Y).
```

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```
reached(X) :- in(0,X).
reached(Y) :- reached(X), in(X,Y).
```

  and we forbid unreached nodes:

```
:- node(X), not reached(X)
```

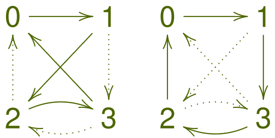
Introducing variables

- Making the call:

  ```
  lparse -n 0 hamilt.txt | smodels
  ```

  We obtain two answers:

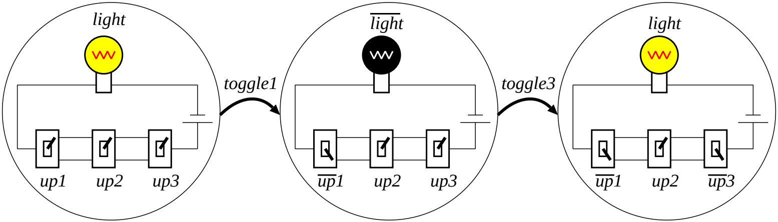
  ```
  Answer: 1
  Stable Model: in(0,1) in(3,0) in(2,3) in(1,2)
  Answer: 2
  Stable Model: in(0,1) in(3,2) in(2,0) in(1,3)
  False
  ```



Answer 1 Answer 2

Reasoning about actions with ASP

- An example of action domain.



Reasoning about actions with ASP

- We begin with some "type declarations".

```
time(0..pathlength).
previoustime(0..pathlength-1).
switch(1..3).
#domain previoustime(I).
#domain time(J).
#domain switch(X).
#domain switch(Y).
```

Reasoning about actions with ASP

```
% Effect axioms
up(X,true,I+1)    :- up(X,false,I), toggle(X,I).
up(X,false,I+1)   :- up(X,true,I), toggle(X,I).
light(true,I+1)   :- light(false,I), toggle(X,I).
light(false,I+1) :- light(true,I), toggle(X,I).

% Inertia
up(X,true,I+1)   :- up(X,true,I), not up(X,false,I+1).
up(X,false,I+1) :- up(X,false,I), not up(X,true,I+1).

light(true,I+1)   :- light(true,I),
                     not light(false,I+1).
light(false,I+1) :- light(false,I),
                     not light(true,I+1).
```

Reasoning about actions with ASP

```
% Constraints: unique value
:- up(X,true,J), up(X,false,J).
:- light(true,J), light(false,J).

% Unique action
:- toggle(X,I), toggle(Y,I), X!=Y.
```

Reasoning about actions with ASP

Prediction example

```
% switches-predict.txt
% Initial state
light(true,0).     up(X,true,0).

% Performed actions
toggle(1,0).
```

Calling lparse/smodels with

```
lparse -c pathlength=1 switches.txt
   switches-predict.txt | smodels
```

we get ...

```
Answer: 1
Stable Model: up(1,false,1) up(2,true,1) up(3,true,1)
light(false,1) ...
```

Reasoning about actions with ASP

Postdiction example:

```
% switches-postdict.txt

% Actions generation
1 { toggle(Z,I) : switch(Z) } 1.
       % generate 1 toggle among all switches Z

% Completing facts about the initial situation
1 {up(X,true,0), up(X,false,0)} 1.
1 {light(true,0), light(false,0)} 1.

% Observations
up(3,true,0). light(true,0). toggle(3,1).
light(false,1). up(1,false,1). up(3,true,1).
```

Reasoning about actions with ASP

Calling lparse with

```
lparse -c pathlength=1 -n 0 switches.txt
   switches-postdict.txt | smodels
```

we get 6 possible explanations. One of them:

```
Answer: 1
Stable Model: toggle(1,0) up(2,false,0) up(1,true,0)
up(2,false,1) ...
```

Reasoning about actions with ASP

Planning example

```
% switches-plan.txt
% Planning problem

% Actions generation
1 { toggle(Z,I) : switch(Z) } 1.

% Initial state
light(true,0). up(X,true,0).

% Goal state
goal :- light(true,pathlength),up(1,false,pathlength),
        up(2,true,pathlength), up(3,false,pathlength).

:- not goal.
```

Reasoning about actions with ASP

Calling lparse with

```
lparse -c pathlength=1 -n 0 switches.txt
   switches-plan.txt | smodels
```

We don't get models. After increasing `pathlength`

```
lparse -c pathlength=2 -n 0 switches.txt
   switches-plan.txt | smodels
```

we get 2 possible plans

```
Answer: 1
Stable Model: toggle(1,0) toggle(3,1) ...
Answer: 2
Stable Model: toggle(3,0) toggle(1,1) ...
```

Extending the syntax

- Disjunctive programs: bodies B as before, but heads allow disjunctions of atoms:

$$p_1 \vee \cdots \vee p_n \leftarrow B$$

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$$p \vee q \leftarrow t, not\ s \qquad\qquad t \leftarrow not\ q$$

Given $I = \{p, t\}$, P^I is the program:

$$p \vee q \leftarrow t \qquad\qquad t \leftarrow$$

whose minimal models are $\{p, t\}$ (stable) and $\{q, t\}$ (non-stable).

Extending the syntax

- The definition is adapted accordingly

Definition (stable model)

I is a stable model of a disjunctive program P if it is a minimal model of P^I.

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- Finding a stable model of a disjunctive program is slightly more complex: Σ_2^P-complete.
- Tools for disjunctive ASP: `DLV`, `GnT`, `cmodels`.

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- Adding default negation in the head [Inoue & Sakama 98]. Rules $H \leftarrow B$ where:
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$$b \vee not\ a \vee not\ d \quad \leftarrow \quad d, not\ e, not\ h$$

$B^- = \neg e \wedge \neg h$ and $\neg H^- = \neg(\neg a \vee \neg d) = (a \wedge d)$.
As $I \models B^- \wedge \neg H^-$, its reduct would correspond to:

$$b \quad \leftarrow \quad h$$

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  becomes the program:

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- But, which is the semantics for $not\ (a \leftarrow b)$ or $a \leftarrow (b \leftarrow c)$?

Equilibrium Logic

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  1. A non-classical monotonic (intermediate) logic called Here-and-There (HT)
  2. A selection of (certain) minimal models that yields nonmonotonicity

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- Satisfaction of formulas
  - $\langle H, T \rangle \models p$ if $p \in H$ (for any atom p)
  - $\wedge, \vee$ as always
  - $\langle H, T \rangle \models \varphi \rightarrow \psi$ if both
    - $\langle H, T \rangle \models \varphi$ implies $\langle H, T \rangle \models \psi$
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    - $\langle T, T\rangle \models \varphi$ implies $\langle T, T\rangle \models \psi$

      This is the same than $T \models \varphi \rightarrow \psi$ in classical logic.
  - Negation $\neg F$ is defined as $F \rightarrow \bot$

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- $\langle H, T \rangle \models \Gamma$ implies $T \models \Gamma$.
- $\langle H, T \rangle \models \neg\varphi$ iff $T \not\models \varphi$ in classical logic.

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- Given $M = \langle H, T \rangle$, we can define a 3-valued mapping $M : Atoms \mapsto \{0, 1, 2\}$ reading:

  2 = $(p \in H)$ = true
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- $\wedge$ returns minimum value, $\vee$ returns maximum and $M(\phi \rightarrow \psi) = 2$ if $M(\phi) \leq M(\psi)$ or returns $M(\psi)$ otherwise.

Equilibrium models

Definition (Equilibrium model)

$\langle T, T \rangle$ is an equilibrium model of a theory Γ if:
$\langle T, T \rangle \models \Gamma$, and there is no $H \subset T$ such that $\langle H, T \rangle \models \Gamma$.

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- Logical techniques available: e.g., methods from many-valued semantics (tableaux, signed logics,. . .)

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- Captures all previous syntax extensions, plus other non-propositional constructions:
  - weight constraints can be represented as nested expressions [Ferraris, Lifschitz 2005];
  - aggregates represented by rules with embedded implications [Ferraris 2004].
  - ordered disjunction from [Brewka et al 2004] (LPOD) can also be captured [Cabalar 2010].

Equilibrium logic

Other interesting features

- In nonmonotonic reasoning, we talk about strong equivalence of Γ_1, Γ_2 when, for any Π:
  $\Gamma_1 \cup \Pi$ and $\Gamma_2 \cup \Pi$ have the same (selected) models.

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  $\Gamma_1 \cup \Pi$ and $\Gamma_2 \cup \Pi$ have the same (selected) models.
- Γ_1, Γ_2 are strongly equivalent iff they are equivalent in HT [Lifschitz et al 2001].

Equilibrium logic

Other interesting features

- Disjunctive programs with negation in the head are a (conjunctive) normal form (CNF) for Equilibrium Logic. [Cabalar & Ferraris 2007].

Theorem

The number of different logic programs (modulo strong equivalence) that can be built for a finite signature of n atoms is:

$$\prod_{i=0}^{n} \left(2^{2^{i}-1}+1\right)^{\binom{n}{i}}$$

With $n = 2$ we get 162, with $n = 3$ around 5 million.

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With $n = 2$ we get 162, with $n = 3$ around 5 million.

- Transformations into this CNF [Cabalar, Pearce & Valverde 2005].

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- Introduction of partial functions [Cabalar 2008].
- Linear temporal equilibrium logic [Cabalar & Pérez 2007].
- Equivalent to the extension of reduct [Ferraris 2005] for arbitrary propositional theories, and general stable model [Ferraris, Lee & Lifschitz 2007] for first order theories.

Minimal logic programs