Linial's Conjecture for Matching-Spine Digraphs*

Jadder Bismarck[1][0000−0003−1729−5856], Cândida Nunes da Silva[2][0000−0002−4649−0274], and Orlando Lee[1][0000−0004−4462−3325]

[1] Universidade Estadual de Campinas, Campinas SP, Brazil
{bismarck, lee}@ic.unicamp.br
[2] Universidade Federal de São Carlos, Sorocaba SP, Brazil
candida@ufscar.br

Abstract. In 1981, Linial conjectured that for every positive integer k, the k-norm of a k-optimal path partition of a digraph D is at most the weight of an optimal partial k-coloring of D. In this work, we present some partial results on this conjecture for the class of matching-spine digraphs.

Keywords: graph theory · digraph · path partition · stable set.

1 Introduction

For a digraph D, let $V(D)$ denote its set of vertices and let $A(D)$ denote its set of arcs. Let P be a path. We denote by $V(P)$ the set of vertices of P. The order of P, denoted by $|P|$, is the number of its vertices and we denote by $\text{ter}(P)$ the terminal vertex v_ℓ of P.

A *path partition* $\mathcal{P}$ of a digraph D is a set of disjoint paths which cover $V(D)$. Let $\pi(D)$ denote the cardinality of a minimum path partition of D. Given a positive integer k, the *k-norm* of $\mathcal{P}$ is defined as $\sum_{P\in\mathcal{P}} \min\{|P|, k\}$. A path partition of minimum k-norm is called *k-optimal* and its k-norm is denoted by $\pi_k(D)$. Note that $\pi(D) = \pi_1(D)$.

A *stable set* S in a digraph D is a subset of vertices of $V(D)$ such that no two vertices of S are adjacent. Let $\alpha(D)$ denote the cardinality of a maximum stable of D. Let k be a positive integer. A *partial k-coloring* $\mathcal{C}$ of D is a set of k disjoint stable sets. The *weight* of $\mathcal{C}$ is defined as $\sum_{C\in\mathcal{C}} |C|$. A partial k-coloring of maximum weight is called *optimal* and its weight is denoted by $\alpha_k(D)$. Note that $\alpha(D) = \alpha_1(D)$.

In 1950, Dilworth [1] proved that the equality $\pi(D) = \alpha(D)$ holds when D is a transitive acyclic digraph. In 1960, Gallai and Milgram [2] generalized Dilworth's Theorem to arbitrary digraphs establishing that $\pi(D) \leq \alpha(D)$ for every digraph D. Later, in 1976, Greene and Kleitman [3] generalized Dilworth's

* This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. The third author was supported by CNPq Proc. 303766/2018-2, CNPq Proc 425340/2016-3 and FAPESP Proc. 2015/11937-9.

theorem, showing that for every transitive acyclic digraph D and every positive integer k, we have $\pi_k(D) = \alpha_k(D)$. In 1981, Linial [4] conjectured that one could generalize Greene-Kleitman's Theorem to arbitrary digraphs as follows.

Conjecture 1 (Linial [4]) *For every digraph D and every positive integer k, we have $\pi_k(D) \leq \alpha_k(D)$.*

2 Matching-Spine Digraphs

We say that $D[X, Y]$ is a *matching-spine* digraph if $D[X]$ has a Hamilton path and the arc set of $D[Y]$ is a matching. In this work we give partial results on the validity of Linial's Conjecture for matching-spine digraphs. The strategy involves finding, for an arbitrary matching-spine digraph, a path partition and a partial k-coloring that give an upper bound for the k-norm and a lower bound for the weight of an optimal path partition and partial k-coloring, respectively.

We define a *canonical path partition* $\mathcal{P}$ of a digraph D as any Hamilton path P of $D[X]$ together with all maximal paths of $D[Y]$; clearly $\pi_k(D) \leq |Y| + \min\{|X|, k\}$. Consider the partition of the vertices of Y into sets Y^0, Y^+, Y^- such that Y^0 contains the isolated vertices, Y^+ the sources and Y^- the sinks in $D[Y]$. We define a *canonical partial k-coloring* $\mathcal{C}$ as the stable sets $Y^0 \cup Y^-$, Y^+ and $\min\{|X|, k-2\}$ singletons of X; clearly $\alpha_k(D) \geq |Y| + \min\{|X|, k-2\}$. Therefore $\pi_k(D) \leq \alpha_k(D) + 2$.

We split the class of matching-spine digraphs into two classes: *k-loose* and *k-tight* digraphs. For the former class, we can show that $\pi_k(D) \leq \alpha_k(D)$ and so Linial's conjecture follows in this case. Our main result is to prove that if D is in the latter class, then $\pi_k(D) \leq |Y| + \min\{|X|, k-1\}$. The technique used in this proof is a (non-trivial) extension of that used in [5] to prove Linial's Conjecture for spine (a subclass of matching-spine) digraphs. Specifically, it involves finding a pair of paths whose union contains $|X| + k + 1$ vertices, thus providing a path partition of k-norm $|Y| + \min\{|X|, k-1\}$. So, when $\alpha_k(D) \leq |Y| + \min\{|X|, k-1\}$, Linial's conjecture also holds. The remaining case occurs when $\alpha_k(D) \leq |Y| + \min\{|X|, k-2\}$ and the conjecture is still open. We conjecture that in this case, $\pi_k(D) \leq |Y| + \min\{|X|, k-2\}$, which would settle Linial's conjecture for matching-spine digraphs.

References

1. Dilworth, R.P.: A decomposition theorem for partially ordered sets. Annals of Mathematics pp. 161–166 (1950)
2. Gallai, T., Milgram, A.: Verallgemeinerung eines graphentheoretischen satzes von Rédei. Acta Sc. Math **21**, 181–186 (1960)
3. Greene, C., Kleitman, D.J.: The structure of Sperner k-families. Journal of Combinatorial Theory, Series A **20**(1), 41–68 (1976)
4. Linial, N.: Extending the Greene-Kleitman theorem to directed graphs. Journal of Combinatorial Theory, Series A **30**(3), 331–334 (1981)
5. Sambinelli, M., da Silva, C.N., Lee, O.: On Linial's conjecture for spine digraphs. Discrete Mathematics **340**(5), 851–854 (2017)