Valid inequalities for the Integer Knapsack Cover polyhedron with setup constraints*

Natália S. Rodrigues[1][0000−0002−3993−8667] and Socorro Rangel[2][0000−0003−3910−8804]

[1] Universidade Estadual de Campinas, Campinas, Brazil
nataliaswrodrigues@gmail.com
[2] Universidade Estadual Paulista, São José do Rio Preto, Brazil
socorro.rangel@unesp.br

Abstract. The Integer Knapsack Cover (IKC) polyhedron can be identified as a substructure of the polyhedron associated to the cutting stock problem (CSP) in the case that the demand can be satisfied with overproduction. Some families of valid inequalities for the IKC polyhedron have been proposed in the literature, however we are particularly interested in valid inequalities for the IKC when setup constraints are also considered (IKC-S). In this paper, we present some theoretical results about the facets of the IKC-S polyhedron. Strong valid inequalities for the IKC-S can be useful to improve cutting plane methods for the CSP with setup constraints.

Keywords: integer knapsack cover problem · setup constraints · valid inequalities.

1 Introduction

The Integer Knapsack Cover set is defined as $X = \{x \in Z^n_+ : \sum_{i \in N} a_i x_i \geq b\}$. We assume that $a_1, \ldots, a_n, b$ are positive integers satisfying $a_1 < a_2 < \ldots < a_n < b$ (see [4]). The inequality defining X is called *cover constraint*.

Let m and M be fixed parameters such that $m < b < M$. We are interested in the convex hull of the sets (1) and (2). These sets appears as a substructure of the cutting stock problem (CSP) polyhedron if the demand can be satisfied with overproduction and setup constraints are considered (X_S) and minimum lot sizes are also imposed (X_{SM}) [3].

$$X_S = \{(x, y) : x \in X, y \in B^n, x_i \leq M y_i, \forall i \in N\} \tag{1}$$

$$X_{SM} = \{(x, y) : x \in X, y \in B^n, m y_i \leq x_i \leq M y_i, \forall i \in N\} \tag{2}$$

* Many thanks to FAPESP (2017/26969-9, 2016/01860-1 and 2013/07375-0).

2 On the dimension of conv(X_S) and conv(X_{SM})

Mazur [1] first proved that conv(X) is full-dimensional. We extend this result for both conv(X_S) and conv(X_{SM}). To prove that conv(X_{SM}) is full-dimensional, as stated in Proposition 1, we choose some set of $2n+1$ points satisfying the cover constraint and show that they are affinely independent. To guarantee the result, M must be large enough so that these points belong to X_{SM}. Since $X_{SM} \subset X_S$, it follows from Proposition 1 that conv(X_S) is also full-dimensional for sufficiently large M.

Proposition 1 *For sufficiently large M, conv(X_{SM}) is full-dimensional.*

3 On the facets of conv(X_S) and conv(X_{SM})

Yaman [4] develops a variety of valid inequalities for conv(X), some of them based on general procedures such as Chvátal-Gomory and *lifting* [2]. In particular, valid inequalities for conv(X) are also valid for conv(X_S) and conv(X_{SM}), even though they are not guaranteed to be efficient.

For the polyhedron conv(X_S), we introduce a class of valid inequalities. We claim that for each $j \in N$, the inequality (3) is valid for conv(X_S).

$$\sum_{i \in N \setminus \{j\}} y_i + \frac{1}{\lceil b/a_j \rceil} x_j \geq 1 \tag{3}$$

Indeed, by the assumption $b > 0$, at least one item must be used so the demand can be satisfied. Fix one item j. If some other item $i \in N \setminus \{j\}$ is used, than $y_i = 1$ and inequality (3) holds. If none of the items $i \in N \setminus \{j\}$ are used, the only item left is j and we need at least ($\lceil b/a_j \rceil$) copies of j to satisfy the demand. In this case, (3) also holds.

Regarding the facets of the polyhedra conv(X_S) and conv(X_{SM}), by choosing an appriated set of affine points we can prove Theorem 1 and 2.

Theorem 1. *For each $j \in N$, the valid inequality (3) is facet-defining for conv(X_S).*

Theorem 2. *For each $j \in N$, the inequality $x_j \geq m y_j$ is facet-defining for conv(X_{SM}).*

References

1. Mazur, D. R., *Integer Programming Approaches to a Multi-Facility Location Problem*, Ph.D. Thesis, John Hopckins University, Baltimore, MD, 1999
2. Nemhauser, G. L; Wolsey, L. A., *Integer and Combinatorial Optimization*, John Wiley & Sons, New York, 1988
3. Toscano, A., Rangel, S., Yanasse, H.H., A heuristic approach to minimize the number of saw cycles in small-scale furniture factories. Ann Oper Res, **258**:719–746 (2017) DOI 10.1007/s10479-015-1955-9
4. Yaman, H., The integer knapsack cover polyhedron, *SIAM Journal on Discrete Mathematics* 21 (2007) 551-572